

$\mathcal{N} = 1$ supersymmetric higher spin holography on AdS_3

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Abstract

We propose a duality between a higher spin $\mathcal{N} = 1$ supergravity on AdS_3 and a large N limit of a family of $\mathcal{N} = (1, 1)$ superconformal field theories. The gravity theory is an $\mathcal{N} = 1$ truncation of the $\mathcal{N} = 2$ supergravity found by Prokushkin and Vasiliev, and the dual conformal field theory is defined by a supersymmetric coset model. We check this conjecture by comparing one loop partition functions and find agreement. Moreover, we study the symmetry of the dual coset model and in particular compute fields of the coset algebra of dimension $3/2, 2, 2$ and $5/2$ explicitly.

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1 Introduction

Recently, holography involving higher spin gauge theories has received a lot of attention. Higher spin gauge theories are believed to be related to the tensionless limit of superstring theory, and indeed higher spin holography may be seen as a simplified, but non-trivial, version of AdS/CFT duality in superstring theory. A famous example is the proposal by Klebanov and Polyakov in [1] which says that a 4 dimensional higher spin gauge theory is dual to the large N limit of the $O(N)$ vector model. Two years ago it was conjectured in [2] that a 3 dimensional higher spin gauge theory is dual to a large N minimal model, see [3] for a review. There are several generalizations of this duality;

a truncated version was considered in [4, 5], and the full $\mathcal{N} = 2$ supersymmetric version was introduced in [6]. In this paper we would like to propose and test the $\mathcal{N} = 1$ supersymmetric version of the duality.

The higher spin gauge theories appearing in these dualities are truncated versions of the $\mathcal{N} = 2$ supergravity by Prokushkin and Vasiliev [7]. A bosonic truncation is used in the original proposal [2] which has an infinite series of higher spin gauge fields with spins $s = 2, 3, \dots$ and two complex scalars. The dual theory is a large N limit of the minimal model with higher spin W_N symmetry [8]. Currently there is much evidence for the duality based on the symmetry [9, 10, 11, 12, 13] and the spectrum [14], and also correlators have been calculated [15, 16, 17, 18, 19]. In the case of finite N , a refined version of the duality has been proposed in [13], however, we will here only be concerned with the strict large N limit. In [4, 5] a further truncation is used for the gravity theory, and it then includes even spin gauge fields with $s = 2, 4, 6, \dots$ and two real scalars with the mass $M^2 = -1 + \lambda^2$. The dual theory is the large N limit of the WD_N minimal model described by the coset theory¹

$$\frac{\widehat{\mathfrak{so}}(2N)_k \oplus \widehat{\mathfrak{so}}(2N)_1}{\widehat{\mathfrak{so}}(2N)_{k+1}} . \quad (1.1)$$

In the limit we also take k to infinity, but fix the 't Hooft parameter

$$\lambda = \frac{2N}{2N + k - 2} , \quad (1.2)$$

which is identified with the λ parameter in the mass of the bulk scalars. This conjecture was supported by the analysis of RG-flow in [4] and the comparison of one-loop partition functions in [5], see also [20] for work on the spin-4 operator in correlators.

In [6] we extended the duality to the case with supersymmetry. The untruncated version of the $\mathcal{N} = 2$ supergravity in [7] is proposed to be dual to the $\mathcal{N} = (2, 2)$ \mathbb{CP}^N Kazama-Suzuki model [21, 22]. The partition functions of the gravity theory and the CFT are shown to match in [23], and the symmetry algebras are analyzed in [24, 25, 26, 27]. Other related works may be found in [28, 29, 30, 31, 32, 33]. By including supersymmetry, quantum effects are known to become more tractable in general. Moreover, the relation to superstring theory could be more transparent as was also mentioned in [24]. In [7] several $\mathcal{N} = 1$ truncations have been also discussed, and we would like to consider the simplest one without any matrix degrees of freedom. The supergravity includes both bosonic and fermionic higher spin gauge fields and also massive scalars and fermions, as in the untruncated case. Our proposal is that the dual CFT is given by the $\mathcal{N} = (1, 1)$ super coset

$$\frac{\widehat{\mathfrak{so}}(2N + 1)_k \oplus \widehat{\mathfrak{so}}(2N)_1}{\widehat{\mathfrak{so}}(2N)_{k+1}} . \quad (1.3)$$

¹As pointed out in [4, 5], the WB_N minimal model is a candidate as well.

There is no enhancement of supersymmetry from $\mathcal{N} = 1$ to $\mathcal{N} = 2$ since this coset does not satisfy the condition in [21, 22]². We need to take the large N limit with 't Hooft parameter³

$$\lambda = \frac{2N}{2N + k - 1} \quad (1.4)$$

kept finite. Again, this parameter is identified with the one in the masses of the bulk matter. We will underpin this conjecture by showing that the supergravity and the CFT partition functions match, i.e. the spectrum is the same on both sides. Further, we will also study the symmetry of the super coset model.

This paper is organized as follows: In the next section we give explicit formulas for one-loop partition functions in the higher spin gravity theories. In section 3, the one-loop partition function of truncated bosonic gravity theory is reproduced by the 't Hooft limit of the coset model (1.1). This was already done in [5], but we obtain the same result by using the different method adopted in [23] as a preparation of the later analysis. In section 4, we introduce the $\mathcal{N} = (1, 1)$ super coset (1.3) and study its torus partition function. We show that in the 't Hooft limit it reproduces the one-loop partition function of the $\mathcal{N} = 1$ supergravity. In section 5, we study the symmetry of the super coset model (1.3). Section 6 is devoted to conclusion and discussions. In appendix A, we review the $\mathcal{N} = 1$ truncation of the $\mathcal{N} = 2$ supergravity found in [7] and show that the $\mathcal{N} = 1$ higher spin algebra is the analytical continuation of the $\text{osp}(2N + 1|2N)$ Lie superalgebra. In appendix B, we summarize useful properties of orthogonal Lie algebras.

2 Higher spin gravity theories

In this section, we study truncated versions of the higher spin $\mathcal{N} = 2$ supergravity in [7] and obtain explicit formulas for their one-loop partition functions. In the first subsection, we review the bosonic gravity theory having only gauge fields of even spin $s = 2, 4, 6, \dots$ which is proposed in [4, 5] as the gravity dual of the WD_N minimal model. In subsection 2.2, we consider the larger $\mathcal{N} = 1$ truncation of $\mathcal{N} = 2$ supergravity via an anti-automorphism which was also introduced in [7], and which we review in appendix A.

2.1 The bosonic truncation

In [2] a bosonic truncation of the $\mathcal{N} = 2$ supergravity introduced in [7] is utilized to construct a simplified version of the AdS/CFT correspondence. The gravity theory

²The condition for enhancement to $\mathcal{N} = 2$ superconformal symmetry is that the coset manifold G/H is a Hermitian symmetric space. This is not true for $\text{SO}(2N + 1)/\text{SO}(2N)$ with $N > 1$.

³For large N, k , the difference between (1.2) and (1.4) is irrelevant.

includes massless gauge fields with spins $s = 2, 3, 4, \dots$ and two complex scalars with mass

$$M^2 = -1 + \lambda^2 . \quad (2.1)$$

The massless sector can be described by a Chern-Simons gauge theory based on the $\mathfrak{hs}[\lambda]$ Lie algebra which can be reduced to $\mathfrak{sl}(N)$ for $\lambda = \pm N$. The dual CFT is proposed to be a large N limit of the WA_N minimal model.

The WD_N (and WB_N) version of [2] was treated by [4] and [5] (the last reference mostly on the former algebra). The bulk side is a truncated version of the $\mathfrak{hs}[\lambda]$ algebra containing only even spins, see [5, 34]. The one-loop partition function for the gauge sector is given by [35, 36, 37]

$$Z_{\text{gauge}} = \prod_{l=1}^{\infty} Z_{\text{gauge}}^{(2l)} = \prod_{l=1}^{\infty} \prod_{n=2l}^{\infty} \frac{1}{|1 - q^n|^2} , \quad (2.2)$$

where $q = \exp(\tau)$ is a modulus for the boundary torus of the Euclidean AdS_3 . The bulk side contains two real scalars with the same mass as in eq. (2.1). The fall-off behaviour at the boundary is chosen oppositely for the two scalars like in the WA_N case. In the dual boundary CFT this gives two real scalars with conformal weights

$$h_+ = \frac{1 + \lambda}{2} , \quad h_- = \frac{1 - \lambda}{2} . \quad (2.3)$$

For each scalar field with the dual conformal weight h , the partition function is [35, 36]

$$Z_{\text{scalar}}^h = \prod_{m,n=0}^{\infty} \frac{1}{1 - q^{h+m} \bar{q}^{h+n}} . \quad (2.4)$$

The one-loop partition function of the gravity theory is then given by

$$Z_{1\text{-loop}}^\lambda = Z_{\text{gauge}} Z_{\text{scalar}}^{h_+} Z_{\text{scalar}}^{h_-} . \quad (2.5)$$

In order to compare the gravity partition function to the dual CFT partition function, it is convenient to rewrite the partition function of the matter sector. We introduce a Young tableau Tab_Λ of shape Λ . Here we assign a non-negative integer $c_{i,j}$ to the box in the Young diagram Λ at the i -th row and the j -th column, see figure 1. The rules for the numbers $c_{i,j}$ are that the entries do not decrease along a row and increase along a column. Then the partition function of a scalar field can be rewritten as [14, 23]

$$Z_{\text{scalar}}^h = \sum_{\Lambda} |\text{ch}_\Lambda(U(h))|^2 , \quad (2.6)$$

where the character of the representation Λ is defined as

$$\text{ch}_\Lambda(U(h)) = \sum_{T \in \text{Tab}_\Lambda} \prod_{j \in T} q^{h+j} , \quad U(h)_{jj} = q^{h+j} . \quad (2.7)$$

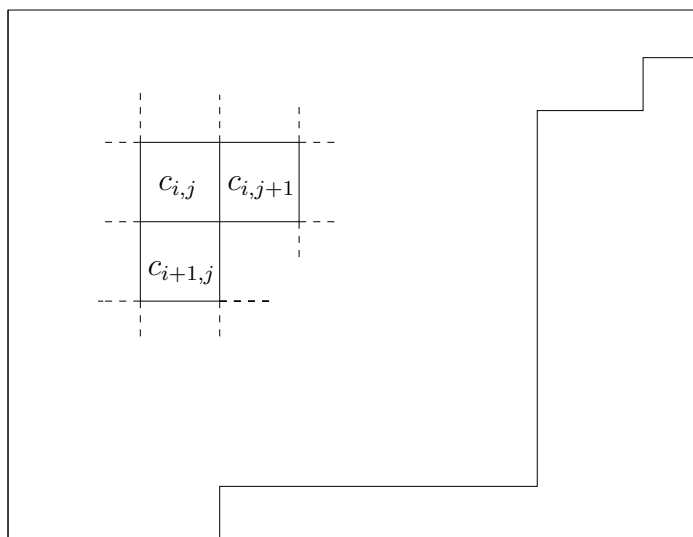


Figure 1: A Young tableau Tab_Λ of a shape Λ . In each box of the Young diagram Λ , we assign a non-negative number $c_{i,j}$ with a rule that $c_{i,j} \leq c_{i,j+1}$ and $c_{i,j} < c_{i+1,j}$. A Young supertableau STab_Λ of a shape Λ are also given by a Young diagram Λ and a non-negative number $c_{i,j}$ in a each box. However, the rules for $c_{i,j}$ are a bit different. The numbers should always satisfy the conditions $c_{i,j} \leq c_{i,j+1}$ and $c_{i,j} \leq c_{i+1,j}$. Further $c_{i,j} < c_{i,j+1}$ if $c_{i,j}$ and $c_{i,j+1}$ are odd, and $c_{i,j} < c_{i+1,j}$ if $c_{i,j}$ and $c_{i+1,j}$ are even.

