Approximations of the Lovász Extension of Pseudo-Boolean Functions;

Applications to Multicriteria Decision Making

by

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Approximations of pseudo-Boolean functions

Preliminary result (Hammer and Rudeanu, 1968)

Any pseudo-Boolean function $f : \{0,1\}^n \to \mathbb{R}$ has a unique expression as a multilinear polynomial in n variables:

$$f(x) = \sum_{T \subseteq N} a_T \prod_{i \in T} x_i, \quad x \in \{0,1\}^n,$$

where $N = \{1, \ldots, n\}$ and $a_T \in \mathbb{R}$.

Definition

Let $f : \{0,1\}^n \to \mathbb{R}$ and $k \in \{0, \ldots, n\}$. The best k-th approximation of f is the multilinear polynomial $f^{(k)} : \{0,1\}^n \to \mathbb{R}$ of degree $\leq k$ defined by

$$f^{(k)}(x) = \sum_{\substack{T \subseteq N \\ |T| \leq k}} a_T^{(k)} \prod_{i \in T} x_i$$

which minimizes

$$\sum_{x \in \{0,1\}^n} [f(x) - f^{(k)}(x)]^2$$

among all multilinear polynomials of degree $\leq k$.

Let $S \subseteq N$. The S-derivative of f at $x \in \{0,1\}^n$, denoted $\Delta_S\, f(x)$, is defined inductively as

$$\begin{aligned} \Delta_i\, f(x) &:= f(x\,|\,x_i = 1) - f(x\,|\,x_i = 0), \\ \Delta_{ij}\, f(x) &:= \Delta_i(\Delta_j\, f)(x) = \Delta_j(\Delta_i\, f)(x), \\ &\;\;\vdots \\ \Delta_S\, f(x) &:= \Delta_i(\Delta_{S\setminus i}\, f)(x) \end{aligned}$$

Theorem (Hammer and Holzman, 1992)

The best k-th approximation $f^{(k)}$ is given by the unique solution of the triangular linear system

$$\frac{1}{2^n} \sum_{x\in\{0,1\}^n} \Delta_S\, f^{(k)}(x) = \frac{1}{2^n} \sum_{x\in\{0,1\}^n} \Delta_S\, f(x), \quad \forall S \subseteq N,\ |S| \leq k.$$

Theorem (Grabisch, Marichal, and Roubens, 1998)

The coefficients $a_S^{(k)}$ of the best k-th approximation $f^{(k)}$ are given from those of f by

$$a_S^{(k)} = a_S + (-1)^{k-|S|} \sum_{\substack{T\supseteq S \\ |T|>k}} \binom{|T\setminus S|-1}{k-|S|} \frac{1}{2^{|T\setminus S|}}\, a_T, \quad S \subseteq N,\ |S| \leq k.$$

Approximations of Lovász extensions

The Lovász extension (Lovász, 1983; Singer, 1985)

Let Π_n denote the family of all permutations π of N. The *Lovász extension* $\hat{f} : [0,1]^n \to \mathbb{R}$ of any pseudo-Boolean function f is defined on each n-simplex

$$\mathcal{B}_\pi = \{x \in [0,1]^n \mid x_{\pi(1)} \leq \cdots \leq x_{\pi(n)}\}, \quad \pi \in \Pi_n,$$

as the unique affine function which interpolates f at the $n+1$ vertices of $\mathcal{B}_\pi$:

$$\hat{f}(x) = \sum_{T \subseteq N} a_T \bigwedge_{i \in T} x_i, \quad x \in [0,1]^n.$$

Definition

Let $\hat{f}$ be the Lovász extension of a pseudo-Boolean function f, and let $k \in \{0, \ldots, n\}$. The best k-th approximation of $\hat{f}$ is the min-polynomial $\hat{f}^{(k)} : [0,1]^n \to \mathbb{R}$ of degree $\leq k$ defined by

$$\hat{f}^{(k)}(x) = \sum_{\substack{T \subseteq N \\ |T| \leq k}} a_T^{(k)} \bigwedge_{i \in T} x_i$$

which minimizes

$$\int_{[0,1]^n} [\hat{f}(x) - \hat{f}^{(k)}(x)]^2 \, dx$$

among all min-polynomials of degree $\leq k$.
We write $\hat{f}^{(k)} = A^{(k)}(\hat{f})$.

