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Two paper volumes, called "Annals of LAMSADE" are planned per year. They can be thematic or representative of the research recently performed in the laboratory.

Foreword

The volume you have in hand includes a number of contributions presented during the joint DIMACS - LAMSADE workshop on Computer Science and Decision Theory, Paris, 27-29 October 2004.

The workshop focused on modern computer science applications of methods developed by decision theorists, in particular methods involving consensus and associated order relations. The broad outlines concern connections between computer science and decision theory, development of new decision-theory-based methodologies relevant to the scope of modern CS problems, and investigation of their applications to problems of computer science and also to problems of the social sciences which could benefit from new ideas and techniques.

The workshop has been organised within the DIMACS - LAMSADE project funded by the NSF and the CNRS aiming to promote join research around the above issues and is expected to be followed by other similar initiatives.

This initiative has been possible thanks to the contribution of NSF, the CNRS and Université Paris-Dauphine.

A special thanks goes to Bruno Escoffier and Meltem Öztürk for their valuable help in organising the workshop and editing this volume.

Denis Bouyssou, Mel Janowitz, Fred Roberts, Alexis Tsoukiàs

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Annales du LAMSADE N°3

The Majority Rule and Combinatorial Geometry (via the Symmetric Group)

James Abello*

Abstract

The Marquis du Condorcet recognized 200 years ago that majority rule can produce intransitive group preferences if the domain of possible (transitive) individual preference orders is unrestricted. We present results on the cardinality and structure of those maximal sets of permutations for which majority rule produces transitive results (consistent sets). Consistent sets that contain a maximal chain in the Weak Bruhat Order inherit from it an upper semimodular sublattice structure. They are intrinsically related to a special class of hamiltonian graphs called persistent graphs. These graphs in turn have a clean geometric interpretation: they are precisely visibility graphs of staircase polygons. We highlight the main tools used to prove these connections and indicate possible social choice and computational research directions.

1 Introduction

Arrow's impossibility theorem [5], says that if a domain of voter preference profiles is sufficiently diverse and if each profile in the domain is mapped into a social order on the alternatives that satisfies a few appealing conditions, then a specific voter is a dictator in the sense that all of his or her strict preferences are preserved by the mapping. One interesting question is how to determine restrictions on sets of voters preference orders which guarantee that every non-empty finite subset of candidates S contains at least one who beats or ties all others under pairwise majority comparisons [14, 15, 17]. When voters express their preferences via linear preference orders over $\{1, \ldots, n\}$ (i.e. permutations in S_n) a necessary and sufficient condition is provided by the following proposition. It identifies embedded 3x3 latin squares as the main reason for intransitivity of the majority rule.

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T_6

123456	135642	351264	531624	563421
123546	312456	351624	536124	564312
123564	312546	356124	531642	564321
132456	312564	351642	536142	653124
132546	315246	356142	536412	653142
132564	315264	356412	536421	653412
135246	315624	356421	563124	653421
135264	315642	531246	563142	654312
135624	351246	531264	563412	654321

Figure 1: A maximal consistent subset of S_6. It is conjectured to be maximum in [15]

Definition 1.1 *A three subset $\{\alpha, \beta, \gamma\} \subset S_n$ contains an embedded 3x3 latin square if there exist $\{i, j, k\} \subset \{1, \ldots, n\}$ such that $\alpha_i = \beta_j = \gamma_k$, $\alpha_j = \beta_k = \gamma_i$ and $\alpha_k = \beta_i = \gamma_j$. $C \subset S_n$ is called consistent if no three subset of C contains an embedded 3x3 latin square.*

Proposition 1.2 *[15] For a finite set of voters P with preference orders in a subset C of S_n , denote by $|aPb|$ the number of voters that prefer a to b. For every subset S of at least three candidates,*

$$\{a \in S : \forall b \in S - a, |aPb| \geq |bPa|\} \neq \emptyset$$

if and only if C does not contain an embedded 3 by 3 latin square (i.e. Consistent sets produce transitive results under majority rule).

It has been conjectured that for every n the maximum cardinality of such consistent sets is not more than 3^{n-1} [1]. Maximal consistent sets that contain a maximal chain in the Weak Bruhat order of S_n are upper semimodular sublattices of cardinality bounded by the n-th Catalan number [4](Theorem 2.2). This result is the basis of an output sensitive algorithm to compute these sublattices (see Remark 2.3 and Corollary 2.4). With such sublattices we associate a class of graphs (called persistent) that offers a bridge from the combinatorics of consistent sets of permutations to non degenerate point configurations (see Section 2.3 and Theorem 2.8). Every graph in this apparently ”new” class can be realized as the visibility graph of a staircase polygon(see Section 3). A colorful way to view these abstract connections is that if the aggregate collection of voters is realizable as a non-degenerate collection of points then majority rule produces transitive results. Under this interpretation point configurations represent the candidates aggregate view provided by the voters rankings (one point per candidate).

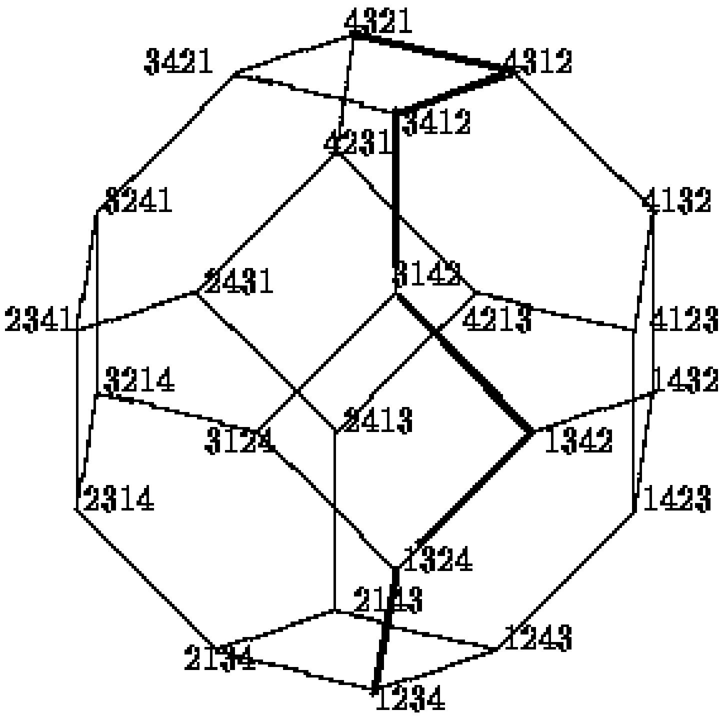

Figure 2: The weak Bruhat order for S_4. A maximal chain is $\{1234, 1324, 1342, 3142, 3412, 4312, 4321\}$.The identity is at the bottom and the identity reverse is at the top. By suitable relabeling we can in fact have any permutation at the top and its reverse at the bottom.

2 The Weak Bruhat Order, Balanced Tableaux and Persistent Graphs

2.1 The Weak Bruhat Order of S_n

For $n \geq 2$, let S_n denote the symmetric group of all permutations of the set $\{1, \ldots, n\}$. As a Coxeter group S_n is endowed with a natural partial order called the weak Bruhat order ([2, 4, 12]. This order is generated by considering a permutation γ an immediate successor of a permutation α if and only if γ can be obtained from α by interchanging a consecutive pair of non inverted elements of α. The partial order $\leq WB$ is the transitive closure of this relation. The unique minimum and maximum elements are the identity and the identity reverse respectively, (Figure 2).

$(S_n, \leq WB)$ is a ranked poset where the rank of a permutation α is its inversion number $i(\alpha)$ = $|\{(\alpha_i, \alpha_j) : i < j$ and $\alpha_i > \alpha_j\}|$. From now on, consider all permutations in S_n written in one line notation and let s_i denote the adjacent transposition of the letters in positions i and $i + 1$. With this convention αs_i is the permutation obtained by switching the symbols α_i and α_{i+1} in α. Every permutation is then representable as a word over the alphabet $\{ s_1, \ldots, s_{n-1} \}$ where the juxtaposition express α as a left to right product of the s_i's. Among these representations, those words that involve exactly $i(\alpha)$ transpositions are called the reduced words for α. Those reduced words that represent the maximum element have length $N = \binom{n}{2}$ = $(n * (n - 1))/2$ and they are the *maximal chains* in $(S_n, \leq WB)$ from the identity permutation to its reverse. They constitute the central combinatorial object in this work. In particular, the majority rule produces transitive results when applied to them. We define now a closure operator that allow us to characterize those maximal consistent sets of permutations that contain maximal chains.

Definition 2.1 *For* $\alpha \in S_n$*, let* $Triples(\alpha) = \{(\alpha_i, \alpha_j, \alpha_k) : i < j < k\}$ *and for* $C \subset S_n$*,* $Triples(C) = \bigcup\{Triples(\alpha) : \alpha \in C\}$*. The Triples closure of a set* $C \subset S_n$ *is* $Closure(C) = \{\alpha \in S_n : Triples(\alpha) \subset Triples(C)\}$.

It is natural to ask how to obtain $Closure(C)$ for a given set $C \subset S_n$. In particular, what is the cardinality and structure of maximal consistent sets? We provide next an answer to these questions for the case that Ch is a maximal chain in $(S_n, \leq WB)$.

2.1.1 Maximal Connected Consistent Sets

It is not difficult to see that any three permutations that contain an embedded 3x3 latin square can not be totally ordered in $(S_n, \leq WB)$. This means that a maximal chain Ch is a consistent set. Moreover, $|Triples(Ch)| = 4\binom{n}{3}$. Therefore $Closure(Ch)$ is a maximal consistent set. The size of $Closure(Ch)$ varies widely depending on Ch. In some cases, it is of $O(n^2)$ and in many others is of size $> 2^{n-1} + 2^{n-2} - 4$ for $n \geq 5$ ([1]). It has been conjectured (since 1985) in [2] that the maximum cardinality of a consistent set in S_n is $\leq 3^{n-1}$. The next result provides information about the structure and maximum cardinality of those consistent sets containing a maximal chain in the weak Bruhat order. It is a useful result because it furnishes an algorithm to generate the Closure of a maximal chain Ch. This allow us to have at our disposal all the possible rankings that are compatible with Ch. They represent in this case the maximum allowable set of ranking choices for the voters if we want to obtain transitivite results from the majority rule. Transitivity conditions like Inada's single peakedness [16] correspond to the choice of a particular maximal chain in $(S_n, \leq WB)$.

Theorem 2.2 *[4]The closure of any maximal chain in* $(S_n, \leq WB)$ *is an upper semi-modular sublattice of* $(S_n, \leq WB)$ *that is maximally consistent. Its cardinality is* $\leq$ *the* n*th Catalan number.*

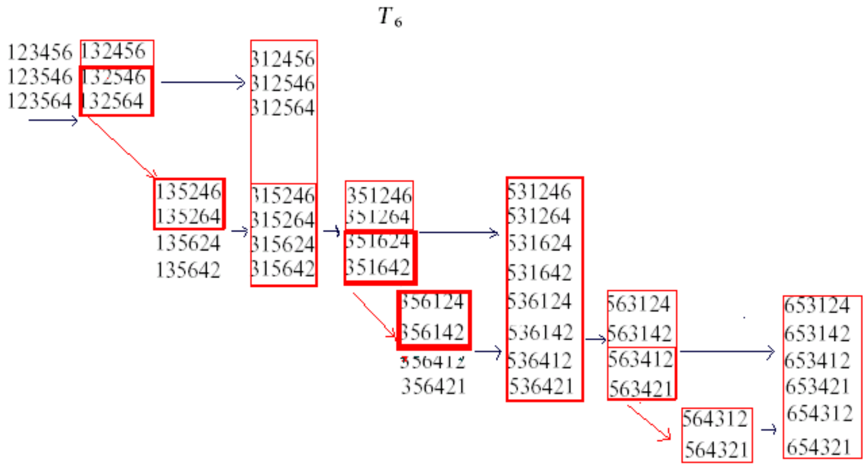

Figure 3: The maximal consistent subset of S_6 of Figure 1 viewed as a sublattice of the Weak Bruhat Order.The subsets enclosed in rectangles are the ones obtained by a projection. The maximal chain is the one defined by the sequence of transpositions $Path(Ch) = \{45, 46, 23, 25, 26, 24, 13, 15, 16, 14, 12, 35, 36, 34, 56\}$. Incoming arrows to a rectangle correspond to a single transposition used to project a previous subset. These transpositions are $\{23, 25, 13, 15, 16, 35, 36, 34, 56\}$.

Remark 2.3 *The question that comes to mind next is where a permutation $\alpha \in Closure(Ch)$ lives in the Hasse diagram of $(S_n, \leq WB)$?. The answer is that it lies close to Ch. Namely, $Closure(Ch)$ is a connected subgraph (the undirected version) in the Hasse diagram of the weak Bruhat order. To see this let $Path(Ch)$ be the labeled ordered path from the identity to the identity reverse, defined by Ch, in the Hasse diagram of $(S_n, \leq WB)$, ie. $Path(Ch) = (t_1, \ldots, t_N)$ where $t_l = (i, j)$ if the symbols i and j were interchanged by the lth transposition in Ch. Notice that this is an alternate notation referring to the actual symbols in a permutation rather than their positions but it is better suited for this portion of the paper. Let $Path_k(Ch)$ denote the set of permutations appearing in the first k steps of $Path(Ch)$, for $k = 1, \ldots, N$. It follows from the proof of the previous theorem that $Closure(Path_k(Ch))$ has a unique maximum element which is precisely the maximum element in $Path_k(Ch)$. Call this element the kth bottom element. Moreover, $Closure(Path_{k+1}(Ch)) - Closure(Path_k(Ch))$ = a projection of certain connected subset of $Closure(Path_k(Ch))$ that is determined by the adjacent transposition t_{k+1}. This is stated more precisely in the following corollary.*

Corollary 2.4 *For a maximal chain Ch in the weak Bruhat order of S_n, let $Projectable_{k+1}(Ch)$ be the set of $\gamma \in Closure(Path_k(Ch))$ for which there exists a downward path from γ to the bottom element of $Closure(Path_k(Ch))$ such that all the adjacent transpositions used in the path are disjoint from t_{k+1}. $Closure(Ch)$ can be computed by* *an iterated application of the following property.*

$$Closure(Path_{k+1}(Ch)) - Closure(Path_k(Ch)) = \{\alpha \in S_n : \\ \exists\gamma \in Projectable_{k+1}(Ch) \ for \ which \ t_{k+1}(\gamma) = \alpha\}$$

Remark 2.5 *The previous corollary can be turned into an algorithm that computes $Closure(Ch)$ in time proportional to $|Closure(Ch)|$, i.e. is an output sensitive algorithm. To our knowledge, no consistent set has been found of cardinality larger than the ones produced by this algorithm. The reason could be that maximal consistent sets that are not connected are not larger than connected ones. Figure 1 is an example of a maximal consistent subset of S_6 with 45 permutations which is conjectured in [15] to be the overall maximum in this case. It was constructed by ad hoc methods but since it contains a maximal chain it can be described succintly as $Closure(Ch)$ where $Path(Ch) = \{45, 46, 23, 25, 26, 24, 13, 15, 16, 14, 12, 35, 36, 34, 56\}$. Its overall structure is illustrated by a coarse drawing of the corresponding sublattice of $(S_6, \leq WB)$ in Figure 3. Each subset obtained by a projection is isomorphic to its pre-image. Incoming arrows into a rectangle depict the pieces that form the preimage of a projection by an adjacent transposition.*

Next we present an alternative encoding of these maximal chains by special tableux of staircase shape called balanced tableaux. These tableaux provide the bridge between the weak Bruhat order and special combinatorial graphs called persistent.

2.2 Balanced Tableaux

A Ferrer's diagram of staircase shape is the figure obtained from $n-1$ left justified columns of squares of lengths $n-1$, $n-2$, ... , 1. A tableau T of staircase shape is a filling of the cells of the Ferrer's diagram of staircase shape with the distinct integers in the set { 1, …, N } where $N = \binom{n}{2}$. We denote by $SS(n)$ the set of tableux of staircase shape and assume for the indexes i,j and k that $i < j < k$. A tableau $T \in SS(n)$ is said to be balanced if for any three entries $T(j,i)$, $T(k,i)$, $T(k,j)$ we have either $T(j,i) < T(k,i) < T(k,j)$ or $T(j,i) > T(k,i) > T(k,j)$. The key property that we exploit is a beautiful bijection due to Edelman and Greene [12]. Namely, given a maximal chain in $(S_n, \leq WB)$, set $T(j,i) = l$ if and only if i and j are the symbols interchanged in going from the $(l-1)$th permutation to the lth permutation in the chain. It is proved in [12] that this mapping defines a one to one correspondence between balanced tableux in $SS(n)$ and maximal chains in $(S_n, \leq WB)$ (The balanced tableau associated with the maximal chain used in Figure 3 is depicted below. In this case $n = 6$ and $N = 15$).

1					
11	2				
7	3	3			
10	6	14	4		
8	4	12	1	5	
9	5	13	2	15	6

With each balanced tableau T we associate a graph $skeleton(T)$ with vertex set $\{1, \ldots, n\}$ and edge set = $\{\ (k, i) : T(k, i) > T(k', i)\ \forall k',\ i < k' < k\ \}$. In other words, the edges in $skeleton(T)$ record those entries in T whose values are larger than all the entries above in its column (i.e. they are restricted local maximum in their columns). By the balanced property this is equivalent to $\{(k, i) : T(k, i) < T(k, i'), \forall i', i < i' < k\}$ (i.e. they are restricted local minimum in their rows). The skeleton corresponding to the above balanced tableau(i.e. the maximal chain used in Figure 3) is

1					
1	2				
0	1	3			
0	1	1	4		
0	0	0	1	5	
0	0	0	1	1	6

The reader may be pondering about the properties of these graphs that arise as skeletons of the balanced tableaux associated with maximal chains in the weak Bruhat order.The next section offers a graph theoretical characterization.

2.3 Persistent Graphs

Chordal graphs are a well studied class with a variety of applications. We introduce now an ordered version of chordality that together with an additional property called inversion completeness define what we call persistent graphs ([8]).

Definition 2.6 *A connected graph $G = (V, E)$ with an specified linear ordering $H = (1, \ldots, n)$ on V is called chordal with respect to H if every H-ordered cycle of length ≥ 4 has a chord. G is called inversion complete with respect to H if for every 4-tuple $i < j < k < l$, it is the case that $\{(H_i, H_k), (H_j, H_l)\} \subset E(G)$ implies that $(H_i, H_l) \in E(G)$.*

In other words, pairs of edges that interlace in the order provided by H force the existence of a third edge joining the minimum and maximum(in the order) of the involved vertices.

Definition 2.7 *A graph $G = (V, E)$ with a Hamiltonian path H is called H-persistent if it is ordered chordal and inversion complete with respect to H.*

The following theorem provides a graph theoretical characterization of the skeletons of balanced tableaux. Namely, they are precisely persistent graphs.

Theorem 2.8 *A graph $G = (V, E)$ is H-persistent if and only if is the skeleton of a balanced tableau $T \in SS(n)$ where $|V| = n$ and H = $(1, 2, \ldots, n)$.*

Proof Sketch: That the skeleton of a balanced tableau $T \in SS(n)$ is hamiltonian with hamiltonian path $H = (1, \ldots, n)$ follows from the definition of the skeleton. That the obtained graph is H-persistent is a consequence of the balanced property. The interesting direction is how to associate with a given H-persistent graph a balanced tableau. The core of the proof relies on the following facts.

1. Any H persistent graph with at least n edges has at least an edge e such that $G - e$ is H-persistent. Call such an edge a reversible edge.
2. The complete graph is H-persistent for $H = (1, 2, \ldots, n)$ and it is the skeleton of the balanced tableau T where for $j > i$, $T(j, i) = (((j - 1) * (j - 2)/2) + i)$ for $i \in \{1, \ldots, j - 1\}$. Each row and column is sorted in increasing order.
3. Given an H-persistent graph G, [3] presents and $O(n^5)$ algorithm that provides a sequence of persistent graphs that starts with the complete graph K_n and ends with G. The algorithm deletes successively a set $\{e_1, \ldots, e_k\}$ of reversible edges and constructs for each $i = 1, \ldots, k$ a maximal chain Ch_i in $(S_n, \leq WB)$ such that $skeleton(Ch_i)$ is isomorphic to the persistent graph G_i = $G - \{e_1, \ldots, e_i\}$.
4. G_k is isomorphic to a persistent graph G given as input.

Items $1, 2, 3, 4$ above allow us to conclude that any persistent graph G is the skeleton of a balanced tableau $T \in SS(n)$ where T is the encoding of the maximal chain Ch_k produced by the algorithm where k is the number of edges that have to be deleted from K_n to obtain G •

Since balanced tableaux and maximal chains in the weak Bruhat order of S_n are just different encodings of the same objects we will abuse notation by using $Skeleton(Ch)$ to refer to the graph associated with the balanced tableau corresponding to Ch. It makes sense them to define an equivalence relation on maximal chains based on the skeletons of their corresponding balanced tableaux. Namely, two maximal chains are related if their corresponding balanced tableaux have the same graph skeleton. The reader may be wandering what this has to do with the majority rule. The answer is that if Ch' is a maximal chain $\subset Closure(Ch)$ then $Skeleton(Ch')$ is identical to $Skeleton(Ch)$, i.e.

each maximal connected consistent set C in the weak Bruhat of S_n has a unique persistent graph associated with it. This graph encodes the local column maximums (and local row minimums)of the tableaux associated with any of the maximal chains appearing in C. The corresponding graph represents a global characteristic of the set of rankings which offers a ”novel” approach to understanding voters profiles. As an example, the well known single peakedness condition for transitivity corresponds to a very special persistent graph. This line of thinking brings immediately the characterization question, i.e. do persistent graphs characterize maximal connected consistent sets? In other words, is the $Closure$ of a maximal chain Ch equal to the union of all maximal chains Ch' which have the same skeleton as Ch?. The answer is not always. For sure we know that $Closure(Ch)$ is contained in the set of all chains that have the same skeleton as Ch but the reverse is not true. However, we can provide a geometric characterization and this is the purpose of the next section.