The one-loop partition function of the gravity theory is thus summarized as

$$Z_{1\text{-loop}}^\Lambda = Z_{\text{gauge}} \sum_{\Lambda, \Xi} |\text{ch}_\Lambda(U(h_+)) \text{ch}_\Xi(U(h_-))|^2 \quad (2.8)$$

with h_\pm given in (2.3).

2.2 The $\mathcal{N} = 1$ truncation

In [7] the truncation from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supergravity has been discussed, see also appendix A. The $\mathcal{N} = 1$ theory also has massless higher spin gauge fields and matter fields. Let us start from the massless sector. As derived in appendix A, the massless sector can be described by a large N limit of $\text{osp}(2N+1|2N) \oplus \text{osp}(2N+1|2N)$ Chern-Simons gauge theory.⁴ In order to see the spin content of the theory, we have to identify an $\text{osp}(1|2)$ subalgebra as the $\mathcal{N} = 1$ supergravity sector. As in [6], we adopt the superprincipal embedding of $\text{osp}(1|2)$, which gives us [38]

$$\text{osp}(2N+1|2N) = \sum_{k=1}^N (R_{2k-1} \oplus R_{2k-1/2}) . \quad (2.9)$$

⁴It might be possible to use $\text{osp}(2N-1|2N) \oplus \text{osp}(2N-1|2N)$ Chern-Simons gauge theory.

Here R_j is the representation of $\text{osp}(1|2)$ decomposing under the $\text{sl}(2)$ as $R_j = D_j \oplus D_{j-1/2}$, where D_j denotes the $2j+1$ dimensional representation of $\text{sl}(2)$. This gives a spin decomposition of the form

$$\text{osp}(2N+1|2N) = \sum_{k=1}^N (D_{2k-1/2} \oplus 2D_{2k-1} \oplus D_{2k-3/2}) . \quad (2.10)$$

We see that we have no odd integer spins. In the infinite N limit we have two fields of each even spin, and one field of each half odd integer spin.

The spin j of the embedded $\text{sl}(2)$ is related to the space-time spin as $s = j + 1$, and a more close examination shows that the integer and the half-integer spin components are bosonic and fermionic elements of $\text{osp}(2N+1|2N)$, respectively. With the help of the results for $\mathcal{N} = 2$ supergravity in [6], we can write down the contribution from higher spin fields as

$$\mathcal{Z}_{\text{gauge}} = \prod_{l=1}^{\infty} (Z_B^{(2l)})^2 \prod_{s=2}^{\infty} Z_F^{(s-1)} , \quad (2.11)$$

where

$$Z_B^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} , \quad Z_F^{(s)} = \prod_{n=s}^{\infty} |1 + q^{n+\frac{1}{2}}|^2 \quad (2.12)$$

are the partition functions of a bosonic field with spin s and a fermionic field with spin $s + 1/2$, respectively.

One advantage of the Chern-Simons description is that we can easily read off the classical asymptotic symmetry near the boundary of AdS_3 [9, 10, 11, 12]. The Chern-Simons theory is a topological theory, and dynamical degrees of freedom exist only at the boundary. The boundary degrees can be described by $\text{osp}(2N+1|2N)$ Wess-Zumino-Novikov-Witten model, whose symmetry is the affine $\text{osp}(2N+1|2N)$ Lie superalgebra. For the application to the AdS/CFT correspondence, we have to assign the boundary conditions ensuring the bulk space being asymptotically AdS space. It was shown in [10, 12] that this condition is the same as for the classical Hamiltonian reduction (see, for instance, [8]). Thus the classical asymptotic symmetry of the $\mathcal{N} = 1$ truncated theory is obtained by the Hamiltonian reduction of affine $\text{osp}(2N+1|2N)$ Lie superalgebra in a large N limit.

The matter sector consists of a single $\mathcal{N} = 1$ hypermultiplet, having two complex scalars with masses respectively

$$(M_-^B)^2 = -1 + \lambda^2 , \quad (M_+^B)^2 = -1 + (\lambda - 1)^2 , \quad (2.13)$$

and two fermions with mass

$$(M_{\pm}^F)^2 = (\lambda - \frac{1}{2})^2 . \quad (2.14)$$

Also for the fermions we can choose two types of boundary conditions. We choose these such that the boundary conformal dimensions simply become a real version of the $\mathcal{N} = 2$ case [6] (see also [23])

$$(\Delta_+^B, \Delta_{\pm}^F, \Delta_-^B) = (2 - \lambda, \frac{3}{2} - \lambda, 1 - \lambda), (\lambda, \frac{1}{2} + \lambda, 1 + \lambda) . \quad (2.15)$$

The one-loop partition function of the matter part is [6]

$$\mathcal{Z}_{\text{matter}} = \mathcal{Z}_{\text{hyper}}^{\frac{\lambda}{2}} \mathcal{Z}_{\text{hyper}}^{\frac{1-\lambda}{2}}, \quad \mathcal{Z}_{\text{hyper}}^h = Z_{\text{scalar}}^{h+\frac{1}{2}} (Z_{\text{spinor}}^h)^2 Z_{\text{scalar}}^h \quad (2.16)$$

where (note no square in the bosonic partition function)

$$Z_{\text{scalar}}^h = \prod_{l,l'=0}^{\infty} \frac{1}{1 - q^{h+l} \bar{q}^{h+l'}} , \quad (Z_{\text{spinor}}^h)^2 = \prod_{l,l'=0}^{\infty} (1 + q^{h+l} \bar{q}^{h+\frac{1}{2}+l'}) (1 + q^{h+\frac{1}{2}+l} \bar{q}^{h+l'}) .$$

As in the bosonic case, it is convenient to define the supercharacter⁵

$$\text{sch}_{\Lambda}(\mathcal{U}(h)) = \sum_{T \in \text{STab}_{\Lambda}} \prod_{j \in T} q^{h+\frac{j}{2}}, \quad \mathcal{U}(h)_{jj} = (-1)^j q^{h+\frac{j}{2}} . \quad (2.17)$$

Here STab_{Λ} represents a Young supertableau of shape Λ . In the supertableau, a non-negative integer is assigned to each box of the Young diagram Λ with rules specified in figure 1. Then following [23], the partition function of the matter sector can be given as

$$\mathcal{Z}_{\text{hyper}}^h = \sum_{\Lambda} |\text{sch}_{\Lambda}(\mathcal{U}(h))|^2 . \quad (2.18)$$

Therefore, the one-loop partition function of the $\mathcal{N} = 1$ truncated theory can be expressed as

$$\mathcal{Z} = \mathcal{Z}_{\text{gauge}} \sum_{\Lambda, \Xi} |\text{sch}_{\Lambda}(\mathcal{U}(h_+)) \text{sch}_{\Xi}(\mathcal{U}(h_-))|^2 , \quad (2.19)$$

where $h_+ = \frac{\lambda}{2}$ and $h_- = \frac{1-\lambda}{2}$.

3 Holography for $\text{SO}(2N)$

It was proposed in [2] that a bosonic truncation of Prokushkin-Vasiliev theory [7] is dual to a large N limit of W_N minimal model. A further consistent truncation is possible in the gravity theory as discussed in section 2.1, and the dual CFT is conjectured to be the WD_N minimal model with the coset description (1.1) [4, 5]. In order to compare with

⁵This is a character of $\text{gl}(\infty|\infty)_+$ considered in [23]. The bosonic one (2.7) is the character of $\text{gl}(\infty)_+$ which is the Lie algebra of infinite dimensional matrices. The generators may be given by $(e_{ij})_{k,l} = \delta_{i,l} \delta_{j,k}$ and only those with finite i, j are considered. The supergroup $\text{gl}(\infty|\infty)_+$ is quite similar to $\text{gl}(\infty)_+$, but now e_{ij} is bosonic for even $i + j$ and fermionic for odd $i + j$.

the classical gravity theory, we need to take a large N limit while keeping the 't Hooft coupling (1.2) finite, and this parameter is identified with λ in (2.1). The equivalence of the spectrum in this limit was shown in [5] by directly applying the method in [14]. In this section, we will obtain the same result by making use of the different method in [23]. In the next section, we will use the same method to show the matching of the spectrum in the $\mathcal{N} = 1$ supersymmetric version of the duality.

3.1 The dual CFT

We would like to reproduce the gravity partition function (2.8) from the viewpoint of the dual coset CFT (1.1)⁶

$$\frac{\widehat{\mathfrak{so}}(2N)_k \oplus \widehat{\mathfrak{so}}(2N)_1}{\widehat{\mathfrak{so}}(2N)_{k+1}}, \quad (3.1)$$

whose central charge is

$$c = N \left[1 - \frac{(2N-1)(2N-2)}{(k+2N-2)(k+2N-1)} \right]. \quad (3.2)$$

We use the diagonal embedding of $\widehat{\mathfrak{so}}(2N)_{k+1}$ into $\widehat{\mathfrak{so}}(2N)_k \oplus \widehat{\mathfrak{so}}(2N)_1$, and the states of the coset (3.1) are obtained by the decomposition

$$\Lambda \otimes \omega = \oplus_{\Xi} (\Lambda, \omega; \Xi) \otimes \Xi. \quad (3.3)$$

The states are thus labeled by (Λ, ω, Ξ) , which are the highest weights of the representations of $\widehat{\mathfrak{so}}(2N)_k$, $\widehat{\mathfrak{so}}(2N)_1$, $\widehat{\mathfrak{so}}(2N)_{k+1}$, respectively. Some basics of $\mathfrak{so}(2N)$ Lie algebra may be found in appendix B. For $\widehat{\mathfrak{so}}(2N)_1$, there are only four representations: The identity, vector, spinor and cospinor representations. The selection rule is

$$\Lambda + \omega - \Xi \in Q_{2N} \quad (3.4)$$

where Q_{2N} is the root lattice of $\mathfrak{so}(2N)$. The congruence class of $\mathfrak{so}(2N)$ is \mathbb{Z}_4 for $N = 2n + 1$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ for $N = 2n$ with $n \in \mathbb{Z}$ (see, e.g., [39]), and this equation uniquely determines ω given Λ and Ξ , and we simply denote the states by $(\Lambda; \Xi)$. Moreover, there are field identifications $(\Lambda; \Xi) \simeq (A\Lambda; A\Xi)$ with an outer automorphism A of the affine algebra $\widehat{\mathfrak{so}}(2N)_k$ and $\widehat{\mathfrak{so}}(2N)_{k+1}$ [40]. The conformal weight of the state $(\Lambda; \Xi)$ is given by

$$h_{(\Lambda; \Xi)} = \frac{C_{2N}(\Lambda)}{k+2N-2} + \frac{C_{2N}(\omega)}{2N-1} - \frac{C_{2N}(\Xi)}{k+2N-1} + n, \quad (3.5)$$

where $C_{2N}(\Lambda)$ is the quadratic Casimir of $\mathfrak{so}(2N)$ and the integer n is the grade at which Ξ appears in $\Lambda \otimes \omega$.

