Aggregation Functions for Multicriteria Decision Aid

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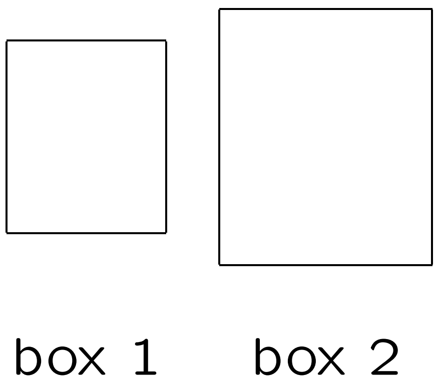
The aggregation problem

Combining several numerical values into a single one

Example (voting theory)
Several individuals form quantifiable judgements about the measure of an object.



$$\frac{\text{area(box 2)}}{\text{area(box 1)}} = ?$$

$$x_1, \ldots, x_n \quad \longrightarrow \quad M(x_1, \ldots, x_n) = x$$

where M = arithmetic mean
geometric mean
median
...

Decision making (voters → criteria)

$$x_1, \ldots, x_n = \text{satisfaction degrees}$$

Aggregation in multicriteria decision making

- Alternatives $A = \{a, b, c, \ldots,\}$
- Criteria $N = \{1, 2, \ldots, n\}$
- Profile $a \in A \longrightarrow$ $(x_1^a, \ldots, x_n^a) \in \mathbb{R}^n$

  $\swarrow$ $\searrow$

  commensurable partial scores

- Aggregation function $M : \mathbb{R}^n \to \mathbb{R}$

  $M : E^n \to \mathbb{R} \quad (E \subseteq \mathbb{R})$

Alternative	crit. 1	$\cdots$	crit. n	global score
a	x_1^a	$\cdots$	x_n^a	$M(x_1^a, \ldots, x_n^a)$
b	x_1^b	$\cdots$	x_n^b	$M(x_1^b, \ldots, x_n^b)$
$\vdots$	$\vdots$		$\vdots$	$\vdots$

Example :

	math.	physics	literature	global
student a	18	16	10	?
student b	10	12	18	?
student c	14	15	15	?

Non-commensurable scales :

	price (to minimize)	consumption (to minimize)	comfort (to maximize)	global
car a	\$10,000	0.15 ℓpm	good	?
car b	\$20,000	0.17 ℓpm	excellent	?
car c	\$30,000	0.13 ℓpm	very good	?
car d	\$20,000	0.16 ℓpm	good	?

Scoring approach

For each $i \in N$, one can define a *net score* :

$$S_i(a) = |\{b \in A \mid b \preccurlyeq_i a\}| - |\{b \in A \mid b \succcurlyeq_i a\}|$$

$$s_i(a) = \frac{S_i(a) + (|A| - 1)}{2(|A| - 1)} \in [0, 1]$$

	price	cons.	comf.	global
car a	1.00	0.66	0.16	?
car b	0.50	0.00	1.00	?
car c	0.00	1.00	0.66	?
car d	0.50	0.33	0.16	?

Aggregation properties

- **Symmetry.** $M(x_1, \ldots, x_n)$ is symmetric.

- **Increasingness.** $M(x_1, \ldots, x_n)$ is nondecreasing in each argument.

- **Idempotency.** $M(x, \ldots, x) = x$ for all x

- **Internality.** $\min x_i \leq M(x_1, \ldots, x_n) \leq \max x_i$
  Note : id. + inc. $\Rightarrow$ int. $\Rightarrow$ id.

- **Associativity.**

$$M(M(x_1, x_2), x_3) = M(x_1, M(x_2, x_3))$$

- **Decomposability.**

$$M(x_1, x_2, x_3) = M(M(x_1, x_3), x_2, M(x_1, x_3))$$

- **Bisymmetry.**

$$M(M(x_1, x_2), M(x_3, x_4)) \\ = M(M(x_1, x_3), M(x_2, x_4))$$

Quasi-arithmetic means

Theorem 1 (Kolmogorov-Nagumo, 1930)
The functions $M_n : E^n \to \mathbb{R}$ are symmetric, continuous, strictly increasing, idempotent, and **decomposable** *if and only if there exists a continuous strictly monotonic function $f : E \to \mathbb{R}$ such that*

$$M_n(x) = f^{-1}\Big[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big] \qquad (n \geq 1)$$