3 Maximal Chains in the Weak Bruhat Order with the same Skeleton and Non-degenerate Point Configurations

Let $Conf$ be a non-degenerate configuration of n points on the plane. Without loss of generality, assume that not two points have the same x-coordinate and label the points from 1 through n in increasing order of their x-coordinates. The points in the configuration determine $N = \binom{n}{2}$ straight lines. We can construct a tableau T of shape $SS(n)$ that encodes the linear order on the slopes of these lines by setting $T(i, j) = l$ if and only if the rank of the slope of the line through i and j in this linear order is l. As the reader may suspect the obtained tableau is a balanced tableau and therefore it encodes a maximal chain in the weak Bruhat Order. This chain is precisely the first half of the Goodman and Pollack circular sequence associated with the configuration ([13]). The question is what is a geometric interpretation of the skeleton of the corresponding tableau?. In other words, what geometric property is encoded by the corresponding persistent graph?. The answer lies in the notion of visibility graphs of staircase polygons (Definition 3.2). This is the subject of the remaining part of this paper. It contains a proof sketch of one of the main results of this work (Theorem 3.4)

Definition 3.1 *Consider a configuration $Conf$ of n points $\{p_1, \ldots, p_n\}$ with coordinates (x_i, y_i) for point p_i. $Conf$ is called a staircase configuration if for every $i < j$, $x_i < x_j$ and $y_i > y_j$. A staircase path consists of a staircase configuration plus the $n-1$ straight line segments joining p_i and p_{i+1}, for $i = 1, \ldots, i = n-1$. A staircase polygon P is a staircase path together with the segments from the origin to p_1 and from the origin to p_n, (Figure 4 illustrates a staircase polygon).*

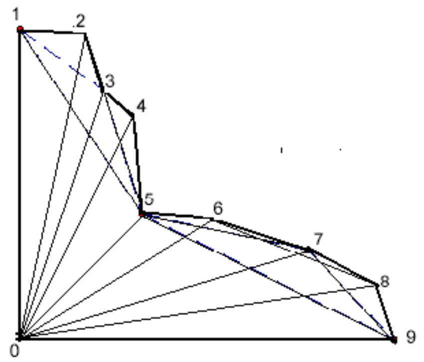

Figure 4: A staircase polygon. Since vertex 0, that is the origin, sees everybody it is removed from consideration.

Definition 3.2 *Two vertices p and q of a simple polygon P are said to be visible if the open line segment (p, q) joining them is completely contained in the interior of P or if the closed segment $[p, q]$ joining them is a segment of P's boundary. The visibility graph of a simple polygon P is denoted by $Vis(P) = (V, E)$ where V is the set of vertices of P and E is the set of polygon vertex pairs that are visible.*

Proposition 3.3 *The visibility graph of a staircase polygon P with ordered vertex set $(p_1, \ldots, p_n)$ is a persistent graph with respect to the hamiltonian path $H = (p_1, \ldots, p_n)$.*

Proof Sketch: The first half period of the Goodman and Pollack circular sequence ([13]) associated with the point configuration, defined by the vertexes of a staircase polygon P, is a maximal chain in the weak Bruhat order. Therefore its associated tableau T which completely encodes the ordering of the slopes is balanced and its associated skeleton is persistent by Theorem 2.8. To see that this graph is identical to the visibility graph of P let m_{ik} denote the magnitude of the slope between points p_i and p_k where $k > i + 1$. p_i is visible from p_k if and only if the open line segment joining them lies in the interior of P. For the case of staircase polygons this implies that there is no $j, k < j < i$ such that $m_{ik} \leq m_{ij}$. Therefore $m_{ik} > m_{jk}$ for $j = i - 1, i - 2, \ldots, k + 1$. Since T encodes this ordering this means that v_i is visible from v_k iff $T(i, k)$ is larger than all entries that lie above it, i.e. $T(i, k)$ is a restricted local maximum. •

From the majority rule view point the previous proposition says that when the voters rankings have a corresponding staircase point configuration the candidates can be placed on a staircase path and each voter's ranking correspond to his/her view, of the candidates in the configuration, when the voter is located outside the convex hull of the point set. The local maximum statistics obtained from the slopes ranking are encoded by geometric visibility among the candidates within the corresponding staircase polygon. What about a converse, i.e. Is it clear when is it that the voters rankings have a corresponding staircase configuration?. The next result states that if the set of voters rankings is the $Closure(Ch)$ for some $Ch \in (S_n, \leq WB)$ then there exists a staircase polygon P on n points so that its $Visibility$ graph is isomorphic to $Skeleton(Ch)$.

Theorem 3.4 *Let M_n denote a maximal consistent set and let Ch be a maximal chain in $(S_n, \leq WB)$. $M_n = Closure(Ch)$ iff $Skeleton(Ch)$ is the visibility graph of a staircase polygon P on n points.*

Proof Sketch:[3] $(\leftarrow)$ The visibility graph of a staircase polygon P is identical to $skeleton(T)$ where T encodes the ranking of the $N = \binom{n}{2}$ slopes determined by the n polygon vertices as in the previous proposition. By letting Ch denote the corresponding maximal chain in $(S_n, \leq WB)$ and using Theorem 2.2 the result follows •

Proof Sketch: $(\rightarrow)$ $M_n = Closure(Ch)$ implies that $Skeleton(Ch)$ is H-persistent where $H = (1, 2, \ldots, n)$ by Theorem 2.8 . The difficult part is to prove that there exists a staircase polygon P such that $Vis(P)$ is identical to $Skeleton(Ch)$. The tricky aspect is that Ch may not be realizable at all as a non-degenerate configuration of points. In fact, deciding if a given Ch is realizable in the sense described in this paper is NP-hard. However, what we are able to prove constructively is that there exists a maximal chain Ch' in $(S_n, \leq WB)$ such that $Skeleton(Ch')$ is identical to $Skeleton(Ch)$ even though Ch may not be realizable. This means that there is a geometric staircase ordering of the candidates whose corresponding set of local maximum is the same as those of any chain in $Closure(Ch)$. In other words by lifting the hard question of direct realizability of maximal chains to persistent graphs we get out of a difficult mathematical stumbling block. The essential tool is an inductive geometric simulation of the main steps followed in the proof of Theorem 2.8. Namely, take $Skeleton(Ch)$ and create corresponding geometric steps that produce from a convex staircase configuration, realizing the complete graph K_n, staircase configurations whose visibility graphs are precisely the intermediate persistent graphs $G_i = G - \{e_i, e_2, \ldots, e_i\}$ where the $e_i's$ are reversible edges. In this way a staircase realization of $G_k = Skeleton(Ch)$ is eventually produced. Full details are deferred to the full paper version•

4 Conclusions

Maximal chains in the weak Bruhat order of the symmetric group are consistent sets that determine structurally maximally connected consistent sets. With each such maximal consistent set we associate a persistent graph that turns out to be a visibility graph of a simple polygon. An interpretation of these results is that these classes of voters profiles can be represented by non-degenerate staircase configuration of points(one point per candidate) where each ranking in the set corresponds to a voter's view of the point configuration. This offers a wide generalization of conditions for transitivity of the majority rule. Among the many interesting questions remaining to be answered we mention the following.

1. Are there any maximal consistent subsets of S_n of larger cardinality than those which are characterized as $Closure(Ch)$ with Ch a maximal chain in $(S_n, \leq WB)$?

2. Given $C \subset S_n$ what is the complexity of determining if $C \subset Closure(Ch)$ for some Ch a maximal chain in $(S_n, \leq WB)$?

3. How to generalize the results obtained here to weak orders instead of linear orders?

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Characterizing Neutral Aggregation on Restricted Domains

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Abstract

Sen [6] proved that by confining voters to value restricted (acyclic) domains voting paradoxes (intransitive relations) can be avoided in aggregation by majority. We generalize this result to any neutral monotone aggregation. In addition, we show that acyclicity is neither necessary nor sufficient for transitivity for neutral non monotone aggregation: we construct a cyclic transitive domain and introduce *strong acyclicity* as a sufficient condition for transitivity. We also show that strong acyclicity is necessary if repeated transitivity is sought. Finally, we present a cyclic domain repeatedly transitively aggregatable by a non neutral function.

1 Introduction

The concept of restricted preference domains was first introduced by Black [2]. In the paper from 1948, he showed that 'single peaked' domains are transitive for majority, namely, by restricting the voters to these domains aggregation by majority will always produce a transitive binary relation. The importance of this concept became more eminent two years later with the publication of Arrow's seminal work [1]. Arrow specified the basic requirements from a social welfare function (SWF): the *Pareto* condition, *independence of irrelevant alternatives* (IIA) and unrestricted domain. He showed that an aggregation satisfying these requirements generates an intransitive relation for at least one profile of voter preferences. Later work by Sen and Pattanaik [6], [7] showed that for any odd number of voters a necessary and sufficient condition for majority to produce a transitive relation is a condition on the domain they called 'value restriction' or 'acyclicity'.[1]

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[1]For an extensive discussion see Sen's book [5]; for an historical perspective see Gaertner [3].

How important is domain acyclicity for SWFs other than majority? Maskin [4] proved that domain acyclicity is a necessary but insufficient condition for transitivity under a neutral and symmetric SWF that is not majority. Maskin conjectured that acyclicity is necessary but insufficient for transitivity under a neutral SWF without dummies, i.e. the symmetry requirement can be substituted with a weaker requirement that every voter has some influence. In this paper we show that acyclicity is necessary and sufficient for transitivity under any neutral monotone SWF and that acyclicity is preserved in the image and therefore the domain is transitive for repeated aggregation. We construct a class of cyclic domains and present a general criteria for transitivity on these domains and give some examples of a non monotone neutral SWFs that satisfy these criteria thus refuting the conjecture. We show that transitivity does not imply repeated transitivity on these domains. For non monotone SWFs that do not satisfy the criteria we show that transitivity requires a condition called *strong acyclicity* and that this condition suffices for repeated aggregation for all neutral SWFs.

2 Preliminaries

We begin by briefly describing the model we will be using. A *voting game* G is a tuple $([n], \mathcal{W})$ where $[n] = [1, \ldots, n]$ is a set of voters and $\mathcal{W}$ is a set of coalitions (subsets of $[n]$) such that $\emptyset \notin \mathcal{W}$, $[n] \in \mathcal{W}$. The set $\mathcal{W}$ designates the winning coalitions. G is a *simple voting game* if either $S \in \mathcal{W}$ or $[n] - S \in \mathcal{W}$ for every coalition $S \subset [n]$. A game is *monotone* if $S \in \mathcal{W}$ and $S \subset T$ imply $T \in \mathcal{W}$. A simple monotone voting game is equivalent to a strong simple game as defined in [8]. Note that we do not require monotonicity in the definition of voting game, indeed non monotonicity is essential for the construction of the example refuting Maskin's conjecture.

A voter is *influential* or *effective* if his or her vote may have some impact on the outcome. In a voting game G this would mean that the voter is a *pivot* for at least one coalition, namely $S \notin \mathcal{W}$ and $S \cup \{i\} \in \mathcal{W}$ for some coalition $S \subset [n] - \{i\}$.

Definition 1 *Let $G = ([n], \mathcal{W})$ and $G' = ([n'], \mathcal{W}')$ be simple voting games. We say that G* ***embeds*** *G' if there exists a surjective function $\varphi : [n] \to [n']$ such that $S \in \mathcal{W}'$ iff $\varphi^{-1}(S) \in \mathcal{W}$ for every $S \subset [n']$.*

We denote embedding by $G' = G \circ \varphi^{-1}$. For a partition B_1, B_2, B_3 of $[n]$ ($B_i \cap B_j = \emptyset$ $i, j = 1, 2, 3$ and $B_1 \cup B_2 \cup B_3 = [n]$) define $\varphi(i) = j$ iff $i \in B_j$ $j = 1, 2, 3$ and denote $G(B_1, B_2, B_3) = G \circ \varphi^{-1}$. Thus for any partition of $[n]$ into three sets there corresponds an embedding of a simple three voter game. There are four such games: dictator D_3, majority Maj_3, parity $Prty_3$ and anti dictator $AntiD_3$ (see table 1). If B_1, B_2, B_3 are three loosing coalitions then G embeds Maj_3 since the union of any two is winning. If B_1, B_2, B_3 are

v_1	v_2	v_3	Maj_3	$Prty_3$	D_3^1	D_3^2	D_3^3	$AntiD_3^1$	$AntiD_3^2$	$AntiD_3^3$
0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	1	1	1	0
0	1	0	0	1	0	1	0	1	0	1
0	1	1	1	0	0	1	1	1	0	0
1	0	0	0	1	1	0	0	0	1	1
1	0	1	1	0	1	0	1	0	1	0
1	1	0	1	0	1	1	0	0	0	1
1	1	1	1	1	1	1	1	1	1	1

Table 1: The three place voting games. 1 for v_1, v_2, v_3 means the variable is in a coalition; 1 in the other columns means the coalition is a winning one

winning coalitions then G embeds $Prty_3$ since any one coalition is winning but the union of any two is loosing. If B_1 is a loosing coalition and B_2, B_3 are winning coalitions then G embeds $AntiD_3$ and if B_1 is winning and B_2, B_3 are loosing then G embeds D_3.

Lemma 1 *Let G be a simple voting game on $n > 3$ effective voters.*

1. *If G is monotone then it embeds Maj_3.*

2. *If G is non monotone then it embeds at least one of $Prty_3$ and $AntiD_3$.*

Proof: (1) Monotonicity and simplicity imply that no two winning coalitions are disjoint. We show that the intersection of all winning coalitions $K = \cap\{S : S \in \mathcal{W}\}$ is empty. If $|K| > 1$ then let K_1 and K_2 be a partition of K and let K_1' and K_2' be a partition of $[n]-K$ hence $K_1 \cup K_1' = [n] - K_2 \cup K_2'$. Neither of $K_1 \cup K_1'$ or $K_2 \cup K_2'$ is a winning coalition (since K is a subset of any winning coalition) contradicting simplicity. If $|K| = 1$ then G is dictatorial, hence the assumption of at least three effective voters implies $K = \emptyset$. A coalition $A \subset [n]$ is a *minimal winning* coalition if $A \in \mathcal{W}$ and $B \notin \mathcal{W}$ if $B \subsetneqq A$. Every winning coalition is a superset of a minimal wining coalition hence the intersection of all minimal winning coalitions is a subset of K and therefore empty. This shows that there exist two distinct minimal coalitions $A_1 \neq A_2$. If $[n] - A_1 \vartriangle A_2$ is a winning coalition then $B_1 = A_1 \cap A_2$, $B_2 = A_1 \vartriangle A_2$ and $B_3 = [n] - A_1 \cup A_2$ ($B_3 \neq \emptyset$ since otherwise $A_1 \cap A_2$ is winning coalition contrary to the minimality assumption) is a partition to three loosing coalitions therefore G embeds Maj_3. If $[n] - A_1 \vartriangle A_2$ is a loosing coalition then the same follows for $B_1 = [n] - A_1 \vartriangle A_2$, $B_2 = A_1 - A_2$ and $B_3 = A_2 - A_1$.

(2) If G is non monotone then there exist a winning coalition A_1 that is a subset of a loosing coalition A'. This implies there exists a winning coalition $A_2 \subset [n] - A'$. Thus we have two disjoint winning coalitions A_1 and A_2. $B_1 = A_1$, $B_2 = A_2$ and

$B_3 = [n] - A_1 \cup A_2$ is a partition with two winning coalitions. Consequentially if B_3 is a winning coalition then G embeds $Prty_3$ and if it is loosing G embeds $AntiD_3$ □

Let $[m]$ be a set of $m > 2$ alternatives. Designate the set of all complete antisymmetric binary relations on $[m]$ by Δ and the set of all linear orders $\Omega \subset \Delta$. In our model a preference is a linear order (we disregard indifference) and a *domain* is a subset of Ω. An n voter *social welfare function* (SWF) on domain $\mathfrak{C} \subset \Omega$ is a function $f : \mathfrak{C}^n \to \Delta$ such that any $P = f(P_1, \ldots, P_n)$ satisfies *independence of irrelevant alternatives* (IIA) : the preference of P on alternatives $a, b \in [m]$ depends only on the individual preferences of each voter between these two alternatives, and the *Pareto* condition: if all voters prefer a to b then so does P. It is implied by these conditions that a function f is a SWF iff there exists a collection of voting games $\{G_{ab}\}_{a,b\in[m]}$ such that aPb iff $\{j \in [n] : aP_jb\}$ is a winning coalition in G_{ab}. Notice that such a collection must satisfy $\mathcal{W}_{ba} = \{[n] - S : S \in \mathcal{W}_{ab} - \{[n]\}\} \cup \{[n]\}$. A SWF is *neutral* if $G_{ab} = G$ for all $a, b \in [m]$ and G is a simple voting game, in this case we shall occasionally identify f with G. A voter $k \in [n]$ is influential in f if it is influential in G_{ab} for at least one pair of alternatives.

Let $P \in \Omega$ be a linear order on m alternatives, we denote by $P(a_1, \ldots, a_k)$ the order induced by P on alternatives $a_1, \ldots, a_k \in [m]$, thus $P(a, b) = [ab]$ if aPb and $P(a, b, c) = [abc]$ if aPb, bPc and aPc. Let $\mathfrak{C}(a_1, \ldots, a_k)$ denote the domain of orders on $\{a_1, \ldots, a_k\}$ induced by $\mathfrak{C}$. A domain is called *cyclic* if there exist three alternatives $a, b, c \in [m]$ such that $\mathfrak{C}(a, b, c)$ contains a *cycle*, namely, a set of the form $\{[abc], [cab], [bca]\}$.

Let $\mathfrak{C} \subset \Omega$ be a preference domain with a SWF f. The *image* $Im(f)$ is the set of all binary relations generated from preferences in the domain. The Pareto principle implies $\mathfrak{C} \subset Im(f)$. We say that $\mathfrak{C}$ is *transitive* for f if the image is a domain of transitive relations, i.e. $Im(f) \subset \Omega$.

Let f and f' be neutral SWF defined by G and G', respectively, such that there is an embedding $G' = G \circ \varphi^{-1}$. For any $P = f'(P_1, \ldots, P_{n'})$ by definition $f(P_{\varphi(1)}, \ldots, P_{\varphi(n)}) = f'(P_1, \ldots, P_{n'}) = P$ and, consequently, $Im(f') \subset Im(f)$. Thus, transitivity for an n-place neutral monotone non dictatorial aggregation implies transitivity for Maj_3 and transitivity for a neutral non monotone aggregation implies transitivity for $Prty_3$ or $AntiD_3$.

3 Neutral Monotone Aggregation

Sen [6] showed that acyclic domains are transitive for majority; we generalize this result to any neutral monotone SWF.

Theorem 1 *Let f be a neutral monotone non dictatorial SWF, and let $\mathfrak{C}$ be a domain of linear orders.*

1. *$\mathfrak{C}$ is transitive for f iff it is acyclic.*

2. $\mathfrak{C}$ *acyclic implies* $Im(f)$ *acyclic.*

Proof: (1) It follows from lemma 1 that $\mathfrak{C}$ is transitive for f if it is transitive for Maj_3. Condorcet's paradox shows Maj_3 aggregates a cycle to an intransitive relation, hence $\mathfrak{C}$ must be acyclic. In the other direction, suppose f is a neutral monotone non dictatorial SWF such that $\mathfrak{C}$ is not transitive for f, thus there exists a profile $P_1, \ldots, P_n \in \mathfrak{C}$ and $P = f(P_1, \ldots, P_n)$ such that $aPbPcPa$. Let $A_1 = \{i : aP_ib\}$, $A_2 = \{i : bP_ic\}$ and $A_3 = \{i : cP_ia\}$ these sets are winning coalitions by definition. Monotinicity implies $A_1 \cap A_2 \neq \emptyset$. If $j \in A_1 \cap A_2$ then aP_jbP_jc, from transitivity follows aP_jc therefore $[abc] \in \mathfrak{C}(a, b, c)$. Likewise $A_1 \cap A_3 \neq \emptyset$ and $A_2 \cap A_3 \neq \emptyset$ imply $[cab], [bca] \in \mathfrak{C}(a, b, c)$ consequently $\mathfrak{C}$ is cyclic.

(2) It suffices to show that $[abc], [cab] \in \mathfrak{C}(a, b, c) \subset Im(f)(a, b, c)$ implies $[bca] \notin Im(f)(a, b, c)$. Assuming the contrary, let $P = f(P_1, \ldots, P_n)$ such that $P(a, b, c) = [bca]$. Let $A_1 = \{j : bP_jc\}$ and $A_2 = \{j : cP_ja\}$, both are winning coalitions since $bPcPa$. If $j \in A_1 \cap A_2$ then bP_jcP_ja, transitivity gives bP_ja implying $P_j(a, b, c) = [bca]$ which by assumption is not in the domain, hence $A_1 \cap A_2 = \emptyset$ contradicting monotinicity $\square$

This shows that acyclicity guarantees transitivity not only for majority but for any monotone neutral SWF and shows that the image remains acyclic. Thus, in a complex multi-tier voting process we know that as long as the voters in the lowest level are restricted to an acyclic preference domain and on each tier the local committees vote via a neutral monotone SWF the *repeated* aggregation will be transitive.

4 A Cyclic Transitive Domain

We define a cyclic domain $\mathfrak{C} = \{[\pi^j(1) \ldots \pi^j(m)] : j = 0, \ldots m - 1\}$ where $\pi^j(i) = i + j \ mod \ m$, called the *unicyclic* domain.

Theorem 2 *The unicyclic domain is transitive for any neutral SWF* f *defined by a game* G *not embedding* Maj_3 *and* $Prty_3$*.*

Proof: Suppose f is a neutral monotone non dictatorial SWF such that $\mathfrak{C}$ is not transitive for f. If there exists a profile $P_1, \ldots, P_n \in \mathfrak{C}$ and $P = f(P_1, \ldots, P_n)$ such that $aPbPcPa$ then the coalitions $A_1 = \{i : aP_ib\}$, $A_2 = \{i : bP_ic\}$ and $A_3 = \{i : cP_ia\}$ are winning coalitions. If $j \in A_1 - A_2 \cap A_3$ then $P_j(a, b, c) = [acb] \notin \mathfrak{C}(a, b, c)$ hence $A_1 \subset A_2 \cap A_3$. Likewise $A_2 \subset A_1 \cup A_3$ and $A_3 \subset A_1 \cap A_2$ thus $A_1 \cap A_2$, $A_1 \cap A_3$ and $A_2 \cap A_3$ is a partition of $[n]$ into three loosing coalitions implying G embeds Maj_3.

If there exists a profile $P_1, \ldots, P_n \in \mathfrak{C}$ and $P = f(P_1, \ldots, P_n)$ such that $aPcPbPa$ then the coalitions $A_1 = \{i : bP_ia\}$, $A_2 = \{i : cP_ib\}$ and $A_3 = \{i : aP_ic\}$ are winning coalitions. If $j \in A_1 \cap A_2$ then $P_j = [cba] \notin \mathfrak{C}(a, b, c)$ hence $A_1 \cap A_2 = \emptyset$. Likewise $A_2 \cap A_3 = \emptyset$ and $A_2 \cap A_3 = \emptyset$ thus A_1, A_2 and A_3 is a partition of $[n]$ into three winning coalitions implying G embeds $Prty_3$ □

We observe that if f is defined by a game G such that $G(B_1, B_2, B_3) = AntiD_3$ for a partition B_1, B_2, B_3 then for a profile $P_1, \ldots, P_n$ such that $P_i(a, b, c) = [abc]$ if $i \in B_1$, $P_i(a, b, c) = [cab]$ if $i \in B_2$ and $P_i(a, b, c) = [bca]$ if $i \in B_3$ it follows that $f(P_1, \ldots, P_n)(a, b, c) = [cba] \notin \mathfrak{C}(a, b, c)$. Thus, $\mathfrak{C}$ is not closed to aggregation by f. Furthermore for a profile $P_1, \ldots, P_n$ such that $P_i(a, b, c) = [cba]$ if $i \in B_1$, $P_i(a, b, c) = [cab]$ if $i \in B_2$ and $P_i(a, b, c) = [bca]$ if $i \in B_3$ it follows that $P = f(P_1, \ldots, P_n)(a, b, c)$ is intransitive since $aPbPcPa$. Consequently, $Im(f)$ is intransitive for f. This shows that if voters in a committee are restricted to the unicyclic domain then a voting process defined by a game that does not embed Maj_3 or $Prty_3$ does not produce paradoxes. However, if this process is repeated on a multi-tiered voting system the restriction is insufficient, since paradoxes may appear in second tier committees.

