

A Complete Description of Comparison Meaningful Functions

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Let F be an aggregation function from \mathbb{R}^n to \mathbb{R} :

$$x_{n+1} = F(x_1, \dots, x_n)$$

where x_1, \dots, x_n are the independent variables and x_{n+1} is the dependent variable.

The general form of F is restricted if we know the *scale type* of the variables x_1, \dots, x_n and x_{n+1} (Luce 1959).

A scale type is defined by the class of *admissible transformations*, transformations which change the scale into an alternative acceptable scale.

x_i defines an *ordinal scale* if the class of admissible transformations consists of the increasing bijections of \mathbb{R} onto \mathbb{R} .

Principle of theory construction (Luce 1959)

Admissible transformations of the independent variables should lead to an admissible transformation of the dependent variable.

Suppose that

$$x_{n+1} = F(x_1, \dots, x_n)$$

where x_{n+1} is an ordinal scale and x_1, \dots, x_n are independent ordinal scales.

Let $A(\mathbb{R})$ be the set of increasing bijections of \mathbb{R} onto \mathbb{R} .

For any $\phi_1, \dots, \phi_n \in A(\mathbb{R})$, there is $\Phi_{\phi_1, \dots, \phi_n} \in A(\mathbb{R})$ such that

$$F[\phi_1(x_1), \dots, \phi_n(x_n)] = \Phi_{\phi_1, \dots, \phi_n}[F(x_1, \dots, x_n)]$$

Assume x_1, \dots, x_n define the *same* ordinal scale.
Then the functional equation simplifies into

$$F[\phi(x_1), \dots, \phi(x_n)] = \Phi_\phi[F(x_1, \dots, x_n)]$$

Equivalently, F fulfills the condition (Orlov 1981)

$$\begin{aligned} F(x_1, \dots, x_n) &\leq F(x'_1, \dots, x'_n) \\ &\iff \\ F[\phi(x_1), \dots, \phi(x_n)] &\leq F[\phi(x'_1), \dots, \phi(x'_n)] \end{aligned}$$

F is said to be *comparison meaningful* (Ovchinnikov 1996)

Assume x_1, \dots, x_n are *independent* ordinal scales.

Recall that the functional equation is

$$F[\phi_1(x_1), \dots, \phi_n(x_n)] = \Phi_{\phi_1, \dots, \phi_n}[F(x_1, \dots, x_n)]$$

Equivalently, F fulfills the condition

$$\begin{aligned} F(x_1, \dots, x_n) \leq F(x'_1, \dots, x'_n) \\ \Downarrow \\ F[\phi_1(x_1), \dots, \phi_n(x_n)] \leq F[\phi_1(x'_1), \dots, \phi_n(x'_n)] \end{aligned}$$

We say that F is *strongly comparison meaningful*

Purpose of the presentation

To provide a complete description of
comparison meaningful functions

To provide a complete description of
strongly comparison meaningful functions

The continuous case

First result (Osborne 1970, Kim 1990)

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ is **continuous** and **strongly comparison meaningful**

$$\Leftrightarrow \left\{ \begin{array}{l} \exists k \in \{1, \dots, n\} \\ \exists g : \mathbb{R} \rightarrow \mathbb{R} \text{ - continuous} \\ \text{ - strictly monotonic or constant} \\ \text{such that} \\ F(x_1, \dots, x_n) = g(x_k) \end{array} \right.$$

+ **idempotent**, i.e., $F(x, \dots, x) = x$

$$\Leftrightarrow \left\{ \begin{array}{l} \exists k \in \{1, \dots, n\} \text{ such that} \\ F(x_1, \dots, x_n) = x_k \end{array} \right.$$

The general case

Second result (Marichal & Mesiar & Růckschlossová 2005)

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ is **strongly comparison meaningful**

$$\Leftrightarrow \left\{ \begin{array}{l} \exists k \in \{1, \dots, n\} \\ \exists g : \mathbb{R} \rightarrow \mathbb{R} \text{ strictly monotonic or constant} \\ \text{such that} \\ F(x_1, \dots, x_n) = g(x_k) \end{array} \right.$$

+ **idempotent**

$$\Leftrightarrow \left\{ \begin{array}{l} \exists k \in \{1, \dots, n\} \text{ such that} \\ F(x_1, \dots, x_n) = x_k \end{array} \right.$$

Comparison meaningful functions

First result (Orlov 1981)

- $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is
- symmetric
 - continuous
 - internal, i.e., $\min_i x_i \leq F(x_1, \dots, x_n) \leq \max_i x_i$
 - comparison meaningful

$$\Leftrightarrow \begin{cases} \exists k \in \{1, \dots, n\} \text{ such that} \\ F(x_1, \dots, x_n) = x_{(k)} \end{cases}$$

where $x_{(1)}, \dots, x_{(n)}$ denote the *order statistics* resulting from reordering x_1, \dots, x_n in the nondecreasing order.

Next step : suppress symmetry and relax internality into idempotency

Lattice polynomials

Definition (Birkhoff 1967)

An n -variable *lattice polynomial* is any expression involving n variables x_1, \dots, x_n linked by the lattice operations

$$\wedge = \min \quad \text{and} \quad \vee = \max$$

in an arbitrary combination of parentheses.

For example,

$$L(x_1, x_2, x_3) = (x_1 \vee x_3) \wedge x_2$$

is a 3-variable lattice polynomial.

Lattice polynomials

Proposition (Ovchinnikov 1998, Marichal 2002)

A lattice polynomial on \mathbb{R}^n is **symmetric** iff it is an order statistic.