⁶We heard that in [34] the duality is refined so as to be applicable for finite N, k . It is pointed out there that an \mathbb{Z}_2 orbifold should be used instead of (1.1), but this difference disappears in the large N limit, like we also found in (5.2). Further, this subtlety does not arise in the WB_N case.

We define the characters of $\widehat{\mathfrak{so}}(2N)_k$ and the branching function as

$$\text{ch}_\Lambda^{2N,k}(q, e^H) = \text{tr}_\Lambda q^{L_0} e^H, \quad b_{\Lambda;\Xi}^{2N,k}(q) = \text{tr}_{(\Lambda;\Xi)} q^{L_0}, \quad (3.6)$$

where L_0 is the zero mode of energy momentum tensor and H is the Casimir element of $\mathfrak{so}(2N)$. From the decomposition (3.3) we find

$$\text{ch}_\Lambda^{2N,k}(q, e^H) \text{ch}_\omega^{2N,1}(q, e^H) = \sum_{\Xi} b_{\Lambda;\Xi}^{2N,k}(q) \text{ch}_\Xi^{2N,k+1}(q, e^H). \quad (3.7)$$

In addition to the chiral sector, the CFT has an anti-chiral sector. We consider the charge-conjugated theory with the Hilbert space $\mathcal{H} = \sum (\Lambda; \Xi) \otimes \overline{(\Lambda; \Xi)}$. We should take care of the field identification when the sum is taken. The partition function is then given by the diagonal modular invariant

$$Z^{2N,k}(q) = |q^{-\frac{c}{24}}|^2 \sum_{[\Lambda;\Xi]} |b_{\Lambda;\Xi}^{2N,k}(q)|^2. \quad (3.8)$$

In order to compare with the classical gravity theory, we take a large N, k limit with the 't Hooft parameter (1.2)

$$\lambda = \frac{2N}{2N + k - 2}$$

kept finite. Since representations of order N^2 decouple in this limit, we keep those whose conformal weights are of order N . With this criterion, the highest weights for the representations that survive can be labeled by Young tableaux [5] (see also appendix B). Moreover, representations are self-conjugate for the orthogonal Lie algebra. Denoting the number of boxes in the i -th row by l_i and in the j -th column by c_j , the quadratic Casimir is expressed as (B.9)

$$C_{2N}(\Lambda) = |\Lambda| \left(N - \frac{1}{2} \right) + \frac{1}{2} \left(\sum_{i=1}^{N-2} l_i^2 - \sum_j c_j^2 \right). \quad (3.9)$$

We consider Young diagrams with only finitely many boxes in the large N limit, and the above equation implies that $C_{2N}(\Lambda) \sim N|\Lambda|$ where $|\Lambda|$ is the number of boxes of the corresponding Young diagram.

3.2 Comparison of partition functions

We would like to compute the branching function $b_{\Lambda;\Xi}^{2N,k}(q)$ in eq. (3.6) in the 't Hooft limit. For q having real part less than one, we can neglect q^N and q^k . Following [5] we first study the large k behavior of the characters of $\widehat{\mathfrak{so}}(2N)_k$ and then compute the branching function utilizing (3.7). For large k , the character becomes [8, 14, 23]

$$\text{ch}_\Lambda^{2N,k}(q, e^H) \simeq \frac{q^{h_\Lambda^{2N,k}} \text{ch}_\Lambda^{2N}(e^H)}{\prod_{n=1}^{\infty} [(1 - q^n)^N \prod_{\alpha \in \Delta_{2N}} (1 - q^n e^{\alpha(H)})]}, \quad (3.10)$$

where we have used the Weyl-Kac formula and Δ_{2N} denotes the roots of $\mathfrak{so}(2N)$. For large k , the affine Weyl group reduces to the finite Weyl group as discussed in [14], which leads to the character of the finite $\mathfrak{so}(2N)$ Lie algebra $\text{ch}_\Lambda^{2N}(e^H)$. The conformal dimension is now

$$h_\Lambda^{2N,k} = \frac{C_{2N}(\Lambda)}{k + 2N - 2} . \quad (3.11)$$

For large k we can reduce the relation (3.7) to

$$\text{ch}_\Lambda^{2N}(e^H) \text{ch}_\omega^{2N,1}(q, e^H) = \sum_{\Xi} a_{\Lambda;\Xi}^{2N}(q) \text{ch}_\Xi^{2N}(e^H) , \quad (3.12)$$

with the help of (3.10). Here the k -independent function $a_{\Lambda;\Xi}^{2N}(q)$ is related to the branching function as

$$b_{\Lambda;\Xi}^{2N,k}(q) \simeq q^{h_\Lambda^{2N,k} - h_\Xi^{2N,k+1}} a_{\Lambda;\Xi}^{2N}(q) . \quad (3.13)$$

As will be shown now, the function $a_{\Lambda;\Xi}^{2N}(q)$ can be written as

$$a_{\Lambda;\Xi}^{2N}(q) = \sum_{\Pi} N_{\Lambda\Pi}^{(2N)\Xi} a_{0;\Pi}^{2N}(q) = \sum_{\Pi} N_{\Lambda\Xi}^{(2N)\Pi} a_{0;\Pi}^{2N}(q) \quad (3.14)$$

with $N_{\Lambda\Xi}^{(2N)\Pi}$ being the Clebsch-Gordan coefficients of $\mathfrak{so}(2N)$. First notice that

$$\text{ch}_\omega^{2N,1}(q, e^H) = \sum_{\Xi} a_{0;\Xi}^{2N}(q) \text{ch}_\Xi^{2N}(e^H) , \quad (3.15)$$

which is obtained from (3.12) with $\Lambda = 0$. Then, we use

$$\text{ch}_\Lambda^{2N}(e^H) \text{ch}_\Pi^{2N}(e^H) = \sum_{\Xi} N_{\Lambda\Pi}^{(2N)\Xi} \text{ch}_\Xi^{2N}(e^H) \quad (3.16)$$

and the fact that the representations are now self-conjugate.

Next, we take the large N limit. As discussed in the previous subsection, the highest weight representations can be labeled by Young diagrams with finitely many boxes in the limit. In the next subsection, we will obtain

$$a_{0;0}(q) = \lim_{N \rightarrow \infty} a_{0;0}^{2N}(q) = \prod_{l=1}^{\infty} \prod_{n=2l}^{\infty} \frac{1}{1 - q^n} , \quad (3.17)$$

$$a_{0;\Xi}(q) = \lim_{N \rightarrow \infty} a_{0;\Xi}^{2N}(q) = a_{0;0}(q) \text{ch}_{\Xi^t}(U(\frac{1}{2})) , \quad (3.18)$$

where the character is defined in (2.7). In the transposed expression Ξ^t , the rows and columns are exchanged. Moreover, the conformal weight becomes

$$h_\Lambda^{2N,k} \simeq \frac{\lambda}{2} |\Lambda| . \quad (3.19)$$

Using these facts, the branching function in the 't Hooft limit can be found as

$$b_{\Lambda;\Xi}^\lambda(q) = q^{\frac{\lambda}{2}(|\Lambda|-|\Xi|)} a_{0;0}(q) \sum_{\Pi} N_{\Lambda\Xi}^{\Pi} \text{ch}_{\Pi^t}(U(\frac{1}{2})) . \quad (3.20)$$

As in [5] (see above the eq. (3.34) of the paper) we assume that

$$|\Lambda| + |\Xi| = |\Pi| , \quad (3.21)$$

which leads to

$$N_{\Lambda\Xi}^{\Pi} = \lim_{N \rightarrow \infty} N_{\Lambda\Xi}^{(2N)\Pi} \quad (3.22)$$

with $N_{\Lambda\Xi}^{\Pi}$ as the Clebsch-Gordan coefficient of $\mathfrak{gl}(\infty)_+$. Using the fact that the Clebsch-Gordan coefficients are the same for the transposed representations, we can show

$$\begin{aligned} \sum_{\Lambda,\Xi} |b_{\Lambda;\Xi}^\lambda(q)|^2 &= Z_{\text{gauge}} \sum_{\Lambda,\Xi} |q^{\frac{\lambda}{2}(|\Lambda|-|\Xi|)} \text{ch}_{\Lambda^t}(U(\frac{1}{2})) \text{ch}_{\Xi^t}(U(\frac{1}{2}))|^2 \\ &= Z_{\text{gauge}} \sum_{\Lambda,\Xi} |\text{ch}_{\Lambda^t}(U(h_+)) \text{ch}_{\Xi^t}(U(h_-))|^2 . \end{aligned} \quad (3.23)$$

This reproduces the gravity partition function (2.8). It was argued in [14, 5] that a large number of null states appear in the 't Hooft limit and the decoupling of these null states is equivalent to the condition (3.21). This assumption is quite important for the equivalence of the partition functions, so we would like to examine it more carefully as a future problem.

3.3 Characters from free fermions

Here we would like to compute (3.17) and (3.18) using free fermions. It is well known that $\widehat{\mathfrak{so}}(2N)_1$ can be expressed by $2N$ real free fermions ψ^a with $a = 1, 2, \dots, 2N$. Thus the task here is to decompose the left hand side of (3.15) by the characters of the zero mode $\mathfrak{so}(2N)$ Lie algebra. The space of free fermions is spanned by

$$\prod_{j=1}^{n_\psi} \psi_{-r_j - \frac{1}{2}}^{a_j} \Omega , \quad (3.24)$$

where $r_j = 0, 1, 2, \dots$ and Ω is the vacuum. The branching function $a_{0;\Xi}$ counts the multiplicity when the representation Ξ of $\mathfrak{so}(2N)$ appears. The representation appears for the first time when $n_\psi = |\Xi|$. Following the argument below (2.49) of [23], the branching function is found to be

$$\text{ch}_{\Xi^t}(U(\frac{1}{2})) \quad (3.25)$$

when summing the possible modes r_j while keeping $n_\psi = |\Xi|$ fixed. Notice here that the modes r_j can be interpreted as the entries of the Young tableau of the shape Ξ since the Fermi statistic explains the rules for the entries (see figure 1).