We set

$$V^{(n)} := \Big\{\hat{f} \,\Big|\, \hat{f}(x) = \sum_{T\subseteq N} a_T \bigwedge_{i\in T} x_i, \;\; a_T \in \mathbb{R}\Big\}$$

$V^{(n)}$ is a vector space isomorphic to $\mathbb{R}^{2^n}$:

$$\hat{f} \quad \longleftrightarrow \quad \big(a_T\big)_{T\subseteq N}$$

In particular,

$$\dim(V^{(n)}) = 2^n$$

In $V^{(n)}$, we define

- a scalar product: $\langle \hat{f}_1, \hat{f}_2\rangle := \int_{[0,1]^n} \hat{f}_1(x)\, \hat{f}_2(x)\, dx$
- a norm: $\|\hat{f}\| := \langle \hat{f}, \hat{f}\rangle^{1/2}$
- a distance: $d(\hat{f}_1, \hat{f}_2) := \|\hat{f}_1 - \hat{f}_2\|$ in $V^{(n)}$

For any $k \in \{0, \ldots, n\}$, we set

$$V^{(k)} := \Big\{\hat{f}^{(k)} \,\Big|\, \hat{f}^{(k)}(x) = \sum_{\substack{T\subseteq N\\ |T|\leq k}} a_T^{(k)} \bigwedge_{i\in T} x_i, \;\; a_T^{(k)} \in \mathbb{R}\Big\}$$

(vector subspace in $V^{(n)}$)

A basis for $V^{(k)}$ is given by

$$B^{(k)} = \Big\{\bigwedge_{j\in S} x_j \,\Big|\, S \subseteq N, |S| \leq k\Big\}$$

In particular,

$$\dim(V^{(k)}) = \sum_{s=0}^{k} \binom{n}{s}$$

The best k-th approximation $\hat{f}^{(k)} = A^{(k)}(\hat{f})$ is given by

$$\begin{aligned} \text{minimize: } & \|\hat{f} - \hat{f}^{(k)}\| \\ \text{subject to: } & \hat{f}^{(k)} \in V^{(k)} \end{aligned}$$

$$\text{(orthogonal projection of } \hat{f} \text{ onto } V^{(k)})$$

$$\Updownarrow$$

$$\int_{[0,1]^n} [\hat{f}(x) - \hat{f}^{(k)}(x)] \bigwedge_{j \in S} x_j \, dx = 0, \qquad \forall S \subseteq N, |S| \leq k$$

$$\Updownarrow$$

$$\sum_{T \subseteq N} a_T \int_{[0,1]^n} \bigwedge_{i \in T} x_i \bigwedge_{j \in S} x_j \, dx = \sum_{\substack{T \subseteq N \\ |T| \leq k}} a_T^{(k)} \int_{[0,1]^n} \bigwedge_{i \in T} x_i \bigwedge_{j \in S} x_j \, dx$$

$$\forall S \subseteq N, |S| \leq k$$

$$\text{with } \int_{[0,1]^n} \bigwedge_{i \in T} x_i \bigwedge_{j \in S} x_j \, dx = \frac{|T| + |S| + 2}{(|T \cup S| + 2)(|T| + 1)(|S| + 1)}$$

Theorem

The coefficients $a_S^{(k)}$ of $A^{(k)}(\hat{f})$ are given from those of $\hat{f}$ by

$$a_S^{(k)} = a_S + (-1)^{k-|S|} \sum_{\substack{T \supseteq S \\ |T| > k}} \frac{\binom{k+|S|+1}{|S|} \binom{|T \setminus S| - 1}{k - |S|}}{\binom{|T|+k+1}{k+1}} a_T, \quad S \subseteq N, |S| \leq k.$$

For $k = 1$ we obtain:

$$\begin{aligned} a_\emptyset^{(1)} &= \sum_{T \subseteq N} \frac{-2(|T| - 1)}{(|T| + 1)(|T| + 2)} a_T \\ a_i^{(1)} &= \sum_{T \ni i} \frac{6}{(|T| + 1)(|T| + 2)} a_T, \quad i \in N \end{aligned}$$

Approximations having two fixed values

We set

$$V^{(k,0,1)} := \left\{\hat{f}^{(k,0,1)} \,\middle|\, \hat{f}^{(k,0,1)} \in V^{(k)},\, \hat{f}^{(k,0,1)}(\mathbf{0}) = 0,\, \hat{f}^{(k,0,1)}(\mathbf{1}) = 1\right\}$$

where $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$.

