Power indices based on ordinal games

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Aggregation of ordinal data
and
open problems related to possibilistic approximation of set functions

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1. Introduction

- Set of players $N = \{1, \dots, n\}$
- Game $v : \mathcal{P}(N) \to \mathbb{R}$

  Particular cases:

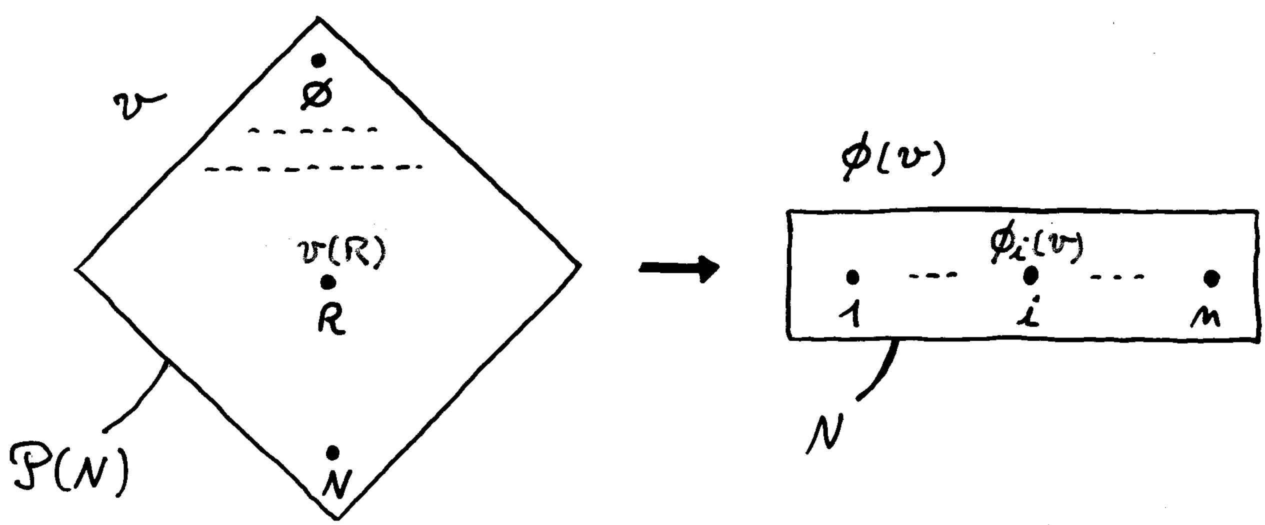
  - Probabilistic game p:

    $p(\emptyset) = 0$, $p(R) = \sum_{i \in R} p(i) \quad \forall R \subseteq N$
  - Possibilistic game π:

    $\pi(\emptyset) = 0$, $\pi(R) = \bigvee_{i \in R} \pi(i) \quad \forall R \subseteq N$
- Power indices $\phi : N \to \mathbb{R}$

1.1 Cardinal context

Given a game v, we are searching for power indices $\phi(v)$.



1.2 Ordinal context

$$v' \equiv v \overset{def}{\iff} \left(v'(R) \geq v'(S) \iff v(R) \geq v(S) \ \forall R, S \subseteq N \right)$$

$$\iff \exists\, \psi : \mathbb{R} \to \mathbb{R} \text{ strictly } \nearrow \text{ s.t. } v' = \psi(v)$$

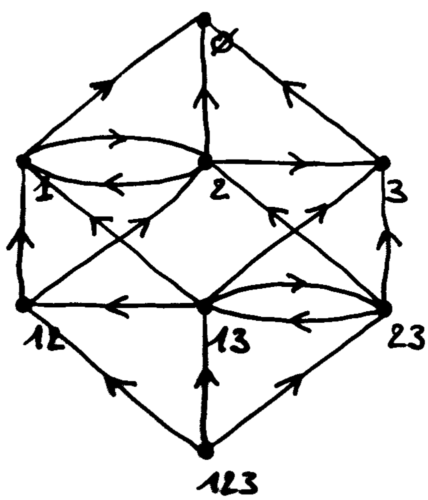
An ordinal game v is defined by a total preorder V on $\mathcal{P}(N)$.
(i.e. a strongly complete and transitive relation V on $\mathcal{P}(N)$)

$$\underset{R}{\overset{v(R)}{\bullet}} \longrightarrow \underset{S}{\overset{v(S)}{\bullet}}$$