To refute Maskin's conjecture, it suffices to show there exists a family of games with an unbounded number of effective voters such that games in the family do not embed neither Maj_3 nor $Prty_3$. We shall give two examples of such games. The *anti dictator game* on n voters $AntiD_n$ is defined by $\mathcal{W} = 2^{[n]-\{1\}} \cup \{[n]\}$ the proper subsets of $[n]$ that do not include voter 1 (the 'anti dictator') and $[n]$. Every voter apart form 1 is a pivot for the coalition $[n]$ and 1 is a pivot for any other non empty coalition therefore no voter is a dummy. For any partition B_1, B_2, B_3 such that $1 \in B_1$ it follows that B_1 is a loosing coalition and B_2 and B_3 are winning coalitions hence $AntiD_n$ embeds only $AntiD_3$. For n odd let $\Omega_1 \subset [n]$ be the odd indices and Ω_2 the even. The *balance* game $G = ([n], \mathcal{W})$ is defined by $\mathcal{W} = \{A : |A \cap \Omega_1| > |A \cap \Omega_2|\}$. Every voter j is a pivot in $A_j = \{i \in [m] : i < j\}$ therefore no voter is a dummy. For a partition B_1, B_2, B_3 there is at least one winning coalition and one loosing coalition hence balance game cannot embed Maj_3 or $Prty_3$.

5 Neutral Non Monotone Aggregation

In the previous section we saw that the image of a transitive domain may be intransitive. We introduce a condition that strengthens the acyclicity requirement and ensures repeated transitivity.

Definition 2 *A domain is called* ***strongly acyclic*** *if* $[abc], [cab] \in \mathfrak{C}(a, b, c)$ *implies* $[acb]$, $[bca] \notin \mathfrak{C}(a, b, c)$.

Obviously strong acyclicity implies acyclicity, thus monotone SWF are repeatedly transitive for such domains. A domain is *mixed* unicyclic and strongly acyclic if $\mathfrak{C}(a, b, c)$ is either unicyclic or strongly acyclic for every $a, b, c \in [m]$.

Theorem 3 *Let f be a neutral non monotone SWF defined by a game G and let $\mathfrak{C}$ be a domain of linear orders.*

1. *If G embeds $Prty_3$ or Maj_3 then:*
   (a) *$\mathfrak{C}$ is transitive iff it is strongly acyclic.*
   (b) *$\mathfrak{C}$ strongly acyclic implies $Im(f)$ strongly acyclic.*
2. *If G does not embed $Prty_3$ or Maj_3 then:*
   (a) *$\mathfrak{C}$ is transitive iff it is mixed unicyclic and strongly acyclic.*
   (b) *$Im(f)$ is transitive iff $\mathfrak{C}$ is strongly acyclic.*

Proof: (1a) If $\mathfrak{C}$ is strongly acyclic and there exists a profile $P_1, \dots, P_n$ and $P = f(P_1, \dots, P_n)$ such that $aPbPcPa$ then the coalitions $A_1 = \{j : aP_jb\}$, $A_2 = \{j : bP_jc\}$ and $A_3 = \{j : cP_ja\}$ are winning. If $j \in A_2 \cap A_3$ then bP_jcP_ja therefore $P_j(a, b, c) = [bca]$, $j \in A_1 - A_2 \cup A_3$ implies aP_jcP_jb hence $P_j(a, b, c) = [acb]$. Strong acyclicity implies $A_2 \cap A_3 = \emptyset$ and $A_2 \cup A_3 = [n]$ thus $A_3 = [n] - A_2$ contradicting simplicity.

If $\mathfrak{C}$ is not strongly acyclic and $[abc], [cab] \in \mathfrak{C}(a, b, c)$ for alternatives $a, b, c \in [m]$ then either $[acb] \in \mathfrak{C}(a, b, c)$ or $[bca] \in \mathfrak{C}(a, b, c)$. If the former case let P_1, P_2, P_3 such that $P_1(a, b, c) = [abc]$, $P_2(a, b, c) = [cab]$ and $P_3(a, b, c) = [acb]$. If G embeds $Prty_3$ then there exists a partition B_1, B_2, B_3 of winning coalitions. Let $P_1, \dots, P_n$ a profile such that $P_i = P_j$ if $i \in B_j$ $j = 1, 2, 3$. Since all three coalitions are winning it follows that $P = f(P_1, \dots, P_n)$ satisfies $aPbPcPa$ and therefore intransitive. If G embeds $AntiD_3$ then there exists a partition where B_1 is a loosing coalition and B_2, B_3 are winning. Let $P_1, \dots, P_n$ a profile such that $P_i = P_j$ if $i \in B_j$ $j = 1, 2, 3$. Again by definition $P = f(P_1, \dots, P_n)$ is intransitive. Lemma 1 shows that any non monotone G embeds either $Prty_3$ or $AntiD_3$ hence f is not transitive for $\mathfrak{C}$. A similar argument shows that f is not transitive if $[bca] \in \mathfrak{C}(a, b, c)$ and f embeds $Prty_3$. Theorem 1 shows the same if f embeds Maj_3.

(1b) Let $\mathfrak{C}$ be a strong acyclic domain such that $[abc], [cab] \in \mathfrak{C}(a, b, c) \subset Im(f)(a, b, c)$. If there exists $P = f(P_1, \dots, P_n)$ such that $P_1, \dots, P_n \in \mathfrak{C}$ and $P(a, b, c) = [bca]$ then $A_1 = \{j : bP_ja\}$, $A_2 = \{j : bP_jc\}$ and $A_3 = \{j : cP_ja\}$ are winning coalitions. $j \in A_1 \cap A_2 \cap A_3$ implies $P_j(a, b, c) = [bca]$ and $j \in [n] - A_1 \cup A_2 \cup A_3$ implies $P_j(a, b, c) = [acb]$ hence strong acyclicity implies $A_1 \cap A_2 \cap A_3 = \emptyset$ and $A_1 \cup A_2 \cup A_3 = [n]$. $j \in A_2 \cap A_3$ implies $bP_jcP_jaP_jb$ thus $A_2 \cap A_3 = \emptyset$. Also $A_1 - A_2 \cup A_3$ implies $bP_jaP_jcP_jb$ therefore $A_1 \subset A_2 \cup A_3$. Consequently, $A_2 \cup A_3 = [n]$ and $A_2 \cap A_3 = \emptyset$ or rather $A_3 = [n] - A_2$ which contradicts simplicity, thus $[bca] \notin Im(f)(a, b, c)$.

If there exists a profile $P_1, \dots, P_n \in \mathfrak{C}$ such that $P = f(P_1, \dots, P_n)$ and $P(a, b, c) = [acb]$, then $A_1 = \{j : aP_jb\}$, $A_2 = \{j : cP_jb\}$ and $A_3 = \{j : aP_jc\}$ are winning coalitions. Strong acyclicity implies $A_1 \cap A_2 \cap A_3 = \emptyset$ and $A_1 \cup A_2 \cup A_3 = [n]$. As before, transitivity of voter preferences implies $A_2 \cap A_3 = \emptyset$ and $A_1 \subset A_2 \cup A_3$ contradicting simplicity, hence $[acb] \notin Im(f)(a,b,c)$.

(2a) If $\mathfrak{C}(a,b,c)$ is unicyclic for alternatives $a,b,c \in [m]$ then it follows from theorem 2 that $P = f(P_1,\ldots,P_n)$ is transitive on a,b,c for any profile $P_1,\ldots,P_n$. The proof of (1a) implies the same if $\mathfrak{C}(a,b,c)$ is strongly acyclic.

(2b) It follows from the proof of (1b) that if $\mathfrak{C}$ is strongly acyclic then $Im(f)$ is transitive. In the other direction, it follows from lemma 1 that f embeds $AntiD_3$, the remark following theorem 2 shows that in this case $Im(f)(a,b,c)$ is intransitive if $\mathfrak{C}(a,b,c)$ is unicyclic □

To summarize: dictatorial SWFs are transitive on any domain, non dictatorial monotone SWFs are transitive only on acyclic domains. Non monotone SWFs embedding $Prty_3$ or Maj_3 are transitive only on strongly acyclic domains. SWFs embedding only $AntiD_3$ and D_3 are transitive on mixed unicyclic/strongly acyclic domains but repeatedly transitive only on strongly acyclic domains.

6 Non Neutral Aggregation

The relation between the domain and aggregation function becomes more complex for transitive SWF when the neutrality condition is relaxed. We give an example of a non neutral monotone SWF that is repeatedly transitive on a cyclic domain. Thus we see that the dichotomy in the neutral case does not hold in the non neutral case.

A domain is called a *P-domain* for $P \in \Omega$ if $P(a,b,c) = [abc]$ implies $\{[acb],[bac]\} \not\subset \mathfrak{C}(a,b,c)$. It follows from this definition that every acyclic domain is a proper subset of some cyclic P-domain for an appropriate $P \in \Omega$. We define a function $f_P : \mathfrak{C}^n \to \Omega$ such that $f_P(P_1,\ldots,P_n) = AntiD_{n+1}(P,P_1,\ldots,P_n)$, it is easy to see that f_P is a non neutral SWF, notice also that it is monotone and symmetric (voters are interchangeable).

Theorem 4 *A P-domain is transitive for f_P.*

Proof: Assume $aQbQcQa$ for $Q = f_P(P_1,\ldots,P_n) = AntiD_{n+1}(P,P_1,\ldots,P_n)$, let $(Q_1,\ldots,Q_{n+1}) = (P,P_1,\ldots,P_n)$ and take $A_1 = \{j : aQ_jb\}$, $A_2 = \{j : bQ_jc\}$ and $A_3 = \{j : cQ_ja\}$ which by assumption are winning coalitions. From the transitivity of $Q_1,\ldots Q_{n+1}$ it follows that $A_1 \cap A_2 \cap A_3 = \emptyset$ and $A_1 \cup A_2 \cup A_3 = [n+1]$. Since $aPb \equiv aQ_1b$ it follows that $1 \in A_1$ and by definition of $AntiD_{n+1}$ it follows that $A_1 = A_2 = [n+1]$, but this implies aQ_jc for all j hence from the Pareto principle it follows that aQc contradicting the assumption on Q. Since $P(a,b,c) = [abc]$ these three alternatives are not arbitrary as before hence we must rule out the other cycle as well. Assume this time

$aQcQbQa$ for $Q, Q_1, \ldots, Q_{n+1}$ as above and take $A_1 = \{j : bQ_ja\}$, $A_2 = \{j : cQ_jb\}$ and $A_3 = \{j : aQ_jc\}$. Since $aPc \equiv aQ_1c$ it follows that $1 \in A_3$ hence $A_3 = [n+1]$. Consequentially $A_1, A_2 \subset A_3$, but $j \in A_1 \cap A_3$ implies $P_j(a, b, c) = [bac]$ and $j \in A_2 \cap A_3$ implies $P_j(a, b, c) = [acb]$. It follows from the P-domain assumption that $A_1 = A_1 \cap A_3 = \emptyset$ or $A_2 = A_2 \cap A_3 = \emptyset$ and in either case this contradicts A_1 and A_2 being winning coalitions $\square$

Theorem 5 *A P-domain is repeatedly transitive for f_P.*

Proof: It suffices to show that $Im(f_p)$ is a P-domain. If $Q(a, b, c) = [acb]$ for $Q = f_P(P_1, \ldots, P_n) = AntiD_{n+1}(P, P_1, \ldots, P_n)$ as above let $(Q_1, \ldots, Q_{n+1}) = (P, P_1, \ldots, P_n)$ and take $A_1 = \{j : aQ_jb\}$, $A_2 = \{j : cQ_jb\}$ and $A_3 = \{j : aQ_jc\}$. $A_1 \cap A_2 \cap A_3 = \emptyset$ since $[acb] \notin \mathfrak{C}(a, b, c)$. From aQc it follows that $1 \in A_3$ thus $A_3 = [n+1]$ and therefore $A_2 \subset A_3$. If $j \in A_2 \cap A_3$ then $aQ_jcQ_jbQ_ja$ contradicting the transitivity of P_j hence $A_2 = \emptyset$. The same argument shows $[bac] \notin Im(f_P)(a, b, c)$ $\square$

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Preference aggregation with multiple criteria of ordinal significance

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Abstract

In this paper we address the problem of aggregating outranking situations in the presence of multiple preference criteria of ordinal signiÞcance. The concept of ordinal concordance of the global outranking relation is deÞned and an operational test for its presence is developed. Finally, we propose a new kind of robustness analysis for global outranking relations taking into account classical dominance, ordinal and classical majority concordance in a same ordinal valued logical framework.

Key words : Multicriteria aid for decision, ordinal signiÞcance weights, robust outranking

1 Introduction

Commonly the problem of aggregating preference situations along multiple points of view is solved with the help of cardinal weights translating the signiÞcance the decision maker gives each criteria (Roy and Bouyssou, 1993). However, determining the exact numerical values of these cardinal weights remains one of the most obvious practical difÞculty in applying multiple criteria aid for decision (Roy and Mousseau, 1996).

To address precisely this problem, we generalize in a Þrst section the classical concordance principle, as implemented in the Electre methods (Roy, 1985), to the context where merely ordinal information concerning these signiÞcance of criteria is available. Basic data and notation is introduced and the classical cardinal concordance principle is reviewed. The ordinal concordance principle is formally introduced and illustrated on a simple car selection problem.

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In a second section, we address theoretical foundations and justiÞcation of the deÞnition of ordinal concordance. By the way, an operational test for assessing the presence or not of the ordinal concordance situation is developed. The core approach involves the construction of a distributional dominance test similar in its design to the stochastic dominance approach.

In a last section we Þnally address the robustness problem of multricriteria decision aid recommendations in the context of the choice problematics. Classical dominance, i.e. unanimous concordance, ordinal as well as cardinal majority concordance are considered altogether in a common logical framework in order to achieve robust optimal choice recommendation. We rely in this approach on previous work on good choices from ordinal valued outranking relations (see Bisdorff and Roubens, 2003).

2 The ordinal concordance principle

We start with setting up the necessary notation and deÞnitions. We follow more or less the notation used in the French multicriteria decision aid community.

2.1 Basic data and notation

As starting point, we require a set A of potential decision actions. To assess binary outranking situations between these actions we consider a coherent family $F = \{g_1, \dots, g_n\}$ of n preference criteria (Roy and Bouyssou, 1993, Chapter 2).

The performance tableau gives us for each couple of decisions actions $a, b \in A$ their corresponding performance vectors $g(a) = \big(g_1(a), ..., g_n(a)\big)$ and $g(b) = \big(g_1(b), ..., g_n(b)\big)$.

A Þrst illustration, shown in Table 1, concerns a simple car selection problem taken from Vincke (1992, pp. 61Ð62)). We consider here a set $A = \{m_1, \dots, m_7\}$ of potential car models which are evaluated on four criteria: *Price*, *Comfort*, *Speed* and *Design*. In this

Table 1: Car selection problem: performance tableau

Cars	q_j	p_j	m_1	m_2	m_3	m_4	m_5	m_6	m_7	w
1: Price	10	50	-300	-270	-250	-210	-200	-180	-150	5/15
2: Comfort	0	1	3	3	2	2	2	1	1	4/15
3: Speed	0	1	3	2	3	3	2	3	2	3/15
4: Design	0	1	3	3	3	2	3	2	2	3/15

Source:Vincke, Ph. 1992, pp. 61Ð62

supposedly coherent family of criteria, the *Price* criterion works in the negative direction of the numerical amounts. The evaluations on the qualitative criteria such as *Comfort*, *Speed* and *Design* are numerically coded as follows: 3 means *excellent* or *superior*, 2 means *average* or *ordinary*, 1 means *weak*.

In general, we may observe on each criterion $g_j \in F$ an indifference threshold $q_j \geq 0$ and a strict preference threshold $p_j \geq q_j$ (see Roy and Bouyssou, 1993, pp. 55Ð59). We suppose for instance that the decision-maker admits on the *Price* criterion an indifference threshold of 10 and a preference threshold of 50 units.

To simplify the exposition, we consider in the sequel that all criteria support the decision makerÕs preferences along a positive direction. Let$\Delta_j(a,b) = g_j(a) - g_j(b)$ denote the difference between the performances of the decision actions a and b on criterion g_j. For each criterion $g_j \in F$, we denote Ò$a \,\mathrm{S}_j\, b$Ó the semiotic restriction of assertion Òa outranks bÓ to the individual criteriong_j.

Definition 1. $\forall a, b, \in A$, the level of credibility $r(a \,\mathrm{S}_j\, b)$ of assertion Ò$a \,\mathrm{S}_j\, b$Ó is deÞned as:

$$r(a \,\mathrm{S}_j\, b) = \begin{cases} 1 & \text{if} \quad \Delta_j(a,b) \geq -q_j \\ \frac{p_j + \Delta_j(a,b)}{p_j - q_j} & \text{if} \quad -p_j \leq \Delta_j(a,b) \leq -q_j \\ 0 & \text{if} \quad \Delta_j(a,b) < -p_j. \end{cases} \tag{1}$$

The level of credibility $r(\overline{a \,\mathrm{S}_j\, b})$ associated with the truthfulness of the negation of the assertion Ò$a \,\mathrm{S}_j\, b$Ó is deÞned as follows:

$$r(\overline{a \,\mathrm{S}_j\, b}) = 1 - r(a \,\mathrm{S}_j\, b). \tag{2}$$

Following these deÞnitions, we Þnd in Table 1 that model m_6 clearly outranks model m_2 on the *Price* criterion ($\Delta_1(m_6, m_2) = 90$ and $r(m_6 \,\mathrm{S}_1\, m_2) = 1$) as well as on the *Speed* criterion ($\Delta_3(m_6, m_2) = 1$ and $r(m_6 \,\mathrm{S}_3\, m_2) = 1$).

Inversely, model m_2 clearly outranks model m_6 on the *Comfort* criterion as well as on the *Design* criterion. Indeed $\Delta_2(m_2, m_6) = 2$ and $r(m_2 \,\mathrm{S}_2\, m_6) = 1$ as well as $\Delta_4(m_2, m_6) = 1$ and $r(m_2 \,\mathrm{S}_4\, m_6) = 1$.

A given performance tableau, if constructed as required by the corresponding decision aid methodology (see Roy, 1985), is warrant for the truthfulness of these Ò*local*Ó, i.e. the individual criterion based preferences of the decision maker. To assess however global preference statements integrating all available criteria, we need to aggregate these local warrants by considering the relative signiÞcance the decision-maker attributes to each individual criterion with respect to his global preference system.

2.2 The classical concordance principle

In the Electre based methods, this issue is addressed by evaluating if, yes or no, a more or less signiÞcant majority of criteria effectively concord on supporting a given global outranking assertion (see Roy and Bouyssou, 1993; Bisdorff, 2002). This classical majority concordance principle for assessing aggregated preferences from multiple criteria was originally introduced by Roy (1968).

Definition 2. Let $w = (w_1, ..., w_n)$ be a set of signiÞcance weights corresponding to the n criteria such that: $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$. For $a, b \in A$, let $a \,\mathrm{S}\, b$ denote the assertion that Òa *globally outranks* bÓ. We denote $r_w(a \,\mathrm{S}\, b)$ the credibility of assertion $a \,\mathrm{S}\, b$ considering given signiÞcance weights w.

$$r_w(a \,\mathrm{S}\, b) \;=\; \sum_{j=1}^{n} \big(w_j \cdot r(a \,\mathrm{S}_j\, b) \big). \tag{3}$$

Assertion Ò$a \,\mathrm{S}\, b$Ó is considered *rather true than false*, as soon as the weighted sum of criterial signiÞcance in favour of the global outranking situation obtains a strict majority, i.e. the weighted sum of criterial signiÞcance is greater than 50%. To clearly show the truth-functional denotation implied by our credibility function r_w, we shall introduce some further notations.

Definition 3. Let Ò$a \,\mathrm{S}\, b$Ó denote the fact that a *globally outranks* b. We denote $\|a \,\mathrm{S}\, b\|_w$ the logical denotation of the credibility calculus taking its truth values in a three valued truth domain $L_3 = \{f_w, u, t_w\}$ where f_w means *rather false than true* considering importance weights w, t_w means *rather true than false* considering importance weights w and u means *logically undetermined*.

$$\|a \,\mathrm{S}\, b\|_w \;=\; \begin{cases} t_w & \text{if} \quad r_w(a \,\mathrm{S}\, b) > 0.5 \;; \\ f_w & \text{if} \quad r_w(a \,\mathrm{S}\, b) < 0.5 \;; \\ u & \text{otherwise}. \end{cases} \tag{4}$$

In our example, let us suppose that the decision-maker admits the signiÞcance weights w shown in Table 1. The *Price* criterion is the most signiÞcant with a weight of $5/15$. Then comes the *Comfort* criterion with $4/15$ and Þnally, both the *Speed* and the *Design*

[1]Readers familiar with the outranking concept will notice the absence of the *veto* issue in our deÞnition of the outranking situation. The veto principle, also called discordance principle by Roy, requires some measurable distance on the criteria scales. For robustness purposes we prefer to keep with solely the sound ordinal properties of the criterion function concept. And the concordance principle already naturally integrates a balancing reasons principle by weighting concordant against discordant arguments (see Bisdorff and Roubens, 2003)

criteria have identical weights $3/15$. By assuming that the underlying family of criteria is indeed coherent, we may thus state that the assertion Ò$m_6 \, \mathrm{S}_w \, m_2$Ó with aggregated signiÞcance of 53.3% is *rather true than false* with respect to the given importance weights w.

The majority concordance approach obviously requires a precise numerical knowledge of the signiÞcance of the criteria, a situation which appears to be difÞcult to achieve in practical applications of multicriteria decision aid.

Substantial efforts have been concentrated on developing analysis and methods for assessing these cardinal signiÞcance weights (see Roy and Mousseau, 1992, 1996). Following this discussion, Dias and Clõmaco (2002) propose to cope with imprecise signiÞcance weights by delimiting sets of potential signiÞcance weights and enrich the proposed decision recommendations with a tolerance in order to achieve robust recommendations.

In this paper we shall not contribute directly to this issue but rely on the fact that in practical application the ordinal weighting of the signiÞcance of the criteria are generally easier to assess and more robust than any precise numerical weights.

2.3 Ordinal concordance principle

Let us assume that instead of a given cardinal weight vector w we observe a complete pre-order π on the family of criteria F which represents the signiÞcance rank each criterion takes in the evaluation of the concordance of the global outranking relation S to be constructed on A.

In our previous car selection example, we may notice for instance that the proposed signiÞcance weights model the following ranking π: *Price* $>$ *Comfort* $>$ $\{$ *Speed, Design*$\}$.