$f(x)$	$M(x)$	name
x	$\frac{1}{n}\sum_{i=1}^{n} x_i$	arithmetic
$\log x$	$\sqrt[n]{\prod_{i=1}^{n} x_i}$	geometric
x^{-1}	$\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}}$	harmonic
x^{α} $(\alpha \in \mathbb{R}_0)$	$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{1}{\alpha}}$	root-power

Proposition 1 (Marichal, 2000)
Symmetry can be removed in the K-N theorem.

Theorem 2 (Fodor-Marichal, 1997)
The functions $M_n : [a,b]^n \to \mathbb{R}$ are symmetric, continuous, **increasing**, *idempotent, and* **decomposable** *if and only if there exist α and β fulfilling $a \le \alpha \le \beta \le b$ and a continuous strictly monotonic function $f : [\alpha, \beta] \to \mathbb{R}$ such that, for any $n \ge 1$,*

$$M_n(x) = \begin{cases} F_n(x) & \text{if } x \in [a,\alpha]^n, \\ G_n(x) & \text{if } x \in [\beta,b]^n, \\ f^{-1}\Big[\frac{1}{n}\sum_{i=1}^{n} f[\text{median}(\alpha, x_i, \beta)]\Big] & \text{otherwise} \end{cases}$$

where F_n and G_n are defined by...

Open problem : remove symmetry !

Theorem 3 (Aczél, 1948)
The function $M : E^n \to \mathbb{R}$ *is symmetric, continuous, strictly increasing, idempotent, and* **bisymmetric** *if and only if there exists a continuous strictly monotonic function* $f : E \to \mathbb{R}$ *such that*

$$M(x) = f^{-1}\Big[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big]$$

When symmetry is removed :
There exist $\omega_1, \ldots, \omega_n > 0$ fulfilling $\sum_i \omega_i = 1$ s.t.

$$M(x) = f^{-1}\Big[\sum_{i=1}^{n} \omega_i f(x_i)\Big]$$

$f(x)$	$M(x)$	name
x	$\sum_{i=1}^{n} \omega_i x_i$	arithmetic
$\log x$	$\prod_{i=1}^{n} x_i^{\omega_i}$	geometric
x^α $(\alpha \in \mathbb{R}_0)$	$\Big(\sum_{i=1}^{n} \omega_i x_i^\alpha\Big)^{\frac{1}{\alpha}}$	root-power

Theorem 4 (Aczél, 1948)
The functions $M_n : E^n \to E$ *are continuous, strictly increasing, and* **associative** *if and only if there exists a continuous and strictly monotonic function* $f : E \to \mathbb{R}$ *such that*

$$M_n(x) = f^{-1}\Big[\sum_{i=1}^{n} f(x_i)\Big]$$

$+$ **idempotency :** $\emptyset$

Open problem : replace str. increasing by increasing

Theorem 5 (Fung-Fu, 1975)
The functions $M_n : E^n \to \mathbb{R}$ *are symmetric, continuous, increasing, idempotent, and* **associative** *if and only if there exists* $\alpha \in E$ *such that*

$$\begin{aligned} M_n(x) &= \text{median}\Big(\bigwedge_{i=1}^{n} x_i, \bigvee_{i=1}^{n} x_i, \alpha\Big) \\ &= \text{median}(x_1, \ldots, x_n, \underbrace{\alpha, \ldots, \alpha}_{n-1}) \end{aligned}$$

Without symmetry :

Theorem 6 (Marichal, 2000)
The functions $M_n : E^n \to \mathbb{R}$ *are continuous, increasing, idempotent, and associative if and only if there exist* $\alpha, \beta \in E$ *such that*

$$M_n(x) = (\alpha \wedge x_1) \vee \Big(\bigvee_{i=1}^{n} (\alpha \wedge \beta \wedge x_i) \Big) \vee (\beta \wedge x_n) \vee \Big(\bigwedge_{i=1}^{n} x_i \Big)$$

Without symmetry and idempotency :