Representative Boolean matrix V:

$$V(R,S) = \begin{cases} 1 & \text{if } v(R) \geq v(S) \\ 0 & \text{else} \end{cases}$$

For instance:

V: 

$$V: \begin{array}{c|cccccccc} & \emptyset & 1 & 2 & 12 & 3 & 13 & 23 & 123 \\ \hline \emptyset & 1 & & & & & & & \\ 1 & 1 & 1 & 1 & & 1 & & & \\ 2 & 1 & 1 & 1 & & 1 & & & \\ 12 & 1 & 1 & 1 & 1 & 1 & & & \\ 3 & 1 & & & & 1 & & & \\ 13 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 23 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 123 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

Example : Ordinal possibilistic game

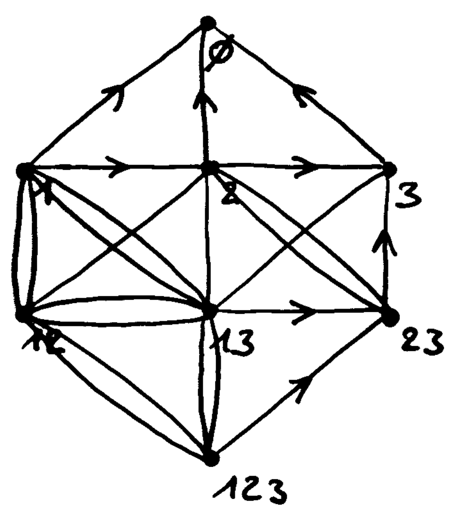
$$\pi(\emptyset) = 0$$
$$\pi(R) = \bigvee_{i \in R} \pi(i)$$

$$\Downarrow$$

- $\Pi(\emptyset, i) = 0 \quad \forall i \in N \qquad (\text{i.e. } \pi(i) > \pi(\emptyset))$
- $\Pi(R,S) = \begin{cases} 1 & \text{if } \bigvee_{i \in R} \pi(i) \geq \bigvee_{i \in S} \pi(i) \\ 0 & \text{else} \end{cases}$

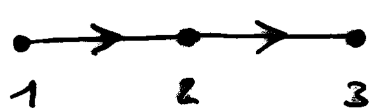
For instance :

Π

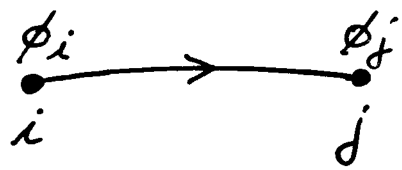


The ordinal possibilistic game Π can be assimilated with its induced total preorder on N :

Π :