There is another contribution to the branching function $a_{0;\Xi}$ with $n_\psi > |\Xi|$, which comes from multiplying with $\mathfrak{so}(2N)$ invariants. From the classical invariant theory, we can see that the $\mathfrak{so}(2N)$ invariants are generated by [41]

$$\prod_{r,s=0}^{\infty} \left(\sum_{a=1}^{2N} \psi_{-r-\frac{1}{2}}^a \psi_{-s-\frac{1}{2}}^a \right)^{M_{rs}} \quad (3.26)$$

and additional invariants that have conformal dimension at least N . Note, that there are also only relations between invariants for conformal dimension of that order due to the first and second fundamental theorems of invariant theory for the vector representation of the orthogonal group. Now that we are dealing with Majorana fermions, there are two differences from (2.51) of [23]. Firstly, we should set $r \neq s$ due to the Fermi statistic. Secondly, we should set $r > s$ since two fermions can be exchanged. In the large N limit, we ignore the finite N effects and obtain

$$a_{0;0}(q) = \prod_{r>s=0}^{\infty} \sum_{M_{r,s}=0}^{\infty} q^{(r+s+1)M_{r,s}} = \prod_{r>s=0}^{\infty} \frac{1}{1 - q^{r+s+1}} = \prod_{l=1}^{\infty} \prod_{n=2l}^{\infty} \frac{1}{1 - q^n}, \quad (3.27)$$

which is (3.17). By multiplying with (3.25), we obtain (3.18).

4 $\mathcal{N} = 1$ supersymmetric holography

In [6] we have proposed the untruncated $\mathcal{N} = 2$ supersymmetric version of the duality in [2], and the equivalence of the spectrum has been shown in [23]. The aim of this paper is to conjecture an $\mathcal{N} = 1$ version of the duality and show that the one-loop partition functions agree. The gravity theory considered is the $\mathcal{N} = 1$ truncation of the $\mathcal{N} = 2$ higher spin supergravity discussed in subsection 2.2 (see also appendix A). We propose that the dual CFT is given by the super coset model (1.3)

$$\frac{\widehat{\mathfrak{so}}(2N+1)_k \oplus \widehat{\mathfrak{so}}(2N)_1}{\widehat{\mathfrak{so}}(2N)_{k+1}}. \quad (4.1)$$

Moreover, we take a large N, k limit while keeping the 't Hooft parameter (1.4) finite, and identify the parameter with λ appearing in the masses (2.13) and (2.14). Geometrically, the quotient by the right action of $\mathrm{SO}(2N)$ is a sphere

$$S^{2N} = \frac{\mathrm{SO}(2N+1)}{\mathrm{SO}(2N)}. \quad (4.2)$$

Our coset, however, is a quotient by the adjoint action of $\mathrm{SO}(2N)$. Introducing $2N$ fermions, given by $\widehat{\mathfrak{so}}(2N)_1$ factor, the model has $\mathcal{N} = (1, 1)$ supersymmetry. This model

should have the symmetry obtained by the Drinfeld-Sokolov reduction of $\mathfrak{osp}(2N+1|2N)$ and this should be studied as a next step. In the following we will show that the gravity partition function is reproduced by the 't Hooft limit of the $\mathcal{N} = (1, 1)$ super coset (4.1).

4.1 The dual CFT

We would like to compute the torus partition function of the coset model (4.1) in the 't Hooft limit and compare it with the gravity partition function. The central charge of the coset model is given by

$$c = \frac{k(2N+1)N}{k+2N-1} + \frac{N(2N-1)}{2N-1} - \frac{(k+1)N(2N-1)}{k+2N-1} = \frac{3Nk}{k+2N-1}. \quad (4.3)$$

The states are labeled as in the bosonic case. Namely, we use Λ, ω, Ξ as the highest weights of representations of $\widehat{\mathfrak{so}}(2N+1)_k, \widehat{\mathfrak{so}}(2N)_1, \widehat{\mathfrak{so}}(2N)_k$, see appendix B for some basics of the orthogonal Lie algebras. The selection rule is now given by

$$\Lambda + \omega - \Xi \in Q_{2N+1} \quad (4.4)$$

where Q_{2N+1} is the root lattice of $\mathfrak{so}(2N+1)$. For the identification of $\mathfrak{so}(2N+1)$ and $\mathfrak{so}(2N)$ weights, see again appendix B. Since the fermions in the gravity theory satisfy the anti-periodic boundary condition along the space-like circle, we have to use the same boundary condition for the fermionic states. We set $\omega = \text{NS}$, where NS means the sum of the identity representation ($\omega = 0$) and the vector representation ($\omega = 2$). With this ω , the selection rule reduces to $\Lambda_N = \Xi_{N-1} + \Xi_N \pmod{2}$ with the notation in appendix B. The states are thus labeled by $(\Lambda; \Xi)$ with the field identification taken into account,⁷ and the conformal weights are

$$h_{(\Lambda; \Xi)} = \frac{1}{k+2N-1} [C_{2N+1}(\Lambda) - C_{2N}(\Xi)] + \frac{\omega}{4} + n. \quad (4.5)$$

Here $C_M(\Lambda)$ is the quadratic Casimir of $\mathfrak{so}(M)$ and the integer n is the grade at which Ξ appears in (Λ, ω) .

We use the embedding $\text{SO}(2N) \hookrightarrow \text{SO}(2N+1)$ as

$$i(v) = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \in \text{SO}(2N+1). \quad (4.6)$$

Then we embed $\widehat{\mathfrak{so}}(2N)_{k+1}$ diagonally into $\widehat{\mathfrak{so}}(2N)_k \otimes \widehat{\mathfrak{so}}(2N)_1$. The states of the coset is obtained by the decomposition

$$\Lambda \otimes \text{NS} = \bigoplus_{\Xi} (\Lambda; \Xi) \otimes \Xi. \quad (4.7)$$

⁷The field identification can be read off from the phases of character modular transformations [40], and it may be written as $(\Lambda, \omega; \Xi) \sim (A\Lambda, \omega + 2; \tilde{A}\Xi)$. Here A is the \mathbb{Z}_2 outer automorphism of $\widehat{\mathfrak{so}}(2N+1)_k$. The group of outer automorphisms of $\widehat{\mathfrak{so}}(2N)$ is \mathbb{Z}_4 or $\mathbb{Z}_2 \times \mathbb{Z}_2$ depending on whether N is odd or even. We set \tilde{A} to be one of the four that exchanges Ξ_0 and Ξ_1 (and possibly others), where the affine Dynkin labels are represented as $[\Xi_0; \Xi_1, \dots, \Xi_N]$.

The branching function of the coset model (4.1) can be defined as in the bosonic case

$$\text{ch}_\Lambda^{2N+1,k}(q, \iota(v)) \text{ch}_{\text{NS}}^{2N,1}(q, v) = \sum_{\Xi} s b_{\Lambda; \Xi}^{2N,k}(q) \text{ch}_{\Xi}^{2N,k+1}(q, v) . \quad (4.8)$$

Here the character of the $2N$ free fermions in the NS-sector is

$$\text{ch}_{\text{NS}}^{2N,1}(q, v) = \prod_{n=0}^{\infty} \prod_{i=1}^N (1 + v_i q^{n+\frac{1}{2}}) (1 + \bar{v}_i q^{n+\frac{1}{2}}) , \quad (4.9)$$

where v is an $\text{SO}(N, N)$ matrix with eigenvalues (v_i, \bar{v}_i) . We consider the Hilbert space spanned as $\mathcal{H} = \sum(\Lambda; \Xi) \otimes \overline{(\Lambda; \Xi)}$ and the partition function is

$$\mathcal{Z}^{2N,k}(q) = |q^{-\frac{c}{24}}|^2 \sum_{[\Lambda; \Xi]} |s b_{\Lambda; \Xi}^{2N,k}(q)|^2 . \quad (4.10)$$

Note that this is not the diagonal modular invariant in the usual sense, since we now take the sum of the identity and the vector representations for $\widehat{\text{so}}(2N)_1$ in the numerator of (4.1) in both the chiral and anti-chiral part.

In order to compare the CFT partition function with the supergravity partition function, we have to take a large N, k limit with the 't Hooft parameter (1.4)

$$\lambda = \frac{2N}{k + 2N - 1} \quad (4.11)$$

kept finite. Now the states of the $\mathcal{N} = (1, 1)$ super coset model (4.1) are labeled by $(\Lambda; \Xi)$, where Λ, Ξ are the highest weights of $\text{so}(2N + 1)$ and $\text{so}(2N)$, respectively. As mentioned in the bosonic case, the highest weight representations for $\text{so}(2N)$ can be labeled by Young diagrams with finitely many boxes in the limit, and the quadratic Casimir becomes $C_{2N}(\Xi) \sim N|\Xi|$ where $|\Xi|$ is the number of boxes of Young diagram denoted by the same letter Ξ . The case of $\text{so}(2N + 1)$ is also studied in appendix B and the same conclusions are obtained in this case. Namely, the highest weights are described by Young diagrams in the limit, which implies that the states of the super coset (4.1) are labeled by sets of two Young diagrams $(\Lambda; \Xi)$. The quadratic Casimir is (B.15)

$$C_{2N+1}(\Lambda) = |\Lambda|N + \frac{1}{2} \left(\sum_{i=1}^{N-1} l_i^2 - \sum_j c_j^2 \right) , \quad (4.12)$$

where the number of boxes in the i -th row is l_i and the number in the j -th column is c_j . In the large N limit, this behaves as $C_{2N+1}(\Lambda) \sim N|\Lambda|$.

We end this subsection by noting that the most fundamental states in the coset, which we believe to generate the remaining states under fusion in the 't Hooft limit, are $(v, 0; 0)$ together with its fermionic partner $(v, v; 0)$, and $(0, 0; v)$ also together with its fermionic partner $(0, v; v)$. Here v denotes the vector representation with only the first Dynkin label

non-zero and equal to one. Using the equation for the conformal weights (4.5) we get

$$\begin{aligned} h_{(v,0;0)} &= \frac{1}{2}\lambda, & h_{(v,v;0)} &= \frac{1}{2}(1+\lambda), \\ h_{(0,v;v)} &= \frac{1}{2}(1-\lambda), & h_{(0,0;v)} &= \frac{1}{2}(2-\lambda), \end{aligned} \quad (4.13)$$

where in the last case we had to use $n = 1$ in (4.5) since we start from the trivial representation in the numerator of the coset. This fits perfectly with the calculated conformal dimensions from the bulk side (2.15). We will thus have two supermultiplets in the CFT generated from $(v, 0; 0) \otimes (v, 0; 0)$ and $(0, v; v) \otimes (0, v; v)$, respectively.

