A Complete Description of Comparison Meaningful Functions

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We assume that

$$F_{f_1,\ldots,f_n}(a) = F[f_1(a), \ldots, f_n(a)] \qquad (a \in S)$$

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x_i defines an *ordinal scale* if the class of admissible transformations consists of the increasing bijections (automorphisms) of $\mathbb{R}$ onto $\mathbb{R}$.

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For any $\phi_1, \ldots, \phi_n \in A(\mathbb{R})$, there is $\Phi_{\phi_1,\ldots,\phi_n} \in A(\mathbb{R})$ such that

$$F[\phi_1(x_1), \ldots, \phi_n(x_n)] = \Phi_{\phi_1,\ldots,\phi_n}[F(x_1, \ldots, x_n)]$$

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Equivalently, F fulfills the condition (Orlov 1981)

$$\begin{gathered} F(x_1, \ldots, x_n) \leqslant F(x_1', \ldots, x_n') \\ \Updownarrow \\ F[\phi(x_1), \ldots, \phi(x_n)] \leqslant F[\phi(x_1'), \ldots, \phi(x_n')] \end{gathered}$$

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F is said to be *comparison meaningful* (Ovchinnikov 1996)

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We say that F is *strongly comparison meaningful*

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$$\Leftrightarrow \left\{ \begin{array}{c} \exists\, k \in \{1, \ldots, n\} \text{ such that} \\ F(x_1, \ldots, x_n) = x_{(k)} \end{array} \right.$$

Comparison meaningful functions

First result (Orlov 1981)

$F : \mathbb{R}^n \to \mathbb{R}$ is
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Next step : suppress symmetry and relax internality into idempotency

Lattice polynomials

Lattice polynomials

Definition (Birkhoff 1967)

An n-variable *lattice polynomial* is any expression involving n variables $x_1, \ldots, x_n$ linked by the lattice operations

$$\wedge = \min \quad \text{and} \quad \vee = \max$$

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For example,

$$L(x_1, x_2, x_3) = (x_1 \vee x_3) \wedge x_2$$

is a 3-variable lattice polynomial.

Lattice polynomials

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We have

$$x_{(k)} = \bigvee_{\substack{T\subseteq\{1,\ldots,n\}\\ |T|=n-k+1}} \bigwedge_{i\in T} x_i = \bigwedge_{\substack{T\subseteq\{1,\ldots,n\}\\ |T|=k}} \bigvee_{i\in T} x_i$$

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Final step : suppress nondecreasing monotonicity (a hard task !)

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- We loose the concept of lattice polynomial
- The description of F is done through a partition of the domain $\mathbb{R}^n$ into particular subsets, called *invariant subsets*

Invariant sets

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Definition (Bartłomiejczyk & Drewniak 2004)

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- A nonempty set $I \subseteq \mathbb{R}^n$ is *invariant* if

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The family $\mathcal{I}(\mathbb{R}^n)$ partitions $\mathbb{R}^n$ into equivalence classes :

$$x \sim y \quad \Leftrightarrow \quad \exists\, \phi \in A(\mathbb{R}) \;:\; y_i = \phi(x_i) \;\; \forall\, i$$

Description of the family $\mathcal{I}(\mathbb{R}^n)$

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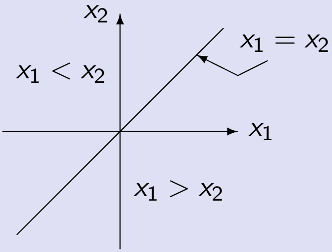
Example : $\mathbb{R}^2$

Minimal invariant sets :

$I_1 = \{(x_1,x_2) \mid x_1 = x_2\}$
$I_2 = \{(x_1,x_2) \mid x_1 < x_2\}$
$I_3 = \{(x_1,x_2) \mid x_1 > x_2\}$



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Invariant functions

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Now, assume that

$$x_{n+1} = F(x_1, \ldots, x_n)$$

where $x_1, \ldots, x_n$ and x_{n+1} define the *same* ordinal scale.

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$$x_{n+1} = F(x_1, \ldots, x_n)$$

where $x_1, \ldots, x_n$ and x_{n+1} define the *same* ordinal scale.

Then the functional equation simplifies into

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F is said to be *invariant* (Bartłomiejczyk & Drewniak 2004)

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Next step : suppress symmetry and nondecreasing monotonicity

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Final step : suppress nondecreasing monotonicity

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These results hold true when F is defined on E^n, where E is any open real interval.

The cases where E is a non-open real interval all have been described and can be found in

J.-L. Marichal, R. Mesiar, and T. Rückschlossová,
A Complete Description of Comparison Meaningful Functions,
Aequationes Mathematicae, in press.

Thank you for your attention