A precise set w of numerical weights may now be compatible or not with such a given signiÞcance ranking of the criteria.

Definition 4. w is a π-compatible set of weights if and only if:

- $w_i = w_j$ for all couples (g_i, g_j) of criteria which are of the same signiÞcance with respect to π;
- $w_i > w_j$ for all couples (g_i, g_j) of criteria such that criterion g_i is certainly more signiÞcant than criterion g_j in the sense of π.

We denote $W(\pi)$ the set of all π-compatible weight vectors w.

Definition 5. For $a, b \in A$, let Ò$a \, \mathrm{S}_\pi \, b$Ó denote the fact that*“a globally outranks b with a significant majority for every π-compatible weight vector”*.

$$a \ \mathrm{S}_\pi \ b \quad \Leftarrow \quad \big(r_w(a \, \mathrm{S} \, b) > 0.5, \ \forall w \in W(\pi) \big). \tag{5}$$

For short, we say that a *globally outranks* b *in the sense of the ordinal concordance principle*.

2.4 Theoretical justification

In other words, the $a \, \mathrm{S}_\pi \, b$ situation is given if for all π-compatible weight vectors w, the aggregated signiÞcance of the assertion $a \, \mathrm{S}_w \, b$ outranks the aggregated signiÞcance of the negation $\overline{a \, \mathrm{S}_w \, b}$ of the same assertion.

Proposition 1.

$$a \, \mathrm{S}_\pi \, b \quad \Leftarrow \quad \big(r_w(a \, \mathrm{S} \, b) \ > \ r_w(\overline{a \, \mathrm{S} \, b}); \ \forall w \in W(\pi) \big). \tag{6}$$

Proof. Implication 6 results immediately from the observation that:

$$\sum_{g_j \in F} w_j \cdot r(a \, \mathrm{S}_j \, b) \ > \ \sum_{g_j \in F} w_j \cdot r(\overline{a \, \mathrm{S}_j \, b}) \quad \Leftrightarrow \quad \sum_{g_j \in F} w_j \cdot r(a \, \mathrm{S}_j \, b) \ > \ \frac{1}{2}.$$

Indeed, $\forall g_j \in F$ we observe that $r(a \, \mathrm{S}_j \, b) + r(\overline{a \, \mathrm{S}_j \, b}) = 1$. This fact implies that:

$$\sum_{g_j \in F} w_j \cdot r(a \, \mathrm{S}_j \, b) \ + \ \sum_{g_j \in F} w_j \cdot r(\overline{a \, \mathrm{S}_j \, b}) \ = \ 1.$$

□

Coming back to our previous car selection problem, we shall later on verify that model m_6 effectively outranks all other 6 car models following the ordinal concordance principle, With any π-compatible set of cardinal weights, model m_6 will always outrank all other car models with a Ô*significant*Õ majority of criteria.

We still need now a constructive approach for computing such ordinal concordance results.

3 Testing for ordinal concordance

In this section, we elaborate general conditions that must be fulÞlled in order to be sure that there exists an ordinal concordance in favour of the global outranking situation. By the way we formulate an operational procedure for constructing a relation S_π on A from a given performance tableau.

3.1 Positive and negative significance

The following condition is identical to the condition of the ordinal concordance principle (see DeÞnition 5).

Proposition 2. $\forall a, b \in A$ *and* $\forall w \in W(\pi)$*:*

$$r_w(a\,\mathrm{S}\,b) > r_w(\overline{a\,\mathrm{S}\,b}) \;\Leftrightarrow\; r_w(a\,\mathrm{S}\,b) \,-\, r_w(\overline{a\,\mathrm{S}\,b}) \;>\; r_w(\overline{a\,\mathrm{S}\,b}) \,-\, r_w(a\,\mathrm{S}\,b). \tag{7}$$

Proof. The equivalence between the right hand side of Equivalence 7 and the right hand side of Implication 6 is obtained with simple algebraic manipulations. □

The inequality in the right hand side of Equivalence 7 gives us the operational key for implementing a test for ordinal concordance of an outranking situation. The same weights w_j and $-w_j$, denoting the Ò*confirming*Ó, respectively the Ò*negating*Ó, signiÞcance of each criterion, appear on each side of the inequality.

Furthermore, the sum of the coefÞcients $r(a\,\mathrm{S}_j\,b)$ and $r(\overline{a\,\mathrm{S}_j\,b})$ on each side of the inequality is a constant equal to n, i.e. the number of criteria in F. Therefore these coefÞcients may appear as some kind of credibility distribution on the set of positive and negative signiÞcance weights.

3.2 Significance distributions

Suppose that the given pre-order π of signiÞcance of the criteria contains k equivalence classes which we are going to denote $\pi_{(k+1)}, ..., \pi_{(2k)}$ in increasing sequence. The same equivalence classes, but in in reversed order, appearing on the ÒnegatingÓ signiÞcance side, are denoted $\pi_{(1)}, ..., \pi_{(k)}$.

Definition 6. For each equivalence class $\pi_{(i)}$, we denote $w_{(i)}$ the cumulated negating, respectively conÞrming, signiÞcance of all equi-signiÞcant criteria gathered in this equivalence class:

$$i = 1, ..., k : w_{(i)} = \sum_{g_j \in \pi_{(i)}} -w_j; \quad i = k+1, ..., 2k : w_{(i)} = \sum_{g_j \in \pi_{(i)}} w_j. \tag{8}$$

We denote $c_{(i)}$ for $i = 1, ..., k$ the sum of all coefÞcients $r(\overline{a\,\mathrm{S}_j\,b})$ such that $g_j \in \pi_{(i)}$ and $c_{(i)}$ for $i = k+1, ..., 2k$ the sum of all coefÞcients $r(a\,\mathrm{S}_j\,b)$ such that $g_j \in \pi_{(i)}$. Similarly, we denote $\overline{c_{(i)}}$ for $i = 1, ..., k$ the sum of all coefÞcients $r(a\,\mathrm{S}_j\,b)$ such that $g_j \in \pi_{(i)}$ and $\overline{c_{(i)}}$ for $i = k+1, ..., 2k$ the sum of all coefÞcients $r(\overline{a\,\mathrm{S}_j\,b})$ such that $g_j \in \pi_{(i)}$.

With the help of this notation, we may rewrite Equivalence 7 as follows:

Proposition 3. $\forall a, b \in A$ *and* $w \in W(\pi)$*:*

$$r_w(a\,\mathrm{S}\,b) > r_w(\overline{a\,\mathrm{S}\,b}) \quad \Leftrightarrow \quad \sum_{i=1}^{2k} c_{(i)} \cdot w_{(i)} > \sum_{i=1}^{2k} \overline{c_{(i)}} \cdot w_{(i)}. \tag{9}$$

CoefÞcients $c_{(i)}$ and $\overline{c_{(i)}}$ represent two distributions, one the negation of the other, on an ordinal scale determined by the increasing signiÞcance $w_{(i)}$ of the equivalence classes in $\pi_{(i)}$.

3.3 Ordinal distributional dominance

We may thus test the right hand side inequality of Equivalence 7 with the classical stochastic dominance principle originally introduced in the context of efÞcient portfolio selection (see Hadar and Russel, 1969; Hanoch and Levy, 1969).

We denote $C_{(i)}$, respectively $\overline{C_{(i)}}$, the increasing cumulative sums of coefÞcients $c_{(1)}$, $c_{(2)}$, ..., $c_{(i)}$, respectively $\overline{c_{(1)}}$, $\overline{c_{(2)}}$, ..., $\overline{c_{(i)}}$.

Lemma 1.

$$\Big(\sum_{i=1}^{2k} c_{(i)} \cdot w_{(i)} > \sum_{i=1}^{2k} \overline{c_{(i)}} \cdot w_{(i)} \Big), \forall w \in W(\pi) \;\Leftrightarrow\; \begin{cases} C_{(i)} \leq \overline{C_{(i)}}, i = 1, ..., 2k; \\ \exists i \in 1, ..., 2k : C_{(i)} < \overline{C_{(i)}}. \end{cases} \tag{10}$$

Proof. Demonstration of this lemma (see for instance Fishburn, 1974) goes by rewriting the right hand inequality of Equivalence 9 with the help of the repartition functions $C_{(i)}$ and $\overline{C_{(i)}}$. It readily appears then that the term by term difference of the cumulative sums is conveniently oriented by the right hand conditions of Equivalence 10. □

This concludes the proof of our main result.

Theorem 1. $\forall a, b \in A$, *let* $C_{(i)}(a, b)$ *represent the increasing cumulative sums of credibilities associated with a given significance ordering of the criteria:*

$$a\,\mathrm{S}_\pi\, b \quad \Leftarrow \quad \begin{cases} C_{(i)}(a, b) \leq \overline{C_{(i)}(a, b)}, i = 1, ..., 2k; \\ \exists i \in 1, ..., 2k : C_{(i)}(a, b) < \overline{C_{(i)}(a, b)}. \end{cases} \tag{11}$$

We observe an ordinal concordant outranking situation between two decision actions a *and* b *as soon as the repartition of credibility on the significance ordering of action* a *dominates the same of action* b*.*

Table 2: Assessing the assertion Òm_4 S_π m_5Ó

$\pi_{(i)}$	-Price	-Comfort	-Speed, Design	Speed,Design	Comfort	Price
$c_{(i)}$	0	0	1	1	1	1
$\overline{c_{(i)}}$	1	1	1	1	0	0
$C_{(i)}$	0	0	1	2	3	4
$\overline{C_{(i)}}$	1	2	3	4	4	4

The preceding result gives us the operational key for testing for the presence of an ordinal concordance situation. Let $L_3 = \{f_\pi, u, t_\pi\}$, where f_π means *rather false than true* with any π-compatible weights w, u means *logically undetermined* and t_π means *rather true than false* with any π-compatible weights w. For each pair of decision actions evaluated in the performance tableau, we may compute such a logical denotation representing truthfulness or falseness of the presence of ordinal concordance in favour of a given outranking situation.

Definition 7. Let π be a signiÞcance ordering of the criteria. $\forall a, b \in A$, let $C_{(i)}(a, b)$ and $\overline{C_{(i)}}(a, b)$ denote the corresponding cumulative sums of increasing sums of credibilities associated with the the relation S_π. We deÞne a logical denotation $\|a \ \mathrm{S} \ b\|_\pi$ in L_3 as follows:

$$\|a \ \mathrm{S} \ b\|_\pi = \begin{cases} t_\pi & \text{if} \quad \begin{cases} C_{(i)}(a,b) \leq \overline{C_{(i)}}(a,b), i = 1, ..., 2k \text{ and} \\ \exists i \in 1, ..., 2k : C_{(i)}(a,b) < \overline{C_{(i)}}(a,b); \end{cases} \\ f_\pi & \text{if} \quad \begin{cases} C_{(i)}(a,b) \geq \overline{C_{(i)}}(a,b), i = 1, ..., 2k \text{ and} \\ \exists i \in 1, ..., 2k : C_{(i)}(a,b) > \overline{C_{(i)}}(a,b); \end{cases} \\ u & \text{otherwise}. \end{cases} \tag{12}$$

Coming back to our simple example, we may now apply this test to car models m_4 and m_5 for instance. In Table 2 we have represented the six increasing equi-signiÞcance classes we may observe. From Table 1 we may compute the credibilities $c_{(i)}$ (respectively $\overline{c_{(i)}}$) associated with the assertion that model m_4 outranks (respectively does not outrank) m_5 as well as the corresponding cumulative distributions $C_{(i)}$ and $\overline{C_{(i)}}$ as shown in Table 2.

Applying our test, we may notice that indeed $\|m_4 \ \mathrm{S} \ m_5\|_\pi = t_\pi$, i.e. it is true that the assertion Òmodelm_4 outranks model m_5Ó will be supported by a more or less signiÞcant majority of criteria for all π-compatible sets of signiÞcance weights.

For information, we may reproduce in Table 3, the complete ordinal outranking relation on A. It is worthwhile noticing that, faithful with the general concordance principle, the outranking situations $a \ \mathrm{S}_\pi \ b$ appearing with value t_π are warranted to be true. Simi-

Table 3: The ordinal concordance of the pairwise outranking

$\|x \mathrm{~S}~ y\|_\pi$	m_1	m_2	m_3	m_4	m_5	m_6	m_7
m_1	-	t_π	u	u	u	u	u
m_2	t_π	-	t_π	f_π	u	f_π	u
m_3	u	t_π	-	u	u	u	u
m_4	t_π	t_π	t_π	-	t_π	t_π	u
m_5	t_π	t_π	t_π	t_π	-	t_π	u
m_6	t_π	t_π	t_π	t_π	t_π	-	t_π
m_7	u	t_π	u	t_π	t_π	t_π	-

larly, the situations showing credibility f_π, are warranted to be false. The other situations, appearing with credibility u are to be considered undetermined (see Bisdorff, 2000).

As previously mentioned, model m_6 gives the unique dominant kernel, i.e. a stable and dominant subset, of the $\{f_\pi, u, t_\pi\}$-valued S_π relation. Therefore this decision action represents a robust good choice decision candidate in the sense that it appears to be a rather true than false good choice with all possible π-compatible sets of signiÞcance weights (see Bisdorff and Roubens, 2003). Indeed, if we apply the given cardinal significance weights, we obtain in this particular numerical setting that model m_6 is not only among the potential good choices but also, and this might not necessarily always be the case, the most signiÞcant one (73%).

Let us now address the robustness issue.

4 Analyzing the robustness of global outrankings

Let us suppose that the decision maker has indeed given a precise set w of signiÞcance weights. The classical majority concordance will thus deliver a mean weighted outranking relation S_w on A.

In our car selection problem the result is shown in Table 4. We may notice here that for instance $r(m_4 \,\mathrm{S}_w\, m_5) = 80\%$. But we know also from our previuos investigation that $\|m_4 \,\mathrm{S}\, m_5\|_\pi = t_\pi$. The outranking situation is thus conÞrmed with any π-compatible weight set w.

Going a step further we could imagine a *dream model* that is the cheapest, the most comfortable, very fast and superior designed model, denoted as m_{top}. It is not difÞcult to see that this model will indeed dominate all the set A with $r(m_{top} \,\mathrm{S}\, x) = 100\%$, i.e. with unanimous concordance $\forall x \in A$. It will naturally also outrank all $x \in A$ in the sense of the ordinal concordance.

Table 4: The cardinal majority concordance of the outranking of the car models

$r_w(\mathrm{S})$	m_1	m_2	m_3	m_4	m_5	m_6	m_7
m_1	-	.83	.67	.67	.67	.67	.67
m_2	.80	-	.72	.47	.67	.47	.67
m_3	.73	.73	-	.75	.67	.67	.67
m_4	.53	.53	.80	-	.80	.63	.67
m_5	.53	.73	.80	.80	-	.72	.67
m_6	.73	.73	.73	.73	.73	-	.83
m_7	.33	.53	.33	.53	.53	.60	-

4.1 Unanimous concordance

Definition 8. $\forall a, b \in A$ we say that Òa *outranks* b *in the sense of the unanimous concordance principle*Ó, denoted Ò$a \, \Delta \, b$Ó, if the outranking assertion considered restricted to each individual criterion is *rather true than false*.

We capture once more the potential truthfulness of this dominance assertion with the help of a logical robustness denotation $\|a \, \mathrm{S} \, b\|_\Delta$ taking its values in $L_3 = \{f_\Delta, u, t_\Delta\}$, where f_Δ means *unanimously false*, t_Δ means *unanimously true* and u means *undetermined* as usual.

$$\forall a, b \in A: \ \|a \, \mathrm{S} \, b\|_\Delta \quad = \quad \begin{cases} t_\Delta & \text{if} \quad \forall g_j \in F: \ r(a \, \mathrm{S}_j \, b) > \frac{1}{2}; \\ f_\Delta & \text{if} \quad \forall g_j \in F: \ r(a \, \mathrm{S}_j \, b) < \frac{1}{2}; \\ u & \text{otherwise}. \end{cases} \tag{13}$$

In our example, neither of the seven models imposes itself on the level of the unanimous concordance principle and the relation Δ remains uniformly undetermined on A.

We are now going to integrate all three outranking relations, i.e. the unanimous, the ordinal and the majority concordance in a common logical framework.

4.2 Integrating unanimous, ordinal and classical majority concordance

Let w represent given numerical signiÞcance weights and π the underlying signiÞcance preorder. We deÞne the following ordinal sequence (increasing from falsity to truth) of logical robustness degrees: f_Δ means *unanimous concordantly false*, f_π means *ordinal concordantly false with any* π*-compatible weights*, f_w means *majority concordantly false*

Table 5: Robustness of the outranking on the car models

$\|S\|$	m_1	m_2	m_3	m_4	m_5	m_6	m_7
m_1	-	t_π	t_w	t_w	t_w	t_w	t_w
m_2	t_π	-	t_π	f_π	t_w	f_π	t_w
m_3	t_w	t_π	-	t_w	t_w	t_w	t_w
m_4	t_π	t_π	t_π	-	t_π	t_π	t_w
m_5	t_w	t_π	t_π	t_π	-	t_π	t_w
m_6	t_π	t_π	t_π	t_π	t_π	-	t_π
m_7	f_w	t_π	f_w	t_π	t_π	t_π	-

with weights w, u means *undetermined*, t_w means *majority concordantly true with weights* w, t_π means *ordinal concordantly true with any* π*-compatible weights* and t_Δ means *unanimous concordantly true*.

On the basis of a given performance tableau, we may thus evaluate the global outranking relation S on A as follows:

Definition 9. Let $L_7 = \{f_\Delta, f_\pi, f_w, u, t_w, t_\pi, t_\Delta\}$. $\forall a, b \in A$, we deÞne an ordinal robustness denotation $\|a\,\mathrm{S}\,b\| \in L_7$ as follows:

$$\|a\,\mathrm{S}\,b\| \quad = \quad \begin{cases} t_\Delta & \text{if} \quad \|a\,\mathrm{S}\,b\|_\Delta = t_\Delta \ ; \\ t_\pi & \text{if} \quad (\|a\,\mathrm{S}\,b\|_\Delta \neq t_\Delta) \wedge (\|a\,\mathrm{S}\,b\|_\pi = t_\pi) \ ; \\ t_w & \text{if} \quad (\|a\,\mathrm{S}\,b\|_\pi \neq t_\pi) \wedge (\|a\,\mathrm{S}\,b\|_w = t_w) \ ; \\ f_\Delta & \text{if} \quad \|a\,\mathrm{S}\,b\|_\Delta = f_\Delta \ ; \\ f_\pi & \text{if} \quad (\|a\,\mathrm{S}\,b\|_\Delta \neq f_\Delta) \wedge (\|a\,\mathrm{S}\,b\|_\pi = f_\pi) \ ; \\ f_w & \text{if} \quad (\|a\,\mathrm{S}\,b\|_\pi \neq f_\pi) \wedge (\|a\,\mathrm{S}\,b\|_w = f_w) \ ; \\ u & \text{otherwise.} \end{cases} \tag{14}$$

On the seven car models, we obtain for instance the results shown in Table 5. If we apply our methodology for constructing good choices from such an ordinal valued outranking relation we obtain a single ordinal concordant good choice: model m_6, and four classical majority concordance based good choices: m_1, m_3, m_4 and m_5. The Þrst good choice remains an admissible good choice with any possible π-compatible set of significance weights, whereas the others are more or less dependent on the precise numerical weights given. Similarly, we discover two potentially bad choices: m_2 at the level t_π and m_5 at the level t_w. The Þrst represents therefore a bad choice on the ordinal concordance level.[2]

[2]Conducting a similar analysis with taking into account the veto principle and thresholds given in Vincke

Table 6: Criteria for selecting a parcel sorting installation

criterion	title	significance weights
g_1	quality of the working place	$3/39$
g_2	quality of operating environment	$2/39$
g_3	operating costs	$5/39$
g_4	throughput	$3/39$
g_5	ease of operation	$3/39$
g_6	quality of maintenance	$5/39$
g_7	ease of installation	$2/39$
g_8	number of sorting bins	$2/39$
g_9	investment costs	$5/39$
g_{10}	bar-code addressing	$1/39$
g_{11}	service quality	$5/39$
g_{12}	development stage	$3/39$

Source: Roy and Bouyssou (1993, p. 527)

4.3 Practical applications

In order to illustrate the practical application of the ordinal concordance principle we present two case studies: the first, a classical historical case, well discussed in the literature and a second, very recent real application at the occasion of the EURO 2004 Conference in Rhodes.

4.3.1 Choosing the best postal parcels sorting machine

Let us first reconsider the problem of choosing a postal parcels sorting machine thoroughly discussed in Roy and Bouyssou (1993, pp 501–541).

We observe a set $A = \{a_1, \ldots, a_9\}$ of 9 potential installations evaluated on the coherent family $F = \{g_1, \ldots, g_{12}\}$ of 12 criteria shown in Table 6. The provided significance weights (see last column) determines the following significance ordering: $w_{10} < w_2 = w_7 = w_8 < w_1 = w_4 = w_5 = w_{12} < w_3 = w_6 = w_9 = w_{11}$. Thus we observe on the pro-

(1992), we find that no ordinal concordance is observed anymore. Applying the given numerical significance weights, one gets however that models m_3 and m_4 appear both as potential good choice. Indeed, model m_6 has a weak evaluation on the *comfort* criterion compared to the excellent evaluation of model m_1 for instance, and the same model m_1 is the most expensive one, therefore a veto appears on this criterion in comparison with the prize of model m_7 for instance. Models m_3 and m_4 represent therefore plausible compromises with respect to the numerical significance weights of the criteria. By the way, our example is a nice justification of the usefulness of the veto principle in suitable practical applications.

Table 7: Qualifying outranking situation $a_1 \, \mathrm{S}_j \, a_5$ and $a_4 \, \mathrm{S}_j \, a_5$

g_j	1	2	3	4	5	6	7	8	9	10	11	12
q_j	5	5	5	5	5	10	8	0	1	10	5	10
$g_j(a_1)$	75	69	68	70	82	72	86	74	-15.23	83	76	29
$g_j(a_4)$	73	57	82	90	75	61	93	60	-15.55	83	71	29
$g_j(a_5)$	76	46	55	90	48	46	93	60	-30.68	83	50	14
$r(a_1 \, \mathrm{S}_j \, a_5)$	1	1	1	0	1	1	1	1	1	1	1	1
$r(a_4 \, \mathrm{S}_j \, a_5)$	1	1	1	1	1	1	1	1	1	1	1	1
$r(\overline{a_1 \, \mathrm{S}_j \, a_5})$	0	0	0	1	0	0	0	0	0	0	0	0
$r(\overline{a_4 \, \mathrm{S}_j \, a_5})$	0	0	0	0	0	0	0	0	0	0	0	0

Source: Roy and Bouyssou (1993, p. 527)

Table 8: cumulative signiÞcance distribution of outranking $a_1 \, \mathrm{S} \, a_5$

$\pi_{(i)}$	$\pi_{(1)}$	$\pi_{(2)}$	$\pi_{(3)}$	$\pi_{(4)}$	$\pi_{(5)}$	$\pi_{(6)}$	$\pi_{(7)}$	$\pi_{(8)}$
$C_{(i)}(a_1, a_5)$	0	1	1	1	2	5	8	12
$\overline{C_{(i)}}(a_1, a_5)$	4	7	10	11	11	12	12	12

posed family of criteria 4 positive equivalence classes: $\pi_{(5)} = \{g_{10}\}$, $\pi_{(6)} = \{g_2, g_7, g_8\}$, $\pi_{(7)} = \{g_1, g_4, g_5, g_{12}\}$, and $\pi_{(8)} = \{g_3, g_6, g_9, g_{11}\}$ and 4 mirrored negative equivalence classes: $\pi_{(1)} = \{g_3, g_6, g_9, g_{11}\}$, $\pi_{(2)} = \{g_1, g_4, g_5, g_{12}\}$, $\pi_{(3)} = \{g_2, g_7, g_8\}$, $\pi_{(4)} = \{g_{10}\}$.