Theorem 7 (Marichal, 2000)
The functions $M_n : [a,b]^n \to [a,b]$ *are continuous, increasing, associative, and have* a *and* b *as idempotent elements if and only if there exist* $\alpha, \beta \in E$ *such that*

$$M_n(x) = \begin{cases} F_n(x) & \text{if } x \in [a, \alpha \wedge \beta]^n, \\ G_n(x) & \text{if } x \in [\alpha \vee \beta, b]^n, \\ (\alpha \wedge x_1) \vee \ldots & \text{otherwise} \end{cases}$$

where F_n *and* G_n *are defined by...*

Interval scales

Example : marks obtained by students
- on a [0,20] scale : 16, 11, 7, 14
- on a [0,1] scale : 0.80, 0.55, 0.35, 0.70
- on a [-1,1] scale : 0.60, 0.10, -0.30, 0.40

Definition. $M : \mathbb{R}^n \to \mathbb{R}$ *is stable for the positive linear transformations if*

$$M(rx_1 + s, \ldots, rx_n + s) = r\, M(x_1, \ldots, x_n) + s$$

for all $x_1, \ldots, x_n \in \mathbb{R}$ *and all* $r > 0$, $s \in \mathbb{R}$.

Theorem 8 (Aczél-Roberts-Rosenbaum, 1986)
The function $M : \mathbb{R}^n \to \mathbb{R}$ *is stable for the positive linear transformations if and only if*

$$M(x) = S(x)\, F\Big(\frac{x_1 - A(x)}{S(x)}, \ldots, \frac{x_n - A(x)}{S(x)}\Big) + A(x)$$

where $A(x) = \frac{1}{n}\sum_i x_i$, $S(x) = \sqrt{\sum_i [x_i - A(x)]^2}$, *and* $F : \mathbb{R}^n \to \mathbb{R}$ *arbitrary.*

Interesting unsolved problem :
Describe increasing and stable functions

Theorem 9 (Marichal-Mathonet-Tousset, 1999) *The function $M : E^n \to \mathbb{R}$ is increasing, stable for the positive linear transformations, and bisymmetric if and only if it is of the form*

$$M(x) = \bigvee_{i \in S} x_i \quad \textit{or} \quad \bigwedge_{i \in S} x_i \quad \textit{or} \quad \sum_{i=1}^{n} \omega_i x_i$$

where $S \subseteq N$, $S \neq \emptyset$, $\omega_1, \ldots, \omega_n \geq 0$, and $\sum_i \omega_i = 1$.

Theorem 10 *The functions $M_n : E^n \to \mathbb{R}$ are increasing, stable for the positive linear transformations, and decomposable if and only if they are of the form*

$$M_n(x) = \bigvee_{i=1}^{n} x_i \ \textit{ or } \ \bigwedge_{i=1}^{n} x_i \ \textit{ or } \ x_1 \ \textit{ or } \ x_n \ \textit{ or } \ \frac{1}{n} \sum_{i=1}^{n} x_i$$

Theorem 11 *The functions $M_n : E^n \to \mathbb{R}$ are increasing, stable for the positive linear transformations, and associative if and only if they are of the form*

$$M_n(x) = \bigvee_{i=1}^{n} x_i \ \textit{ or } \ \bigwedge_{i=1}^{n} x_i \ \textit{ or } \ x_1 \ \textit{ or } \ x_n$$

An illustrative example (Grabisch, 1996)

Evaluation of students w.r.t. three subjects: mathematics, physics, and literature.

Student	M	P	L	global
a	0.90	0.80	0.50	?
b	0.50	0.60	0.90	?
c	0.70	0.75	0.75	?