4.2 Comparison of partition functions

In order to compare the CFT partition function with its gravity dual, we have to take the 't Hooft limit. For large k , the leading terms of the characters are

$$\text{ch}_\Lambda^{2N+1,k}(q, e^{\iota(H)}) \simeq \frac{q^{h_\Lambda^{2N+1,k}} \text{ch}_\Lambda^{2N+1}(e^{\iota(H)})}{\prod_{n=1}^{\infty} [(1-q^n)^N \prod_{\alpha \in \Delta_{2N+1}} (1 - e^{\alpha(\iota(H))} q^n)]}, \quad (4.14)$$

$$\text{ch}_\Lambda^{2N,k+1}(q, e^H) \simeq \frac{q^{h_\Lambda^{2N,k+1}} \text{ch}_\Lambda^{2N}(e^H)}{\prod_{n=1}^{\infty} [(1-q^n)^N \prod_{\alpha \in \Delta_{2N}} (1 - e^{\alpha(H)} q^n)]} \quad (4.15)$$

as in (3.10), where Δ_{2N+1} and Δ_{2N} denote the root systems of $\text{so}(2N+1)$ and $\text{so}(2N)$, respectively. As before, the Lie algebra characters appear in the limit. The conformal dimensions are now

$$h_\Lambda^{2N+1,k} = \frac{C_{2N+1}(\Lambda)}{k+2N-1}, \quad h_\Lambda^{2N,k+1} = \frac{C_{2N}(\Lambda)}{k+2N-1}. \quad (4.16)$$

Defining the leading term for large k as

$$sb_{\Lambda;\Xi}^{2N,k}(q) \simeq q^{h_\Lambda^{2N+1,k} - h_\Xi^{2N,k+1}} sa_{\Lambda;\Xi}^{2N}(q), \quad (4.17)$$

we have k -independent relations

$$\text{ch}_\Lambda^{2N+1}(\iota(v))\vartheta(q, v) = \sum_{\Xi} sa_{\Lambda;\Xi}^{2N}(q) \text{ch}_\Xi^{2N}(v). \quad (4.18)$$

The roots in Δ_{2N+1} include $\pm e_j$ ($j = 1, \dots, N$) in the orthogonal basis in addition to those in Δ_{2N} , and from this fact we have

$$\vartheta(q, v) = \prod_{n=0}^{\infty} \prod_{i=1}^N \frac{(1 + v_i q^{n+\frac{1}{2}})(1 + \bar{v}_i q^{n+\frac{1}{2}})}{(1 - v_i q^{n+1})(1 - \bar{v}_i q^{n+1})}. \quad (4.19)$$

Setting $\Lambda = 0$ in (4.18), we obtain

$$\vartheta(q, v) = \sum_{\Xi} sa_{0;\Xi}^{2N}(q) \text{ch}_\Xi^{2N}(v). \quad (4.20)$$

If we use the decomposition

$$\text{ch}_\Lambda^{2N+1}(\iota(v)) = \sum_{\Phi} R_{\Lambda\Phi}^{(2N)} \text{ch}_\Phi^{2N}(v) \quad (4.21)$$

and (3.16), then we find

$$sa_{\Lambda;\Xi}^{2N} = \sum_{\Phi,\Psi} R_{\Lambda\Phi}^{(2N)} N_{\Phi\Psi}^{(2N)\Xi} sa_{0;\Psi}^{2N}(q) = \sum_{\Phi,\Psi} R_{\Lambda\Phi}^{(2N)} N_{\Phi\Xi}^{(2N)\Psi} sa_{0;\Psi}^{2N}(q) . \quad (4.22)$$

Now we take the 't Hooft limit. In the limit, the highest weights of $\text{so}(2N+1)$ and $\text{so}(2N)$ are expressed by Young diagrams. In the next subsection, we will find

$$sa_{0;0}(q) = \lim_{N \rightarrow \infty} sa_{0;0}^{2N}(q) = \prod_{l=1}^{\infty} \prod_{n=2l}^{\infty} \left[\frac{1}{1-q^n} \right]^2 \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} (1+q^{n+\frac{1}{2}}) , \quad (4.23)$$

$$sa_{0;\Xi}(q) = \lim_{N \rightarrow \infty} sa_{0;\Xi}^{2N}(q) = sa_{0;0}(q) \text{sch}_{\Xi^t}(\mathcal{U}(\frac{1}{2})) , \quad (4.24)$$

where the supercharacter is defined in (2.17). The conformal dimensions are

$$h_\Lambda^{2N+1,k} \simeq \frac{\lambda}{2} |\Lambda| , \quad h_\Xi^{2N,k+1} \simeq \frac{\lambda}{2} |\Xi| . \quad (4.25)$$

As in [5], we assume the decoupling of null states in the 't Hooft limit, which means that only the terms with

$$|\Phi| + |\Xi| = |\Psi| \quad (4.26)$$

contribute to the fusion rules. In the large N limit, the coefficients stabilize as

$$\lim_{N \rightarrow \infty} N_{\Phi\Xi}^{(2N)\Psi} = N_{\Phi\Xi}^\Psi , \quad \lim_{N \rightarrow \infty} R_{\Lambda\Xi}^{(2N)} = R_{\Lambda\Xi} , \quad (4.27)$$

where $N_{\Phi\Xi}^\Psi$ are the Clebsch-Gordan coefficients of $\text{gl}(\infty)_+$. Just like the $\text{u}(M)$ case discussed in [23], the restriction functions for the $\text{so}(M)$ case become $R_{\Lambda\Phi} = N_{\Lambda|\Lambda/\Xi}^\Xi$ as shown in [42]. Here $|\Lambda/\Xi|$ represents the Young diagram with a single row and with $|\Lambda| - |\Xi|$ number of boxes. Then eqs. (3.57) and (3.59) of [23]

$$\text{sch}_\Lambda \text{sch}_\Xi = \sum_{\Pi} N_{\Lambda\Xi}^\Pi \text{sch}_\Pi , \quad \text{sch}_\Lambda(\mathcal{U}(0)) = \sum_{\Xi} R_{\Lambda\Xi} \text{sch}_{\Xi^t}(\mathcal{U}(\frac{1}{2})) , \quad (4.28)$$

which were proven in appendix A of that paper, lead to

$$\begin{aligned} sb_{\Lambda;\Xi}^\lambda &= q^{\frac{\lambda}{2}(|\Lambda|-|\Xi|)} \sum_{\Phi,\Psi} R_{\Lambda\Phi} N_{\Phi\Xi}^\Psi sa_{0;0}(q) \text{sch}_{\Psi^t}(\mathcal{U}(\frac{1}{2})) \\ &= sa_{0;0}(q) \text{sch}_\Lambda(\mathcal{U}(h_+)) \text{sch}_{\Xi^t}(\mathcal{U}(h_-)) . \end{aligned} \quad (4.29)$$

Combining the anti-chiral part, we have

$$\sum_{\Lambda,\Xi} |sb_{\Lambda;\Xi}^\lambda(q)|^2 = \mathcal{Z}_{\text{gauge}} \sum_{\Lambda,\Xi} |\text{sch}_\Lambda(\mathcal{U}(h_+)) \text{sch}_{\Xi^t}(\mathcal{U}(h_-))|^2 , \quad (4.30)$$

which reproduces the supergravity result (2.19).

4.3 Characters from free bosons and fermions

Let us now derive (4.23) and (4.24) using free bosons and fermions. The character (4.19) is that of $2N$ real fermions ψ^a ($a = 1, 2, \dots, 2N$) and $2N$ real bosons J^c ($c = 1, 2, \dots, 2N$). The Fock space is spanned by

$$\prod_{j=1}^{n_\psi} \psi_{-r_j - \frac{1}{2}}^{a_j} \prod_{l=1}^{n_J} J_{-t_l - 1}^{c_l} \Omega, \quad (4.31)$$

where r_j, t_l run over non-negative integers.

From (4.20), we can see that the branching function $sa_{0;\Xi}$ counts the multiplicity when the representation Ξ of $\mathfrak{so}(2N)$ appears. The representation appears for the first time when $|\Xi| = n_\psi + n_J$. Following the argument above (3.69) of [23], the branching function is found to be

$$\text{sch}_{\Xi^t}(U(\frac{1}{2})) \quad (4.32)$$

when summing over the possible modes and n_ψ, n_J while keeping the sum $n_\psi + n_J = |\Xi|$ fixed. We also need to consider the $\mathfrak{so}(2N)$ invariant states

$$\prod_{r,s=0}^{\infty} \left(\sum_{a=1}^{2N} \psi_{-r-\frac{1}{2}}^a \psi_{-s-\frac{1}{2}}^a \right)^{K_{rs}} \prod_{t,u=0}^{\infty} \left(\sum_{a=1}^{2N} J_{-t-1}^a J_{-u-1}^a \right)^{L_{tu}} \prod_{r,u=0}^{\infty} \left(\sum_{a=1}^{2N} \psi_{-r-\frac{1}{2}}^a J_{-u-1}^a \right)^{P_{ru}}. \quad (4.33)$$

Here we should set K_{rs} non-zero only for $r > s$ and L_{tu} non-zero only for $t \geq u$. We also set $P_{ru} = 0, 1$ for all r, u since they are fermionic operators. Invariant states can be constructed using Weyl's fundamental theorems of invariant theory for the orthogonal group [41]. Invariants that are not polynomials of the above states appear only for conformal dimensions larger than N , and also non-trivial relations between invariant states appear for the first time at that order of conformal dimension. In the large N limit, we can ignore these finite N effects and obtain

$$\begin{aligned} sa_{0;0}(q) &= \prod_{r>s=0}^{\infty} \sum_{K=0}^{\infty} q^{(r+s+1)K} \prod_{t \geq u=0}^{\infty} \sum_{L=0}^{\infty} q^{(t+u+2)L} \prod_{r,u=0}^{\infty} \sum_{P=0}^1 q^{(r+u+\frac{3}{2})P} \\ &= \prod_{r>s=0}^{\infty} \frac{1}{1-q^{r+s+1}} \prod_{t \geq u=0}^{\infty} \frac{1}{1-q^{t+u+2}} \prod_{r,u=0}^{\infty} (1+q^{r+u+\frac{3}{2}}) \\ &= \prod_{l=1}^{\infty} \prod_{n=2l}^{\infty} \left[\frac{1}{1-q^n} \right]^2 \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} (1+q^{n+\frac{1}{2}}) \end{aligned} \quad (4.34)$$

as in (4.23). By multiplication with (4.32), we get (4.24).