A previous decision aid analysis has eventually produced a performance tableau of which we show an extract in Table 7. The evaluations on each criterion, except g_9 (*costs of investment* in millions of French francs), are normalized such that $0 \leq g_j(a_i) \leq 100$. If we consider for instance the installations a_1 and a_5, we may deduce the local outranking credibility coefÞcients $r(a_1 \, \mathrm{S}_j \, a_5)$ shown in Table 7. There is no unanimous concordance in favour of $a_1 \, \mathrm{S} \, a_5$. Indeed we observe on criterion g_4 (*throughput*) a signiÞcant negative difference in performance. We may nevertheless observe an ordinal concordance situation $a_1 \; \mathrm{S}_\pi \; a_5$ as distribution $C_{(i)}(a_1, a_5)$ is entirely situated to the right of distribution $\overline{C_{(i)}}(a_1, a_5)$ (see Table 8).

On the complete set of pairwise outrankings of potential installations, we observe the robustness denotation shown in Table 9. We may notice the presence of one unanimous concordance situation $a_4 \Delta a_5$ qualifying the outranking of a_4 over a_5 (see Table 7). Computing from this ordinally valued robust outranking relation all robust good choices, i.e. minimal dominant sets in the sense of the robust concordance, we obtain that installations a_1, a_2, a_3 and a_4 each one gives a robust good choice at level t_π, whereas the installations a_5 and a_9 give each one a robust bad choice again at level t_π. If we apply in particular the

Table 9: Robustness degrees of outranking situations

$\|a_i \, \mathrm{S} \, a_j\|$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
a_1	-	t_π	t_π	t_π	t_π	t_π	t_π	t_π	t_π
a_2	t_π	-	t_π	t_π	t_π	t_π	t_π	t_π	t_π
a_3	t_π	t_π	-	t_π	t_π	t_π	t_π	t_π	t_π
a_4	t_π	t_π	t_π	-	t_Δ	t_π	t_π	t_π	t_π
a_5	f_π	f_π	f_π	f_π	-	f_π	f_π	f_w	t_π
a_6	t_w	f_w	t_w	t_π	t_π	-	t_w	t_w	t_π
a_7	t_π	t_π	t_w	t_π	t_π	t_π	-	t_π	t_π
a_8	t_w	t_π	t_π	t_π	t_π	t_π	t_π	-	t_π
a_8	f_w	t_π	t_π	t_π	t_π	f_w	f_w	t_π	-

given numerical signiÞcance weights (see Table 6), we furthermore obtain that a_1 gives among the four potential good choices the most credible (67%) one whereas among the admissible bad choices it is installation a_5 which gives the most credible (67%) worst one. This result precisely conÞrms and even formally validates the robustness discussion reported in Roy and Bouyssou (1993, p. 538).

4.3.2 The Euro Best Poster Award 2004: finding a robust consensual ranking

The Programme Committee of the 20th European Conference on Operational Research, Rhodes 2004 has introduced a new type of EURO K conference participation consisting in a daily poster session linked with an oral 30 minutes presentation in front of the poster, a presentation style similar to poster sessions in traditional natural sciences congresses. In order to promote these new discussion presentations,the organizers of the conference proposed a EURO Best Poster Award (EBPA) consisting of a diploma and a prize of 1000 €. Each contributor accepted in the category of the discussion presentations was invited to submit a pdf image of his poster to a Þve member jury.

The Programme Committee retained the following evaluation criteria: *scientific quality* (sq), *contribution to OR theory and/or practice* (ctp), *originality* (orig) and *presentation quality* (pq) in decreasing order of importance. 13 candidates actually submitted a poster in due time and the Þve jury members were asked to evaluate the 13 posters on each criteria with the help of an ordinal scale : 0 (very weak) to 10 (excellent) and to propose a global ranking of the posters.

Table 10: Global outranking of the posters

$r_w(S)$	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}
p_1	-	.58	.24	.12	.46	.68	.34	.76	.65	.04	.63	.08	.28
p_2	.42	-	.34	.34	.34	.42	.42	.40	.61	.24	.45	.34	.26
p_3	.82	.74	-	.54	.66	.98	.86	.96	.69	.16	.81	.58	.46
p_4	.98	.68	.62	-	.76	.98	.82	.98	.69	.28	.75	.70	.54
p_5	.64	.68	.72	.48	-	1.0	.78	.98	.69	.26	.75	.52	.0
p_6	.54	.58	.10	.10	.34	-	.42	.86	.65	.0	.63	.04	.0
p_7	.68	.72	.32	.46	.30	.86	-	.82	.65	.10	.69	.50	.36
p_8	.50	.60	.16	.20	.30	.66	.40	-	.71	.02	.67	.16	.0
p_9	.43	.49	.35	.35	.41	.49	.37	.49	-	.0	.39	.37	.35
p_{10}	1.0	.80	1.0	.84	1.0	1.0	.90	1.0	.71	-	.81	.88	.80
p_{11}	.71	.61	.37	.29	.29	.43	.39	.59	.69	.0	-	.31	.43
p_{12}	.98	.66	.70	.62	.64	.96	.78	.94	.69	.32	.75	-	.56
p_{13}	.1.0	.76	.70	.60	.80	.80	.70	.96	.69	.48	.81	.64	-

As all Þve jury members were ofÞcially equal in signiÞcance, we may consider to be in the presence of a family of $5 \times 4 = 20$ criteria gathered into four equivalence classes listed hereafter in decreasing order of signiÞcance: $\pi_{(1)} = \{sq_1, sq_2, sq_3, sq_4, sq_5\}$, $\pi_{(2)} = \{pct_1, pct_2, pct_3, pct_4, pct_5\}$, $\pi_{(3)} = \{orig_1, orig_2, orig_3, orig_4, orig_5\}$ and $\pi_{(4)} = \{pq_1, pq_2, pq_3, pq_4, pq_5\}$.

The cardinal signiÞcance weights associated with the four classes of equi-signiÞcant criteria were eventually the following: $w_{sq_i} = 4$, $w_{ctp_i} = 3$, $w_{orig_i} = 2$ and $w_{pq_i} = 1$, for $i = 1$ to 4.

The decision problem we are faced with is to aggregate the 20 rankings of the 13 posters on the basis of the given performance tableau. To do so we Þrst computed the credibility index r_w of the global outranking relation S shown in Table 10 using the given signiÞcance weights w.

Considering the ordinal character of the criterial scales involved, indifference and preference thresholds were considered to be identically zero, respectively one, on all criteria and no veto thresholds were to be considered.

Applying our bipolar ranking approach (see Bisdorff, 1999) to this classical outranking relation gives the following ranking of the posters:

Table 11: Robust outranking of the posters

$\|S\|$	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}
p_1	-	t_π	f_π	f_π	f_w	t_π	f_π	t_π	t_π	f_π	t_π	f_π	f_π
p_2	f_π	-	f_π	f_π	f_π	f_π	f_π	f_π	t_π	f_π	f_w	f_π	f_π
p_3	t_π	t_π	-	t_w	t_w	t_π	t_π	t_π	t_π	f_π	t_π	t_w	f_w
p_4	t_π	t_π	t_π	-	t_π	t_π	t_π	t_π	t_π	f_w	t_π	t_π	t_π
p_5	t_π	t_π	t_π	f_w	-	t_Δ	t_π	t_π	t_π	f_π	t_π	t_w	f_Δ
p_6	t_w	t_π	f_π	f_π	f_π	-	f_π	t_π	t_π	f_Δ	t_π	f_π	f_Δ
p_7	t_π	t_π	f_π	f_w	f_π	t_π	-	t_π	t_π	f_π	t_π	u	f_π
p_8	u	t_π	f_π	f_π	f_π	t_π	f_π	-	t_π	f_π	t_π	f_π	f_Δ
p_9	f_π	f_π	f_π	f_π	f_π	f_π	f_π	f_π	-	f_Δ	f_π	f_π	f_π
p_{10}	t_Δ	t_π	t_Δ	t_π	t_Δ	t_Δ	t_π	t_Δ	t_π	-	t_π	t_π	t_π
p_{11}	t_π	t_π	f_π	f_π	f_π	f_π	f_π	t_π	t_π	f_Δ	-	f_π	f_π
p_{12}	t_π	t_π	t_π	t_w	t_π	t_π	t_π	t_π	t_π	f_w	t_π	-	t_π
p_{13}	t_Δ	t_π	t_π	t_w	t_π	t_π	t_π	t_π	t_π	f_w	t_π	t_π	-

Bipolar ranking of the 13 posters from relation S

Best choice	p_{10}
2nd best choice	p_{13}
3rd best choice	p_4, p_{12}
4th best choice	p_3
5th best choice	p_5
6th best choice	p_7
6th worst choice	p_1
5th worst choice	p_6
4th worst choice	p_8
3rd worst choice	p_{11}
2nd worst choice	p_2
Worst choice	p_9

Poster p_{10} appears majoritarian as the best candidate as it globally outranks all other poster with a comfortable weighted signiÞcance of 80%, followed in a second position by poster p_{13} and posters p_4 and p_{12} ex eaquo in a third position. On the other side, poster p_9 appears to be the least appreciated by the judges (overall signiÞcance: 60%), preceded by poster p_2 in the second worst position. But is this precise consensual ordering not an artifact induced by our more or less arbitrarily chosen cardinal importance weights: 4, 3, 2, 1 ? To check this point, we compute the robustness degrees of the previous outranking relation as shown in Table 11. Directly applying the same bipolar ranking approach to the ordinal valued $\|S\|$ outranking relation, we obtain the following ordering:

Bipolar ranking of the 13 posters from relation $\|S\|$

Best choice	p_{10}
2nd best choice	p_4
3rd best choice	p_{12}, p_{13}
4th best choice	p_5
5th best choice	p_3
6th best choice	p_7
6th worst choice	p_7
5th worst choice	p_1
4th worst choice	p_6
3rd worst choice	p_8, p_{11}
2nd worst choice	p_2
Worst choice	p_9

Previous results get well conÞrmed on the whole. Indeed with a robustness degree of t_π, i.e. rather true than false with any π-compatible weights, poster p_{10} is conÞrmed in the Þrst[3] and poster p_9 in the last position [4].

Attributing the EBPA 2004 to poster p_{10} was therefore indeed independent of the choice of any precise numerical signiÞcance weights verifying the signiÞcance ordering of the four criteria as imposed by the Programme Committee.

5 Conclusion

In this paper we have presented a formal approach for assessing binary outranking situations on the basis of a performance tableau involving criteria of solely ordinal signiÞcance. The concept of ordinal concordance has been introduced and a formal testing procedure based on distributional dominance is developed. Thus we solve a major practical problem concerning the precise numerical knowledge of the individual signiÞcance weights that is required by the classical majority concordance principle as implemented for instance in the Electre methods. Applicability of the concordance based aggregation of preference is extended to the case where only ordinal signiÞcance of the criteria is available.

[3] Poster p_{10}, which obtained the EBPA 2004, was submitted by Federica RICCA, Bruno SIMEONE and Isabella LARI on *Political Districting via Weighted Voronoï Regions* from the University of Rome ÒLa SapienzaÓ.

[4] It is worthwhile noticing that our bipolar ranking method was not designed to be necessarily stable with respect to the above robustness analysis. And indeed, we may notice a slight order reversal concerning respective positions of posters p_4 and p_{13}. But otherwise there appears no major divergence between both orderings.

Furthermore, even if precise numerical signiÞcance is available, we provide a robustness analysis of the observed preferences by integrating unanimous, ordinal and majority based concordance in a same logical framework.

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On some ordinal models for decision making under uncertainty

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Abstract

In the field of Artificial Intelligence many models for decision making under uncertainty have been proposed that deviate from the traditional models used in Decision Theory, i.e. the Subjective Expected Utility (SEU) model and its many variants. These models aim at obtaining simple decision rules that can be implemented by efficient algorithms while based on inputs that are less rich than what is required in traditional models. One of these models, called the likely dominance (LD) model, consists in declaring that an act is preferred to another as soon as the set of states on which the first act gives a better outcome than the second act is judged more likely than the set of states on which the second act is preferable. The LD model is at much variance with the SEU model. Indeed, it has a definite ordinal flavor and it may lead to preference relations between acts that are not transitive. This paper proposes a general model for decision making under uncertainty tolerating intransitive and/or incomplete preferences that will contain both the SEU and the LD models as particular cases. Within the framework of this general model, we propose a characterization of the preference relations that can be obtained with the LD model. This characterization shows that the main distinctive feature of such relations lies in the very poor relation comparing preference differences that they induce on the set of outcomes.

Key words : Decision under uncertainty, Subjective Expected Utility, Conjoint measurement, Nontransitive preferences, Likely Dominance model.

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1 Introduction

The specific needs of Artificial Intelligence techniques have led many Computer Scientists to propose models for decision under uncertainty that are at variance with the classical models used in Decision Theory, i.e. the Subjective Expected Utility (SEU) model and its many variants (see Fishburn, 1988; Wakker, 1989, for overviews). This gives rise to what is often called "qualitative decision theory" (see Boutilier, 1994; Brafman and Tennenholtz, 1997, 2000; Doyle and Thomason, 1999; Dubois et al., 1997, 2001; Lehmann, 1996; Tan and Pearl, 1994, for overviews). These models aim at obtaining simple decision rules that can be implemented by efficient algorithms while based on inputs that are less rich than what is required in traditional models. This can be achieved, e.g. comparing acts only on the basis of their consequences in the most plausible states (Boutilier, 1994; Tan and Pearl, 1994) or refining the classical criteria (Luce and Raiffa, 1957; Milnor, 1954) for decision making under complete ignorance (see Brafman and Tennenholtz, 2000; Dubois et al., 2001).

One such model, called the "likely dominance" (LD) model, was recently proposed by Dubois et al. (1997) and later studied in Dubois et al. (2003a, 2002) and Fargier and Perny (1999). It consists in declaring that an act a is preferred to an act b as soon as the set of states for which a gives a better outcome than b is judged "more likely" than the set of states for which b gives a better outcome than a. Such a way of comparing acts has a definite ordinal flavor. It rests on a simple "voting" analogy and can be implemented as soon as a preference relation on the set of outcomes and a likelihood relation between subsets of states (i.e. events) are known. Contrary to the other models mentioned above, simple examples inspired from Condorcet's paradox (see Sen, 1986) show that the LD model does not always lead to preference relations between acts that are complete or transitive. Such relations are therefore quite different from the ones usually dealt with in Decision Theory.

Previous characterizations (see Dubois et al., 2003a, 2002; Fargier and Perny, 1999) of the relations that can be obtained using the LD model (of, for short, LD relations) have emphasized their "ordinal" character via the use of variants of a "noncompensation" condition introduced in Fishburn (1975, 1976, 1978) that have been thoroughly studied in the area of multiple criteria decision making (see Bouyssou, 1986, 1992; Bouyssou and Vansnick, 1986; Dubois et al., 2003b; Fargier and Perny, 2001; Vansnick, 1986). Since this condition is wholly specific to such relations, these characterizations are not perfectly suited to capture their essential distinctive features within a more general framework that would also include more traditional preference relations.

The purpose of this paper is twofold. We first introduce a general axiomatic framework for decision under uncertainty that will contain both the SEU and LD models as particular cases. This general framework tolerating incomplete and/or intransitive preferences is based on related work in the area of conjoint measurement (see Bouyssou and

Pirlot, 2002). The second aim of this paper is to propose an alternative characterization of the preference relations that can be obtained using the likely dominance rule within this general framework. This characterization will allow us to emphasize the main distinctive feature of such relations, i.e. the poor relation comparing preference differences that they induce on the set of outcomes. This analysis specializes the one in Bouyssou and Pirlot (2004b) to the case of decision making under uncertainty.

It should be noticed that the interest of studying models tolerating intransitive preferences was forcefully argued by Fishburn (1991). It has already generated much work (see, e.g. Fishburn, 1982, 1984, 1988, 1989, 1990, 1991; Fishburn and Lavalle, 1987a,b, 1988; Lavalle and Fishburn, 1987; Loomes and Sugden, 1982; Nakamura, 1998; Sugden, 1993). These models all use some form of an additive nontransitive model. The originality of our approach is to replace additivity by a mere decomposability requirement which, at the cost of much weaker uniqueness results, allows for a very simple axiomatic treatment.

This paper is organized as follows. Section 2 introduces our setting and notation. The LD model is introduced in section 3. Our general framework for decision making under uncertainty is presented and analyzed in section 4. Section 5 characterize the relations that can be obtained using the LD model within our general framework. A final section discusses our results and presents several extensions of our analysis. An appendix contains examples showing the independence of the conditions used in the paper. The rest of this section is devoted to our, classical, vocabulary concerning binary relations.

A *binary relation* $\mathcal{R}$ on a set X is a subset of $X \times X$; we write $a \ \mathcal{R} \ b$ instead of $(a, b) \in \mathcal{R}$. A binary relation $\mathcal{R}$ on X is said to be:

- *reflexive* if $[a \ \mathcal{R} \ a]$,
- *complete* if $[a \ \mathcal{R} \ b$ or $b \ \mathcal{R} \ a]$,
- *symmetric* if $[a \ \mathcal{R} \ b] \Rightarrow [b \ \mathcal{R} \ a]$,
- *asymmetric* if $[a \ \mathcal{R} \ b] \Rightarrow [Not[b \ \mathcal{R} \ a]]$,
- *transitive* if $[a \ \mathcal{R} \ b$ and $b \ \mathcal{R} \ c] \Rightarrow [a \ \mathcal{R} \ c]$,
- *Ferrers* if $[(a \ \mathcal{R} \ b$ and $c \ \mathcal{R} \ d) \Rightarrow (a \ \mathcal{R} \ d$ or $c \ \mathcal{R} \ b)]$,
- *semi-transitive* if $[(a \ \mathcal{R} \ b$ and $b \ \mathcal{R} \ c) \Rightarrow (a \ \mathcal{R} \ d$ or $d \ \mathcal{R} \ c)]$

for all $a, b, c, d \in X$.

A *weak order* (resp. an *equivalence*) is a complete and transitive (resp. reflexive, symmetric and transitive) binary relation. If $\mathcal{R}$ is an equivalence on X, $X/\mathcal{R}$ will denote the set of equivalence classes of $\mathcal{R}$ on X. An *interval order* is a complete and Ferrers binary relation. A *semiorder* is a semi-transitive interval order.

2 The setting

We adopt a classical setting for decision under uncertainty with a finite number of states. Let $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be the set of *outcomes* and $N = \{1, 2, \ldots, n\}$ be the set of states. It is understood that the elements of N are exhaustive and mutually exclusive: one and only one state will turn out to be true. An *act* is a function from N to Γ. The set of all acts is denoted by $\mathcal{A} = \Gamma^N$. Acts will be denoted by lowercase letters $a, b, c, d, \ldots$. An act $a \in \mathcal{A}$ therefore associates to each state $i \in N$ an outcome $a(i) \in \Gamma$. We often abuse notation and write a_i instead of $a(i)$.

Among the elements of $\mathcal{A}$ are constant acts, i.e. acts giving the same outcome in all states. We denote $\overline{\alpha}$ the constant act giving the outcome $\alpha \in \Gamma$ in all states $i \in N$. Let $E \subseteq N$ and $a, b \in \mathcal{A}$. We denote $a_E b$ the act $c \in \mathcal{A}$ such that $c_i = a_i$, for all $i \in E$ and $c_i = b_i$, for all $i \in N \setminus E$. Similarly $\alpha_E b$ will denote the act $d \in \mathcal{A}$ such that $d_i = \alpha$, for all $i \in E$ and $d_i = b_i$, for all $i \in N \setminus E$. When $E = \{i\}$ we write $a_i b$ and $\alpha_i b$ instead of $a_{\{i\}} b$ and $\alpha_{\{i\}} b$.

In this paper $\succsim$ will always denote a binary relation on the set $\mathcal{A}$. The binary relation $\succsim$ is interpreted as an "at least as good as" preference relation between acts. We note $\succ$ (resp. $\sim$) the asymmetric (resp. symmetric) part of $\succsim$. A similar convention holds when $\succsim$ is starred, superscripted and/or subscripted. The relation $\succsim$ induces a relation $\succsim_\Gamma$ on the set Γ of outcomes via the comparison of constant acts letting:

$$\alpha \succsim_\Gamma \beta \Leftrightarrow \overline{\alpha} \succsim \overline{\beta}.$$

Let E be a nonempty subset of N. We define the relation $\succsim_E$ on $\mathcal{A}$ letting, for all $a, b \in \mathcal{A}$,

$$a \succsim_E b \Leftrightarrow [a_E c \succsim b_E c, \text{ for all } c \in \mathcal{A}].$$

When $E = \{i\}$ we write $\succsim_i$ instead of $\succsim_{\{i\}}$.

If, for all $a, b \in \mathcal{A}$, $a_E c \succsim b_E c$, for some $c \in \mathcal{A}$, implies $a \succsim_E b$, we say that $\succsim$ is independent for E. If $\succsim$ is independent for all nonempty subsets of states we say that $\succsim$ is *independent*. It is not difficult to see that a binary relation is independent if and only if it is independent for $N \setminus \{i\}$, for all $i \in N$ (see Wakker, 1989). Independence as defined here is therefore nothing else than the Sure Thing Principle (postulate $P2$) introduced by Savage (1954).

We say that state $i \in N$ is *influent* (for $\succsim$) if there are $\alpha, \beta, \gamma, \delta \in \Gamma$ and $a, b \in \mathcal{A}$ such that $\alpha_i a \succsim \beta_i b$ and $Not[\gamma_i a \succsim \delta_i b]$ and *degenerate* otherwise. It is clear that a degenerate state has no influence whatsoever on the comparison of the elements of $\mathcal{A}$ and may be suppressed from N. In order to avoid unnecessary minor complications, we suppose henceforth that *all states in N are influent*. Note that this does not rule out the existence of *null* events $E \subseteq N$, i.e. such that $a_E c \sim b_E c$, for all $a, b, c \in \mathcal{A}$. This is exemplified below.

