Ordinal aggregation with qualitative scales*

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We consider a finite set of potential actions that are compared using different points of view. The evaluation of each action for a given point of view is measured on a qualitative (ordinal) scale. We suppose that "utilities" are associated to each measure related to a given qualitative scale in such a way that all these monotonic utilities are commensurable.

The utilities are thus observed on a common ordinal scale, i.e. admissible transformations are monotonic increasing functions defined on a support which is supposed to be the unit interval.

The satisfaction degree related to each point of view and to any subset of all different points of view is expressed in terms of a Choquet capacity also called fuzzy measure (monotonic set function such that the measure of the empty set is equal to zero and the measure of the whole set is one). This degree is measured on the same ordinal scale that is used to observe the utilities.

If one accepts these rather strong assumptions, the multiple criteria decision making framework can be linked to the theory of preference between acts for decision under uncertainty. The states of the world correspond to the points of view and the set of acts corresponds to the set of potential actions.

We will consider an aggregation operator that determines a consensus among the different points of view called a Sugeno integral, i.e. a max-min combination of utilities and satisfaction degrees.

Particular cases of Sugeno integrals are Boolean max-min operators (in that case, Choquet capacities are supposed to be equal either to zero or one) and the weighted

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max (Choquet capacities are then identified to possibility measures).

In the case of Boolean Choquet capacities, the Sugeno integral and the corresponding Choquet integral are confounded.

Sugeno integrals and weighted max can be easily expressed in terms of the median of the utilities and some satisfaction degrees. Clearly the median is the qualitative counterpart to the averaging operators (classical weighted means and Choquet integrals) underlying the cardinal expected utility theory (Bernoulli (1938), von Neumann and Morgenstern (1944), Savage (1954), Jaffray (1992), Sarin and Wakker (1992)).

The first characterization of qualitative consensus functions is due to Sabbadin (1998) in the spirit of the work of Savage. Five conditions are considered :

1. Ranking (as in Savage first axiom) : a complete preorder on the set of potential actions is supposed to exist and to be revealed by the descriptive process).

2. Non triviality (corresponds to the Savage fifth axiom) : non trivial comparisons between utilities exist.

3. Weakened order over constant actions (weaker condition but close to Savage third axiom).

4. Non compensation : the consensus of a binary action reflects one of its two consequences or the satisfaction of the subsets of points of view which create the dichotomy.

5. Commensurability : the satisfaction degree scale can be projected on the common ordinal preference scale.

These five axioms characterize the monotonic utility of a potential action, i.e. a consensus function expressed in terms of a Sugeno integral.

The Sugeno integral consensus function has been axiomatized in an equivalent way by Marichal (1998).

Let us consider the following conditions over a consensus function that aggregates all utilities and satisfaction degrees :

(i) continuity

(ii) monotonicity

(iii) independence of constant actions with respect to satisfaction degrees : the consensus on equal utilities does not depend on satisfaction degrees.

(iv) comparison meaningfulness : the linear order on the set of actions induced by the consensus function is preserved for any continuous monotonic transformations of the common ordinal scale used to evaluate the actions for all criteria and the satisfaction degrees.

Conditions (i)–(iv) characterize consensus functions which are either constant either a strictly increasing transformation of the Sugeno integral.

The following example is constantly used to illustrate the concepts that are introduced in the presentation.

Consider that students are evaluated on three different topics (M : mathematics, φ : physics, L : literature). For each topic, a mark is given that corresponds to a 6-point scale. ($g_{(6)}$: excellent, $g_{(5)}$: very good, $g_{(4)}$: good, $g_{(3)}$: fair, $g_{(2)}$: satisfactory, $g_{(1)}$: weak).

Three prototypes are considered : a, b, c.
a is excellent in M, very good in φ, good in L.
b is good in M, satisfactory in φ, excellent in L.
c is fair in M, good in φ and in L.

The decision maker is able to rank the prototypes according to the following order : $c > a > b$.

The decision maker also gives the following marks to the following vectors of scores :
(excellent in M, excellent in L, weak in φ) receives a global mark "very good"
(excellent in M and φ, weak in L) receives a global mark "fair"
(excellent in M, weak in φ and L) is considered to be globally "satisfactory"
(excellent in L, weak in M and φ) corresponds to "globally weak".

The decision maker also considers that M and φ should be treated in an equivalent way.