5 Symmetries of the dual conformal field theory

The symmetry algebra of the coset

$$\frac{\widehat{\mathfrak{so}}(2N+1)_k \oplus \widehat{\mathfrak{so}}(2N)_1}{\widehat{\mathfrak{so}}(2N)_{k+1}}, \quad (5.1)$$

is the commutant subalgebra $\text{Com}(\widehat{\mathfrak{so}}(2N)_{k+1}, \widehat{\mathfrak{so}}(2N+1)_k \oplus \widehat{\mathfrak{so}}(2N)_1)$ of $\widehat{\mathfrak{so}}(2N+1)_k \oplus \widehat{\mathfrak{so}}(2N)_1$ that commutes with $\widehat{\mathfrak{so}}(2N)_{k+1}$. Here, by $\widehat{\mathfrak{so}}(2N)_1$ we mean rank $2N$ free fermions \mathcal{F}_{2N} . Since they transform in the $2N$ -dimensional vector (standard) representation of $\mathfrak{so}(2N)$, they contain a homomorphic image of $\widehat{\mathfrak{so}}(2N)_1$ as subalgebra. In addition, as we will see, the coset algebra is too large as it also contains additional fields with spin of order N . In order to get rid of these additional fields, one needs an orbifold projection by improper orthogonal transformations. At finite N , the candidate coset for the $\mathcal{N} = 1$ super W-algebra is thus

$$\text{Orb}\left(\frac{\widehat{\mathfrak{so}}(2N+1)_k \oplus \mathcal{F}_{2N}}{\widehat{\mathfrak{so}}(2N)_{k+1}}\right), \quad (5.2)$$

but since the additional fields appear at conformal dimension at least N , these are invisible in the large N limit, in other words, the spin content for the coset and its orbifold is the same for large N .

5.1 The dimension $3/2, 2, 2, 5/2$ fields of the coset algebra

Before we discuss the complete coset algebra, we will explicitly compute the fields of lowest conformal dimension. For this we need some preparation, we write $\mathfrak{so}(2N+1) = \mathfrak{so}(2N) \oplus m$, where m carries the standard representation of $\mathfrak{so}(2N)$. We denote the currents of $\widehat{\mathfrak{so}}(2N+1)_k$ that belong to m with the Roman indices J^i, J^j, \dots and those associated to $\mathfrak{so}(2N)$ with Greek indices J^α, J^β, \dots . In addition, the fermions of $\widehat{\mathfrak{so}}(2N)_1$ also transform in the standard representation and we denote them by ψ^i, ψ^j, \dots . Then, the operator products expansions are

$$\begin{aligned} J^\alpha(z)J^\beta(w) &\sim \frac{k\delta_{\alpha,\beta}}{(z-w)^2} + \frac{f^{\alpha\beta\gamma}J^\gamma(w)}{(z-w)}, \\ J^\alpha(z)J^i(w) &\sim \frac{f^{\alpha ij}J^j(w)}{(z-w)}, \\ J^i(z)J^j(w) &\sim \frac{k\delta_{i,j}}{(z-w)^2} + \frac{f^{ij\alpha}J^\alpha(w)}{(z-w)}. \end{aligned} \quad (5.3)$$

We also need the operator product with normal ordered products of currents. Denote by $(J^\alpha J^\beta)(z)$ the normal ordered product of two currents, then we e.g. have

$$\begin{aligned} J^\alpha(z)(J^\beta J^\gamma)(w) &\sim \frac{k f^{\alpha\beta\gamma}}{(z-w)^3} + \frac{k\delta_{\alpha,\beta} J^\gamma(w) + k\delta_{\alpha,c} J^\beta(w) + f^{\alpha\beta\delta} f^{\delta\gamma\epsilon} J^\epsilon(w)}{(z-w)^2} + \\ &+ \frac{f^{\alpha\beta\delta}(J^\delta J^\gamma)(w) + f^{\alpha\gamma\delta}(J^\beta J^\delta)(w)}{(z-w)}. \end{aligned} \quad (5.4)$$

With the help of this formula, we compute

$$\begin{aligned} J^i(z)(J^\alpha J^\alpha)(w) &\sim \frac{(2N-1)J^i(w)}{(z-w)^2} + \frac{f^{i\alpha j}((J^j J^\alpha)(w) + (J^\alpha J^j)(w))}{(z-w)}, \\ J^\alpha(z)(J^i J^i)(w) &\sim \frac{4J^\alpha(w)}{(z-w)^2}, \\ J^i(z)(J^j J^j)(w) &\sim \frac{(2k+2N-1)J^i(w)}{(z-w)^2} - \frac{f^{i\alpha j}((J^j J^\alpha)(w) + (J^\alpha J^j)(w))}{(z-w)}. \end{aligned} \quad (5.5)$$

The fermionic fields are denoted by ψ^i with operator product expansion

$$\psi^i(z)\psi^j(w) \sim \frac{\delta_{i,j}}{(z-w)}, \quad (5.6)$$

and the corresponding currents are

$$j^\alpha = -\frac{1}{2} f^{\alpha ij} \psi^i \psi^j. \quad (5.7)$$

Moreover ψ^i are primaries in the standard representation for these currents

$$j^\alpha(z)\psi^i(w) \sim \frac{f^{\alpha ij} \psi^j(w)}{(z-w)}. \quad (5.8)$$

Let $\gamma = 2k + 4N - 2$, then the Virasoro field of the coset algebra is

$$\begin{aligned} T &= T_{\widehat{\mathfrak{so}}(2N+1)_k} + T_{\text{fermion}} - T_{\widehat{\mathfrak{so}}(2N)_{k+1}} \\ &= \frac{1}{\gamma} ((J^i J^i) - 2(J^\alpha j^\alpha) + (2k-3)T_{\text{fermion}}). \end{aligned} \quad (5.9)$$

The coset symmetry algebra is the algebra that commutes with the $K^\alpha = J^\alpha + j^\alpha$, this is certainly true for

$$G = \sqrt{\frac{2}{\gamma}} (J^i \psi^i). \quad (5.10)$$

The fields G and T obey the operator product algebra of the $\mathcal{N} = 1$ super Virasoro algebra, that is

$$\begin{aligned} T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}, \\ T(z)G(w) &\sim \frac{3G(w)/2}{(z-w)^2} + \frac{\partial G(w)}{(z-w)}, \\ G(z)G(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{(z-w)}. \end{aligned} \quad (5.11)$$

Next, there is an additional dimension two field that commutes with both j^α and J^α , hence also with K^α , and this is the Virasoro field \tilde{T} of the coset

$$\frac{\widehat{\mathfrak{so}}(2N+1)_k}{\widehat{\mathfrak{so}}(2N)_k}. \quad (5.12)$$

Explicitly, in terms of currents, it reads

$$\begin{aligned} \tilde{T} &= T_{\widehat{\mathfrak{so}}(2N+1)_k} - T_{\widehat{\mathfrak{so}}(2N)_k} \\ &= \frac{1}{\gamma}(J^i J^i) - \frac{1}{\gamma(\gamma-2)}(J^\alpha J^\alpha). \end{aligned} \quad (5.13)$$

This field is not a Virasoro primary, but the following linear combination

$$W_2 = c\tilde{T} - \beta T, \quad \beta = \frac{4kN}{\gamma^2(\gamma-2)}((\gamma-2)(2k+2N-1) - 4N+2) \quad (5.14)$$

is. This statement is a straightforward computation using the above operator product expansions. The dimension 5/2 partner $G_{5/2}$ of W_2 can be computed as

$$\begin{aligned} G(z)W_2(w) &\sim \frac{G_{5/2}(w)}{(z-w)}, \\ G_{5/2} &= \frac{1}{\sqrt{2}\gamma} \left(\beta(2(J^i \partial \psi^i) - (\psi^i \partial J^i)) + a f^{ij\alpha}(((J^\alpha J^j) \psi^i) + ((J^j J^\alpha) \psi^i)) \right), \\ a &= \frac{12kN}{\gamma^2} \left((2k+2N-1) + 2\frac{\gamma}{\gamma-2} \right). \end{aligned} \quad (5.15)$$

The operator product of W_2 with itself does not generate new fields, and it is

$$\begin{aligned} W_2(z)W_2(w) &\sim \frac{(c_b - \beta)^2 c/2}{(z-w)^4} + \frac{2(c-2\beta)W_2(w) + 2\beta(c-\beta)T(w)}{(z-w)^2} + \\ &+ \frac{(c-2\beta)\partial W_2(w) + \beta(c-\beta)\partial T(w)}{(z-w)}, \end{aligned} \quad (5.16)$$

where c_b is the central charge of the bosonic coset (5.12). The computation of further operator products becomes very complicated, but in a large k limit they simplify just as explained in section 6.5 of [33] for the case of the $\mathcal{N} = 2$ coset. We find

$$\begin{aligned} \lim_{k \rightarrow \infty} W_2(z)G_{5/2}(w) &\sim -\frac{6N^2G(w)}{(z-w)^3} - \frac{2N^2\partial G(w) + NG_{5/2}(w)}{(z-w)^2} \\ &\quad - \frac{\frac{N^2}{2}\partial^2G(w) + \frac{2N}{5}\partial G_{5/2}(w) + G_{7/2}(w) + \frac{3N}{2}A_{7/2}(w) - \frac{12N^2}{5}B_{7/2}(w)}{(z-w)}. \end{aligned} \quad (5.17)$$

Here $G_{7/2}$ is a fermionic primary field of conformal dimension $7/2$, and $A_{7/2}$ and $B_{7/2}$ are dimension $7/2$ descendants that together with $G_{7/2}$, $\partial G_{5/2}$ and ∂^2G form an orthogonal basis of dimension $7/2$ fields. The precise form of these fields in the large k limit is

$$\begin{aligned} G_{7/2} &= \frac{3N}{2}\partial G_{5/2} + \frac{5N}{2}(W_2G) + \frac{3N^2}{5}\partial^2G - \frac{9N^2}{\sqrt{k}}(J^i\partial^2\psi^i) - 4N(W_2G) + \frac{12N^2}{5}(TG), \\ A_{7/2} &= \frac{N}{2}(W_2G) + \frac{3N}{10}\partial G_{5/2}, \quad B_{7/2} = (TG) - \frac{3}{8}\partial^2G. \end{aligned} \quad (5.18)$$

We believe that the fields G, T, W_2 already generate the full symmetry algebra under iterated operator products.

5.2 The field content of the coset algebra

We now consider the field content of the coset algebra. Note, that the following analysis is in many respects similar to the one of last section, in particular, it relies on the classical invariant theory.