Example 1
Let $N = \{1, 2, 3, 4\}$ and $\Gamma = \mathbb{R}$. Let $p_1 = p_2 = p_3 = p_4 = 1/4$. Define $\succsim$ on $\mathcal{A}$ letting

$$a \succsim b \Leftrightarrow \sum_{i \,\in\, S(a,b)} p_i \geq \sum_{j \,\in\, S(b,a)} p_j - 1/4.$$

for all $a, b \in \mathcal{A}$, where $S(a, b) = \{i \in N : a_i \geq b_i\}$. With such a relation, it is easy to see that all states are influent while they are all null. Observe that $\succsim$ is complete but is not transitive. We shall shortly see that this relation can be obtained with the LD model. $\diamond$

3 The likely dominance model

The following definition, building on Dubois et al. (1997) and Fargier and Perny (1999), formalizes the idea of a LD relation, i.e., of a preference relation that has been obtained comparing acts by pairs on the basis of the "likelihood" of the states favoring each element of the pair.

Definition 1 (LD relations)
Let $\succsim$ be a reflexive *binary relation on $\mathcal{A}$. We say that $\succsim$ is a LD relation if there are:*

- *a* complete *binary relation $\mathcal{S}$ on Γ,*
- *a binary relation $\trianglerighteq$ between subsets of N having N for union that is* monotonic *w.r.t. inclusion, i.e. such that for all $A, B, C, D \subseteq N$,*

$$[A \trianglerighteq B, C \supseteq A, B \supseteq D, C \cup D = N] \Rightarrow C \trianglerighteq D, \tag{1}$$

such that, for all $a, b \in \mathcal{A}$,

$$a \succsim b \Leftrightarrow \mathcal{S}(a, b) \trianglerighteq \mathcal{S}(b, a), \tag{2}$$

where $\mathcal{S}(a, b) = \{i \in N : a_i \;\mathcal{S}\; b_i\}$. We say that $\langle \trianglerighteq, \mathcal{S} \rangle$ is a representation *of $\succsim$.*

Hence, when $\succsim$ is a LD relation, the preference between a and b only depends on the subsets of states favoring a or b in terms of the complete relation $\mathcal{S}$. It does not depend on "preference differences" between outcomes besides what is indicated by $\mathcal{S}$. A major advantage of the LD model is that it can be applied to compare acts as soon as there is a binary relation allowing to compare outcomes and a relation allowing to compare events in terms of likelihood.

Let $\succsim$ be a LD relation with a representation $\langle \trianglerighteq, \mathcal{S} \rangle$. We denote by $\mathcal{I}$ (resp. $\mathcal{P}$) the symmetric part (resp. asymmetric part) of $\mathcal{S}$. For all $A, B \subseteq N$, we define the relations $\triangleq$, $\triangleright$ and $\bowtie$ between subsets of N having N for union letting: $A \triangleq B \Leftrightarrow [A \trianglerighteq B$ and $B \trianglerighteq A]$, $A \triangleright B \Leftrightarrow [A \trianglerighteq B$ and $Not[B \trianglerighteq A]]$, $A \bowtie B \Leftrightarrow [Not[A \trianglerighteq B]$ and $Not[B \trianglerighteq A]]$.

The following lemma takes note of some elementary properties of LD relations; it uses the hypothesis that all states are influent.

Lemma 1
If $\succsim$ is a LD relation with a representation $\langle \trianglerighteq, \mathcal{S} \rangle$, then:

1. *$\mathcal{P}$ is nonempty,*
2. *for all $A, B \subseteq N$ such that $A \cup B = N$ exactly one of $A \triangleright B$, $B \triangleright A$, $A \triangleq B$ and $A \bowtie B$ holds and we have $N \triangleq N$,*
3. *for all $A \subseteq N$, $N \trianglerighteq A$,*
4. *$N \triangleright \varnothing$,*
5. *$\succsim$ is independent,*
6. *$\succsim$ is marginally complete, i.e., for all $i \in N$, all $\alpha, \beta \in \Gamma$ and all $a \in \mathcal{A}$, $\alpha_i a \succsim \beta_i a$ or $\beta_i a \succsim \alpha_i a$,*
7. *$\mathcal{S} = \succsim_\Gamma$,*
8. *for all $i \in N$ and all $a, b \in \mathcal{A}$, either $a \succsim_i b \Leftrightarrow a_i \mathcal{S} b_i$ or $a \sim_i b$,*
9. *$\succsim$ has a* unique *representation.*

PROOF
Part 1. If $\mathcal{P}$ is empty, then, since $\mathcal{S}$ is complete, $\mathcal{S}(a, b) = N$, for all $a, b \in \mathcal{A}$. Hence, for all $i \in N$, all $\alpha, \beta, \gamma, \delta \in \Gamma$, and all $a, b \in \mathcal{A}$,

$$\mathcal{S}(\alpha_i a, \beta_i b) = \mathcal{S}(\gamma_i a, \delta_i b) \text{ and}$$
$$\mathcal{S}(\beta_i b, \alpha_i a) = \mathcal{S}(\delta_i b, \gamma_i a).$$

This implies, using (2), that state $i \in N$ is degenerate, contrarily to our hypothesis.

Part 2. Since the relation $\mathcal{P}$ is nonempty and $\mathcal{S}$ is complete, for all $A, B \subseteq N$ such that $A \cup B = N$, there are $a, b \in \mathcal{A}$ such that $\mathcal{S}(a, b) = A$ and $\mathcal{S}(b, a) = B$. We have, by construction, exactly one of $a \succ b$, $b \succ a$, $a \sim b$ and $[Not[a \succsim b]$ and $Not[b \succsim a]]$. Hence, using (2), we have exactly one of $A \triangleright B$, $B \triangleright A$, $A \triangleq B$ and $A \bowtie B$. Since the relation $\mathcal{S}$ is complete, we have $\mathcal{S}(a, a) = N$. Using the reflexivity of $\succsim$, we know that $a \sim a$, so that (2) implies $N \triangleq N$.

Parts 3 and 4. Let $A \subseteq N$. Because $N \triangleq N$, the monotonicity of $\trianglerighteq$ implies $N \trianglerighteq A$. Suppose that $\varnothing \trianglerighteq N$. Then the monotonicity of $\trianglerighteq$ would imply that $A \trianglerighteq B$, for all $A, B \subseteq N$ such that $A \cup B = N$. This would contradict the fact that each state is influent.

Part 5. Using the completeness of $\mathcal{S}$, we have, for all $\alpha, \beta, \gamma, \delta \in \Gamma$ and all $a, b \in \mathcal{A}$,

$$\mathcal{S}(\alpha_i a, \alpha_i b) = \mathcal{S}(\beta_i a, \beta_i b) \text{ and}$$
$$\mathcal{S}(\alpha_i b, \alpha_i a) = \mathcal{S}(\beta_i b, \beta_i a).$$

Using (2), this implies that, for all $i \in N$, all $\alpha, \beta \in \Gamma$ and all $a, b \in \mathcal{A}$, $\alpha_i a \succsim \alpha_i b \Leftrightarrow \beta_i a \succsim \beta_i b$. Therefore, $\succsim$ is independent for $N \setminus \{i\}$ and, hence, independent.

Part 6 follows from the fact that $\mathcal{S}$ is complete, $N \triangleq N$ and $N \trianglerighteq N \setminus \{i\}$, for all $i \in N$.

Part 7. Suppose that $\alpha \succsim_\Gamma \beta$ so that $\overline{\alpha} \succsim \overline{\beta}$ and $Not[\alpha \ \mathcal{S} \ \beta]$. Since $\mathcal{S}$ is complete, we have $\beta \ \mathcal{P} \ \alpha$. Using (2) and $N \triangleright \varnothing$, we have $\overline{\beta} \succ \overline{\alpha}$, a contradiction. Conversely, if $\alpha \ \mathcal{S} \ \beta$ we obtain, using (2) and the fact that $N \trianglerighteq A$, for all $A \subseteq N$, $\overline{\alpha} \succsim \overline{\beta}$ so that $\alpha \succsim_\Gamma \beta$.

Part 8. Let $i \in N$. We know that $N \triangleq N$ and $N \trianglerighteq N \setminus \{i\}$. If $N \triangleq N \setminus \{i\}$, then (2) implies $a \succsim_i b$ for all $a, b \in \mathcal{A}$. Otherwise we have $N \triangleright N \setminus \{i\}$ and $N \triangleq N$. It follows that $\alpha \ \mathcal{S} \ \beta \Rightarrow \overline{\alpha} \succsim_i \overline{\beta}$ and $\alpha \ \mathcal{P} \ \beta \Rightarrow \overline{\alpha} \succ_i \overline{\beta}$. Since $\mathcal{S}$ and $\succsim_i$ are complete, it follows that $\mathcal{S} = \succsim_i$.

Part 9. Suppose that $\succsim$ is a LD relation with a representation $\langle \trianglerighteq, \mathcal{S} \rangle$. Suppose that $\succsim$ has another representation $\langle \trianglerighteq', \mathcal{S}' \rangle$. Using part 7, we know that $\mathcal{S} = \mathcal{S}' = \succsim_\Gamma$. Using (2), it follows that $\trianglerighteq = \trianglerighteq'$. □

4 A general framework for decision under uncertainty tolerating intransitive preferences

We consider in this section binary relations $\succsim$ on $\mathcal{A}$ that can be represented as:

$$a \succsim b \Leftrightarrow F(p(a_1, b_1), p(a_2, b_2), \ldots, p(a_n, b_n)) \geq 0 \qquad \text{(UM)}$$

where p is a real-valued function on Γ^2 that is *skew symmetric* (i.e. such that $p(\alpha, \beta) = -p(\beta, \alpha)$, for all $\alpha, \beta \in \Gamma$) and F is a real-valued function on $\prod_{i=1}^n p(\Gamma^2)$ being *nondecreasing* in all its arguments and such that, abusing notation, $F(\mathbf{0}) \geq 0$.

It is useful to interpret p as a function measuring preference differences between outcomes. The fact that p is supposed to be skew symmetric means that the preference difference between α and β is the opposite of the preference difference between β and α, which seems a reasonable hypothesis for preference differences. With this interpretation in mind, the acts a and b are compared as follows. In each state $i \in N$, the preference difference between a_i and b_i is computed. The synthesis of these preference differences is performed applying the function F. If this synthesis is positive, we conclude that $a \succsim b$. Given this interpretation, it seems reasonable to suppose that F is nondecreasing in each of its arguments. The fact that $F(\mathbf{0}) \geq 0$ simply means that the synthesis of null preference differences in each state should be nonnegative; this ensures that $\succsim$ will be reflexive. Model (UM) is the specialization to the case of decision making under uncertainty of conjoint measurement models studied in Bouyssou and Pirlot (2002).

It is not difficult to see that model (UM) encompasses preference relations $\succsim$ on $\mathcal{A}$ that are neither transitive nor complete. It is worth noting that this model is sufficiently flexible to contain many others as particular cases including:

- the SEU model (see, e.g. Wakker, 1989) in which:

$$a \succsim b \Leftrightarrow \sum_{i=1}^{n} w_i u(a_i) \geq \sum_{i=1}^{n} w_i u(b_i) \tag{SEU}$$

  where w_i are nonnegative real numbers that add up to one and u is a real-valued function on Γ,

- the Skew Symmetric Additive model (SSA) (see Fishburn, 1988, 1990) in which

$$a \succsim b \Leftrightarrow \sum_{i=1}^{n} w_i \Phi(a_i, b_i) \geq 0 \tag{SSA}$$

  where w_i are nonnegative real numbers that add up to one and Φ is a skew symmetric ($\Phi(\alpha, \beta) = -\Phi(\beta, \alpha)$) real-valued function on Γ^2.

We will show in the next section that model (UM) also contains all LD relations. As shown below, model (UM) implies that $\succsim$ is independent. It is therefore not suited to cope with violations of the Sure Thing Principle that have been widely documented in the literature (Allais, 1953; Ellsberg, 1961; Kahneman and Tversky, 1979), which can be done, e.g. using Choquet Expected Utility or Cumulative Prospect Theory (see Chew and Karni, 1994; Gilboa, 1987; Karni and Schmeidler, 1991; Luce, 2000; Nakamura, 1990; Schmeidler, 1989; Wakker, 1989, 1994, 1996; Wakker and Tversky, 1993).

The flexibility of model (UM) may obscure some of its properties. We summarize what appears to be the most important ones in the following.

Lemma 2
Let $\succsim$ be a binary relation on $\mathcal{A}$ that has a representation in model (UM). *Then:*

1. *$\succsim$ is reflexive, independent and marginally complete,*

2. $[a \succ_i b$ *for all* $i \in J \subseteq N] \Rightarrow [a \succ_J b]$*,*

3. $\succsim_\Gamma$ *is complete.*

PROOF
Part 1. The reflexivity of $\succsim$ follows from the skew symmetry of p and $F(\mathbf{0}) \geq 0$. Independence follows from the fact that $p(\alpha, \alpha) = 0$, for all $\alpha \in \Gamma$. $Not[\alpha_i a \succsim \beta_i a]$ and $Not[\beta_i a \succsim \alpha_i a]$ imply, abusing notation, $F([p(\alpha, \beta)]_i, [\mathbf{0}]_{-i}) < 0$ and $F([p(\beta, \alpha)]_i, [\mathbf{0}]_{-i}) < 0$. Since $F(\mathbf{0}) \geq 0$ and F is nondecreasing, we have $p(\alpha, \beta) < 0$ and $p(\beta, \alpha) < 0$, which contradicts the skew symmetry of p. Hence, $\succsim$ is marginally complete.

Part 2. Observe that $\alpha \succ_i \beta$ is equivalent to $F([p(\alpha, \beta)]_i, [\mathbf{0}]_{-i}) \geq 0$ and $F([p(\beta, \alpha)]_i, [\mathbf{0}]_{-i}) < 0$. Since $F(\mathbf{0}) \geq 0$ we know that $p(\beta, \alpha) < 0$ using the nondecreasingness of F. The skew symmetry of p implies $p(\alpha, \beta) > 0 > p(\beta, \alpha)$ and the desired property easily follows using the nondecreasingness of F.

Part 3. Because p is skew symmetric, we have, for all $\alpha, \beta \in \Gamma$, $p(\alpha, \beta) \geq 0$ or $p(\beta, \alpha) \geq 0$. Since $F(\mathbf{0}) \geq 0$, the completeness of $\succsim_\Gamma$ follows from the nondecreasingness of F. □

The analysis of model (UM) heavily rests on the study of induced relations comparing preference differences on the set of outcomes. The interest of such relations was already powerfully stressed by Wakker (1988, 1989) (note however that, although we use similar notation, our definitions differs from his).

Definition 2 (Relations comparing preference differences)
Let $\succsim$ *be a binary relation on* $\mathcal{A}$*. We define the binary relations* $\succsim^*$ *and* $\succsim^{**}$ *on* Γ^2 *letting, for all* $\alpha, \beta, \gamma, \delta \in \Gamma$*,*

$$(\alpha, \beta) \succsim^* (\gamma, \delta) \Leftrightarrow [\textit{for all } a, b \in \mathcal{A} \textit{ and all } i \in N, \gamma_i a \succsim \delta_i b \Rightarrow \alpha_i a \succsim \beta_i b],$$

$$(\alpha, \beta) \succsim^{**} (\gamma, \delta) \Leftrightarrow [(\alpha, \beta) \succsim^* (\gamma, \delta) \textit{ and } (\delta, \gamma) \succsim^* (\beta, \alpha)].$$

The asymmetric and symmetric parts of $\succsim^*$ are respectively denoted by $\succ^*$ and $\sim^*$, a similar convention holding for $\succsim^{**}$. By construction, $\succsim^*$ and $\succsim^{**}$ are reflexive and transitive. Therefore, $\sim^*$ and $\sim^{**}$ are equivalence relations. Note that, by construction, $\succsim^{**}$ is *reversible*, i.e. $(\alpha, \beta) \succsim^{**} (\gamma, \delta) \Leftrightarrow (\delta, \gamma) \succsim^{**} (\beta, \alpha)$.

We note a few useful connections between $\succsim^*$ and $\succsim$ in the following lemma.

Lemma 3
1. $\succsim$ *is independent if and only if (iff)* $(\alpha, \alpha) \sim^* (\beta, \beta)$*, for all* $\alpha, \beta \in \Gamma$

2. *For all* $a, b, c, d \in \mathcal{A}$*, all* $i \in N$ *and all* $\alpha, \beta \in \Gamma$

$$[a \succsim b \text{ and } (c_i, d_i) \succsim^* (a_i, b_i)] \Rightarrow c_i a \succsim d_i b, \quad (3)$$

$$[(c_j, d_j) \sim^* (a_j, b_j), \text{ for all } j \in N] \Rightarrow [a \succsim b \Leftrightarrow c \succsim d]. \quad (4)$$

PROOF
Part 1. It is clear that [$\succsim$ is independent] $\Leftrightarrow$ [$\succsim$ is independent for $N \setminus \{i\}$, for all $i \in N$]. Observe that [$\succsim$ is independent for $N \setminus \{i\}$, for all $i \in N$] $\Leftrightarrow$ [$\alpha_i a \succsim \alpha_i b \Leftrightarrow \beta_i a \succsim \beta_i b$, for all $\alpha, \beta \in \Gamma$, all $i \in N$ and all $a, b \in \mathcal{A}$] $\Leftrightarrow$ [$(\alpha, \alpha) \sim^* (\beta, \beta)$ for all $\alpha, \beta \in \Gamma$].

Part 2. (3) is clear from the definition of $\succsim^*$, (4) follows. □

The following conditions are an adaptation to the case of decision making under uncertainty of conditions used in Bouyssou and Pirlot (2002) in the context of conjoint measurement. They will prove will prove central in what follows.

Definition 3 (Conditions URC1 and URC2)
Let $\succsim$ *be a binary relation on* $\mathcal{A}$*. This relation is said to satisfy:*

$$\textit{URC1 if} \quad \left.\begin{array}{c} \alpha_i a \succsim \beta_i b \\ \textit{and} \\ \gamma_j c \succsim \delta_j d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \gamma_i a \succsim \delta_i b \\ \textit{or} \\ \alpha_j c \succsim \beta_j d, \end{array}\right.$$

$$\textit{URC2 if} \quad \left.\begin{array}{c} \alpha_i a \succsim \beta_i b \\ \textit{and} \\ \beta_j c \succsim \alpha_j d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \gamma_i a \succsim \delta_i b \\ \textit{or} \\ \delta_j c \succsim \gamma_j d, \end{array}\right.$$

for all $i, j \in N$*, all* $a, b, c, d \in \mathcal{A}$ *and all* $\alpha, \beta, \gamma, \delta \in \Gamma$.

Condition URC1 suggests that, independently of the state $i \in N$, either the difference (α, β) is at least as large as the difference (γ, δ) of *vice versa*. Indeed, suppose that $\alpha_i a \succsim \beta_i b$ and $Not[\gamma_i a \succsim \delta_i b]$. This is the sign that the preference difference between α and β appears to be larger than the preference difference between γ and δ. Therefore if $\gamma_j c \succsim \delta_j d$, we should have $\alpha_j c \succsim \beta_j d$, which is URC1. Similarly, condition URC2 suggests that the preference difference (α, β) is linked to the "opposite" preference difference (β, α). Indeed if $\alpha_i a \succsim \beta_i b$ and $Not[\gamma_i a \succsim \delta_i b]$, so that the difference between γ and δ is not larger than the difference between α and β, URC2 implies that $\beta_j c \succsim \alpha_j d$ should imply $\delta_j c \succsim \gamma_j d$, so that the difference between δ and γ is not smaller than the difference between β and α. The following lemma summarizes the main consequences of URC1 and URC2.

Lemma 4

1. *URC1* $\Leftrightarrow$ [$\succsim^*$ *is complete*],

2. *URC2* $\Leftrightarrow$
[*for all* $\alpha, \beta, \gamma, \delta \in \Gamma, Not[(\alpha, \beta) \succsim^* (\gamma, \delta)] \Rightarrow (\beta, \alpha) \succsim^* (\delta, \gamma)$],

3. *[URC1 and URC2]* $\Leftrightarrow$ *[$\succsim^{**}$ is complete]*.

4. *In the class of reflexive relations, URC1 and URC2 are independent conditions.*

5. *URC2* $\Rightarrow$ *[$\succsim$ is independent]*.

PROOF
Part 1. Suppose that URC1 is violated so that $\alpha_i a \succsim \beta_i b$, $\gamma_j c \succsim \delta_j d$, $Not[\gamma_i a \succsim \delta_i b]$ and $Not[\alpha_j c \succsim \beta_j d]$. This is equivalent to $Not[(\alpha, \beta) \succsim^* (\gamma, \delta)]$ and $Not[(\gamma, \delta) \succsim^* (\alpha, \beta)]$.

Part 2. Suppose that URC2 is violated so that $\alpha_i a \succsim \beta_i b$, $\beta_j c \succsim \alpha_j d$, $Not[\gamma_i a \succsim \delta_i b]$ and $Not[\delta_j c \succsim \gamma_j d]$. This is equivalent to $Not[(\gamma, \delta) \succsim^* (\alpha, \beta)]$ and $Not[(\delta, \gamma) \succsim^* (\beta, \alpha)]$. Part 3 easily follows from parts 1 and 2.

Part 4: see examples 2 and 3 in appendix.

Part 5. Suppose that $\alpha_i a \succsim \alpha_i b$. Using URC2 implies $\beta_i a \succsim \beta_i b$, for all $\beta \in \Gamma$. Hence, $\succsim$ is independent. □

The following lemma shows that all relations satisfying model (UM) satisfy URC1 and URC2; this should be no surprise since within model (UM) the skew symmetric function p induces on Γ^2 a reversible weak order.

Lemma 5
Let $\succsim$ be a binary relation on $\mathcal{A}$. If $\succsim$ has a representation in model (UM) *then $\succsim$ satisfies URC1 and URC2.*

PROOF
[URC1]. Suppose that $\alpha_i a \succsim \beta_i b$ and $\gamma_j c \succsim \delta_j d$. Using model (UM) we have:

$$F([p(\alpha, \beta)]_i, [p(a_k, b_k)]_{k \neq i}) \geq 0 \text{ and } F([p(\gamma, \delta)]_j, [p(c_\ell, d_\ell)]_{\ell \neq j}) \geq 0,$$

with $[\cdot]_i$ denoting the ith argument of F. If $p(\alpha, \beta) \geq p(\gamma, \delta)$ then using the nondecreasingness of F, we have $F([p(\alpha, \beta)]_j, [p(c_\ell, d_\ell)]_{\ell \neq j}) \geq 0$ so that $\alpha_j c \succsim \beta_j d$. If $p(\alpha, \beta) < p(\gamma, \delta)$ we have $F([p(\gamma, \delta)]_i, [p(a_k, b_k)]_{k \neq i}) \geq 0$ so that $\gamma_i a \succsim \delta_i b$. Hence URC1 holds.

