WE ARE CONCERNED WITH A BASIC QUESTION IN MCDA

How do we aggregate ordinal information ?

N $(k \in N)$: set of points of view

A $(a, b, \ldots \in A)$: set of potential actions

g_k :(mapping from A to X_k)

X_k : ordinal scale related to k
{set of all possible linguistic variables for g_k}

Ex :

$+ \quad 0 \quad -$

X_1 : (3 pt. scale)

Excellent Average Weak

X_2 : (3 pt. scale)

Excel. V.Good Good Satisf. Weak V.Weak

X_3 :
(6pt. scale)

PROFILE RELATED TO ACTION a

$$g(a) : (\underset{\in X_1}{g_1(a)}, \ldots \underset{\in X_k}{g_k(a)}, \ldots , \underset{\in X_n}{g_n(a)}) \in \prod_{i=1}^{n} X_i$$

We will assume the commensurability among diff. scales i.e. we determine

ordinal utilities : $U_k : g_k \to L$ (common ordinal scale)

and we define an aggregation function M that determines the

consensus among points of view

i.e. an ordinal global utility

$$U(g_1, \ldots , g_n) = M[\underset{\in L}{U_1(g_1)}, \underset{\in L}{U_2(g_2)}, \ldots , \underset{\in L}{U_n(g_n)}] \in L$$

As a consequence,

all actions are comparable in terms of a WEAK ORDER (partial preorder) defined on A

TYPICAL PROBLEMS

Ph.D. students selection (R. Fuller, Ph.D. th., 1998)

Research Interests

	Excellent	Average	Weak
Fit in research group	☐	☐	☐
On the frontier of research	☐	☐	☐
Contributions	☐	☐	☐

Academic Background

	Excellent	Average	Weak
University	☐	☐	☐
Grade average	☐	☐	☐
Time for acquiring degree	☐	☐	☐

Letters of recommendation

Yes	No
☐	☐

Question : what is the global evaluation of candidate a :

Excellent	Average	Weak
☐	☐	☐

APPLICATION FOR AN ACADEMIC POSITION AT ULg (1998)