The generic field content of a coset algebra can under certain circumstances be computed using classical invariant theory [43]. The coset, we are interested in, is of this favourable type. As mentioned before, the coset algebra is the commutant subalgebra

$$\text{Com}(\widehat{\mathfrak{so}}(2N)_{k+1}, \widehat{\mathfrak{so}}(2N+1)_k \oplus \mathcal{F}_{2N}). \quad (5.19)$$

The algebra $\widehat{\mathfrak{so}}(2N+1)_k \oplus \mathcal{F}_{2N}$ is generated as a conformal field theory by the fields J^α generating the $\widehat{\mathfrak{so}}(2N)_k$ subalgebra of $\widehat{\mathfrak{so}}(2N+1)_k$, the fields J^i which are primaries in the vector representation of $\widehat{\mathfrak{so}}(2N)_k$, and the fermions ψ^i . An alternative set of generators is K^α, J^i and ψ^i , where the fields K^α generate the $\widehat{\mathfrak{so}}(2N)_{k+1}$ subalgebra of the commutant problem. Note that J^i as well as ψ^i are primaries in the vector representation of K^α . In such a situation, it was argued that the fields of the commutant subalgebra are those that can be identified with the $\text{SO}(2N)$ invariant products of the vector representation as follows. Let

$$p(J, \partial J, \dots, \partial^m J, \partial\psi, \dots, \partial^m\psi) \quad (5.20)$$

be a normally ordered polynomial in J, ψ and their derivatives that is invariant under the natural action of $\text{SO}(2N)$. Then

$$p(J, \mathcal{D}J, \dots, \mathcal{D}^m J, \mathcal{D}\psi, \dots, \mathcal{D}^m \psi) \quad (5.21)$$

is a generator of the coset algebra, and all generators are of such a form. Here, the covariant derivative is

$$\mathcal{D}I^i = \partial I^i + \frac{1}{k+1} f^{\alpha ij} (J^\alpha I^j), \quad I^i \in \{J^i, \psi^i\}. \quad (5.22)$$

We are thus left with determining all invariants in the vector representation. Weyl's first fundamental theorem for the orthogonal group [41] tells us that all such invariants are expressible in terms of the basic invariants which are traces of two vectors and determinants of matrices whose columns are $2N$ vectors. Clearly, the determinants have spin at least N and thus they are invisible in the large N limit. Also, note that the determinants are improper invariants, this means they change sign under transformations by orthogonal matrices with determinant minus one. The traces are proper invariants and all proper invariants can be expressed in terms of traces [41]. We have three types of traces

$$A_{(n,m)} = \mathcal{D}^n J^i \mathcal{D}^m J^i, \quad B_{(n,m)} = \mathcal{D}^n \psi^i \mathcal{D}^m \psi^i, \quad C_{(n,m)} = \mathcal{D}^n J^i \mathcal{D}^m \psi^i. \quad (5.23)$$

The spins are

$$\Delta(A_{(n,m)}) = n + m + 2, \quad \Delta(B_{(n,m)}) = n + m + 1, \quad \Delta(C_{(n,m)}) = n + m + \frac{3}{2}. \quad (5.24)$$

Now, if there were no relations between products of the fields, then we can count that the algebra we found has a generating set of fields whose bosonic fields have spin $2, 2, 4, 4, 6, 6, \dots$ while the fermionic generators have spin $3/2, 5/2, 7/2, \dots$. Note, that these fields are multiplets of the $\mathcal{N} = 1$ super algebra as $(3/2, 2), (2, 5/2), (7/2, 4), \dots$. The second fundamental theorem of invariant theory for the orthogonal group [41] states that all relations between invariants either involve a determinant of a matrix whose columns are vectors or they are determinants of $(2N - 1) \times (2N - 1)$ matrices whose entries are traces. All these relations concern invariants whose spin is at least N which implies that the spin content of the coset algebra agrees with the proposed higher spin supergravity in the large N limit.

We would like to remark, that due to the determinants, the coset algebra at finite N is larger than the $\mathcal{N} = 1$ super W-algebra. In [44] it has been proposed in an analogous situation to consider an orbifold in order to obtain a smaller coset algebra. It is certainly possible that the invariant subalgebra, invariant under all improper orthogonal transformations, is the $\mathcal{N} = 1$ super W-algebra. As mentioned before, this is an issue which becomes invisible in the large N limit and is thus of minor importance for the present purpose.

6 Conclusion and outlook

In this work, we propose that the $\mathcal{N} = 1$ truncation of Prokushkin and Vasiliev's $\mathcal{N} = 2$ higher spin supergravity on AdS_3 [7] is dual to a limit of a family of conformal field theories given by the $\mathcal{N} = (1, 1)$ super cosets

$$\frac{\widehat{\mathfrak{so}}(2N+1)_k \oplus \widehat{\mathfrak{so}}(2N)_1}{\widehat{\mathfrak{so}}(2N)_{k+1}}. \quad (6.1)$$

We need to take the large N limit with 't Hooft parameter

$$\lambda = \frac{2N}{2N+k-1} \quad (6.2)$$

kept finite. We have supported this conjecture by showing that the supergravity and the CFT partition functions match, i.e. the spectrum is the same on both sides. Further, we also studied the symmetry of the super coset model, especially we provided explicit formulae for the fields of dimension $3/2, 2, 2$ and $5/2$.

It often happens that seemingly very different cosets possess the same symmetry algebra. We would like to remark that there are other cosets whose spin content of the symmetry algebra seems to coincide with the one of the coset of the present work. For example consider

$$\frac{\widehat{\mathfrak{osp}}(1|2N)_k \oplus \mathcal{B}_N}{\widehat{\mathfrak{sp}}(2N)_{k-1/2}}. \quad (6.3)$$

Here \mathcal{B}_N denotes rank N $\beta\gamma$ -ghosts which contain as a subalgebra a homomorphic image of $\mathfrak{sp}(2N)_{-1/2}$. The central charge of this coset is

$$c = -\frac{3Nk}{k+N+1/2}. \quad (6.4)$$

This means that only for some negative levels k , we get a positive central charge. The spin content of this coset can be studied as in the last section and with the help of Weyl's fundamental theorems of invariant theory for the symplectic group [41]. Indeed in the large N limit, the spin content of this coset coincides with the spin content of the coset algebra studied in last section. More cosets are constructed as follows. Let $S_{M|P}$ be the algebra generated by $2M$ free real fermions and P $\beta\gamma$ ghosts, then $S_{M|P}$ contains a homomorphic image of $\widehat{\mathfrak{osp}}(2M|2P)_1$ as subalgebra. The symmetry algebra of the cosets

$$\frac{\widehat{\mathfrak{osp}}(2M+1|2P)_k \oplus \mathcal{S}_{M|P}}{\widehat{\mathfrak{osp}}(2M|2P)_{k+1}} \quad (6.5)$$

can be studied as before in the large k limit and again seems to have the same spin content. Conformal field theories of supercosets are usually not unitary and hence we expect a tentative dual higher spin supergravity to be less interesting.

Further work is needed to obtain a better understanding of the duality. In order to compare the partition functions, we assumed that some states in the CFT decouple from

the others in the large N limit. It is thus necessary to examine whether this assumption is true or not. Also, it would be desirable to show that the asymptotic symmetry of the $\mathcal{N} = 1$ supergravity can be reproduced by the 't Hooft limit of the $\mathcal{N} = (1, 1)$ super coset, in particular one should also study the Hamiltonian reduction of the $\widehat{\mathfrak{osp}}(2N + 1|2N)_k$ affine Lie algebra. Furthermore, like in the cases of the other holographies on AdS_3 , important checks of the duality would be to calculate and compare correlators, and to consider the RG-flow.

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A $\mathcal{N} = 1$ truncation of Prokushkin-Vasiliev theory

The field equations of higher spin gravity theory by Prokushkin and Vasiliev can be expressed in terms of generating functions (W_μ, B, S_α) [7]. Here W_μ is a space-time one-form including the higher spin gauge fields, B is a zero-form including matter fields, and S_α is an auxiliary field. The generating functions depend on the parameters $(z_\alpha, y_\alpha; \psi_{1,2}, k, \rho|x_\mu)$ where x_μ are the space-time coordinates. The spinor index α takes values 1, 2. The generating functions are expanded as

$$A(z, y; \psi_{1,2}, k, \rho|x) = \sum_{B,C,D,E=0}^1 \sum_{m,n=0}^{\infty} A_{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n}^{BCDE} k^B \rho^C \psi_1^D \psi_2^E z^{\alpha_1} \dots z^{\alpha_m} y^{\beta_1} \dots y^{\beta_n} .$$

The Grassmann parity $\pi = 0, 1$ is determined by the number of spinor indices as

$$\begin{aligned} \pi(W_{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n}) &= \frac{1}{2}(1 - (-1)^{|m+n|}) , & \pi(B_{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n}) &= \frac{1}{2}(1 - (-1)^{|m+n|}) , \\ \pi(S_{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n}) &= \frac{1}{2}(1 - (-1)^{|m+n+1|}) . \end{aligned} \quad (\text{A.1})$$

Moreover, we define a map σ by

$$\sigma[A(z, y; \psi_{1,2}, k, \rho)] = A^{\text{rev}}(-iz, iy; \psi_{1,2}, k, \rho) \quad (\text{A.2})$$

where the order of all generating elements is reversed in A^{rev} . As shown in [7] the following transformation is a symmetry of the field equations

$$\eta(W_\mu) = -i^{\pi(W)} \sigma(W_\mu) , \quad \eta(B) = i^{\pi(B)} \sigma(B) , \quad \eta(S_\alpha) = i^{\pi(S)+1} \sigma(S_\alpha) . \quad (\text{A.3})$$

We can thus consistently truncate the fields to those invariant under this transformation, and this gives us the $\mathcal{N} = 1$ supersymmetric theory.

We consider vacuum solutions with $B = \nu$. In [7] they obtained three types of vacuum solution for S_α , but we chose $S_{\alpha,0}^{\text{sym}}$ in eq. (6.6) of that paper. The vacuum solution $W = W_0$ depends only on $(\tilde{y}_\alpha; \psi_1, k)$. Here $\tilde{y}_\alpha = \tilde{y}_\alpha^{\text{sym}}$ is defined in eq. (6.11) of [7], but all we need is that they obey the following fundamental commutator

$$[\tilde{y}_\alpha, \tilde{y}_\beta] = 2i\epsilon_{\alpha,\beta}(1 + \nu k), \quad \{\tilde{y}_\alpha, k\} = 0. \quad (\text{A.4})$$

Explicit forms of $S_{\alpha,0}^{\text{sym}}$ and $\tilde{y}_\alpha^{\text{sym}}$ will not be used, but the following properties are important

$$\sigma[S_{\alpha,0}^{\text{sym}}] = -iS_{\alpha,0}^{\text{sym}}, \quad \sigma[\tilde{y}_\alpha^{\text{sym}}] = i\tilde{y}_\alpha^{\text{sym}}. \quad (\text{A.5})$$

In particular, $S_{\alpha,0}^{\text{sym}}$ is invariant under the action of η defined in eq. (A.3). Defining A, \bar{A} as

$$W_0 = -\frac{1 + \psi_1}{2}A - \frac{1 - \psi_1}{2}\bar{A}, \quad (\text{A.6})$$

the field equations for A, \bar{A} are given by those of the Chern-Simons theory for the algebra generated by (\tilde{y}_α, k) . This algebra was called $\text{shs}[\lambda]$ algebra in [6] where $\lambda = (1 - \nu)/2$. If we consider the sub-sector with even number of \tilde{y}_α and $k = 1$, then $\text{shs}[\lambda]$ is reduced to its bosonic sub-algebra $\text{hs}[\lambda]$. The generators of $\text{shs}[\lambda]$ may be given by [33]

$$V_m^{(s)+} = \left(\frac{-i}{4}\right)^{s-1} S_m^s, \quad V_m^{(s)-} = \left(\frac{-i}{4}\right)^{s-1} S_m^s k, \quad V^{(1)-} = k + \nu \quad (\text{A.7})$$

with $s = 2, 3, \dots$ for bosonic generators and $s = 3/2, 5/2, \dots$ for fermionic generators. Here S_m^s is the symmetric product of \tilde{y}_α s, where $2s - 2$ is the number of \tilde{y}_α and $2m = N_1 - N_2$ with $N_{1,2}$ being the number of $\tilde{y}_{1,2}$. For the bosonic generators, even s generators are invariant under the action of (A.3), and for the fermionic generators, $V_m^{(s)\pm}$ generators with $s = 2n \mp 1/2$ ($n = 1, 2, \dots$) survive.

