Characterization of some aggregation functions stable for positive linear transformations

by

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1. Basic definitions

Let $[a,b] \subseteq \mathbb{R}$ and $m \in \mathbb{N}_0$.

Definition 1. An *aggregation function* defined on $[a,b]^m$ is a function

$$M^{(m)} : [a,b]^m \to \mathbb{R}.$$

Example : the arithmetic mean

$$M^{(m)}(x_1, \dots, x_m) = \frac{1}{m} \sum_{i=1}^{m} x_i$$

Definition 2. An *aggregation operator* is a sequence $M = (M^{(m)})_{m \in \mathbb{N}_0}$ of aggregation functions defined on $[a,b]^m$.

Example : the arithmetic mean

$$M^{(1)}(x_1) = x_1$$

$$M^{(2)}(x_1, x_2) = \frac{x_1 + x_2}{2}$$

$$M^{(3)}(x_1, x_2, x_3) = \frac{x_1 + x_2 + x_3}{3}$$

...

2. Aggregation properties

Natural properties

- symmetry (Sy)
- increasing monotonicity ($\nearrow$)

Stability properties (Aczél, Roberts (1986))

- stability for the positive linear transformations (SPL):

$$M^{(m)}(rx_1+t, \dots, rx_m+t) = r M^{(m)}(x_1, \dots, x_m) + t$$

$$\forall (x_1, \dots, x_m) \in [a,b]^m \quad \forall r, t \in \mathbb{R}, \ r>0$$

$$\begin{array}{ccc} x_1, \dots, x_m & \xrightarrow{\quad M \quad} & M(x_1, \dots, x_m) \\ \downarrow \phi & & \downarrow \phi \\ \phi(x_1), \dots, \phi(x_m) & \xrightarrow{\quad M \quad} & \phi M(x_1, \dots, x_m) \\ & & \parallel \\ & & M(\phi x_1, \dots, \phi x_m) \end{array}$$

$$\phi(x) = rx + t, \ r>0$$

- stability for the standard negation N (SSN):

$$M^{(m)}(1-x_1, \dots, 1-x_m) = 1 - M^{(m)}(x_1, \dots, x_m)$$

$$\forall (x_1, \dots, x_m) \in [a,b]^m$$

Algebraic properties (Aczél (1966))

The aggregation function $M^{(m)}$ defined on $[a,b]^m$ is:

- associative (A) if $m=2$ and $\forall x \in [a,b]^3$

$$M^{(2)}(M^{(2)}(x_1,x_2),x_3) = M^{(2)}(x_1, M^{(2)}(x_2,x_3))$$

- autodistributive (AD) if $m=2$ and $\forall x \in [a,b]^3$

$$M^{(2)}(x_1, M^{(2)}(x_2,x_3)) = M^{(2)}(M^{(2)}(x_1,x_2), M^{(2)}(x_1,x_3))$$

$$M^{(2)}(M^{(2)}(x_1,x_2),x_3) = M^{(2)}(M^{(2)}(x_1,x_3), M^{(2)}(x_2,x_3))$$

- bisymmetric (B) if $m \geq 2$ and

$$M^{(m)}(M^{(m)}(x_{11},\dots,x_{1m}),\dots,M^{(m)}(x_{m1},\dots,x_{mm}))$$
$$= M^{(m)}(M^{(m)}(x_{11},\dots,x_{m1}),\dots,M^{(m)}(x_{1m},\dots,x_{mm}))$$

for all square matrices

$$\begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mm} \end{pmatrix} \in [a,b]^{m \times m}$$

The aggregation operator M is :

- **associative (A)** if $\forall \underline{x} \in [a,b]^m$,

$$M^{(m)}(x_1, \dots, x_j, \underbrace{\boxed{x_{j+1}, \dots, x_k}}, x_{k+1}, \dots, x_m)$$

$$= M^{(m-k+j+1)}(x_1, \dots, x_j, \bar{x}, x_{k+1}, \dots, x_m)$$

- **decomposable (D)** if $\forall x \in [a,b]^m$

$$M^{(m)}(x_1, \dots, \boxed{x_i'}, \dots, x_j, \dots, \boxed{x_k'}, \dots, \boxed{x_\ell'}, \dots)$$

$$= M^{(m)}(x_1, \dots, \bar{x}', \dots, x_j, \dots, \bar{x}', \dots, \bar{x}', \dots)$$

- **strongly bisymmetric (SB)** if

$$M^{(p)}(M^{(m)}(x_{11}, \dots, x_{1m}), \dots, M^{(m)}(x_{p1}, \dots, x_{pm}))$$

$$= M^{(m)}(M^{(p)}(x_{11}, \dots, x_{p1}), \dots, M^{(p)}(x_{1m}, \dots, x_{pm}))$$

for all (~~square~~) matrices

$$\begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{p1} & \cdots & x_{pm} \end{pmatrix} \in [a,b]^{p \times m}$$