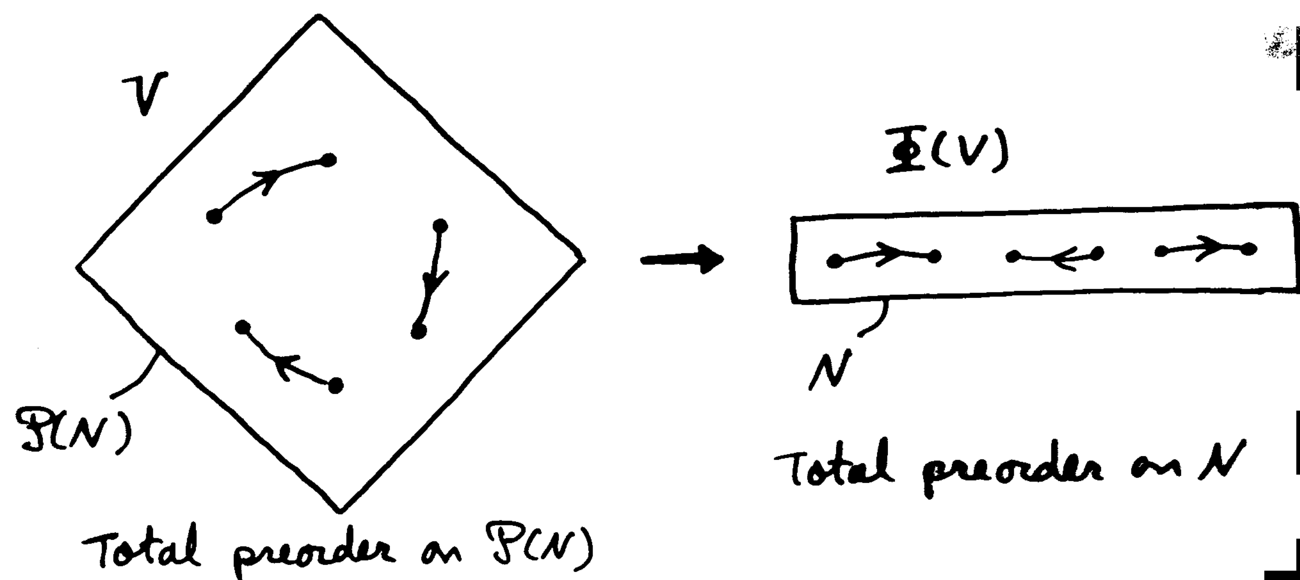


The power indices ϕ are defined by a total preorder Φ on N.



$$\Phi_{ij} = \begin{cases} 1 & \text{if } \phi_i \geq \phi_j \\ 0 & \text{else} \end{cases}$$

Given an ordinal game V, we are searching for ordinal power indices $\Phi(V)$:



<u>Numerical representation of Φ:</u>

Let Φ be a binary relation on N. Then Φ is a total preorder if and only if $\exists\, \phi : N \to \mathbb{R}$ s.t.

$$\Phi_{ij} = \begin{cases} 1 & \text{if } \phi_i \geq \phi_j \\ 0 & \text{else} \end{cases}$$

ϕ can be defined as $\phi_i = \sum_{j \in N} \Phi_{ij}$

2. Possibilistic approximation

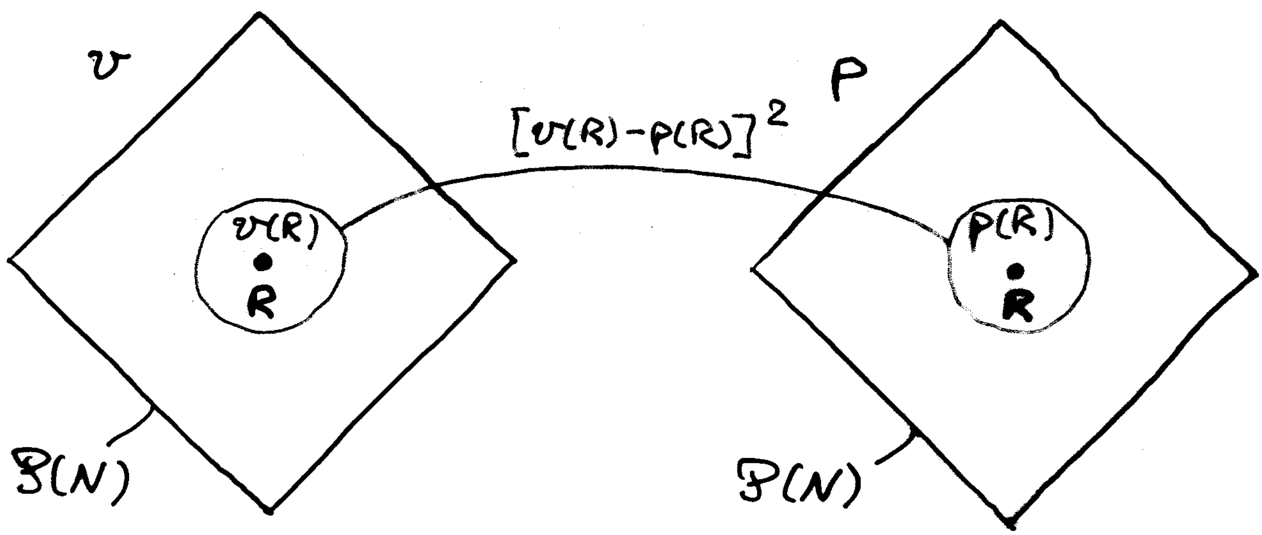
2.1 Cardinal context

The best linear approximation to v, defined as

$$p(R) = p_0 + \sum_{i \in R} p_i$$

corresponds to minimizing

$$d(v, p) = \sum_{R \subseteq N} [v(R) - p(R)]^2$$



It has been characterized by Hammer and Holzman (1992):