Any element in $V^{(k,0,1)}$ is of the form

$$\hat{f}^{(k,0,1)}(x) = \sum_{\substack{T \subseteq N \\ |T| \leq k}} a_T^{(k,0,1)} \bigwedge_{i \in T} x_i$$

with

$$\begin{cases} a_{\emptyset}^{(k,0,1)} = 0 \\ \sum\limits_{\substack{T \subseteq N \\ |T| \leq k}} a_T^{(k,0,1)} = 1 \end{cases}$$

For a fixed $j \in N$, these functions can also be written as

$$\hat{f}^{(k,0,1)}(x) = x_j + \sum_{\substack{T \subseteq N, T \neq j \\ 1 \leq |T| \leq k}} a_T^{(k,0,1)} \Big(\bigwedge_{i \in T} x_i - x_j \Big)$$

$V^{(k,0,1)}$ is an affine subspace in $V^{(k)}$. A basis for $V^{(k,0,1)}$ is given by

$$B_j^{(k,0,1)} = \left\{ \bigwedge_{i \in S} x_i - x_j \,\middle|\, S \subseteq N, S \neq \{j\}, 1 \leq |S| \leq k \right\}$$

for one $j \in N$. In particular

$$\dim(V^{(k,0,1)}) = \dim(V^{(k)}) - 2.$$

Problem

Given $\hat{f} \in V^{(n)}$, we search for the solution $\hat{f}^{(k,0,1)} = A^{(k,0,1)}(\hat{f})$ of

$$\begin{aligned} \text{minimize:} &\ \|\hat{f} - \hat{f}^{(k,0,1)}\| \\ \text{subject to:} &\ \hat{f}^{(k,0,1)} \in V^{(k,0,1)} \end{aligned}$$

(orthogonal projection of $\hat{f}$ onto $V^{(k,0,1)}$)

Since $V^{(k,0,1)} \subset V^{(k)}$, we have $A^{(k,0,1)}(\hat{f}) = A^{(k,0,1)}(A^{(k)}(\hat{f}))$ and the problem becomes

$$\begin{aligned} \text{minimize:} &\ \|A^{(k)}(\hat{f}) - \hat{f}^{(k,0,1)}\| \\ \text{subject to:} &\ \hat{f}^{(k,0,1)} \in V^{(k,0,1)} \end{aligned}$$

(orthogonal projection of $A^{(k)}(\hat{f})$ onto $V^{(k,0,1)}$)

$$\Updownarrow$$

$$\int_{[0,1]^n} [(A^{(k)}\hat{f})(x) - \hat{f}^{(k,0,1)}(x)] \, (\bigwedge_{i \in S} x_i - x_j) \, dx = 0$$
$$\forall S \subseteq N, S \neq \{j\}, 1 \leq |S| \leq k$$

For $k = 1$ we obtain:

$$\begin{aligned} a_{\emptyset}^{(1,0,1)} &= 0 \\ a_i^{(1,0,1)} &= a_i^{(1)} + \frac{1}{n}\Big(1 - \sum_{j \in N} a_j^{(1)}\Big), \quad i \in N \end{aligned}$$

Increasing approximations having two fixed values

We set

$$V^{[k,0,1]} := \left\{ \hat{f}^{[k,0,1]} \,\middle|\, \hat{f}^{[k,0,1]} \in V^{(k,0,1)},\, \hat{f}^{[k,0,1]} \text{ is increasing} \right\}$$

Any element in $V^{[k,0,1]}$ is of the form

$$\hat{f}^{[k,0,1]}(x) = \sum_{\substack{T \subseteq N \\ |T| \leq k}} a_T^{[k,0,1]} \bigwedge_{i \in T} x_i$$

with

$$\begin{cases} a_\emptyset^{[k,0,1]} = 0 \\ \sum\limits_{\substack{T \subseteq N \\ |T| \leq k}} a_T^{[k,0,1]} = 1 \\ \sum\limits_{\substack{T:\, i \in T \subseteq S \\ |T| \leq k}} a_T^{[k,0,1]} \geq 0, \quad S \subseteq N, i \in S. \end{cases}$$

$V^{[k,0,1]}$ is a non-empty closed convex polyhedron in $V^{(k,0,1)}$.