We have

$$x_{(k)} = \bigvee_{\substack{T \subseteq \{1, \dots, n\} \\ |T| = n - k + 1}} \bigwedge_{i \in T} x_i = \bigwedge_{\substack{T \subseteq \{1, \dots, n\} \\ |T| = k}} \bigvee_{i \in T} x_i$$

The nonsymmetric case

Second result (Yanovskaya 1989)

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ is

- continuous
- idempotent
- comparison meaningful

$\Leftrightarrow \exists$ a lattice polynomial $L : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $F = L$.

+ symmetric

$\Leftrightarrow \exists k \in \{1, \dots, n\}$ such that $F = OS_k$ (k th order statistic).

Next step : suppress idempotency

The nonidempotent case

Third result (Marichal 2002)

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ is - continuous
- comparison meaningful

$$\Leftrightarrow \left\{ \begin{array}{l} \exists L : \mathbb{R}^n \rightarrow \mathbb{R} \text{ lattice polynomial} \\ \exists g : \mathbb{R} \rightarrow \mathbb{R} \text{ - continuous} \\ \text{- strictly monotonic or constant} \\ \text{such that} \\ F = g \circ L \end{array} \right.$$

+ symmetric

$$F = g \circ OS_k$$

Next step : replace continuity with nondecreasing monotonicity

The noncontinuous case

Fourth result (Marichal & Mesiar & Růckschlossová 2005)

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ is - **nondecreasing**
- **comparison meaningful**

$$\Leftrightarrow \left\{ \begin{array}{l} \exists L : \mathbb{R}^n \rightarrow \mathbb{R} \text{ lattice polynomial} \\ \exists g : \mathbb{R} \rightarrow \mathbb{R} \text{ strictly increasing or constant} \\ \text{such that} \\ F = g \circ L \end{array} \right.$$

These functions are continuous up to possible discontinuities of function g

Final step : suppress nondecreasing monotonicity (a hard task !)

The general case

... is much more complicated to describe

- We lose the concept of lattice polynomial
- The description of F is done through a partition of the domain \mathbb{R}^n into particular subsets, called *invariant subsets*

Invariant subsets

Let us consider the subsets of \mathbb{R}^n of the form

$$I = \{x \in \mathbb{R}^n \mid x_{\pi(1)} \triangleleft_1 \cdots \triangleleft_{n-1} x_{\pi(n)}\}$$

where π is any permutation on $\{1, \dots, n\}$ and $\triangleleft_i \in \{<, =\}$.

Denote this class of subsets by $\mathcal{I}(\mathbb{R}^n)$.

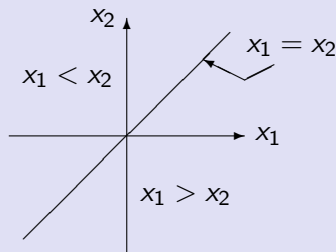
Example : \mathbb{R}^2

Description of $\mathcal{I}(\mathbb{R}^2)$:

$$I_1 = \{(x_1, x_2) \mid x_1 = x_2\}$$

$$I_2 = \{(x_1, x_2) \mid x_1 < x_2\}$$

$$I_3 = \{(x_1, x_2) \mid x_1 > x_2\}$$



Invariant subsets

Proposition (Bartłomiejczyk & Drewniak 2004)

The class $\mathcal{I}(\mathbb{R}^n)$ consists of the *minimal invariant* subsets of \mathbb{R}^n .

That is,

- Each subset $I \in \mathcal{I}(\mathbb{R}^n)$ is *invariant* in the sense that
$$(x_1, \dots, x_n) \in I \Rightarrow (\phi(x_1), \dots, \phi(x_n)) \in I \quad \forall \phi \in A(\mathbb{R})$$
- Each subset $I \in \mathcal{I}(\mathbb{R}^n)$ is *minimal* in the sense that it has no proper invariant subset

The family $\mathcal{I}(\mathbb{R}^n)$ partitions \mathbb{R}^n into equivalence classes :

$$x \sim y \Leftrightarrow \exists \phi \in A(\mathbb{R}) : y_i = \phi(x_i) \quad \forall i$$

The general case

Fifth result (Marichal & Mesiar & Růckschlossová 2005)

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ is **comparison meaningful**

$$\Leftrightarrow \forall I \in \mathcal{I}(\mathbb{R}^n), \left\{ \begin{array}{l} \exists k_I \in \{1, \dots, n\} \\ \exists g_I : \mathbb{R} \rightarrow \mathbb{R} \text{ strictly monotonic or constant} \\ \text{such that} \\ F|_I(x_1, \dots, x_n) = g_I(x_{k_I}) \\ \\ \text{where } \forall I, I' \in \mathcal{I}(\mathbb{R}^n), \\ \bullet \text{ either } g_I = g_{I'} \\ \bullet \text{ or } \text{ran}(g_I) = \text{ran}(g_{I'}) \text{ is a singleton} \\ \bullet \text{ or } \text{ran}(g_I) < \text{ran}(g_{I'}) \\ \bullet \text{ or } \text{ran}(g_I) > \text{ran}(g_{I'}) \end{array} \right.$$

Conclusion

We have described all the possible merging functions

$$F : \mathbb{R}^n \rightarrow \mathbb{R},$$

which map n ordinal scales into an ordinal scale.

These results hold true when F is defined on E^n , where E is any open real interval.

The cases where E is a non-open real interval all have been described and can be found in

J.-L. Marichal, R. Mesiar, and T. Růckschlossová,
A Complete Description of Comparison Meaningful Functions,
Aequationes Mathematicae 69 (2005) 309–320.

Thank you for your attention