	Exc	V.G	G	Sat	Weak
Scientific value of CV	☐	☐	☐	☐	☐

	Exc	V.G	Sat	Weak	V.W
Teaching effectiveness	☐	☐	☐	☐	☐

	Positive	Neutral	Neg.
Interview	☐	☐	☐

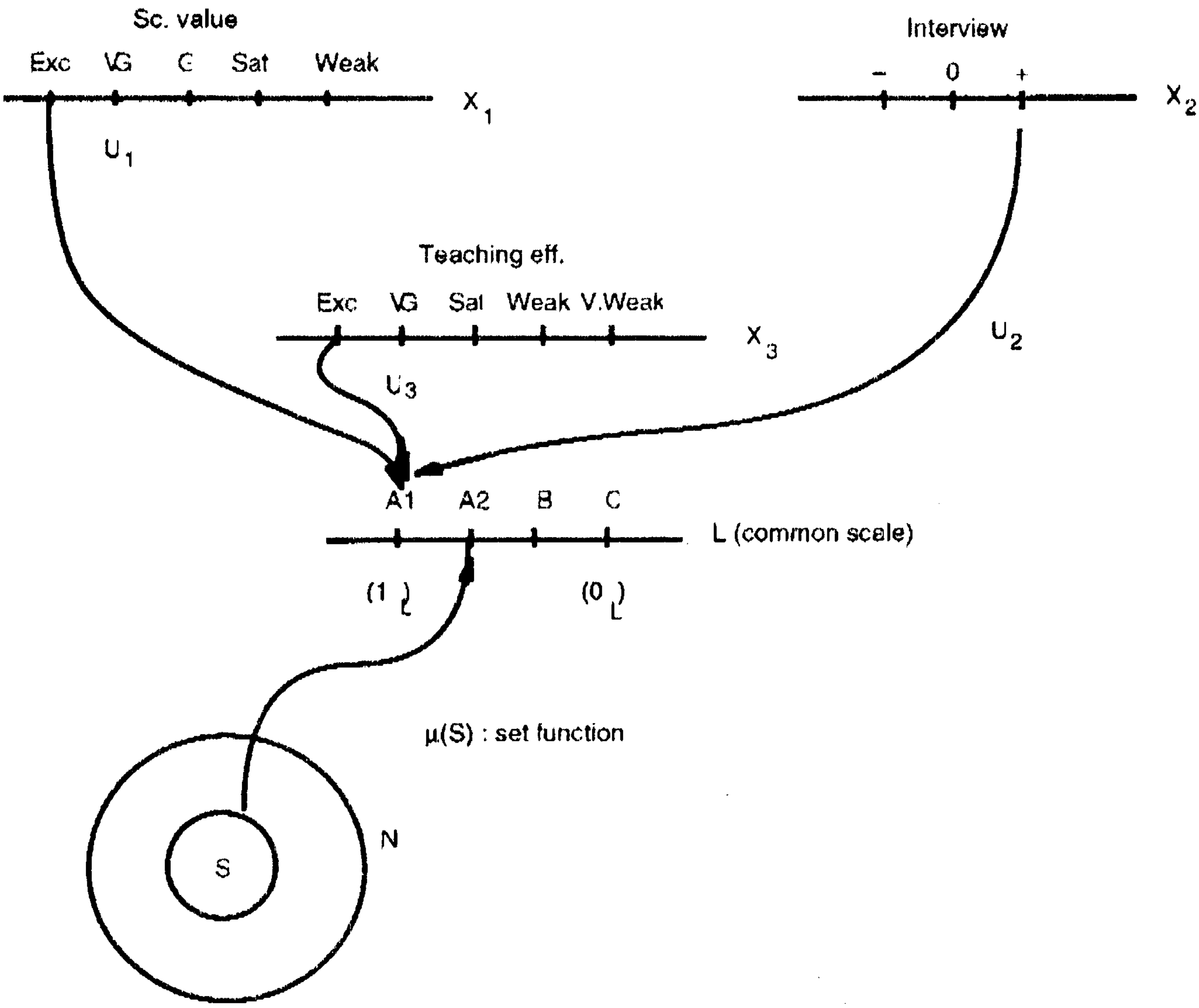
One has to deliver a global evaluation

A1	A2	B	C
☐	☐	☐	☐

We assume the commensurability among

- the ordinal scales
- degree of importance of subsets of points of view

Using ordinal utilities : $U_i(g_i) \in L$



POSSIBLE PROBLEMS :
INTERACTIVITY AMONG CRITERIA
(i.e. violation of preferential independence)

	Maths	Physics	Literature
a	VG	G	VG
b	VG	VG	G

If "very good" in Maths then rank acc. to Lit. ⇒

$$a > b$$

	Maths	Physics	Literature
c	Weak	G	VG
d	Weak	VG	G

If "weak" in Maths then rank acc. to Phys. ⇒

$$d > c$$

By monotonicity : $b > d$

$$\Rightarrow \boxed{a > b > d > c}$$

PREFERENTIAL INDEPENDENCE

Notations :

profile f	$(f_1, \ldots, f_n)$
profile g	$(g_1, \ldots, g_n)$
profile h	$(h_1, \ldots, h_n)$
profile k	$(k_1, \ldots, k_n)$
profile fAh	$(\underbrace{f_1, \ldots, f_a}_{A} \underbrace{g_{a+1}, \ldots, g_n}_{\bar{A}})$

$$f \succeq g \quad \text{if} \quad U(f) \geq U(g)$$

PREF. IND. MEANS

$$fAh \succeq gAh \Rightarrow fAk \succeq gAk$$

$$\forall A \subset N, \ \forall f, g, h, k \in \prod_k X_k$$

In classical expected utility theory, this property was criticized firstly by Allais (1952) and was the starting point of contributions related to nonexpected utility (see Edwards (92) for a survey).

WE WILL

- PROPOSE AS CONSENSUS FUNCTION THE SUGENO INTEGRAL
- CHARACTERIZE THIS AGGREGATOR AND
- SHOW SOME PROPERTIES INCLUDING AN IMPOSSIBILITY THEOREM WHICH RELATES CONSENSUS TO ARROW'S THEOREM

SUGENO INTEGRAL

Consider $U_k[g_k]$ defined on $[0,1]$

$\mu(T)$: Choquet capacity
fuzzy measure (Sugeno measure)
$\mu(S) \leq \mu(T)$ if $S \subset T$
$\mu(\emptyset) = 0_L \quad \mu(N) = 1_L$

measured on the same ordinal scale.

We briefly write $U_k[g_k] : x_k$.

