CONSENSUS WITH ORDINAL INFORMATION

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CLASSICAL MAUT (Cardinal utility theory)

$k \in N$: set of points of view $\quad |N| = n$
$a \in A$: set of potential actions

g_k : mapping from A to $\mathbb{R}$
$g_k(a)$: evaluation of a for k,
X_k : scale related to k (set of all possible values for g_k)

$$g : (g_1, \cdots, g_n) \in \prod_k X_k$$

$g(a) : (g_1(a), \cdots, g_n(a))$: profile related to a

We suppose that

- all actions are comparable

$(\succcurlyeq, A)$: total preorder defined on the set A

From the classical numerical representation theorem, there exists U : mapping from $\prod_k X_k$ to $\mathbb{R}$ such that

$$a \succcurlyeq b \quad \text{iff} \quad U[g(a)] \geq U[g(b)]$$

Under classical conditions

$\lessdot$ Kranz, Luce, Suppes and Tversky (1978)
Wakker (1989)

U can be expressed in terms of an additive model

$$U(a) = \sum_i U_i(a) = \sum_i U_i[g_i(a)]$$

U : decomposable utility function

U_i are not unique

$U_i \rightarrow V_i = \alpha U_i + \beta_i$ (admissible transformations)

$$U(a) = \sum_i p_i W_i(a) = \sum_i p_i W_i[g_i(a)]$$

with $p_i \geq 0$

$$\begin{aligned} \sum p_i &= 1 \\ W_i(g_{i*}) &= 0 \\ W_i(g_i^*) &= 1 \end{aligned}$$

W_i : normalized utility functions

$$p_i = U[g_{1*}, \ldots, g_{i-1*}, g_i^*, g_{i+1*}, \ldots, g_{n*}]$$

Commensurability in the cardinal case

g_1 : prof. experience (in years)
g_2 : level of studies (BAC + ... years)

$$\begin{array}{ccccccccc} 0 & \le g_1 \le & 20 & & 0 & \le g_2 \le & 6 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ g_{1*} & & g_1^* & & g_{2*} & & g_2^* \end{array}$$

If one considers the normalization operation

$$g_1' = \frac{g_1 - g_{1*}}{g_1^* - g_{1*}} \qquad g_2' = \frac{g_2 - g_{2*}}{g_2^* - g_{2*}},$$

we might accept that g_1 and g_2 are measured on the same scale S

$$U_1(g_1) = S(g_1') \qquad U_2(g_2) = S(g_2')$$

and it is meaningful to consider

$$U_1(g_1) \ge U_2(g_2)$$

EXTENSIONS

- delete the condition of transitivity in the symmetric part of $\succcurlyeq$

$\lessdot$ Tversky (1969) : additive difference model

- delete the condition of completeness of $\succcurlyeq$

$\lessdot$ Bouyssou (1986)
Fishburn (1990)
Vind (1991)
Bouyssou and Pirlot (1996)
: "differences of preference" model

- introduce interactions among criteria.

- consider that evaluations are measured on ordinal scales.

PREFERENTIAL INDEPENDENCE

In the additive model, a key property is known as preferential independence (identical to the sure thing principle of Savage in decision theory under uncertainty)

$$\begin{aligned} f &: (f_1, \ldots, f_n) \\ g &: (g_1, \ldots, g_n) \\ fAg &: (\underbrace{f_1 \ldots f_a}_{A} \underbrace{g_{a+1} \ldots g_n}_{\overline{A}}) \end{aligned}$$

$$fAh \succcurlyeq gAh \Rightarrow fAk \succcurlyeq gAk, \quad \forall f, g, h, k \in \prod_k X_k$$

If the principle is violated (it is a necessary condition to consider the additive utility theory as an adequate tool), interactions among points of view will appear which might be encompassed by the use of

a Choquet integral

What is a Choquet integral ?

$$U_C(a)$$
$$= \sum_{i=1}^{n} \{U_{(i)}(g_{(i)}(a)) - U_{(i-1)}(g_{(i-1)}(a))\}\, \mu((i), \ldots, (n)).$$

This implies that we need to determine

$\mu(T)$: set function from $\mathcal{P}(N)$ to $\mathbb{R}$
Choquet capacity

$$\begin{cases} \mu(\emptyset) = 0 \\ \mu(N) = 1 \\ \mu(A) \leq \mu(B) \text{ if } A \subset B \end{cases}$$

$$U_C(a)$$
$$= \sum_{i=1}^{n} U_{(i)}(g_{(i)}(a))[\mu((i), \ldots, (n)) - \mu((i+1), \ldots, (n))]$$
$$= \sum_{i=1}^{n} U_{(i)}(g_{(i)}(a))\delta_{(i)}[\mu((i), \ldots, (n))]$$

i.e. a weighted sum on ordered values.