Problem

Given $\hat{f} \in V^{(n)}$, we search for the solution $\hat{f}^{[k,0,1]} = A^{[k,0,1]}(\hat{f})$ of

$$\begin{aligned} \text{minimize: } & \|\hat{f} - \hat{f}^{[k,0,1]}\| \\ \text{subject to: } & \hat{f}^{[k,0,1]} \in V^{[k,0,1]} \end{aligned}$$

(projection of $\hat{f}$ onto the polyhedron $V^{[k,0,1]}$)

Since $V^{[k,0,1]} \subset V^{(k,0,1)}$, we can replace $\hat{f}$ by $A^{(k,0,1)}(\hat{f})$.

Case of $k = 1$: the closest weighted arithmetic mean to a Lovász extension

Problem

Find the solution $(a_1^{[1,0,1]}, \ldots, a_n^{[1,0,1]}) \in \mathbb{R}^n$ of:

$$\text{minimize:} \qquad \int_{[0,1]^n} [\sum_{i=1}^{n} a_i^{(1,0,1)} x_i - \sum_{i=1}^{n} a_i^{[1,0,1]} x_i]^2 \, dx$$

subject to:

$$\begin{cases} \sum_{i=1}^{n} a_i^{[1,0,1]} = 1 \\ a_i^{[1,0,1]} \geq 0, \qquad i \in N. \end{cases}$$

Recall that

$$\begin{aligned} V^{(1,0,1)} &= \Big\{ \sum_{i=1}^{n} \omega_i \, x_i \,\Big|\, \sum_{i=1}^{n} \omega_i = 1, \omega_i \in \mathbb{R} \Big\} \\ V^{[1,0,1]} &= \Big\{ \sum_{i=1}^{n} \omega_i \, x_i \,\Big|\, \sum_{i=1}^{n} \omega_i = 1, \omega_i \geq 0 \Big\} \end{aligned}$$

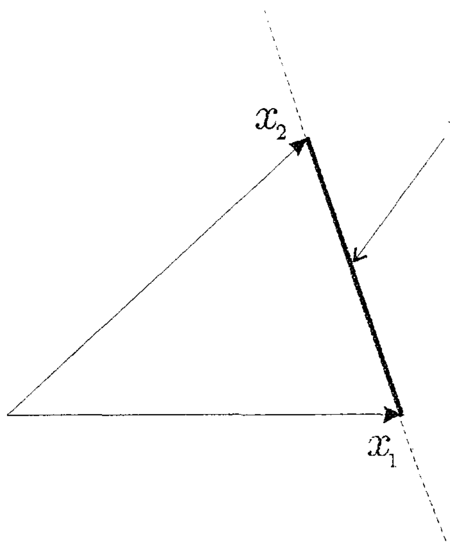
$V^{(1,0,1)}$ is the affine hull of $x_1, \ldots, x_n$
$V^{[1,0,1]}$ is the convex hull of $x_1, \ldots, x_n$

$$\dim(V^{(1,0,1)}) = \dim(V^{[1,0,1]}) = n - 1$$

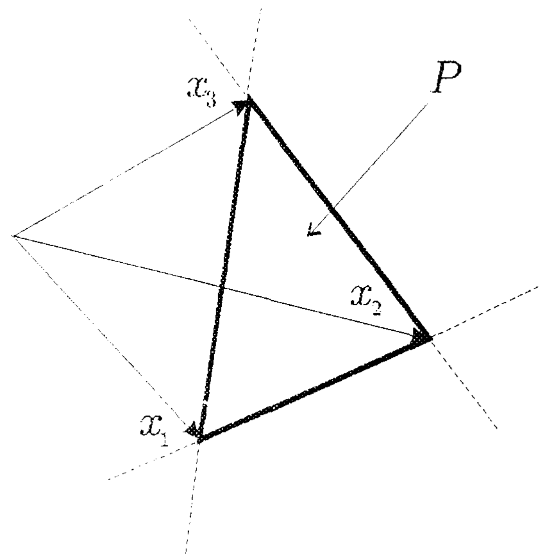
$$\|x_i - x_j\| = \frac{1}{\sqrt{6}}, \quad \forall i, j \in N, i \neq j$$