We define a consensus function $M_\mu(x_1, \ldots, x_n)$ of Sugeno integral type as

$$
\begin{aligned}
& U_S(x_1, \ldots, x_n) \\
&= \bigvee_{T \subset N} \left[\mu(T) \wedge \left(\bigwedge_{i \in T} x_i \right) \right] \qquad \text{max-min form} \\
&= \bigvee_{i=1}^{n} \left[x_{(i)} \wedge \mu((i), \ldots, (n)) \right] \\
&= \bigwedge_{T \subset N} \left[\mu(N \setminus T) \vee \left(\bigvee_{i \in T} x_i \right) \right] \qquad \text{max-min form} \\
&= \bigwedge_{i=1}^{n} \left[x_{(i)} \vee \mu((i+1), \ldots, (n)) \right] \\
&= \text{median} \left[x_1, \ldots, x_n, \mu((2), \ldots, (n)), \ldots, \mu((n)) \right] \\
& \qquad \text{median form}
\end{aligned}
$$

Instead of dealing with

WEIGHTED MEANS

used in the cardinal utility theory

- $M(x_1, \dots, x_n) = \sum_i p_i x_i = \sum_i p_i W_i(g_i)$
- $U_C(x_1, \dots, x_n) = \dots$

Choquet integral : weighted sum of ordered values $x_{(i)} \leq \cdots \leq x_{(n)}$ non additive in terms of $W_i(g_i)$.

WE WILLL CONSIDER MEDIANS

which is the classical statistical estimator of the mean when dealing with ordinal values

$$U_S(x_1 \dots x_n) = \text{median}(x_{(1)}, \dots, x_{(n)}, \mu((2) \dots (n)), \dots, \mu((n))) \in L$$

Particular cases of Sugeno integrals

Boolean max-min, min-max

$$
\begin{aligned}
U_S &= B_\mu^{\vee\wedge} = \bigvee_{T\subset N}\left[\mu(T)\wedge\left(\bigwedge_{i\in T}x_i\right)\right]\\
&= \bigwedge_{T\subset N}\left[\mu(N\setminus T)\vee\left(\bigvee_{i\in T}x_i\right)\right]
\end{aligned}
$$

$$\mu(T)\in\{0,1\}$$

Ordinal OWA operator (Dubois et al., 1994) (Yager, 1994)

$$\mu((i),\ldots,(n)) = v(T) = w_{n-t+1}$$

independent of the ordering depends only on the cardinality

$$
\begin{aligned}
U_S &= OOWA_\mu\\
&= \bigvee_{i=1}^{n}(x_{(i)}\wedge w_i),\ w_1 = 1_L,\ w_1\geq w_2\geq\cdots\geq w_n\\
&= \text{median}(x_1,\ldots,x_n,w_2,\ldots,w_n)
\end{aligned}
$$

Weighted max

If μ is a possibility measure Π i.e. defined by $(p_1, \ldots, p_n)$

$$\bigvee_i p_i = 1_L, \qquad \mu(T) = \bigvee_{i \in T} p_i$$

$$U_S(x) = \bigvee_{i=1}^{n} [x_i \wedge p_i]$$

Weighted min

If μ is a necessity measure N i.e. defined by $(n_1, \ldots, n_n)$

$$\bigwedge_i n_i = 0_L, \qquad \mu(N \setminus T) = \bigwedge_{i \in T} n_i$$

$$U_S(x) = \bigwedge_{i=1}^{n} [x_i \vee n_i]$$

Some desirable properties of the Sugeno aggregator

- $U_S(\overline{x};\mu) = U_S(x,\ldots,x;\mu) = x$,
$\overline{x}$: constant action. U_S is idempotent.

- Consider $(\overline{1}_L A \overline{0}_L) = (\underbrace{1_L \ldots 1_L}_{A} \underbrace{0_L \ldots 0_L}_{\overline{A}})$

$$U_S(\overline{1_L} A \overline{0_L}) = \mu(A).$$

$\mu(A)$ is interpreted as the utility of the profile $(\overline{1}_L A \overline{0}_L)$.

- Consider now a binary action

$$\overline{x} A \overline{y} = (\underbrace{x \ldots x}_{A} \underbrace{y \ldots y}_{\overline{A}})$$

$$\begin{aligned} U_S(\overline{x} A \overline{y}) &= \text{median } (x, y, \mu(\overline{A})) \text{ if } x < y \\ &= \text{median } (x, y, \mu(A)) \text{ if } x > y \end{aligned}$$

$U_S(\overline{x} A \overline{y})$ is either equal to $x, y, \mu(A), \mu(\overline{A})$.