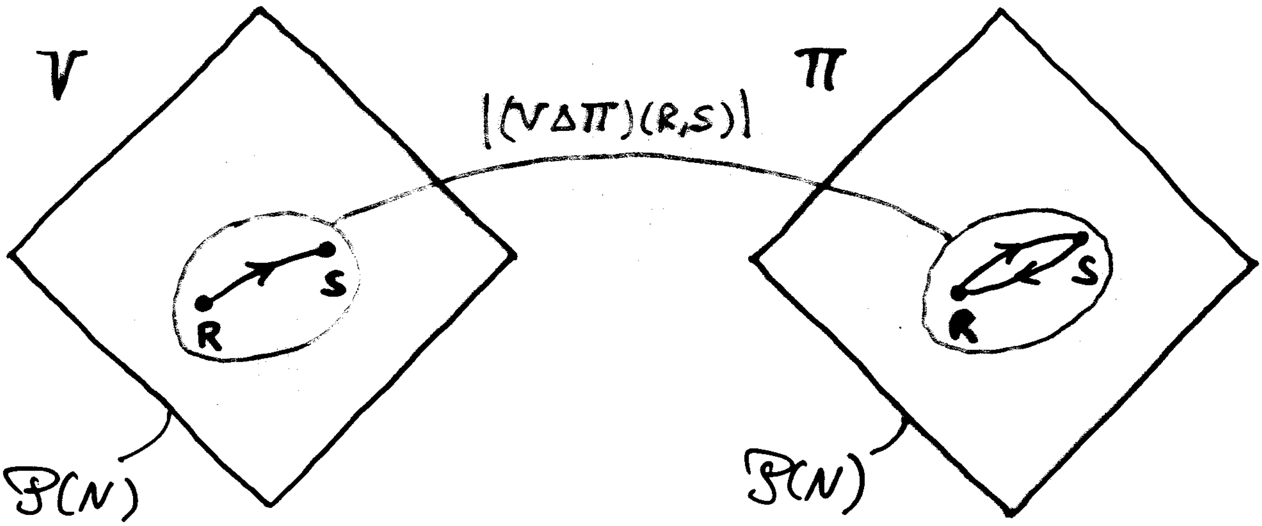
$$\begin{cases} p_0 = \dfrac{1}{2^m} \displaystyle\sum_{R \subseteq N} (m - 2r + 1)\, v(R), \quad r = |R| \\ p_i = b_i(v) = \dfrac{1}{2^{m-1}} \displaystyle\sum_{R \subseteq N \setminus i} [v(Ri) - v(R)] \quad \forall i \in N \end{cases}$$

2.2 Ordinal context

The best possibilistic approximation Π to $\mathcal{V}$ corresponds to minimizing

$$d(\mathcal{V}, \Pi) = |\mathcal{V} \,\Delta\, \Pi| = |(\mathcal{V} \cup \Pi) \setminus (\mathcal{V} \cap \Pi)|$$