3. Examples

Aggregation functions $M^{(m)}$	Aggregation operators M
Weighted arithmetic mean	Weighted arithmetic mean
$WAM^{(m)}_{\omega^{(m)}}(x_1,\dots,x_m) = \sum_i \omega_i^{(m)} x_i$	$WAM_\omega = \left(WAM^{(m)}_{\omega^{(m)}}\right)_{m\in\mathbb{N}_0}$
$\sum \omega_i^{(m)} = 1 \;,\; \omega_i^{(m)} \geq 0$	$\omega = \left(\omega^{(m)}\right)_{m\in\mathbb{N}_0} \;,\; \sum \omega_i^{(m)} = 1 ,\; \omega_i^{(m)} \geq 0$
Arithmetic mean	Arithmetic mean
$AM^{(m)}(x_1,\dots,x_m) = \frac{1}{m}\sum_i x_i$	$AM = \left(AM^{(m)}\right)_{m\in\mathbb{N}_0}$

Aggregation functions $M^{(m)}$	Aggregation operators M
Projection associated to the i-th argument	First projection and last projection
$P_i^{(m)}(x_1,\dots,x_m) = x_i$	$FP = (P_1^{(m)})_{m\in\mathbb{N}_0}$ $LP = (P_m^{(m)})_{m\in\mathbb{N}_0}$
Minimum and Maximum	Minimum and maximum
$MIN^{(m)}(x_1,\dots,x_m) = \min_i x_i$ $MAX^{(m)}(x_1,\dots,x_m) = \max_i x_i$	$MIN = (MIN^{(m)})_{m\in\mathbb{N}_0}$ $MAX = (MAX^{(m)})_{m\in\mathbb{N}_0}$
Partial minimum and maximum	Partial minimum and maximum
$MIN^{(m)}_{N^{(m)}}(x_1,\dots,x_m) = \min_{i\in N^{(m)}} x_i$ $MAX^{(m)}_{N^{(m)}}(x_1,\dots,x_m) = \max_{i\in N^{(m)}} x_i$	$MIN_N = (MIN^{(m)}_{N^{(m)}})_{m\in\mathbb{N}_0}$ $MAX_N = (MAX^{(m)}_{N^{(m)}})_{m\in\mathbb{N}_0}$
$N^{(m)} \neq \emptyset$	$N = (N^{(m)})_{m\in\mathbb{N}_0}$

4 Characterizations

Aggregation functions

$m=2$	$\underline{\nearrow, \text{SPL}, A}$ $\text{MIN}^{(2)}, \text{MAX}^{(2)}, P_1^{(2)}, P_2^{(2)}$	$\underline{+\ \text{Sy}}$ $\text{MIN}^{(2)}, \text{MAX}^{(2)}$	$\underline{+\ \text{SSN}}$ $P_1^{(2)}, P_2^{(2)}$
$m=2$	$\underline{\nearrow, \text{SPL}, \text{AD}}$ $\text{MIN}^{(2)}, \text{MAX}^{(2)}$ $\{\text{WAM}^{(2)}_{\omega^{(2)}} \mid \omega^{(2)} \in [0,1]^2\}$	$\underline{+\ \text{Sy}}$ $\text{MIN}^{(2)}, \text{MAX}^{(2)}, \text{AM}^{(2)}$	$\underline{+\ \text{SSN}}$ $\{\text{WAM}^{(2)}_{\omega^{(2)}} \mid \omega^{(2)} \in [0,1]^2\}$
$m \geq 2$	$\underline{\nearrow, \text{SPL}, B}$ $\{\text{MIN}^{(m)}_{N^{(m)}} \mid N^{(m)} \subseteq \{1,\dots,m\}\}$ $\{\text{MAX}^{(m)}_{N^{(m)}} \mid N^{(m)} \subseteq \{1,\dots,m\}\}$ $\{\text{WAM}^{(m)}_{\omega^{(m)}} \mid \omega^{(m)} \in [0,1]^m\}$	$\underline{+\ \text{Sy}}$ $\text{MIN}^{(m)}, \text{MAX}^{(m)}, \text{AM}^{(m)}$	$\underline{+\ \text{SSN}}$ $\{\text{WAM}^{(m)}_{\omega^{(m)}} \mid \omega^{(m)} \in [0,1]^m\}$

Aggregation operators

$\nearrow$, SPL, A MIN, MAX, FP, LP	+ Sy MIN, MAX	+ SSN FP, LP
$\nearrow$, SPL, D MIN, MAX, FP, LP, AM	+ Sy MIN, MAX, AM	+ SSN FP, LP, AM
$\nearrow$, SPL, SB $\{MIN_N \mid N = (N^{(m)})_{m \in \mathbb{N}_0}\}$ $\{MAX_N \mid N = (N^{(m)})_{m \in \mathbb{N}_0}\}$ $\{WAM_w \mid w = (w^{(m)})_{m \in \mathbb{N}_0}\}$	+ Sy MIN, MAX, AM	+ SSN $\{WAM_w \mid w = (w^{(m)})_{m \in \mathbb{N}_0}\}$