(marks are expressed on a scale from 0 to 1)

Often used: the weighted arithmetic mean

$$\text{WAM}_\omega(x) = \sum_{i=1}^{n} \omega_i x_i$$

with $\sum_i \omega_i = 1$ and $\omega_i \geq 0$ for all $i \in N$

$$\left.\begin{array}{r} \omega_{\text{M}} = 0.35 \\ \omega_{\text{P}} = 0.35 \\ \omega_{\text{L}} = 0.30 \end{array}\right\} \Rightarrow$$

Student	global
a	0.74
b	0.65
c	0.73

$$\boxed{a \succ c \succ b}$$

Suppose we want to favor student c

Student	M	P	L	global
a	0.90	0.80	0.50	0.74
b	0.50	0.60	0.90	0.65
c	0.70	0.75	0.75	0.73

No weight vector $(\omega_{\mathsf{M}}, \omega_{\mathsf{P}}, \omega_{\mathsf{L}})$ satisfying

$$\omega_{\mathsf{M}} = \omega_{\mathsf{P}} > \omega_{\mathsf{L}}$$

is able to provide $c \succ a$!

$$\begin{aligned} c \succ a \quad \Leftrightarrow \quad & 0.70\omega_{\mathsf{M}} + 0.75\omega_{\mathsf{P}} + 0.75\omega_{\mathsf{L}} \\ & > 0.90\omega_{\mathsf{M}} + 0.80\omega_{\mathsf{P}} + 0.50\omega_{\mathsf{L}} \\ \Leftrightarrow \quad & -0.20\omega_{\mathsf{M}} - 0.05\omega_{\mathsf{P}} + 0.25\omega_{\mathsf{L}} > 0 \\ \Leftrightarrow \quad & -0.25\omega_{\mathsf{M}} + 0.25\omega_{\mathsf{L}} > 0 \\ \Leftrightarrow \quad & \omega_{\mathsf{L}} > \omega_{\mathsf{M}} \end{aligned}$$

What's wrong ?

$$\begin{aligned} \mathsf{WAM}_\omega(1,0,0) &= \omega_{\mathsf{M}} = 0.35 \\ \mathsf{WAM}_\omega(0,1,0) &= \omega_{\mathsf{P}} = 0.35 \\ \mathsf{WAM}_\omega(1,1,0) &= 0.70 \text{ !!!} \end{aligned}$$

What is the importance of $\{\mathsf{M}, \mathsf{P}\}$?

Definition (Choquet, 1953; Sugeno, 1974)
A fuzzy measure on N is a set function $v : 2^N \to [0,1]$ such that

$i)$ $v(\emptyset) = 0, v(N) = 1$

$ii)$ $S \subseteq T \Rightarrow v(S) \leq v(T)$

$$\begin{aligned} v(S) &= \text{weight of } S \\ &= \text{degree of importance of } S \end{aligned}$$

A fuzzy measure is additive if

$$v(S \cup T) = v(S) + v(T) \quad \text{if } S \cap T = \emptyset$$

$\rightarrow$ independent criteria

$$v(\mathsf{M}, \mathsf{P}) = v(\mathsf{M}) + v(\mathsf{P}) \ \ (= 0.70)$$

Question : how can we extend the weighted arithmetic mean by taking into account the interaction among criteria ?

The discrete Choquet integral

Definition
Let $v \in \mathcal{F}_N$. The (discrete) Choquet integral of $x \in \mathbb{R}^n$ w.r.t. v is defined by

$$\mathcal{C}_v(x) := \sum_{i=1}^{n} x_{(i)}[v(A_{(i)}) - v(A_{(i+1)})]$$

with the convention that $x_{(1)} \leq \cdots \leq x_{(n)}$.
Also, $A_{(i)} = \{(i), \ldots, (n)\}$.

Example: If $x_3 \leq x_1 \leq x_2$, we have

$$\begin{aligned} \mathcal{C}_v(x_1, x_2, x_3) &= x_3\,[v(3,1,2) - v(1,2)] \\ &\quad + x_1\,[v(1,2) - v(2)] \\ &\quad + x_2\, v(2) \end{aligned}$$

Particular case:

$$\boxed{v \text{ additive} \quad \Rightarrow \quad \mathcal{C}_v = \text{WAM}_\omega}$$

Indeed,

$$\mathcal{C}_v(x) = \sum_{i=1}^{n} x_{(i)} v((i)) = \sum_{i=1}^{n} x_i \underbrace{v(i)}_{\omega_i}$$