$$\Downarrow$$

$$P := V^{[1,0,1]} \text{ is a regular simplex in } V^{(1,0,1)}$$



$n = 2$



$n = 3$

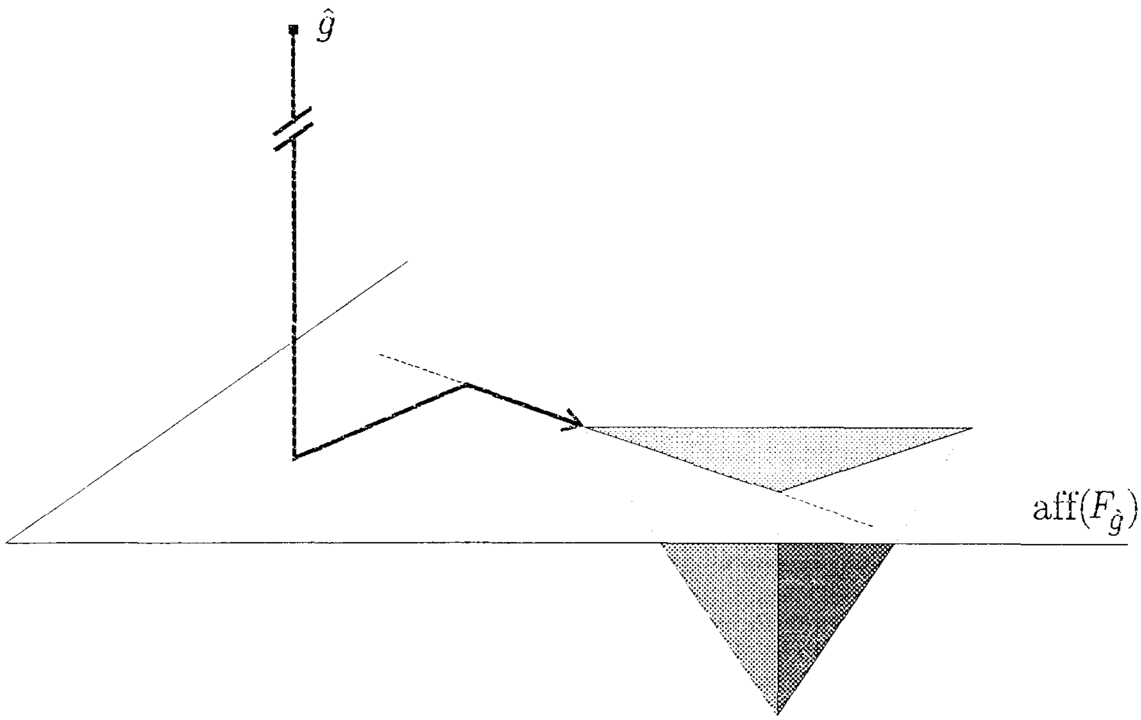
Assume $\hat{g} := A^{(1,0,1)}(\hat{f}) \in V^{(1,0,1)} \setminus P$.
Then $A^{[1,0,1]}(\hat{f})$ can be obtained by projecting $\hat{g}$ onto P.

There exists a facet $F_{\hat{g}}$ of P such that the affine hull $\mathrm{aff}(F_{\hat{g}})$ of its vertices contains $\hat{g}$ or separates $\hat{g}$ from P.

Theorem

Let $\hat{g} \in V^{(1,0,1)} \setminus P$. Then the projection of $\hat{g}$ onto P is in $F_{\hat{g}}$.

The projection of $\hat{g}$ onto P can be obtained by first projecting $\hat{g}$ onto $\mathrm{aff}(F_{\hat{g}})$ and then projecting, if necessary, the obtained projection onto $F_{\hat{g}}$.