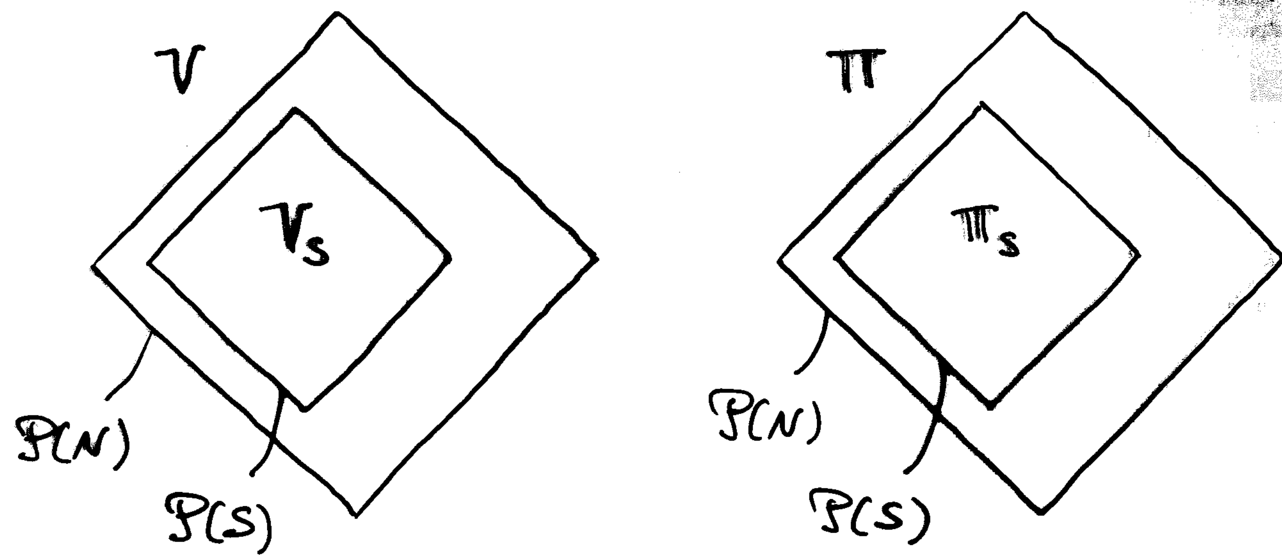
and can be computed with a polynomial time algorithm.



$$|(\mathcal{V} \Delta \Pi)(R,S)| = \begin{cases} 0 & \text{if} \quad \mathcal{V}: R \rightarrow S, \quad \Pi: R \rightarrow S \\ 1 & \text{if} \quad \mathcal{V}: R \rightarrow S, \quad \Pi: R \rightleftarrows S \\ 1 & \text{if} \quad \mathcal{V}: R \rightleftarrows S, \quad \Pi: R \rightarrow S \\ 2 & \text{if} \quad \mathcal{V}: R \rightarrow S, \quad \Pi: R \leftarrow S \end{cases}$$

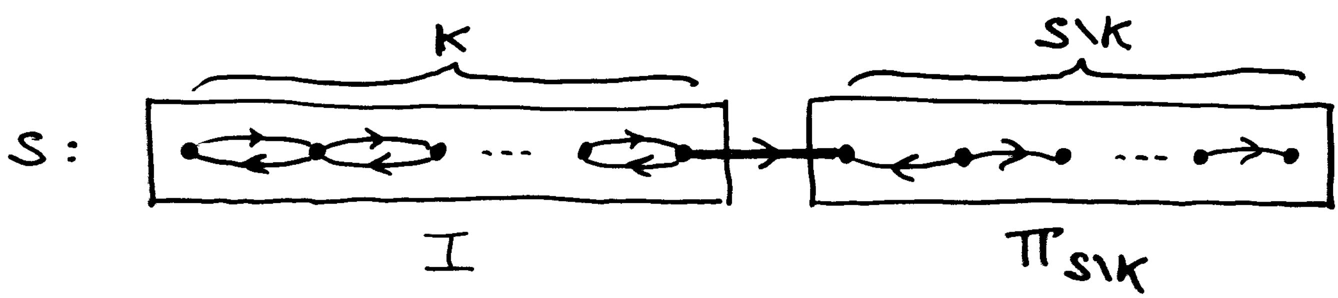
Some notations :

Let $S \subseteq N$. We set :



- $V_S :=$ restriction of V to S
- $z_S^* := \min\limits_{\Pi_S} |V_S \,\Delta\, \Pi_S|$
- $\Pi_S^* := \arg\min\limits_{\Pi_S} |V_S \,\Delta\, \Pi_S|$
- $\forall K \subseteq S$, $[I_K, \Pi_{S\setminus K}]$ denotes the total preorder Π_S on S such that

$$\begin{cases} \Pi_S(i,j) = 1 & \text{if } i,j \in K \\ \Pi_S(i,j) = 0 & \text{if } i \in S\setminus K,\ j \in K \\ \Pi_S = \Pi_{S\setminus K} & \text{on } S\setminus K \end{cases}$$