If $x < y$

$$U_S(\overline{x}A\overline{y}) \text{ is either}$$

$\min(x, y)$
$\max(x, y)$
compensative value $\mu(\overline{A})$ if such that

$$x < \mu(\overline{A}) < y$$

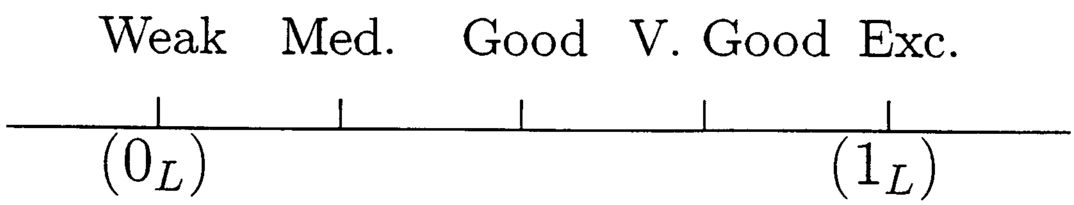
(This property is called "non compensation" by Dubois, Prade, Sabbadin (1998)).

Some "imposed" (maybe undesirable) properties of Sugeno integrals

- If k is a " veto criterion" at level x, i.e. $U(x\{k\}f) = x$, $f_{h \neq k} > x$,
THEN $U(x\{k\}\overline{1}_L) = x$.

(Excellency on N/k cannot compensate the weakness on k).

Suppose L-scale :



If U (Weak Med Med Med) = Weak
then U (Weak Exc. Exc. Exc.) = Weak

- If A is a veto coalition on level x, i.e. $U(\overline{x}Af) = x$, $f_{k \notin A} > x$,
THEN $U(\overline{x}A\overline{1}_L) = x$

(Excellency on $\overline{A}$ cannot compensate weakness on A)

Dual property

- If k is a "favor criterion" at level x, i.e. $U(x\{k\}f) = x$, $f_{h\neq k} < x$,
THEN $U(x\{k\}\overline{0}_L) = x$.

Ex.
If U (Exc. V.Good V.Good V.Good) = Exc.
Then U(Exc. Weak Weak Weak) = Exc. (!!!)

- If $f \succ g$ and $f \succ \overline{x}$
  THEN $f \succ g \vee \overline{x}$

(If f is preferred to g and also to constant profile x, then even if the worst scores of g are improved to x, f is still preferred to the modified boosted g)

- If $f \prec g$ and $f \prec \overline{x}$
  THEN $f \prec g \wedge \overline{x}$

IMPOSSIBILITY THEOREM

Introduction

Choquet integral is used in order to

- deal with interactions among criteria (basic work done by Schmeidler (1986), Wakker (1989))
- overcome the classical problem of preferential independence (sure-thing principle)

Murofushi and Sugeno (1992) have shown that the use of Choquet integral as a consensus function implies that

preferential independence

$\Rightarrow \mu$ is additive

(decomposable utility function)

$\Rightarrow$ classical expected utility.

Central question is

Sugeno integral + preferential independence

$\Rightarrow$?

Sugeno integral

$+$

preferential independence

$$\Rightarrow \boxed{\text{One criterion is a dictator.}}$$

Relaxation of preferential independence in terms of <u>weak</u> preferential independence modifies this result.

Directional preferential independence in coordinates means

$$x\{k\}y \succ x'\{k\}y \Rightarrow x\{k\}z \succeq x'\{k\}z, \quad \forall k$$

Directional mutual independence means

$$fAh \succ gAh \Rightarrow fAk \succeq gAk$$

Aggregation according to Sugeno integral

Directional preferential independence in coordinates and directional mutual independence might be violated.

Ex. with directional preferential independence in coordinates that is not violated
($\Rightarrow$ dictator)

C_1 \ C_2	+	0	-
+	+	0	0
0	0	0	0
-	-	-	-

Decision table:

$$\underset{f\;h}{(+\,+)} \succ \underset{g\;h}{(0\,+)}$$

but

$$\underset{f\;k}{(+\,0)} \sim \underset{g\;k}{(0\,0)}$$

$$\underset{f\;k'}{(+\,-)} \sim \underset{g\;k'}{(0\,-)}$$

1) The table can be obtained with Sugeno integral and $\mu(2) = \underset{(0_L)}{-}$, $\mu(1) = 0$,

$$\mu(1,2) = \underset{(1_L)}{+}.$$

2) No dictator.