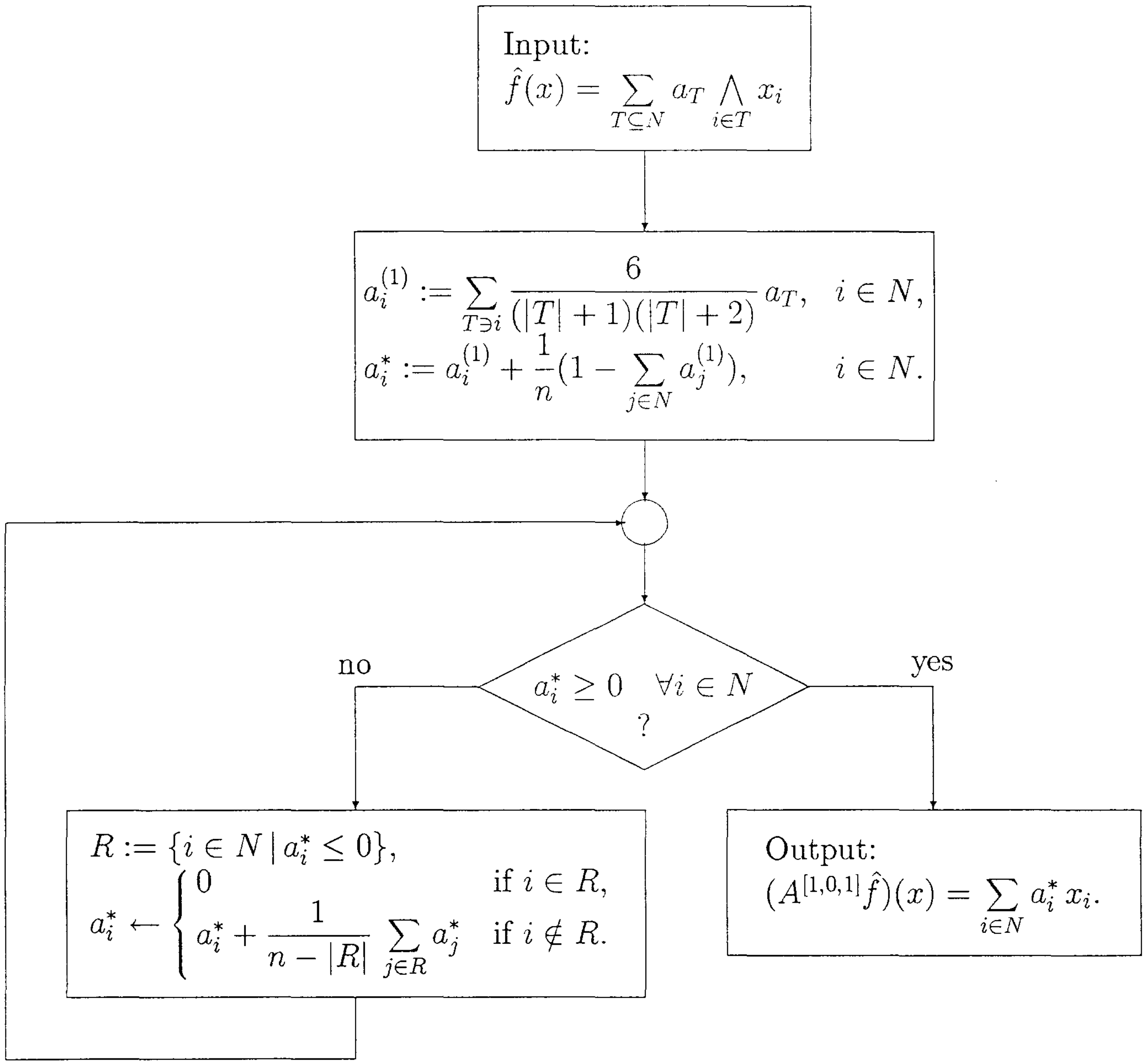
If more than one affine hull contain $\hat{g}$ or separate it from P then the projection onto P is clearly in the intersection of the corresponding facets.

Prestep. $P := V^{[1,0,1]}$, $\hat{g} := A^{(1,0,1)}(\hat{f})$.

Step 1. $F_{\hat{g}}^{\cap} :=$ intersection of all the facets of P whose affine hull contains $\hat{g}$ or separates it from P.

Step 2. $\hat{h} :=$ projection of $\hat{g}$ onto $\mathrm{aff}(F_{\hat{g}}^{\cap})$.

Step 3. If $\hat{h} \in F_{\hat{g}}^{\cap}$ then $\hat{h}$ is the projection of $A^{(1,0,1)}(\hat{f})$ onto P, $\longrightarrow$ stop, else $P \leftarrow F_{\hat{g}}^{\cap}$, $\hat{g} \leftarrow \hat{h}$, return to Step 1.