The result (Crama, Marichal, Mathonet)

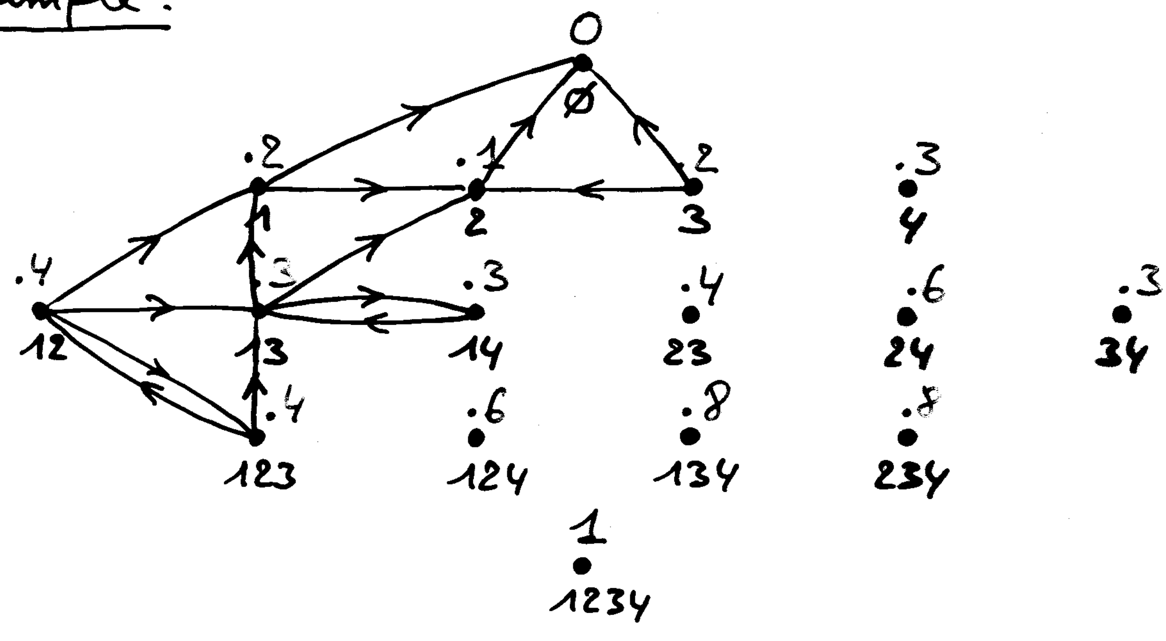
$$\boxed{z_S^* = \min_{K\subseteq S} \left| V_S \,\Delta\, [I_K, \pi^*_{S\setminus K}] \right| \qquad \forall S\subseteq N}$$

Let $S\subseteq N$ and assume that z_T^*, π_T^* are known $\forall\, T\subset S,\ T\neq S$.

$\Rightarrow$ One can compute z_S^*, π_S^* in $O(\underbrace{2^n}_{K\subseteq S}\underbrace{4^n}_{\Delta}) = O(8^n)$.

$\Rightarrow$ One can compute $\pi^*_\emptyset, \pi^*_i, \pi^*_{ij}, \dots, \pi^*_N$ in $O(2^n\cdot 8^n) = O(16^n) = O(k^4)$ where $k = 2^n = |\mathcal{P}(N)|$.