Properties of the Choquet integral

- **Linearity w.r.t. the fuzzy measures**
  There exist 2^n functions $f_T : \mathbb{R}^n \to \mathbb{R}$ $(T \subseteq N)$ such that

$$\mathcal{C}_v = \sum_{T \subseteq N} v(T)\, f_T \qquad (v \in \mathcal{F}_N)$$

  Indeed, on can show that

$$\mathcal{C}_v(x) = \sum_{T \subseteq N} v(T) \underbrace{\sum_{K \supseteq T} (-1)^{|K|-|T|} \bigwedge_{i \in K} x_i}_{f_T(x)}$$

- **Stability w.r.t. positive linear transformations**
  For any $x \in \mathbb{R}^n, r > 0, s \in \mathbb{R}$,

$$\mathcal{C}_v(r\,x_1 + s, \ldots, r\,x_n + s) = r\,\mathcal{C}_v(x_1, \ldots, x_n) + s$$

  Example : marks obtained by students
  - on a $[0, 20]$ scale : 16, 11, 7, 14
  - on a $[0, 1]$ scale : 0.80, 0.55, 0.35, 0.70
  - on a $[-1, 1]$ scale : 0.60, 0.10, -0.30, 0.40

  Remark : The partial scores may be embedded in $[0, 1]$

- **Increasing monotonicity**
  For any $x, x' \in \mathbb{R}^n$, one has

$$x_i \leq x'_i \quad \forall i \in N \qquad \Rightarrow \qquad \mathcal{C}_v(x) \leq \mathcal{C}_v(x')$$

- $\mathcal{C}_v$ **is properly weighted by** v

$$\mathcal{C}_v(e_S) = v(S) \qquad\qquad (S \subseteq N)$$

$e_S =$ characteristic vector of S in $\{0,1\}^n$
Example : $e_{\{1,3\}} = (1,0,1,0,\ldots)$

Independent criteria	Dependent criteria
$\mathrm{WAM}_\omega(e_{\{i\}}) = \omega_i$	$\mathcal{C}_v(e_{\{i\}}) = v(i)$
$\mathrm{WAM}_\omega(e_{\{i,j\}}) = \omega_i + \omega_j$	$\mathcal{C}_v(e_{\{i,j\}}) = v(i,j)$

Example :

$$\begin{array}{ccccc} v(\mathsf{M},\mathsf{P}) & < & v(\mathsf{M}) & + & v(\mathsf{P}) \\ \| & & \| & & \| \\ \mathcal{C}_v(1,1,0) & & \mathcal{C}_v(1,0,0) & & \mathcal{C}_v(0,1,0) \end{array}$$

Axiomatic characterization of the class of Choquet integrals with n arguments

Theorem (Marichal, 2000)
The operators $M_v : \mathbb{R}^n \to \mathbb{R}$ ($v \in \mathcal{F}_N$) are

- **linear w.r.t. the underlying fuzzy measures** v
  M_v is of the form
  $$M_v = \sum_{T \subseteq N} v(T)\, f_T \qquad (v \in \mathcal{F}_N)$$
  where f_T's are independent of v

- **stable for the positive linear transformations**
  $$M_v(r\, x_1 + s, \ldots, r\, x_n + s) = r\, M_v(x_1, \ldots, x_n) + s$$
  for all $x \in \mathbb{R}^n, r > 0, s \in \mathbb{R}$, and all $v \in \mathcal{F}_N$

- **increasing**

- **properly weighted by** v
  $$M_v(e_S) = v(S) \qquad (S \subseteq N, v \in \mathcal{F}_N)$$

if and only if $M_v = \mathcal{C}_v$ for all $v \in \mathcal{F}_N$

Back to the example

Assumptions :
- M and P are more important than L
- M and P are somewhat substitutive

Non-additive model : $\mathcal{C}_v$

$$v(\text{M}) = 0.35$$
$$v(\text{P}) = 0.35$$
$$v(\text{L}) = 0.30$$

$$v(\text{M}, \text{P}) = 0.60 \qquad \text{(redundancy)}$$
$$v(\text{M}, \text{L}) = 0.80 \qquad \text{(complementarity)}$$
$$v(\text{P}, \text{L}) = 0.80 \qquad \text{(complementarity)}$$

$$v(\emptyset) = 0$$
$$v(\text{M}, \text{P}, \text{L}) = 1$$

Student	M	P	L	WAM	Choquet
a	0.90	0.80	0.50	0.74	0.71
b	0.50	0.60	0.90	0.65	0.67
c	0.70	0.75	0.75	0.73	0.74

Now : $\boxed{c \succ a \succ b}$

Another example (Marichal, 2000)

Student	M	P	L	global
a	0.90	0.70	0.80	?
b	0.90	0.80	0.70	?
c	0.60	0.70	0.80	?
d	0.60	0.80	0.70	?