[URC2]. Similarly, suppose that $\alpha_i a \succsim \beta_i b$ and $\beta_j c \succsim \alpha_j d$. We thus have:

$$F([p(\alpha, \beta)]_i, [p(a_k, b_k)]_{k \neq i}) \geq 0 \text{ and } F([p(\beta, \alpha)]_j, [p(c_\ell, d_\ell)]_{\ell \neq j}) \geq 0.$$

If $p(\alpha, \beta) \geq p(\gamma, \delta)$, the skew symmetry of p implies $p(\delta, \gamma) \geq p(\beta, \alpha)$. Using the nondecreasingness of F, we have $F([p(\delta, \gamma)]_j, [p(c_\ell, d_\ell)]_{\ell \neq j}) \geq 0$, so that $\delta_j c \succsim \gamma_j d$. Similarly, if $p(\alpha, \beta) < p(\gamma, \delta)$, we have, using the nondecreasingness of F, $F([p(\gamma, \delta)]_i, [p(a_k, b_k)]_{k \neq i})$ ≥ 0 so that $\gamma_i a \succsim \delta_i b$. Hence URC2 holds. □

It turns out that conditions URC1 and URC2 allow to completely characterize model (UM) when $\Gamma/\sim^{**}$ is finite or countably infinite.

Theorem 1
Let $\succsim$ be a binary relation on $\mathcal{A}$. If $\Gamma/\sim^{**}$ is finite or countably infinite, then $\succsim$ has a representation (UM) *iff it is reflexive and satisfies URC1 and URC2.*

PROOF
Necessity follows from lemmas 2 and 5. We establish sufficiency.

Since URC1 and URC2 hold, we know from lemma 4 that $\succsim^{**}$ is complete so that it is a weak order. This implies that $\succsim^{*}$ is a weak order. Since $\Gamma/\sim^{**}$ is finite or countably infinite, it is clear that $\Gamma/\sim^{*}$ is finite or countably infinite. Therefore, there is a real-valued function q on Γ^2 such that, for all $\alpha, \beta, \gamma, \delta \in \Gamma$, $(\alpha, \beta) \succsim^{*} (\gamma, \delta) \Leftrightarrow q(\alpha, \beta) \geq q(\gamma, \delta)$. Given a particular numerical representation q of $\succsim^{*}$, let $p(\alpha, \beta) = q(\alpha, \beta) - q(\beta, \alpha)$. It is obvious that p is skew symmetric and represents $\succsim^{**}$.

Define F as follows:

$$F(p(a_1, b_1), p(a_2, b_2), \ldots, p(a_n, b_n)) = \begin{cases} \exp(\sum_{i=1}^{n} p(a_i, b_i)) & \text{if } a \succsim b, \\ -\exp(-\sum_{i=1}^{n} p(a_i, b_i)) & \text{otherwise.} \end{cases}$$

The well-definedness of F follows from (4). To show that F is nondecreasing, suppose that $p(\alpha, \beta) \geq p(\gamma, \delta)$, i.e. that $(\alpha, \beta) \succsim^{**} (\gamma, \delta)$. If $\gamma_i a \succsim \delta_i b$, we know from (3) that $\alpha_i a \succsim \beta_i b$ and the conclusion follows from the definition of F. If $Not[\gamma_i a \succsim \delta_i b]$, we have either $Not[\alpha_i a \succsim \beta_i b]$ or $\alpha_i a \succsim \beta_i b$. In either case, the conclusion follows from the definition of F. Since $\succsim$ is reflexive, we have $F(\mathbf{0}) \geq 0$, as required. This completes the proof. □

Remark 1
Following Bouyssou and Pirlot (2002), it is not difficult to extend theorem 1 to sets of arbitrary cardinality adding a, necessary, condition implying that the weak order $\succsim^{*}$ (and, hence, $\succsim^{**}$) has a numerical representation. This will not be useful here and we leave the details to the interested reader.

We refer to Bouyssou and Pirlot (2002) for an analysis of the, obviously quite weak, uniqueness properties of the numerical representation of model (UM). Observe that, if $\succsim$ has a representation in model (UM), we must have that:

$$(\alpha, \beta) \succ^{**} (\gamma, \delta) \Rightarrow p(\alpha, \beta) > p(\gamma, \delta). \tag{5}$$

Hence, the number of distinct values taken by p in a representation in model (UM) is an upper bound of the number of distinct equivalence classes of $\succsim^{**}$. •

Remark 2
Following the analysis in Bouyssou and Pirlot (2002), it is not difficult to analyze variants of model (UM). For instance, when Γ is finite or countably infinite:

- the weakening of model (UM) obtained considering a function p that may not be skew symmetric but is such that $p(\alpha, \alpha) = 0$, for all $\alpha \in \Gamma$, is equivalent to supposing that $\succsim$ is reflexive, independent and satisfies URC1,

- the weakening of model (UM) obtained considering a function F that may not be nondecreasing is equivalent to supposing that $\succsim$ is reflexive and independent,

- the strengthening of model (UM) obtained considering a function F that is odd ($F(\mathbf{x}) = -F(\mathbf{x})$) is equivalent to supposing that $\succsim$ is complete and satisfies URC1 and URC2.

In Bouyssou and Pirlot (2004c), we study the strengthening of model (UM) obtained requiring that F that is odd and strictly increasing in each of its arguments. In the finite or countably infinite case, this model is shown to be characterized by the completeness of $\succsim$ and the "Cardinal Coordinate Independence" condition introduced in Wakker (1984, 1988, 1989) in order to derive the SEU model. This condition implies both URC1 and URC2 for complete relations.

All the above results are easily generalized to cover the case of an arbitrary set of consequences adding appropriate conditions guaranteeing that $\succsim^*$ has a numerical representation (on these conditions, see Fishburn, 1970; Krantz et al., 1971) •

5 A new characterization of LD relations

We have analyzed in Bouyssou and Pirlot (2004c) the relations between model (UM) and models (SEU) and (SSA). We show here what has to be added to the conditions of theorem 1 in order to characterize LD relations. The basic intuition behind this analysis is quite simple. Consider a binary relation $\succsim$ that has a representation in model (UM) in which the function p takes at most three distinct values, i.e. a positive value, a null value and a negative value. In such a case, it is tempting to define the relation $\mathcal{S}$ letting $\alpha \; \mathcal{P} \; \beta \Leftrightarrow p(\alpha, \beta) > 0$ and $\alpha \; \mathcal{I} \; \beta \Leftrightarrow p(\alpha, \beta) = 0$. Since p takes only three distinct values, the relation $\mathcal{S}$ summarizes without any loss the information contained in the skew symmetric function p. This brings us quite close to a LD relation. We formalize this intuition below. This will require the introduction of conditions that will limit the number of equivalence classes of $\sim^*$ and, therefore, $\sim^{**}$.

Definition 4 (Conditions UM1 and UM2)
Let $\succsim$ be a binary relation on a set $\mathcal{A}$. This relation is said to satisfy:

$$\textit{UM1 if} \quad \left.\begin{array}{c} \alpha_i a \succsim \beta_i b \\ \textit{and} \\ \gamma_j c \succsim \delta_j d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \beta_i a \succsim \alpha_i b \\ \textit{or} \\ \delta_i a \succsim \gamma_i b \\ \textit{or} \\ \alpha_j c \succsim \beta_j d, \end{array}\right.$$

$$\textit{UM2 if} \quad \left.\begin{array}{c} \alpha_i a \succsim \beta_i b \\ \textit{and} \\ \beta_j c \succsim \alpha_j d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \beta_i a \succsim \alpha_i b \\ \textit{or} \\ \gamma_i a \succsim \delta_i b \\ \textit{or} \\ \gamma_j c \succsim \delta_j d, \end{array}\right.$$

for all $i, j \in N$, all $a, b, c, d \in \mathcal{A}$ and all $\alpha, \beta, \gamma, \delta \in \Gamma$.

In order to analyze these two conditions, it will be useful to introduce the following two conditions:

$$\left.\begin{array}{c} \alpha_i a \succsim \beta_i b \\ \text{and} \\ \gamma_j c \succsim \delta_j d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \beta_i a \succsim \alpha_i b \\ \text{or} \\ \alpha_j c \succsim \beta_j d, \end{array}\right. \tag{6}$$

$$\left.\begin{array}{c} \alpha_i a \succsim \beta_i b \\ \text{and} \\ \beta_j c \succsim \alpha_j d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \beta_i a \succsim \alpha_i b \\ \text{or} \\ \gamma_j c \succsim \delta_j d, \end{array}\right. \tag{7}$$

for all $i, j \in N$, all $a, b, c, d \in \mathcal{A}$ and all $\alpha, \beta, \gamma, \delta \in \Gamma$. Condition (6) has a simple interpretation. Suppose that $\alpha_i a \succsim \beta_i b$ and $Not[\beta_i a \succsim \alpha_i b]$. This is the sign that the preference difference between α and β is strictly larger than the preference difference between β and α. Because with LD relations there can be only three types of preference differences (positive, null and negative) and preference differences are compared in a reversible way, this implies that the preference difference between α and β must be at least as large as any other preference difference. In particular, if $\gamma_j c \succsim \delta_j d$, it must follow that $\alpha_j c \succsim \beta_j d$. This is what condition (6) implies. Condition (7) has an obvious dual interpretation: if a difference is strictly smaller than its opposite then any other preference must be at least as large as this difference. Conditions UM1 and UM2 are respectively deduced from (6) and (7) by adding a conclusion to these conditions. This additional conclusion ensures that these new conditions are independent from URC1 and URC2. This is formalized below.

Lemma 6

1. (6) $\Leftrightarrow [Not[(\beta, \alpha) \succsim^* (\alpha, \beta)] \Rightarrow (\alpha, \beta) \succsim^* (\gamma, \delta)]$,

2. (7) $\Leftrightarrow [Not[(\beta, \alpha) \succsim^* (\alpha, \beta)] \Rightarrow (\gamma, \delta) \succsim^* (\beta, \alpha)]$, *3.* (6) $\Rightarrow$ *UM1,*

4. (7) $\Rightarrow$ *UM2,*

5. URC2 and UM1 $\Rightarrow$ (6),

6. URC1 and UM2 $\Rightarrow$ (7),

7. $[$*URC1*$,$ *URC2*$,$ *UM1 and UM2*$] \Rightarrow [\succsim^{**}$ *is a weak order having at most three equivalence classes*$]$.

8. In the class of reflexive relations, URC1, URC2, UM1 and UM2 are independent conditions.

PROOF

Part 1. We clearly have $Not[$(6)$] \Leftrightarrow [Not[(\beta,\alpha) \succsim^* (\alpha,\beta)]$ and $Not[(\alpha,\beta) \succsim^* (\gamma,\delta)]]$. The proof of part 2 is similar. Parts 3 and 4 are obvious since UM1 (resp. UM2) amounts to adding a possible conclusion to (6) (resp. (7)).

Part 5. Suppose that $\alpha_i a \succsim \beta_i b$ and $\gamma_j c \succsim \delta_j d$. If $Not[\delta_j a \succsim \gamma_j b]$, UM1 implies $\beta_i a \succsim \alpha_i b$ or $\alpha_j c \succsim \beta_j d$. Suppose now that $\delta_j a \succsim \gamma_j b$. Using URC2 $\delta_i a \succsim \gamma_i b$ and $\gamma_j a \succsim \delta_j b$ imply $\beta_i a \succsim \alpha_i b$ or $\alpha_j a \succsim \beta_j b$. Hence, (6) holds.

Part 6. Suppose that $\alpha_i a \succsim \beta_i b$ and $\beta_j c \succsim \alpha_j d$. If $Not[\gamma_i a \succsim \delta_i b]$, UM2 implies $\beta_i a \succsim \alpha_i b$ or $\gamma_j c \succsim \delta_j d$. Suppose now that $\gamma_i a \succsim \delta_i b$. Using URC1 $\gamma_i a \succsim \delta_i b$ and $\beta_j c \succsim \alpha_j d$ imply $\beta_i a \succsim \alpha_i b$ or $\gamma_j c \succsim \delta_j d$. Hence, (7) holds.

Part 7. Since URC1 and URC2 hold, we know that $\succsim^{**}$ is complete. Since $\succsim^{**}$ is reversible, the conclusion will be false iff there are $\alpha,\beta,\gamma,\delta \in \Gamma$ such that $(\alpha,\beta) \succ^{**} (\gamma,\delta) \succ^{**} (\alpha,\alpha)$.

1. Suppose that $(\alpha,\beta) \succ^* (\gamma,\delta)$ and $(\gamma,\delta) \succ^* (\alpha,\alpha)$. Using URC2, we know that $(\alpha,\alpha) \succsim^* (\delta,\gamma)$. Using the transitivity of $\succsim^*$ we have $(\gamma,\delta) \succ^* (\delta,\gamma)$. Since $(\alpha,\beta) \succ^* (\gamma,\delta)$, this contradicts (6).

2. Suppose that $(\alpha,\beta) \succ^* (\gamma,\delta)$ and $(\alpha,\alpha) \succ^* (\delta,\gamma)$. Using URC2, we know that $(\gamma,\delta) \succsim^* (\alpha,\alpha)$. Using the transitivity of $\succsim^*$ we have $(\gamma,\delta) \succ^* (\delta,\gamma)$. Since $(\alpha,\beta) \succ^* (\gamma,\delta)$, this contradicts (6).

3. Suppose that $(\delta,\gamma) \succ^* (\beta,\alpha)$ and $(\gamma,\delta) \succ^* (\alpha,\alpha)$. Using URC2, we know that $(\alpha,\alpha) \succsim^* (\delta,\gamma)$ so that $(\gamma,\delta) \succ^* (\delta,\gamma)$. Since $(\delta,\gamma) \succ^* (\beta,\alpha)$, this contradicts (7).

4. Suppose that $(\delta,\gamma) \succ^* (\beta,\alpha)$ and $(\alpha,\alpha) \succ^* (\delta,\gamma)$. Using URC2 we have $(\gamma,\delta) \succsim^* (\alpha,\alpha)$ so that $(\gamma,\delta) \succ^* (\delta,\gamma)$. Since $(\delta,\gamma) \succ^* (\beta,\alpha)$, this contradicts (7).

Part 8: see examples 4, 5, 6 and 7 in appendix. □

In view of the above lemma, conditions UM1 and UM2 seem to adequately capture the ordinal character of the aggregation at work in a LD relation within the framework of model (UM). Indeed, the following lemma shows that all LD relations satisfy UM1 and UM2 while having a representation in model (UM).

Lemma 7
Let $\succsim$ be a binary relation on $\mathcal{A}$. If $\succsim$ is a LD relation then,

1. *$\succsim$ satisfies URC1 and URC2,*
2. *$\succsim$ satisfies UM1 and UM2.*

PROOF
Let $\langle \trianglerighteq, \mathcal{S} \rangle$ be the representation of $\succsim$.

Part 1. Let us show that URC1 holds, i.e. that $\alpha_i a \succsim \beta_i b$ and $\gamma_j c \succsim \delta_j d$ imply $\gamma_i a \succsim \delta_i b$ or $\alpha_j c \succsim \beta_j d$.

There are 9 cases to envisage:

	$\gamma \, \mathcal{P} \, \delta$	$\gamma \, \mathcal{I} \, \delta$	$\delta \, \mathcal{P} \, \gamma$
$\alpha \, \mathcal{P} \, \beta$	(*i*)	(*ii*)	(*iii*)
$\alpha \, \mathcal{I} \, \beta$	(*iv*)	(*v*)	(*vi*)
$\beta \, \mathcal{P} \, \alpha$	(*vii*)	(*viii*)	(*ix*)

Cases (*i*), (*v*) and (*ix*) clearly follow from (2). All other cases easily follow from (2) and the monotonicity of $\trianglerighteq$. The proof for URC2 is similar.

Part 2. Let us show that UM1 holds, i.e. that $\alpha_i a \succsim \beta_i b$ and $\gamma_j c \succsim \delta_j d$ imply $\beta_i a \succsim \alpha_i b$ or $\gamma_i a \succsim \delta_i b$ or $\alpha_j c \succsim \beta_j d$.

If $\alpha \, \mathcal{P} \, \beta$ then, using (2) and the monotonicity of $\trianglerighteq$, $\gamma_j c \succsim \delta_j d$ implies $\alpha_j c \succsim \beta_j d$. If $\beta \, \mathcal{P} \, \alpha$ then, using (2) and the monotonicity of $\trianglerighteq$, $\alpha_i a \succsim \beta_i b$ implies $\beta_i a \succsim \alpha_i b$. If $\alpha \, \mathcal{I} \, \beta$, then $\beta \, \mathcal{I} \, \alpha$ so that, using (2), $\alpha_i a \succsim \beta_i b$ implies $\beta_i a \succsim \alpha_i b$. The proof for UM2 is similar. □

We are now in position to present the main result of this section.

Theorem 2
Let $\succsim$ be a binary relation on $\mathcal{A}$. Then $\succsim$ is a LD relation iff it is reflexive and satisfies URC1, URC2, UM1 and UM2.

PROOF
Necessity follows from lemma 7 and the definition of a LD relation. We show that if $\succsim$ satisfies URC1 and URC2 and is such that $\sim^{**}$ has at most three distinct equivalence classes then $\succsim$ is a LD relation. In view of lemma 6, this will establish sufficiency.

Define $\mathcal{S}$ letting, for all $\alpha, \beta \in \Gamma$, $\alpha \ \mathcal{S} \ \beta \Leftrightarrow (\alpha, \beta) \succsim^{**} (\beta, \beta)$. By hypothesis, we know that $\succsim^{**}$ is complete and $\succsim$ is independent. It easily follows that $\mathcal{S}$ is complete.

The relation $\succsim^{*}$ being complete, the influence of $i \in N$ implies that there are $\gamma, \delta, \alpha, \beta \in \Gamma$ such that $(\alpha, \beta) \succ^{*} (\gamma, \delta)$. Since $\succsim^{**}$ is complete, this implies $(\alpha, \beta) \succ^{**} (\gamma, \delta)$. If $(\alpha, \beta) \succ^{**} (\beta, \beta)$ then $\alpha \ \mathcal{P} \ \beta$. If not, then $(\beta, \beta) \succsim^{**} (\alpha, \beta)$ so that $(\beta, \beta) \succ^{**} (\gamma, \delta)$ and, using the reversibility of $\succsim^{**}$ and the independence of $\succsim$, $\delta \ \mathcal{P} \ \gamma$. This shows that $\mathcal{P}$ is not empty. This implies that $\succsim^{**}$ has exactly three distinct equivalence classes, since $\alpha \ \mathcal{P} \ \beta \Leftrightarrow (\alpha, \beta) \succ^{**} (\beta, \beta) \Leftrightarrow (\beta, \beta) \succ^{**} (\beta, \alpha)$. Therefore, $\alpha \ \mathcal{P} \ \beta$ iff (α, β) belongs to the first equivalence class of $\succsim^{**}$ and (β, α) to its last equivalence class. Consider any two subsets $A, B \subseteq N$ such that $A \cup B = N$ and let:

$$A \trianglerighteq B \Leftrightarrow [a \succsim b, \text{ for some } a, b \in \mathcal{A} \text{ such that } \mathcal{S}(a, b) = A \text{ and } \mathcal{S}(b, a) = B].$$

If $a \succsim b$ then, by construction, we have $\mathcal{S}(a, b) \trianglerighteq \mathcal{S}(b, a)$. Suppose now that $\mathcal{S}(a, b) \trianglerighteq \mathcal{S}(b, a)$, so that there are $c, b \in \mathcal{A}$ such that $c \succsim d$ and $(c_i, d_i) \sim^{**} (a_i, b_i)$, for all $i \in N$. Using (4), we have $a \succsim b$. Hence (2) holds. The monotonicity of $\trianglerighteq$ easily follows from (3). This completes the proof. $\square$

We have therefore obtained a complete characterization of LD relation within the general framework of model (UM). Conditions UM1 and UM2 implying that $\succsim^{**}$ has at most three distinct equivalence classes appear as the main distinctive characteristic of LD relations. Clearly a binary relation $\succsim$ having a representation in models (SEU) or (SSA) will, in general, have a much richer relation $\succsim^{**}$.

6 Discussion and extensions

The purpose of this paper was twofold. We have first introduced a general axiomatic framework for decision under uncertainty that contains both the SEU and the LD models as particular cases. This model, while tolerating intransitive and/or incomplete preferences, has a simple and intuitive interpretation in terms of preference differences. It is nontrivial unlike, e.g., the general model introduced in Chu and Halpern (2003). We showed that it can be characterized using simple conditions, while avoiding the use of any unnecessary structural assumptions. The second aim of this paper was to put our general framework to work, using it to propose an alternative characterization of the preference relations that can be obtained using the likely dominance rule. This characterization has emphasized the main specific feature of LD relations, i.e. the fact that they use a very poor information concerning preference differences admitting only "positive", "null" and "negative" differences.

6.1 Comparison with Fargier and Perny (1999) and Dubois et al. (2003a)

We compare below our characterization of LD relations with the one proposed in Fargier and Perny (1999); closely related results are found in Dubois et al. (2003a, 2002). Their characterization is based on a condition called "qualitative independence" (and later called "ordinal invariance" in Dubois et al. (2003a, 2002)) that is a slight variant (using a reflexive relation instead of an asymmetric one) of the "noncompensation" condition introduced in Fishburn (1975, 1976, 1978) which, in turn, is a "single profile" analogue of the independence condition used in Arrow's theorem (see Sen, 1986).

Since our definition of LD relations differs from the one used in Fargier and Perny (1999) (they do not impose that $\trianglerighteq$ is necessarily monotonic w.r.t. inclusion) we reformulate their result below. For any $a, b \in \mathcal{A}$, let $R(a, b) = \{i \in N : a_i \succsim_\Gamma b_i\}$.

Definition 5
Let $\succsim$ be a binary relation on $\mathcal{A}$. This relation is said to satisfy monotonic qualitative independence *(MQI) if,*

$$\left.\begin{array}{c} R(a,b) \supseteq R(c,d) \\ \textit{and} \\ R(b,a) \subseteq R(d,c) \end{array}\right\} \Rightarrow [c \succsim d \Rightarrow a \succsim b],$$

for all $a, b, c, d \in \mathcal{A}$.

Condition MQI is strengthens the "qualitative independence" condition used in Fargier and Perny (1999) (this condition is obtained replacing inclusions by equalities in the expression of MQI; as observed in Dubois et al. (2003a, 2002), it is also possible to use instead of MQI the original qualitative independence condition together with a condition imposing that $\succsim$ is monotonic w.r.t. $\succsim_\Gamma$) to include an idea of monotonicity. Condition MQI is a "single profile" analogue of the NIM (i.e., Neutrality, Independence, Monotonicity) condition that is classical in Social Choice Theory (see Sen, 1986, p. 1086).

As shown below, in what is an adaptation of Fargier and Perny (1999, proposition 5), this condition allows for a very simple characterization of LD relations.

Proposition 1
Let $\succsim$ be a binary relation on $\mathcal{A}$. The relation $\succsim$ is a LD relation iff

- *$\succsim$ is reflexive,*
- *$\succsim_\Gamma$ is complete,*
- *$\succsim$ satisfies MQI.*

PROOF
Necessity. Reflexivity holds by definition of a LD relation. That $\succsim_\Gamma$ must be complete follows from part 3 of lemma 2. The necessity of MQI follows from (2), using the monotonicity of $\trianglerighteq$ and part 7 of lemma 1.