Input:
$\hat{f}(x) = \sum_{T \subseteq N} a_T \bigwedge_{i \in T} x_i$
$a_i^{(1)} := \sum_{T \ni i} \frac{6}{(|T|+1)(|T|+2)} a_T, \quad i \in N,$
$a_i^* := a_i^{(1)} + \frac{1}{n}(1 - \sum_{j \in N} a_j^{(1)}), \quad i \in N.$
no
$a_i^* \geq 0 \quad \forall i \in N$?
yes
$R := \{i \in N \,|\, a_i^* \leq 0\},$
$a_i^* \leftarrow \begin{cases} 0 & \text{if } i \in R, \\ a_i^* + \frac{1}{n-|R|} \sum_{j \in R} a_j^* & \text{if } i \notin R. \end{cases}$
Output:
$(A^{[1,0,1]}\hat{f})(x) = \sum_{i \in N} a_i^* x_i.$

Example:

Let $\hat{f} : [0,1]^4 \to \mathbb{R}$ be given by

$$\begin{aligned}\hat{f}(x) &= \frac{3}{10}\left[x_1 + x_2 + x_3 + (x_1 \wedge x_2) + (x_1 \wedge x_3) + (x_2 \wedge x_3)\right] \\ &\quad -\frac{21}{25}(x_1 \wedge x_2 \wedge x_3) + \frac{1}{25}(x_1 \wedge x_2 \wedge x_3 \wedge x_4).\end{aligned}$$

The best linear approximation is given by

$$(A^{(1)}\hat{f})(x) = \frac{1}{100} + \frac{89}{250}(x_1 + x_2 + x_3) + \frac{1}{125}x_4$$

and the best min-quadratic approximation by

$$\begin{aligned}(A^{(2)}\hat{f})(x) &= -\frac{27}{700} + \frac{803}{1750}(x_1 + x_2 + x_3) - \frac{8}{875}x_4 \\ &\quad -\frac{19}{175}\left[(x_1 \wedge x_2) + (x_1 \wedge x_3) + (x_2 \wedge x_3)\right] \\ &\quad +\frac{2}{175}\left[(x_1 \wedge x_4) + (x_2 \wedge x_4) + (x_3 \wedge x_4)\right].\end{aligned}$$

We also have

$$(A^{(1,0,1)}\hat{f})(x) = \frac{337}{1000}(x_1 + x_2 + x_3) - \frac{11}{1000}x_4$$

$$\begin{aligned}(A^{(2,0,1)}\hat{f})(x) &= \frac{29419}{67000}(x_1 + x_2 + x_3) - \frac{1937}{67000}x_4 \\ &\quad -\frac{181}{1675}\left[(x_1 \wedge x_2) + (x_1 \wedge x_3) + (x_2 \wedge x_3)\right] \\ &\quad +\frac{4}{335}\left[(x_1 \wedge x_4) + (x_2 \wedge x_4) + (x_3 \wedge x_4)\right].\end{aligned}$$

and

$$(A^{[1,0,1]}\hat{f})(x) = \frac{1}{3}(x_1 + x_2 + x_3).$$

Applications to Multicriteria Decision Making

Example (Grabisch, 1995)

3 students: a, b, c
3 criteria: mathematics (M), physics (P), and literature (L)
Aggregation operator: weighted arithmetic mean
Weights: 3, 3, 2.

student	M	P	L	global evaluation
a	18	16	10	15.25
b	10	12	18	12.75
c	14	15	15	14.62

No weight vector $(\omega_M, \omega_P, \omega_L)$ satisfying $\omega_M = \omega_P > \omega_L$ is able to favor student c :

$$c \succ a \quad \Longleftrightarrow \quad \omega_L > \omega_M$$

We substitute a non-additive measure to the weight vector (additive measure)

Definition (Choquet, 1953)

A Choquet capacity on N is a set function $\mu : 2^N \to [0,1]$ satisfying

i) $\mu_\emptyset = 0, \mu_N = 1$

ii) $S \subseteq T \;\Rightarrow\; \mu_S \leq \mu_T$

For example, we can define

$$\begin{array}{llll} \mu_\emptyset = 0 & \mu_{\mathrm{M}} = 0.45 & \mu_{\mathrm{MP}} = 0.50 & \mu_{\mathrm{MPL}} = 1 \\ & \mu_{\mathrm{P}} = 0.45 & \mu_{\mathrm{ML}} = 0.90 & \\ & \mu_{\mathrm{L}} = 0.30 & \mu_{\mathrm{PL}} = 0.90 & \end{array}$$