Conditions related to consensus functions of that type have been studied by

Wakker (1989) in the framework of decision under uncertainty

and revisited by

Modave and Grabisch (1998) in MCDM

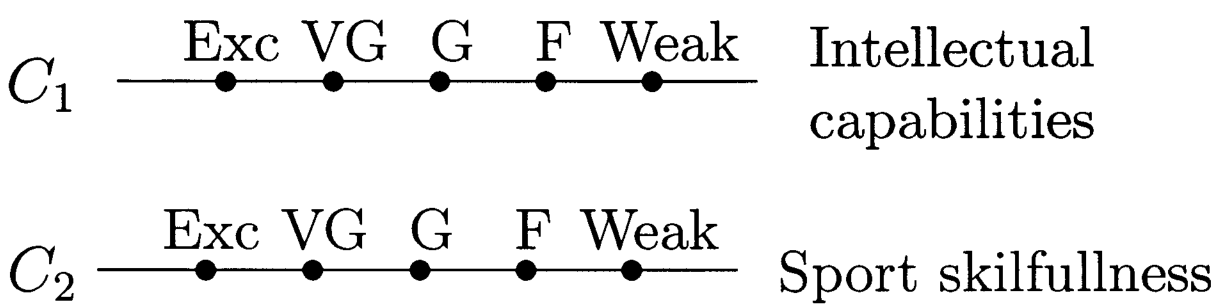
with the use of commensurability hypotheses.

Ordinal MAUT

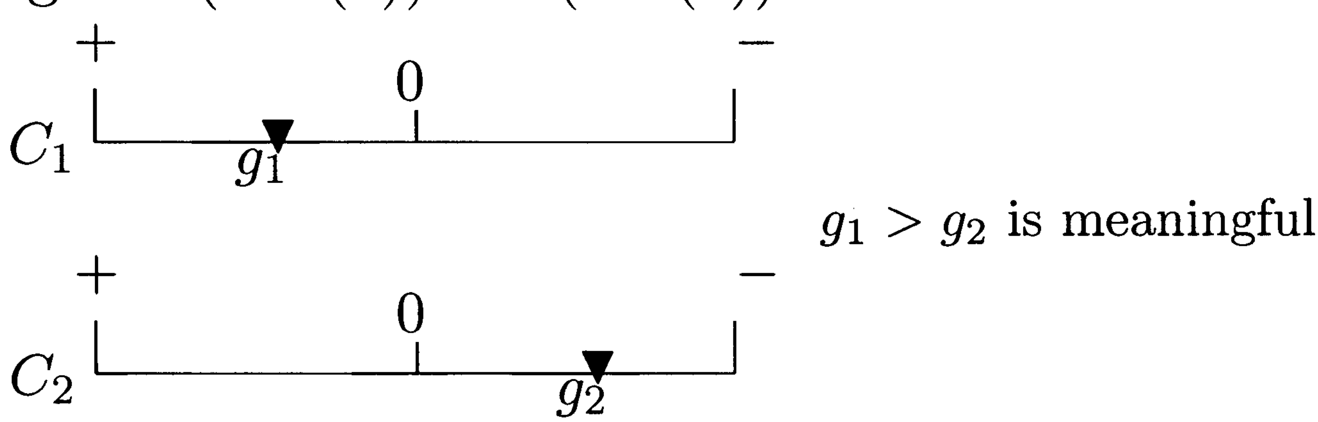
When utilities are of ordinal nature, every consensus functions dealing with weight summation (classical or of Choquet type) is meaningless.

We consider the ordinal commensurability hypothesis :

$$\{g_k\} \text{ are expressed on a common ordinal scale } X : \bigcup_k X_k$$



Common scale S gives meaningful understanding of $S(\text{Exc}(1)) > S(\text{VG}(2))$



Sugeno integral

Consider $U_k[g_k]$ defined on $[0,1]$

$$\mu(T), \quad T \subset N$$

measured on the same ordinal scale.

We briefly write $U_k[g_k] : x_k$.