We may define a different basis for the bosonic generators as

$$U_m^{(s)\pm} = \left(\frac{-i}{4}\right)^{s-1} S_m^s \frac{1 \pm k}{2}. \quad (\text{A.8})$$

Without the $\mathcal{N} = 1$ truncation, $U_m^{(s)+}$ generate $\text{hs}[\lambda]$ while $U_m^{(s)-}$ generate $\text{hs}[1 - \lambda]$. It is known that the infinite dimensional Lie algebra $\text{hs}[\lambda]$ can be truncated at $\lambda = \pm n$ with integer n and the reduced algebra becomes $\text{sl}(n)$ [45]. In the same way, the even spin sub-algebra of $\text{hs}[\lambda]$ is reduced at $\lambda = \pm n$ to $\text{sp}(n)$ for even n and $\text{so}(n)$ for odd n (see, e.g., [5]). Thus, the $\mathcal{N} = 1$ truncation of $\text{shs}[\lambda]$ can be reduced at $\lambda = 2N + 1$ to a superalgebra whose bosonic sub-algebra is given by $\text{so}(2N + 1) \oplus \text{sp}(2N)$. Notice that $(V_m^{(2)+}, V_r^{(3/2)+})$ with $m = 0, \pm 1$ and $r = \pm 1/2$ are the generators of the $\text{osp}(1|2)$ sub-algebra. In terms

of the superprincipal embedding of $\text{osp}(1|2)$, the generators of $\text{osp}(2N+1|2N)$ can be decomposed by the representation of $\text{osp}(1|2)$. The action of $V_{\pm 1/2}^{(3/2)^+}$ produces fermionic generators from bosonic ones in the same representation of $\text{osp}(1|2)$. This implies that the $\mathcal{N} = 1$ truncation of $\text{shs}[\lambda]$ can be reduced to the $\text{osp}(2N+1|2N)$ superalgebra since the bosonic sub-algebra of $\text{osp}(2N+1|2N)$ is $\text{so}(2N+1) \oplus \text{sp}(2N)$. In other words, the symmetry for the massless gauge sector of the $\mathcal{N} = 1$ truncated theory is given by an analytic continuation of $\text{osp}(2N+1|2N)$ with $\lambda = 1+2N$. Or put differently, when λ takes integer values, the \mathbb{Z}_2 automorphism η defined in (A.3) becomes the \mathbb{Z}_2 automorphism acting on supermatrices by a combination of (minus) supertransposition and conjugation with a special matrix and this defines $\text{osp}(2N+1|2N)$ in terms of $\text{gl}(2N+1|2N)$, see [38].

The small perturbation by matter fields can be obtained by setting $B = \nu + \mathcal{C}$. Studying the dynamical parts of \mathcal{C} , we can read off the matter content. The $\mathcal{N} = 1$ truncation of the matter fields is discussed around eq. (10.8) in [7], and it is given by an $\mathcal{N} = 1$ hypermultiplet with two complex scalars having masses

$$(M_-^B)^2 = -1 + \lambda^2, \quad (M_+^B)^2 = -1 + (\lambda - 1)^2, \quad (\text{A.9})$$

and two fermions with mass

$$(M_{\pm}^F)^2 = (\lambda - \frac{1}{2})^2. \quad (\text{A.10})$$

B Orthogonal Lie algebras

Some basics of $\text{so}(2N)$ and $\text{so}(2N+1)$ Lie algebra are summarized.

B.1 $\text{so}(2N)$ Lie algebra

It will be convenient to introduce an orthogonal basis e_i ($i = 1, 2, \dots, N$) with $e_i \cdot e_j = \delta_{ij}$. In this basis, the roots of the $\text{so}(2N)$ Lie algebra are of the form $\pm e_i \pm e_j$ ($i \neq j$) and the simple roots are

$$\alpha_i = e_i - e_{i+1} \quad (i = 1, \dots, N-1), \quad \alpha_N = e_{N-1} + e_N. \quad (\text{B.1})$$

The fundamental weights are

$$\begin{aligned} \lambda_i &= \sum_{l=1}^i e_l \quad (i = 1, \dots, N-2), \\ \lambda_{N-1} &= \frac{1}{2}(e_1 + e_2 + \dots + e_{N-1} - e_N), \\ \lambda_N &= \frac{1}{2}(e_1 + e_2 + \dots + e_{N-1} + e_N), \end{aligned} \quad (\text{B.2})$$

and the Weyl vector is

$$\rho = \sum_{i=1}^N \lambda_i = \sum_{i=1}^N (N-i)e_i . \quad (\text{B.3})$$

We consider a representation with the highest weight

$$\Lambda = \sum_{i=1}^N \Lambda_i \lambda_i , \quad (\text{B.4})$$

where the coefficients are the Dynkin labels $\Lambda_i \geq 0$. In the orthogonal basis, the highest weight can be expressed as

$$\Lambda = \sum_{i=1}^N l_i e_i . \quad (\text{B.5})$$

with

$$l_i = \sum_{l=i}^{N-2} \Lambda_l + \frac{1}{2}(\Lambda_{N-1} + \Lambda_N) \quad (i = 1, \dots, N-2) , \quad (\text{B.6})$$

$$l_{N-1} = \frac{1}{2}(\Lambda_{N-1} + \Lambda_N) , \quad l_N = \frac{1}{2}(\Lambda_{N-1} - \Lambda_N) .$$

In this basis, the quadratic Casimir is computed as

$$C_{2N}(\Lambda) = \frac{1}{2}\Lambda \cdot (\Lambda + 2\rho) = \frac{1}{2} \sum_{i=1}^N l_i^2 + \sum_{i=1}^N l_i(N-i) . \quad (\text{B.7})$$

As shown in [5], the quadratic Casimir for the representation with $\Lambda_{N-1} \neq 0$ and/or $\Lambda_N \neq 0$ is of order N^2 and these representations will be neglected. In other words, we set $\Lambda_{N-1} = \Lambda_N = 0$. Then the highest weight is labeled by a Young tableau with l_i boxes in the i -th row. Notice that now $l_i \geq l_{i+1}$ and $l_{N-1} = l_N = 0$. Denoting the number of boxes in the j -th column by c_j , we have (see (A.9) of [14])

$$\sum_i i l_i = \frac{1}{2} \sum_j c_j^2 + \frac{|\Lambda|}{2} \quad (\text{B.8})$$

where the total number of boxes is denoted by $|\Lambda|$. The quadratic Casimir is now

$$C_{2N}(\Lambda) = |\Lambda| \left(N - \frac{1}{2} \right) + \frac{1}{2} \left(\sum_{i=1}^{N-2} l_i^2 - \sum_j c_j^2 \right) , \quad (\text{B.9})$$

and thus $C_{2N}(\Lambda) \sim N|\Lambda|$ in the large N limit.

B.2 $\mathfrak{so}(2N + 1)$ Lie algebra

We use the orthogonal basis e_i ($i = 1, \dots, N$) as in the $\mathfrak{so}(2N)$ case. The roots of the $\mathfrak{so}(2N + 1)$ Lie algebra are $\pm e_j$ in addition to $\pm e_i \pm e_j$ ($i \neq j$) with $i, j = 1, \dots, N$. The simple roots are

$$\alpha_i = e_i - e_{i+1} \quad (i = 1, \dots, N - 1), \quad \alpha_N = e_N, \quad (\text{B.10})$$

and the fundamental weights are

$$\lambda_i = \sum_{l=1}^i e_l \quad (i = 1, \dots, N - 1), \quad \lambda_N = \frac{1}{2} \sum_{l=1}^N e_l. \quad (\text{B.11})$$

The Weyl vector is now

$$\rho = \sum_{i=1}^N (N + \frac{1}{2} - i) e_i. \quad (\text{B.12})$$

The Dynkin labels $\Lambda_i \geq 0$ are the coefficients of the highest weight

$$\Lambda = \sum_{i=1}^N \Lambda_i \lambda_i = \sum_{i=1}^N l_i e_i, \quad (\text{B.13})$$

$$l_i = \sum_{l=i}^{N-1} \Lambda_l + \frac{1}{2} \Lambda_N \quad (i = 1, \dots, N - 1), \quad l_N = \frac{1}{2} \Lambda_N. \quad (\text{B.14})$$

The quadratic Casimir for the highest weight representation is

$$C_{2N+1}(\Lambda) = \frac{1}{2} \sum_{i=1}^N l_i^2 + \sum_{i=1}^N l_i (N + \frac{1}{2} - i). \quad (\text{B.15})$$

It is easy to see that all the elements of the inverse of Cartan matrix $C_{ij}^{-1} = \lambda_i \cdot \lambda_j$ are non-negative. Thus we have

$$C_{2N+1}(\Lambda) \geq C_{2N+1}(\Lambda^{(s)}) = N^2 \cdot \frac{a}{4} + \frac{a^2}{8}, \quad (\text{B.16})$$

where $\Lambda_i^{(s)} = 0$ for $i = 1, \dots, N - 1$ and $\Lambda_N^{(s)} = a$. This implies that $C_{2N+1}(\Lambda)$ is of order N^2 for representations with $\Lambda_N \neq 0$ as in the $\mathfrak{so}(2N)$ case, so we again set $\Lambda_N = 0$. The highest weight representation is now labeled by a Young tableau with l_i boxes in the i -th row. We denote the number of boxes in the j -th column by c_j , The quadratic Casimir is now

$$C_{2N+1}(\Lambda) = |\Lambda|N + \frac{1}{2} \left(\sum_{i=1}^{N-1} l_i^2 - \sum_j c_j^2 \right), \quad (\text{B.17})$$

which again leads to $C_{2N+1}(\Lambda) \sim N|\Lambda|$ for large N .

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