Any real valued set function can be assimilated unambiguously with a pseudo-Boolean function. Particularly, to any Choquet capacity μ corresponds a unique increasing pseudo-Boolean function $f : \{0,1\}^n \to \mathbb{R}$ such that $f(\mathbf{0}) = 0$ and $f(\mathbf{1}) = 1$:

$$\mu \longleftrightarrow f_\mu(x) = \sum_{T\subseteq N} a_T \prod_{i\in T} x_i$$

with

$$a_S = \sum_{T\subseteq S} (-1)^{|S\setminus T|} \mu_T, \quad S \subseteq N.$$

$$\begin{array}{llll} a_\emptyset = 0 & a_{\mathrm{M}} = 0.45 & a_{\mathrm{MP}} = -0.40 & a_{\mathrm{MPL}} = -0.10 \\ & a_{\mathrm{P}} = 0.45 & a_{\mathrm{ML}} = 0.15 & \\ & a_{\mathrm{L}} = 0.30 & a_{\mathrm{PL}} = 0.15 & \end{array}$$

Definition (Choquet, 1953)

Let μ be a Choquet capacity on N. The (discrete) Choquet integral of a function $x : N \to [0,1]$ w.r.t. μ is defined by

$$\mathcal{C}_\mu(x) := \sum_{i=1}^{n} x_{(i)} \left[\mu_{\{(i),\ldots,(n)\}} - \mu_{\{(i+1),\ldots,(n)\}}\right],$$

with the convention that $x_{(1)} \leq \cdots \leq x_{(n)}$.

We also have

$$\mathcal{C}_\mu(x) = \hat{f}_\mu(x) = \sum_{T\subseteq N} a_T \bigwedge_{i\in T} x_i$$

Theorem

Let $M_\mu : [0,1]^n \to \mathbb{R}$ be an aggregation operator depending on a Choquet capacity μ on N. Then M_μ is

- *linear w.r.t. the Choquet capacity :*
  there exist functions $g_T(x) : [0,1]^n \to \mathbb{R}$, $T \subseteq N$, such that

$$M_\mu(x) = \sum_{T \subseteq N} a_T \, g_T(x), \quad \forall \mu.$$

- *increasing in each variable*
- *stable for the positive linear transformations*

$$M_\mu(r\,x_1 + s, \ldots, r\,x_n + s) = r\,M_\mu(x_1, \ldots, x_n) + s$$

  for all $x \in [0,1]^n$ and all $r > 0, s \in \mathbb{R}$.

- *an extension of μ :*

$$M_\mu(e_T) = \mu_T, \quad T \subseteq N.$$

if and only if $M_\mu = \mathcal{C}_\mu$.

Back to the example :

student	M	P	L	WAM_ω	$\mathcal{C}_\mu$
a	18	16	10	15.25	13.90
b	10	12	18	12.75	13.60
c	14	15	15	14.62	14.60

$$\mathcal{C}_\mu : c \succ a \succ b$$

Linear approximation :

$$A^{[1,0,1]}(\mathcal{C}_\mu) = 0.29\, x_M + 0.29\, x_P + 0.42\, x_L$$

Proposition

When $n \leq 3$, the weights of the linear approximation identify with the Shapley value :

$$a_i^{[1,0,1]} = \phi_\mu(i) = \sum_{T \ni i} \frac{1}{|T|}\, a_T, \quad i \in N.$$

However, for

$$\begin{aligned}\hat{f}(x) &= \frac{3}{10}\,[x_1 + x_2 + x_3 + (x_1 \wedge x_2) + (x_1 \wedge x_3) + (x_2 \wedge x_3)] \\ &\quad -\frac{21}{25}\,(x_1 \wedge x_2 \wedge x_3) + \frac{1}{25}\,(x_1 \wedge x_2 \wedge x_3 \wedge x_4),\end{aligned}$$

we have

$$\begin{aligned}&a_1^{[1,0,1]} = a_2^{[1,0,1]} = a_3^{[1,0,1]} = \frac{1}{3} \text{ and } a_4^{[1,0,1]} = 0 \\ &\phi(1) = \phi(2) = \phi(3) = \frac{33}{100} \text{ and } \phi(4) = \frac{1}{100}\end{aligned}$$

Min-quadratic approximation :

$$\begin{aligned}A^{[2,0,1]}(\mathcal{C}_\mu) &= 0.47\, x_M + 0.47\, x_P + 0.31\, x_L \\ &\quad -0.45\,(x_M \wedge x_P) + 0.10\,[(x_M \wedge x_L) + (x_P \wedge x_L)]\end{aligned}$$

student	M	P	L	WAM_ω	$\mathcal{C}_\mu$	$A^{[2,0,1]}(\mathcal{C}_\mu)$
a	18	16	10	15.25	13.90	13.83
b	10	12	18	12.75	13.60	13.67
c	14	15	15	14.62	14.60	14.88