We define a consensus function $M_\mu(x_1, \ldots, x_n)$ of Sugeno integral type as

$$\begin{aligned}
& U_S(x_1, \ldots, x_n) \\
&= \bigvee_{T \subset N} \left[\mu(T) \wedge \left(\bigwedge_{i \in T} x_i\right)\right] \qquad \text{max-min form} \\
&= \bigvee_{i=1}^{n} \left[x_{(i)} \wedge \mu((i), \ldots, (n))\right] \\
&= \bigwedge_{T \subset N} \left[\mu(N \setminus T) \vee \left(\bigvee_{i \in T} x_i\right)\right] \qquad \text{max-min form} \\
&= \bigwedge_{i=1}^{n} \left[x_{(i)} \vee \mu((i+1), \ldots, (n))\right] \\
&= \text{median}\left[x_1, \ldots, x_n, \mu((2), \ldots, (n)), \ldots, \mu((n))\right.
\end{aligned}$$

median form

Comparison between Choquet and Sugeno integrals

$$\begin{aligned} & U_C(x_1, \ldots, x_n; \mu) \\ = & \sum_i \left[x_{(i)} - x_{(i-1)} \right] \mu((i), \ldots, (n)) \end{aligned}$$

$$U_S(x_1, \ldots, x_n; \mu) = \bigvee_i \left[x_{(i)} \wedge \mu((i), \ldots, (n)) \right]$$

$$U_C(x_1, \ldots, x_n; \mu) = \sum_{T \subset N} \left[a(T) \bigvee_{i \in T} x_i \right]$$

a : Möbius transform of μ

$$U_S(x_1, \ldots, x_n; \mu) = \bigvee_T \left[\mu(T) \wedge \left(\bigwedge_{i \in T} x_i \right) \right]$$

If $\mu(T) \in \{0, 1\}$,

$U_C = U_S$: Boolean max-min aggregator

Some properties of the Sugeno aggregator

- $U_S(\overline{x};\mu) = U_S(x,\ldots,x;\mu) = x$,

$\overline{x}$: constant action. U_S is idempotent.

- Consider $(\overline{1}A\overline{0}) = (\underbrace{1\ldots1}_{A}\underbrace{0\ldots0}_{\overline{A}})$

$$U_S(\overline{1}A\overline{0}) = \mu(A).$$

$\mu(A)$ is interpreted as the utility of the profile $(\overline{1}A\overline{0})$.

- Consider now a binary action

$$\overline{x}A\overline{y} = (\underbrace{x\ldots x}_{A}\underbrace{y\ldots y}_{\overline{A}})$$

$$\begin{aligned} U_S(\overline{x}A\overline{y}) &= \text{median }(x,y,\mu(\overline{A})) \text{ if } x < y \\ &= \text{median }(x,y,\mu(A)) \text{ if } x > y \end{aligned}$$

$U_S(\overline{x}A\overline{y})$ is either equal to $x, y, \mu(A), \mu(\overline{A})$.

No compensation is allowed between x and y.

Particular cases of Sugeno integrals

Boolean max-min, min-max

$$
\begin{aligned}
U_S \quad = B_\mu^{\vee\wedge} \quad &= \bigvee_{T\subset N} \left[\mu(T) \wedge \left(\bigwedge_{i\in T} x_i\right)\right] \\
&= \bigwedge_{T\subset N} \left[\mu(N\setminus T) \vee \left(\bigvee_{i\in T} x_i\right)\right] \\
&\mu(T) \in \{0,1\}
\end{aligned}
$$

Weighted max

If μ is a possibility measure Π i.e. defined by $(p_1, \ldots, p_n)$

$$\bigvee_i p_i = 1, \qquad \mu(T) = \bigvee_{i \in T} p_i$$

$$U_S(x) = \bigvee_{i=1}^{n} [x_i \wedge p_i]$$

Weighted min

If μ is a necessity measure N i.e. defined by $(n_1, \ldots, n_n)$

$$\bigwedge_i n_i = 0, \qquad \mu(N \setminus T) = \bigwedge_{i \in T} n_i$$

$$U_S(x) = \bigwedge_{i=1}^{n} [x_i \vee n_i]$$

Why is the commensurability hypothesis needed ?

- If one uses the Sugeno integral, it is obvious

$$U_S(x) = \bigvee_T \left[\mu(T) \wedge \left(\bigwedge_{i \in T} x_i \right) \right]$$

- Consider the very general result of KIM (1990, MSS).