Behavior of the decision maker :
When a student is good at M (0.90), it is preferable that he/she is better at L than P, so

$$\boxed{a \succ b}$$

When a student is not good at M (0.60), it is preferable that he/she is better at P than L, so

$$\boxed{d \succ c}$$

Additive model : WAM_ω

$$\left.\begin{array}{lcl} a \succ b & \Leftrightarrow & \omega_{\mathsf{L}} > \omega_{\mathsf{P}} \\ d \succ c & \Leftrightarrow & \omega_{\mathsf{L}} < \omega_{\mathsf{P}} \end{array}\right\} \quad \text{No solution !}$$

Non additive model : $\mathcal{C}_v$

Student	M	P	L	global
a	0.90	0.70	0.80	0.81
b	0.90	0.80	0.70	0.79
c	0.60	0.70	0.80	0.71
d	0.60	0.80	0.70	0.72

Particular cases of Choquet integrals

- **Weighted arithmetic mean**

$$\mathrm{WAM}_\omega(x) = \sum_{i=1}^n \omega_i x_i\,, \quad \sum_{i=1}^n \omega_i = 1, \quad \omega_i \geq 0$$

Proposition
Let $v \in \mathcal{F}_N$.
The following assertions are equivalents
$i)$ v is additive
$ii)$ $\exists$ a weight vector ω such that $\mathcal{C}_v = \mathrm{WAM}_\omega$
$iii)$ $\mathcal{C}_v$ is additive: $\mathcal{C}_v(x+x') = \mathcal{C}_v(x) + \mathcal{C}_v(x')$

- **Ordered weighted averaging** (Yager, 1988)

$$\mathrm{OWA}_\omega(x) = \sum_{i=1}^n \omega_i x_{(i)}\,, \quad \sum_{i=1}^n \omega_i = 1, \quad \omega_i \geq 0$$

with the convention that $x_{(1)} \leq \cdots \leq x_{(n)}$.

Proposition (Grabisch-Marichal, 1995)
Let $v \in \mathcal{F}_N$.
The following assertions are equivalents
$i)$ v is cardinality-based
$ii)$ $\exists$ a weight vector ω such that $\mathcal{C}_v = \mathrm{OWA}_\omega$
$iii)$ $\mathcal{C}_v$ is a symmetric function.

Ordinal scales

Example : Evaluation of a scientific journal paper on importance

1=Poor, 2=Below average, 3=Average,
4=Very Good, 5=Excellent

Values :	1, 2, 3, 4, 5
or :	2, 7, 20, 100, 246
or :	-46, -3, 0, 17, 98

Numbers assigned to an ordinal scale are defined up to an increasing bijection $\varphi : \mathbb{R} \to \mathbb{R}$.

Definition. *A function $M : E^n \to \mathbb{R}$ is comparison meaningful if for any increasing bijection $\varphi : E \to E$ and any $x, x' \in E^n$,*

$$M(x_1, \ldots, x_n) \leq M(x'_1, \ldots, x'_n)$$
$$\Updownarrow$$
$$M(\varphi(x_1), \ldots, \varphi(x_n)) \leq M(\varphi(x'_1), \ldots, \varphi(x'_n))$$

Means on ordered sets

Example. The arithmetic mean is a meaningless function. Consider

$$4 = \frac{3+5}{2} < \frac{1+8}{2} = 4.5$$

and any bijection φ such that $\varphi(1) = 1$, $\varphi(3) = 4$, $\varphi(5) = 7$, $\varphi(8) = 8$. We have

$$5.5 = \frac{4+7}{2} \not< \frac{1+8}{2} = 4.5$$

Theorem 12 (Ovchinnikov, 1996)
The function $M : E^n \to \mathbb{R}$ is symmetric, continuous, internal, and comparison meaningful if and only if there exists $k \in N$ such that

$$M(x) = x_{(k)} \qquad (x \in E^n)$$

Note : $x_{(k)} = \text{median}(x)$ if $n = 2k - 1$

Lattice polynomials

Definition. A lattice polynomial defined in $\mathbb{R}^n$ is any expression constructed from the variables $x_1, \ldots, x_n$ and the symbols $\wedge$, $\vee$.