Aggregation according to Sugeno integral where Directional preferential independence in coordinates is violated

Ex.

$$\beta \quad a \quad \mu(3) \quad \mu(2) \quad \mu(1) \quad \mu(12) \quad \mu(13) \quad \mu(123) \quad b \quad \alpha$$

$$\begin{aligned}(ab\alpha) &\succ (ba\alpha) \qquad (1)\\ (ab\beta) &\prec (ba\beta) \qquad (2)\end{aligned}$$

$$\begin{aligned}(ab\alpha) &= \text{median}(a, b, \alpha, \mu(23), \mu(3)) = \mu(23)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vee\\ (ba\alpha) &= \text{median}(b, a, \alpha, \mu(13), \mu(3)) = \mu(13)\end{aligned}$$

$$\begin{aligned}(ab\beta) &= \text{median}(a, b, \beta, \mu(12), \mu(2)) = \mu(2)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wedge\\ (ba\beta) &= \text{median}(a, b, \beta, \mu(12), \mu(1)) = \mu(1)\end{aligned}$$

LINKS WITH DECISION UNDER UNCERTAINTY

In von Neumann and Morgenstern and Savage pioneer works, different "acts" under various "states of nature" are considered.

One can evaluate the consequence of an act a under state of nature k : $U[g_k(a)]$.

There is a common evaluation scale for events (state of nature) and acts and it is possible to evaluate uncertainty and preference by means of a totally ordered scale $(L, \succeq)$.

Different measures of uncertainty have been considered :

$k \longrightarrow p_k$ probability measures v. N & M (1944)
Savage (1953)

$S \subset N \longrightarrow \mu(S)$:belief functions

Jaffray & Wakker (1994)
Sarin & Wakker (1992)

$\mu(S)$: possibility measures

Dubois & Prade (1995)

: capacities

Dubois, Prade, Sabbadin (1998)

Characterization of Sugeno integral consensus (Marichal (1998))

Consider $\{x_i\}$ all being defined on the same ordinal scale.
Admissible transformations : bijections φ.

(i) M is continuous.
(ii) M is idempotent.
(iii) M satisfies the "ordinal comparison meaningfulness" condition

$$M(x) \leq M(y) \Leftrightarrow M(\varphi x) \leq M(\varphi y)$$

$$[(\mathrm{i}) + (\mathrm{ii}) + (\mathrm{iii})] \Leftrightarrow$$

$$\begin{aligned} M(x) & \\ = & \ \mathrm{median}(x_1, \ldots, x_n, \underbrace{\mu((i)\ldots(n)), \ldots, \mu((n))}_{\in\{0,1\}}) \\ = & \ B_\mu^{\vee\wedge}, \text{ boolean max-min} \end{aligned}$$

μ does not show up !

If (iv) : preferential independence is introduced,

$$[(\mathrm{i}) + (\mathrm{ii}) + (\mathrm{iii}) + (\mathrm{iv})] \Rightarrow \exists \text{ dictator}.$$

Second result : Marichal (1998)

Consider $M(x, \mu)$:

(P1) : M is continuous

(P2) : M is idempotent on X

(P3) : M satisfies the "ordinal comparison meaningfulness"

$$[(P1) + (P2) + (P3)] \Leftrightarrow$$

$$M \text{ is } M_S(x, \mu)$$

(P4) : $M_S(e_S, \mu) = \mu(S)$

Characterization of Sugeno integrals

Sabbadin (1998) in the spirit of the work by Savage on decision under uncertainty.

Consider $(x_1, \dots , x_n)$ commensurable evaluations.

(P1) Ranking $(\succcurlyeq, A)$ (Savage first axiom).
A complete preorder on the set A is supposed to exist.

(P2) Non triviality : $\exists g_j, g_\ell$ such that $g_k < g_\ell$ (Savage fifth axiom).
Non trivial comparisons between evaluation exist.

(P3) Weakened order over constant actions (weaker than Savage third axiom)

$$x < y \Rightarrow \overline{x}Ah \prec \overline{y}Ah$$

(P4) "Non compensation" : $(\overline{x}A\overline{y})$ is either equal to $x, y, \mu(A), \mu(\overline{A})$.
The consensus of a binary action reflects one of its two evaluations or the satisfaction of the subsets which create the dichotomy.

(P5) Commensurability : $\exists g \in X$, such that $\overline{g} \sim (\overline{1}A\overline{0})$.
The satisfaction level scale can be projected on the common ordinal preference scale.
You can exchange a constant.

$[(P1) + (P2) + (P3) + (P4) + (P5)] \Leftrightarrow$

$$\exists U,\ \mu : \text{Choquet capacity}$$

such that

$$M(x_1, \ldots, x_n) = \bigvee_{T \subset N} \left[\mu(T) \wedge \left(\bigvee_{i \in T} x_i \right) \right]$$