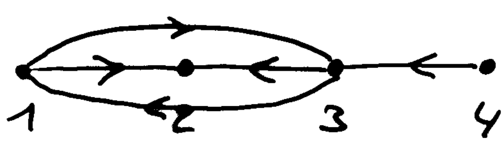
Example:



Optimal solution (unique):

$$\pi(4) > \pi(3) = \pi(1) > \pi(2)$$

π :



3 Axiomatic approach

3.1 Cardinal context

$\boxed{\text{Given } v, \text{ one is searching for } \phi(v).}$

Axioms

- Linearity (L): $\begin{cases} \phi(v+w) = \phi(v) + \phi(w) \\ \phi(c.v) = c\,\phi(v) \end{cases}$

- Dummy (D):

$v(Ri) = v(R) + v(i) \quad \forall R \subseteq N \setminus i \implies \phi_i(v) = v(i)$

- Monotonicity (M):

v monotonic $\implies \phi_i(v) \geq 0 \quad \forall i \in N$

- Symmetry (S):

$\forall \sigma =$ permutation on N, $\phi_i(v) = \phi_{\sigma(i)}(\sigma v) \quad \forall i \in N$

Weber (1988):

If $\phi(v)$ fulfils L, D, M, S, then

$$\phi_i(v) = \sum_{R \subseteq N \setminus i} p_r \left[v(Ri) - v(R) \right] \quad \forall i \in N$$

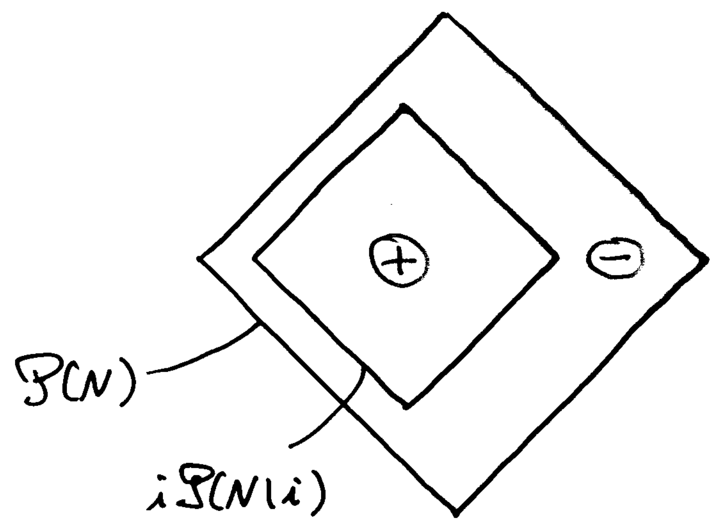
with $p_r \geq 0$, $\sum_{R \subseteq N \setminus i} p_r = 1$, $r = |R|$

For instance :

- <u>Banzhaf power indices :</u>

$$b_i(v) = \frac{1}{2^{n-1}} \sum_{R \subseteq N \setminus i} \left[v(Ri) - v(R) \right]$$

$$= \frac{1}{2^{n-1}} \left[\underbrace{\sum_{R \subseteq N \setminus i} v(Ri)}_{\oplus} - \underbrace{\sum_{R \subseteq N \setminus i} v(R)}_{\ominus} \right]$$



- <u>Shapley power indices</u>

$$s_i(v) = \frac{1}{n!} \sum_{R \subseteq N \setminus i} r!\,(n-r-1)!\,\left[v(Ri) - v(R) \right]$$

3.2 Ordinal context

$$\boxed{\text{Given } V \text{, one is searching for } \Phi(V).}$$

(V is monotonic)

Axioms

- **$\Phi(V)$ = total preorder**

  – strong completeness:

  $\forall i,j \in N : \Phi_{ij}(V) = 1$ or $\Phi_{ji}(V) = 1$

  – transitivity:

  $\forall i,j,k \in N : \Phi_{ij}(V) = 1,\ \Phi_{jk}(V) = 1 \Rightarrow \Phi_{ik}(V) = 1$

- **Null player (N):** $\quad (v(Ri) = v(R) \ \ \forall R \subseteq N \setminus i)$

  $V(Ri, R) = V(R, Ri) = 1 \quad \forall R \subseteq N \setminus i$

  $\Rightarrow \ \Phi_{ji}(V) = 1 \quad \forall j \in N \setminus i$

  $$i \xleftarrow{\ \Phi\ } j$$

- **Symmetry (S):**

  $\forall \sigma$ = permutation on N,

  $$\Phi_{ij}(V) = \Phi_{\sigma(i)\sigma(j)}(\sigma V) \quad \forall i,j \in N$$

- <u>Increasingness (I)</u>:

  Given V, let $V' = V$ except that
  $\exists R \subseteq N \setminus ij$ s.t. $v'(Ri) > v(Ri)$ or $v'(Rj) < v(Rj)$.
  Then $\Phi_{ij}(V') \geq \Phi_{ij}(V)$ and $\Phi_{ji}(V') \leq \Phi_{ji}(V)$

- <u>Dominance (D)</u>:

  $V(Ri, Rj) = 1 \quad \forall R \subseteq N \setminus ij \quad$ (i.e. $v(Ri) \geq v(Rj)$) $\Rightarrow \Phi_{ij}(V) = 1$

- <u>Strong dominance (SD)</u>:

  $$\left.\begin{array}{l} V(Ri, Rj) = 1 \quad \forall R \subseteq N \setminus ij \\ \exists R' : V(R'j, R'i) = 0 \end{array}\right\} \Rightarrow \Phi_{ji}(V) = 0$$

- <u>Independance of irrelevant coalitions (IC)</u>:

  $\Phi_{ij}(V)$ does not depend on pairs $(R, S) \subseteq N \times N$ for which
  $i, j \notin R \cup S$ or $i, j \in R \cap S$ or $ij \in R \setminus S$ or $ij \in S \setminus R$

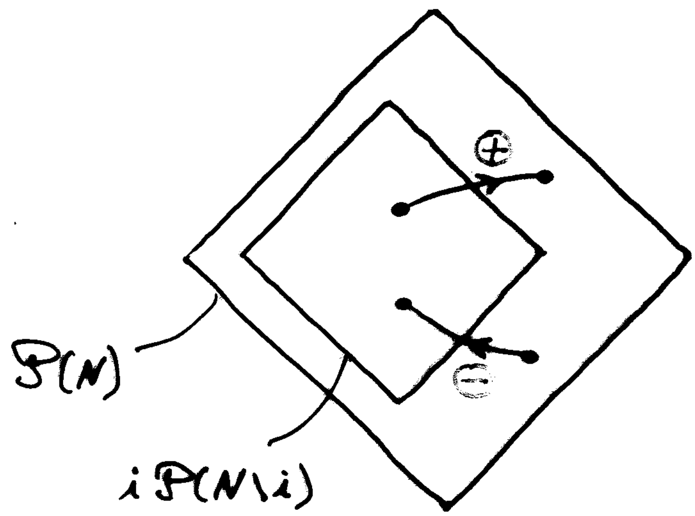
<u>Example</u> (Marichal, Mathieu)

$$\boxed{\begin{aligned} \phi_i(V) &= \sum_{R, S \subseteq N \setminus i} p_r^s \left[V(Ri, S) - V(S, Ri) \right] \\ &= \sum_{R, S \subseteq N \setminus i} p_r^s \operatorname{sgn}\left[v(Ri) - v(S) \right] \\ &(r = |R|,\ s = |S|) \qquad p_r^s > 0 \\ &\text{fulfils } N, S, I, D, SD, IC. \end{aligned}}$$

For instance :

- <u>Ordinal Banzhaf power indices</u>

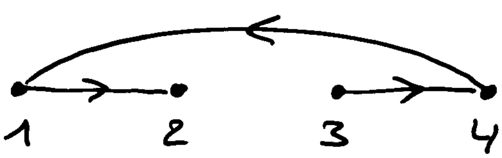
$$b_i(V) = \sum_{R,S \subseteq N \setminus i} \operatorname{sgn}\left[v(Ri) - v(\overset{S}{\cancel{R}}) \right]$$



Back to the example $(\pi(4) > \pi(3) = \pi(1) > \pi(2))$:

$$b_3(V) > b_4(V) > b_1(V) > b_2(V)$$

$\Phi(V)$:



- <u>Ordinal Shapley power indices :</u>

$$s_i(V) = \sum_{R,S \subseteq N \setminus i} r!\, s!\, (n-r-1)!\, (n-s-1)!\, \operatorname{sgn}\left[v(Ri) - v(S) \right]$$