Power Indices Based on Ordinal Games

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Abstract

Shapley (1953) and Banzhaf (1965) solved multiperson cooperative games by assessing a value (power index) to each player of a game. Shapley value can be interpreted as giving each player his average marginal contribution to all coalitions of players. Banzhaf power index related to a player i corresponds to the probability that a coalition wins when the player i joins randomly the coalition. Given a game $v : 2^N \to \mathbb{R}$ (i.e. a real valued set function on N), where N is a set of players, the Shapley power index related to the player $i \in N$ is given by

$$S(i) = \sum_{R \subseteq N \setminus \{i\}} \frac{(n-r-1)!\, r!}{n!} \, [v(R \cup \{i\}) - v(R)], \quad \forall i \in N,$$

and the Banzhaf power index by

$$B(i) = \frac{1}{2^{n-1}} \sum_{R \subseteq N \setminus \{i\}} [v(R \cup \{i\}) - v(R)], \quad \forall i \in N.$$

Those two indices have been axiomatically characterized by several authors (see e.g. Grabisch and Roubens (1997)).

In some practical situations, the numbers $\{v(S) \,|\, S \subseteq N\}$, defining the game, can be determined only up to the order. An *ordinal game* is a linear preorder $(2^N, \preceq)$ defined on 2^N. The antisymmetric part of the binary relation $\preceq$ is denoted by $\prec$ and the symmetric part by $\sim$. A preference matrix P can be defined as follows (assuming a fixed order in 2^N):

$$P(R,S) = \begin{cases} 1, & \text{if } R \succ S, \\ 0, & \text{if } R \sim S, \\ -1, & \text{if } R \prec S, \end{cases}$$

for all $R, S \subseteq N$. The problem is to rank the players (i.e. to give a linear preorder $\preceq$ on N) with the use of this ordinal information. We propose the uniform ranking value:

$$R(i) = \sum_{R,S \subseteq N \setminus \{i\}} P(R \cup \{i\}, S), \quad \forall i \in N,$$

and the preorder on N is obtained by the classical rule:

$$i \preceq j \quad \Leftrightarrow \quad R(i) \leq R(j)$$

for all $i, j \in N$. As done for the Shapley and Banzhaf power indices, we propose some natural properties fulfilled by this ranking method, which could lead to an axiomatical characterization.

Keywords: game theory, ordinal scales, power indices.