Consider $M(x_1, \dots, x_n)$ with x_k measured on independent ordinal scales (admissible transformations are φ_i st. increasing continuous bijection from $[0,1]$ to $[0,1]$)
(i) M is continuous
(ii) M maps independent ordinal scales into an ordinal scale

$$M(x_1, \dots, x_n) \leq M(y_1, \dots, y_n) \Leftrightarrow$$
$$M(\varphi_1 x_1, \dots, \varphi_n x_n) \leq M(\varphi_1 y_1, \dots, \varphi_n y_n)$$

[(i) + (ii)] $\Leftrightarrow$ M is either a constant
either a dictator consensus $g(x_j)$ for some j
g : continuous st. monotonic function

OPEN PROBLEM : Suppress (i) !

Characterization of Sugeno integral consensus (Marichal (1998))

Consider $\{x_i\}$ all being defined on the same ordinal scale.
Admissible transformation : bijection φ.

(i) M is continuous.
(ii) M is idempotent.
(iii) M satisfies the "ordinal comparison meaningfulness" condition

$$M(x) \leq M(y) \Leftrightarrow M(\varphi x) \leq M(\varphi y)$$

$$[(\mathrm{i}) + (\mathrm{ii}) + (\mathrm{iii})] \Leftrightarrow$$

$$\begin{aligned} M(x) & \\ = & \quad \mathrm{median}(x_1, \ldots, x_n), \underbrace{\mu((i)\ldots(n)), \ldots, \mu((n))}_{\in\{0,1\}} \\ = & \quad B_\mu^{\vee\wedge}, \text{ boolean max-min} \end{aligned}$$

μ does not show up !

Second result : Marichal (1998)

Commensurability is assumed for $\{x_i\}$ and $\{\mu(T), T \subset N\}$.

$\mu(T)$ can be considered as $x'_1, \ldots, x'_{2^n-2}$.

$$M(x_1, \ldots, x_n; \mu) : [0,1]^{n+2^n-2} \longrightarrow \mathbb{R}$$

(1) M is *continuous*

(2) M presents the *independence* of constant actions with respect to satisfaction degrees

$$M(x, \ldots, x; \mu) = M(x, \ldots, x; \mu')$$

(weaker condition than idempotency).

(3) M presents the "ordinal comparison meaningfulness" property.

$[(1) + (2) + (3)] \Leftrightarrow M$ is constant or
is $g \circ M_S(x, \mu)$
(a g transform of the Sugeno integral)
g : st. monotonic bijection

Third result : Marichal (1998)

Consider $M(x, \mu)$:

(P1) : M is continuous

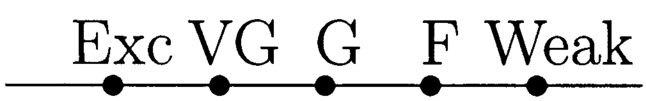
(P2) : M is idempotent

(P3) : M satisfies the "ordinal comparison meaningfulness"

$$[(P1) + (P2) + (P3)] \Leftrightarrow$$

$$M \text{ is } M_S(x, \mu)$$

Continuity is questionable for pre-defined finite scales



Characterization of Sugeno integrals

Sabbadin (1998) in the spirit of the work by Savage on decision under uncertainty.

Consider $(x_1, \dots, x_n)$ commensurable evaluations.

(P1) Ranking $(\succcurlyeq, A)$ (Savage first axiom).
A complete preorder on the set A is supposed to exist.

(P2) Non triviality : $\exists g_j, g_\ell$ such that $g_k < g_\ell$ (Savage fifth axiom).
Non trivial comparisons between evaluation exist.

(P3) Weakened order over constant actions (weaker than Savage third axiom)

$$x \prec y \Rightarrow \overline{x}Ah \prec \overline{y}Ah$$

(P4) Non compensation : $(\overline{x}A\overline{y})$ is either equal to $x, y, \mu(A), \mu(\overline{A})$.
The consensus of a binary action reflects one of its two evaluations or the satisfaction of the subsets which create the dichotomy.

(P5) Commensurability : $\exists g \in X$, such that $\overline{g} \sim (\overline{1}A\overline{0})$.
The satisfaction level scale can be projected on the common ordinal preference scale.
You can exchange a constant.

$[(\text{P1}) + (\text{P2}) + (\text{P3}) + (\text{P4}) + (\text{P5})] \Leftrightarrow$

$$\exists U,\ \mu : \text{Choquet capacity}$$

such that

$$M(x_1, \dots, x_n) = \bigvee_{T \subset N} \left[\mu(T) \wedge \left(\bigvee_{i \in T} x_i \right) \right]$$