Example : $(x_2 \vee (x_1 \wedge x_3)) \wedge (x_4 \vee x_2)$

It can be proved that such an expression can always be put in the form

$$L_c(x) = \bigvee_{\substack{T \subseteq N \\ c(T)=1}} \bigwedge_{i \in T} x_i$$

where $c : 2^N \to \{0, 1\}$ is a nonconstant set function such that $c(\emptyset) = 0$.

In particular,

$$x_{(k)} = \bigvee_{\substack{T \subseteq N \\ |T|=n-k+1}} \bigwedge_{i \in T} x_i$$

$$\text{median}(x_1, \ldots, x_{2k-1}) = \bigvee_{\substack{T \subseteq N \\ |T|=k}} \bigwedge_{i \in T} x_i$$

Without symmetry :

Theorem 13 (Marichal-Mathonet, 2001)
The function $M : E^n \to \mathbb{R}$ *is continuous, idempotent, and comparison meaningful if and only if there exists a nonconstant set function* $c : 2^N \to \{0, 1\}$, *with* $c(\emptyset) = 0$, *such that*

$$M(x) = L_c(x) \qquad (x \in E^n)$$

Without symmetry and idempotency :

Theorem 14 *The function* $M : E^n \to \mathbb{R}$ *is nonconstant, continuous, and comparison meaningful if and only if there exist a nonconstant set function* $c : 2^N \to \{0, 1\}$, *with* $c(\emptyset) = 0$, *and a continuous and strictly monotonic function* $g : E \to \mathbb{R}$ *such that*

$$M(x) = g(L_c(x)) \qquad (x \in E^n)$$

Replacing continuity by increasing monotonicity:

Theorem 15 (Marichal-Mathonet, 2001)
Assume that E is open. The function $M : E^n \to \mathbb{R}$ is increasing, idempotent, and comparison meaningful if and only if there exists a nonconstant set function $c : 2^N \to \{0, 1\}$, with $c(\emptyset) = 0$, such that

$$M(x) = L_c(x) \qquad (x \in E^n)$$

Open problem : Describe increasing and comparison meaningful functions.

Connection with Choquet integral

Proposition 2 (Murofushi-Sugeno, 1993)
If $v \in \mathcal{F}_N$ is $\{0,1\}$-valued then $\mathcal{C}_v(x) = L_v(x)$

Conversely, we have $L_c(x) = \mathcal{C}_c(x)$

Proposition 3 (Radojević, 1998)
A function $M : E^n \to \mathbb{R}$ is a Choquet integral if and only if it is a weighted arithmetic mean of lattice polynomials

$$\mathcal{C}_v(x) = \sum_{i=1}^{q} \omega_i \, L_{c_i}(x)$$

This decomposition is not unique !

$$\begin{aligned} & 0.2x_1 + 0.6x_2 + 0.2(x_1 \wedge x_2) \\ = \; & 0.4x_2 + 0.4(x_1 \wedge x_2) + 0.2(x_1 \vee x_2) \end{aligned}$$

Proposition 4 (Marichal, 2001)
Any Choquet integral can be expressed as a lattice polynomial of weighted arithmetic means

$$\mathcal{C}_v(x) = L_c(g_1(x), \ldots, g_n(x))$$

Example (continued)

$$\begin{aligned} & 0.2x_1 + 0.6x_2 + 0.2(x_1 \wedge x_2) \\ = & (0.4x_1 + 0.6x_2) \wedge (0.2x_1 + 0.8x_2) \end{aligned}$$

The converse is not true ! The function

$$\Big(\frac{x_1 + x_2}{2}\Big) \wedge x_3$$

is not a Choquet integral.

Unsolved problem : Give conditions under which a lattice polynomial of weighted arithmetic means is a Choquet integral.