Sufficiency. Let $\mathcal{S} = \succsim_\Gamma$. By hypothesis, $\mathcal{S}$ is complete. If $\succ_\Gamma$ is empty, we have $R(a,b) = N$ for all $a, b \in \mathcal{A}$. Using the reflexivity of $\succsim$ and MQI this implies that $a \succsim b$, for all $a, b \in \mathcal{A}$ and, hence, that all states $i \in N$ are degenerate, contrary to our hypothesis. Hence $\succ_\Gamma = \mathcal{P}$ is nonempty.

Let $A, B \subseteq N$ such that $A \cup B = N$. Since $\mathcal{P}$ is nonempty there are $a, b \in \mathcal{A}$ such that $\mathcal{S}(a,b) = A$ and $\mathcal{S}(b,a) = B$. Define $\trianglerighteq$ letting:

$$A \trianglerighteq B \Leftrightarrow [a \succsim b, \text{ for some } a, b \in \mathcal{A} \text{ such that } \mathcal{S}(a,b) = A \text{ and } \mathcal{S}(b,a) = B].$$

If $a \succsim b$ then, by construction, we have $\mathcal{S}(a,b) \trianglerighteq \mathcal{S}(b,a)$. Suppose now that $\mathcal{S}(a,b) \trianglerighteq \mathcal{S}(b,a)$. By construction, there are $c, d \in \mathcal{A}$ such that $c \succsim d$ and $\mathcal{S}(c,d) = A$ and $\mathcal{S}(d,c) = B$. Using MQI, it follows that $a \succsim b$. That $\trianglerighteq$ is monotonic w.r.t. inclusion clearly follows from MQI. □

We refer to Dubois et al. (2002); Fargier and Perny (1999) for a thorough analysis of this result, including a careful comparison of the above conditions with the classical ones used in Savage (1954).

Although proposition 1 offers a simple characterization of LD relations, condition MQI appears at the same time quite strong (this will be apparent if one tries to reformulate MQI in terms of $\succsim$) and wholly specific to LD relations. In our view, the characterization of LD relations within model (UM) proposed above allows to better isolate what appears to be the specific features of LD relations while showing their links with more classical preference relations used in the field of decision under uncertainty.

It should also be stressed that the characterization of LD relations is far from being the only objective of the above-mentioned papers. Rather, their aim is to study the, drastic, consequences of supposing that $\succsim$ is a LD relation and has nice transitivity properties (e.g. $\succ$ being transitive or without circuits). This analysis, that is closely related to Arrow-like theorems in Social Choice Theory (see Campbell and Kelly, 2002; Sen, 1986, for overviews), illuminates the relations between the LD rule, possibility theory and nonmonotonic reasoning. Such an analysis is clearly independent from the path followed to characterize LD relations.

6.2 Extensions

As already mentioned, model (UM) is the specialization to the case of decision making under uncertainty of the conjoint measurement models proposed in Bouyssou and Pirlot

(2002). It is not difficult to see that model (UM) not only allows for intransitive relations $\succsim$ between acts but also for intransitive relation $\succsim_\Gamma$ between outcomes. This may be seen as a limitation of model (UM). Indeed, whereas intransitivities are not unlikely when comparing acts (see Fishburn, 1991), one would expect a much more well behaved relation when it turns to comparing outcomes. We show in this section how to extend our results to cover this case. Before doing so, let us stress that it is quite remarkable that any transitivity hypothesis is unnecessary to obtain a complete characterization of LD relations. As forcefully argued in Saari (1998), this seems to be an essential feature of "ordinal" models.

Adapting the analysis in Bouyssou and Pirlot (2004a) to the case of decision under uncertainty, let us first show that it is possible to specialize model (UM) in order introduce a linear arrangement of the elements of Γ. We consider binary relations $\succsim$ on $\mathcal{A}$ that can be represented as:

$$a \succsim b \Leftrightarrow F(\varphi(u(a_1), u(b_1)), \dots, \varphi(u(a_n), u(b_n))) \geq 0 \qquad \text{(UM*)}$$

where u is a real-valued function on Γ, φ is a real-valued function on $u(\Gamma)^2$ that is skew symmetric, nondecreasing in its first argument (and, therefore, nonincreasing in its second argument) and F is a real-valued function on $\prod_{i=1}^{n} \varphi(u(\Gamma)^2)$ being nondecreasing in all its arguments and such that $F(\mathbf{0}) \geq 0$.

Comparing models (UM*) and (UM), it is clear that (UM*) is the special case of model (UM) in which the function p measuring preference differences between outcomes may be factorised using a function u measuring the "utility" of the outcomes and a skew symmetric function φ measuring preference differences between outcomes on the basis of u. It is easy to see that model (UM*) implies that $\succsim_\Gamma$ is complete and that $\succ_\Gamma$ is transitive. The analysis below will, in fact, show that model (UM*) implies that $\succsim_\Gamma$ is a semiorder.

The analysis of model (UM*) will require the introduction of three new conditions inspired from Bouyssou and Pirlot (2004a).

Definition 6 (Conditions UAC1, UAC2 and UAC3)
We say that $\succsim$ *satisfies:*

$$\textit{UAC1 if} \quad \left.\begin{array}{c} \alpha_i a \succsim b \\ \textit{and} \\ \beta_j c \succsim d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \beta_i a \succsim b \\ \textit{or} \\ \alpha_j c \succsim d, \end{array}\right.$$

$$\textit{UAC2 if} \quad \left.\begin{array}{c} a \succsim \alpha_i b \\ \textit{and} \\ c \succsim \beta_j d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \succsim \beta_i b \\ \textit{or} \\ c \succsim \alpha_j d, \end{array}\right.$$

$$\textit{UAC3 if} \quad \left.\begin{array}{c} a \succsim \alpha_i b \\ \textit{and} \\ \alpha_j c \succsim d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \succsim \beta_i b \\ \textit{or} \\ \beta_j c \succsim d, \end{array}\right.$$

for all $a, b, c, d \in \mathcal{A}$*, all* $i, j \in N$ *and all* $\alpha, \beta \in \Gamma$*.*

Condition UAC1 suggests that the elements of Γ can be linearly ordered considering "upward dominance": if α "upward dominates" β then $\beta_i a \succsim b$ entails $\alpha_i a \succsim b$, for all $a, b \in \mathcal{A}$ and all $i \in N$. Condition UAC2 has a similar interpretation considering now "downward dominance". Condition UAC3 ensures that the linear arrangements of the elements of Γ obtained considering upward and downward dominance are not incompatible. The study of the impact of these new conditions on model (UM) will require an additional definition borrowed from Doignon et al. (1988).

Definition 7 (Linearity)
Let $\mathcal{R}$ be a binary relation on a set X^2. We say that:

- $\mathcal{R}$ *is* right-linear *iff* $[Not[(y,z) \mathcal{R} (x,z)] \Rightarrow (x,w) \mathcal{R} (y,w)]$,
- $\mathcal{R}$ *is* left-linear *iff* $[Not[(z,x) \mathcal{R} (z,y)] \Rightarrow (w,y) \mathcal{R} (w,x)]$,
- $\mathcal{R}$ *is* strongly linear *iff* $[Not[(y,z) \mathcal{R} (x,z)]$ *or* $Not[(z,x) \mathcal{R} (z,y)]] \Rightarrow [(x,w) \mathcal{R} (y,w)$ *and* $(w,y) \mathcal{R} (w,x)]$,

for all $x, y, z, w \in X$.

The impact of our new conditions on the relations $\succsim^*$ and $\succsim^{**}$ comparing preference differences between outcomes are noted below.

Lemma 8

1. *UAC1* $\Leftrightarrow \succsim^*$ *is right-linear,*
2. *UAC2* $\Leftrightarrow \succsim^*$ *is left-linear,*
3. *UAC3* $\Leftrightarrow$ $[[Not[(\alpha,\gamma) \succsim^* (\beta,\gamma)]$ *for some* $\gamma \in \Gamma] \Rightarrow [(\delta,\alpha) \succsim^* (\delta,\beta),$ *for all* $\delta \in \Gamma]]$,
4. $[$*UAC1, UAC2 and UAC3*$] \Leftrightarrow \succsim^*$ *is strongly linear* $\Leftrightarrow \succsim^{**}$ *is strongly linear.*
5. *In the class of reflexive relations satisfying URC1 and URC2, UAC1, UAC2 and UAC3 are independent conditions.*

PROOF
Part 1. $\succsim^*$ is not right-linear iff for some $\alpha, \beta, \gamma, \delta \in \Gamma$, we have $Not[(\gamma,\beta) \succsim^* (\alpha,\beta)]$ and $Not[(\alpha,\delta) \succsim^* (\gamma,\delta)]$. This equivalent to

$$[\alpha_i a \succsim \beta_i b] \text{ and } Not[\gamma_i a \succsim \beta_i b] \text{ and}$$
$$[\gamma_j c \succsim \delta_j d] \text{ and } Not[\alpha_j c \succsim \delta_j d],$$

for some $a, b, c, d \in \mathcal{A}$ and some $i, j \in N$. This is exactly $Not[$UAC1$]$. Parts 2 and 3 are established similarly.

Part 4. The first equivalence is immediate from parts 1 to 3. The second equivalence directly results from the definitions of $\succsim^*$ and $\succsim^{**}$.

Part 5: see examples 8, 9 and 10 in appendix. □

We summarize some useful consequences of model (UM*) in the following:

Lemma 9
Let $\succsim$ be a binary relation on $\mathcal{A}$. If $\succsim$ has a representation in (UM*) *then:*

1. *it satisfies URC1 and URC2,*
2. *it satisfies UAC1, UAC2 and UAC3,*
3. *the binary relation T on Γ defined by $\alpha\ T\ \beta \Leftrightarrow (\alpha, \beta) \succsim^{**} (\alpha, \alpha)$ is a semiorder.*

PROOF
Part 1 follows from the definition of model (UM*) and theorem 1.

Part 2. Suppose that $\alpha_i a \succsim b$ and $\beta_j c \succsim d$. This implies, abusing notation,

$$F([\varphi(u(\alpha), u(b_i))]_i, [\varphi(u(a_k), u(b_k))]_{k \neq i}) \geq 0 \text{ and}$$
$$F([\varphi(u(\beta), u(d_j))]_j, [\varphi(u(c_\ell), u(d_\ell))]_{\ell \neq j}) \geq 0.$$

If $u(\beta) < u(\alpha)$, since φ is nondecreasing in its first argument and F is nondecreasing in all its arguments, we obtain

$$F([\varphi(u(\alpha), u(d_j))]_j, [\varphi(u(c_\ell), u(d_\ell))]_{\ell \neq j}) \geq 0,$$

so that $\alpha_j c \succsim d$. If $u(\beta) \geq u(\alpha)$, since φ is nondecreasing in its first argument and F is nondecreasing in all its arguments, we obtain

$$F([\varphi(u(\beta), u(b_i))]_i, [\varphi(u(a_k), u(b_k))]_{k \neq i}) \geq 0,$$

so that $\beta_i a \succsim b$. Hence, UAC1 holds. The proof is similar for UAC2 and UAC3.

Part 3. Since URC1 and URC2 hold, we know from lemma 4 that $\succsim^{**}$ is complete. It is reversible by construction. From lemma 8, we know that $\succsim^{**}$ is strongly linear. From the proof of theorem 2, we know that T is complete. It remains to show that it is Ferrers and semi-transitive.

[Ferrers]. Suppose that $\alpha\ T\ \beta$ and $\gamma\ T\ \delta$ so that $(\alpha, \beta) \succsim^{**} (\beta, \beta)$ and $(\gamma, \delta) \succsim^{**} (\delta, \delta)$. In contradiction with the thesis, suppose that $Not[\alpha\ T\ \delta]$ and $Not[\gamma\ T\ \beta]$ so that $(\delta, \delta) \succ^{**} (\alpha, \delta)$ and $(\beta, \beta) \succ^{**} (\gamma, \beta)$. Using the fact that $\succsim^{**}$ is a weak order, this implies $(\alpha, \beta) \succ^{**} (\gamma, \beta)$ and $(\gamma, \delta) \succ^{**} (\alpha, \delta)$. This violates the strong linearity of $\succsim^{**}$.

[Semi-transitivity]. Suppose that $\alpha \ T \ \beta$ and $\beta \ T \ \gamma$ so that $(\alpha,\beta) \succsim^{**} (\beta,\beta)$ and $(\beta,\gamma) \succsim^{**} (\gamma,\gamma)$. In contradiction with the thesis, suppose that $Not[\alpha \ T \ \delta]$ and $Not[\delta \ T \ \gamma]$ so that $(\delta,\delta) \succ^{**} (\alpha,\delta)$ and $(\gamma,\gamma) \succ^{**} (\delta,\gamma)$. Using the fact that $\succsim^{**}$ is a reversible weak order, we obtain $(\alpha,\beta) \succ^{**} (\alpha,\delta)$ and $(\beta,\gamma) \succ^{**} (\delta,\gamma)$. This violates the strong linearity of $\succsim^{**}$. Hence, T is semi-transitive. □

The conditions introduced so far allow us to characterize model (UM*) when Γ and, hence, $\mathcal{A}$, is at most denumerable.

Theorem 3
Suppose that Γ is finite or countably infinite and let $\succsim$ be a binary relation on $\mathcal{A}$. Then $\succsim$ has a representation (UM*) *iff it is reflexive and satisfies URC1, URC2, UAC1, UAC2 and UAC3.*

PROOF
Necessity results from lemmas 2, 5 and 9. The proof of sufficiency rests on the following claim proved in Bouyssou and Pirlot (2004a, Proposition 2).

CLAIM Let $\mathcal{R}$ be a weak order on a finite or countably infinite set X^2. There is a real-valued function u on X and a real-valued function φ on $u(X)^2$ being nondecreasing in its first argument and nonincreasing in its second argument, such that, for all $x, y, z, w \in X$,

$$(x,y) \ \mathcal{R} \ (z,w) \Leftrightarrow \varphi(u(x),u(y)) \geq \varphi(u(z),u(w))$$

iff $\mathcal{R}$ is strongly linear. In addition, the function φ can be chosen to be skew-symmetric iff $\mathcal{R}$ is reversible.

Sufficiency follows from combining theorem 1 with lemma 8 and the above claim. □

Remark 3
The above result can be extended without much difficulty to sets of arbitrary cardinality. Note however that, contrary to theorem 1, theorem 3 is only stated here for finite or countably infinite sets $\mathcal{A}$. This is no mistake. In fact, as shown in Fishburn (1973, Theorem A(ii)), it may well happen that $\mathcal{R}$ is a strongly linear weak order on X^2, that the set of equivalence classes induced by $\mathcal{R}$ is finite or countably infinite while the above claim fails. •

We now use the framework of model (UM*) to analyze LD relations in which $\mathbb{S}$ is a semiorder. Let us first show that all such relations have a representation in model (UM*).

Lemma 10
Let $\succsim$ be a binary relation on $\mathcal{A}$. If $\succsim$ is a LD relation with a representation $\langle \trianglerighteq, \mathbb{S} \rangle$ in which $\mathbb{S}$ is a semiorder then $\succsim$ satisfies UAC1, UAC2 and UAC3.

PROOF
[UAC1]. Suppose that $\alpha_i a \succsim b$ and $\beta_j c \succsim d$. We want to show that either $\beta_i a \succsim b$ or $\alpha_j c \succsim d$.

If $b_i \mathcal{P} \alpha$ or $d_j \mathcal{P} \beta$, the conclusion follows from the monotonicity of $\trianglerighteq$.

If $\alpha \mathcal{P} b_i$ and $\beta \mathcal{P} d_j$, we have, using the fact that $\mathcal{P}$ is Ferrers, $\alpha \mathcal{P} d_j$ or $\beta \mathcal{P} b_i$. In either case the desired conclusion follows using the fact that $\succsim$ is a LD relation.

This leaves three exclusive cases: $[\alpha \mathcal{I} b_i$ and $\beta \mathcal{P} d_j]$ or $[\alpha \mathcal{P} b_i$ and $\beta \mathcal{I} d_j]$, or $[\alpha \mathcal{I} b_i$ and $\beta \mathcal{I} d_j]$. Using Ferrers, either case implies $\alpha \mathcal{S} d_j$ or $\beta \mathcal{S} b_i$. If either $\alpha \mathcal{P} d_j$ or $\beta \mathcal{P} b_i$, the desired conclusion follows from monotonicity. Suppose therefore that $\alpha \mathcal{I} d_j$ and $\beta \mathcal{I} b_i$. Since we have either $\alpha \mathcal{I} b_i$ or $\beta \mathcal{I} d_j$, the conclusion follows using the fact that $\succsim$ is a LD relation.

Hence UAC1 holds. The proof for UAC2 is similar, using Ferrers.

[UAC3]. Suppose that $a \succsim \alpha_i b$ and $\alpha_j c \succsim d$. We want to show that either $a \succsim \beta_i b$ or $\beta_j c \succsim d$.

If either $\alpha \mathcal{P} a_i$ or $d_j \mathcal{P} \alpha$, the conclusion follows from monotonicity.

If $a_i \mathcal{P} \alpha$ and $\alpha \mathcal{P} d_j$, then semi-transitivity implies $a_i \mathcal{P} \beta$ or $\beta \mathcal{P} d_j$. In either case, the conclusion follows from monotonicity.

This leaves three exclusive cases: $[a_i \mathcal{I} \alpha$ and $\alpha \mathcal{P} d_j]$ or $[a_i \mathcal{P} \alpha$ and $\alpha \mathcal{I} d_j]$ or $[a_i \mathcal{I} \alpha$ and $\alpha \mathcal{I} d_j]$. In either case, semi-transitivity implies $a_i \mathcal{S} \beta$ or $\beta \mathcal{S} d_j$. If either $a_i \mathcal{P} \beta$ or $\beta \mathcal{P} d_j$. the desired conclusion follows from monotonicity. Suppose therefore that $a_i \mathcal{I} \beta$ or $\beta \mathcal{I} d_j$. Since in each of the remaining cases we have either $a_i \mathcal{I} \alpha$ or $\alpha \mathcal{I} d_j$, the conclusion follows because $\succsim$ is a LD relation. □

Although lemma 8 shows that in the class of reflexive binary relations satisfying URC1 and URC2, UAC1, UAC2 and UAC3 are independent conditions, the situation is more delicate when we bring conditions UM1 and UM2 into the picture since they impose strong requirements on $\succsim^*$ and $\succsim^{**}$. We have:

Lemma 11

1. *Let $\succsim$ be a reflexive binary relation on $\mathcal{A}$ satisfying URC1, URC2, UM1 and UM2. Then $\succsim$ satisfies UAC1 iff it satisfies UAC2.*

2. *In the class of reflexive binary relations satisfying URC1, URC2, UM1 and UM2, conditions UAC1 and UAC3 are independent.*

PROOF
Part 1. The proof uses the following claim.

CLAIM When URC1, URC2, UM1 and UM2 hold then we have one of the following:

1. $(\alpha,\beta) \succ^* (\beta,\beta) \succ^* (\beta,\alpha)$, for all $\alpha,\beta \in \Gamma$ such that $(\alpha,\beta) \succ^{**} (\beta,\beta)$,
2. $(\alpha,\beta) \succ^* (\beta,\beta)$ and $(\beta,\beta) \sim^* (\beta,\alpha)$, for all $\alpha,\beta \in \Gamma$ such that $(\alpha,\beta) \succ^{**} (\beta,\beta)$,
3. $(\alpha,\beta) \sim^* (\beta,\beta)$ and $(\beta,\beta) \succ^* (\beta,\alpha)$, for all $\alpha,\beta \in \Gamma$ such that $(\alpha,\beta) \succ^{**} (\beta,\beta)$,

PROOF OF THE CLAIM
Using part 3 of lemma 4 and part 8 of lemma 8, we know that $\succsim^{**}$ is a weak order having at most three distinct equivalence classes. Let $\alpha,\beta \in \Gamma$ be such that $(\alpha,\beta) \succ^{**} (\beta,\beta)$. By construction, we have either $(\alpha,\beta) \succ^* (\beta,\beta)$ or $(\beta,\beta) \succ^* (\beta,\alpha)$. There are three cases to examine.

1. Suppose first that $(\alpha,\beta) \succ^* (\beta,\beta)$ and $(\beta,\beta) \succ^* (\beta,\alpha)$. Consider $\gamma,\delta \in \Gamma$ such that $(\gamma,\delta) \succ^{**} (\delta,\delta)$. If either $(\gamma,\delta) \sim^* (\delta,\delta)$ or $(\delta,\gamma) \sim^* (\delta,\delta)$, it is easy to see, using the independence of $\succsim$ and the definition of $\succsim^{**}$, that we must have:

   $$(\alpha,\beta) \succ^{**} (\gamma,\delta) \succ^{**} (\beta,\beta) \succ^{**} (\delta,\gamma) \succ^{**} (\beta,\alpha),$$

   violating the fact that $\sim^{**}$ has at most three distinct equivalence classes. Hence we have, for all $\gamma,\delta \in \Gamma$ such that $(\gamma,\delta) \succ^{**} (\delta,\delta)$, $(\gamma,\delta) \succ^* (\delta,\delta)$ and $(\delta,\delta) \succ^* (\delta,\gamma)$.

2. Suppose that $(\alpha,\beta) \succ^* (\beta,\beta)$ and $(\beta,\beta) \sim^* (\beta,\alpha)$ and consider any $\gamma,\delta \in \Gamma$ such that $(\gamma,\delta) \succ^{**} (\delta,\delta)$. If $(\gamma,\delta) \succ^* (\delta,\delta)$ and $(\delta,\delta) \succ^* (\delta,\gamma)$, we have, using the independence of $\succsim$ and the definition of $\succsim^{**}$,

   $$(\gamma,\delta) \succ^{**} (\alpha,\beta) \succ^{**} (\beta,\beta) \succ^{**} (\beta,\alpha) \succ^{**} (\delta,\gamma),$$

   violating the fact that $\sim^{**}$ has at most three distinct equivalence classes. If $(\gamma,\delta) \sim^* (\delta,\delta)$ and $(\delta,\delta) \succ^* (\delta,\gamma)$, then URC2 is violated since we have $(\alpha,\beta) \succ^* (\gamma,\delta)$ and $(\beta,\alpha) \succ^* (\delta,\gamma)$. Hence, it must be true that $(\gamma,\delta) \succ^{**} (\delta,\delta)$ implies $(\gamma,\delta) \succ^* (\delta,\delta)$ and $(\delta,\delta) \sim^* (\delta,\gamma)$.

3. Suppose that $(\alpha,\beta) \sim^* (\beta,\beta)$ and $(\beta,\beta) \succ^* (\beta,\alpha)$ and consider any $\gamma,\delta \in \Gamma$ such that $(\gamma,\delta) \succ^{**} (\delta,\delta)$. If $(\gamma,\delta) \succ^* (\delta,\delta)$ and $(\delta,\delta) \succ^* (\delta,\gamma)$, we have, using the independence of $\succsim$ and the definition of $\succsim^{**}$,

   $$(\gamma,\delta) \succ^{**} (\alpha,\beta) \succ^{**} (\beta,\beta) \succ^{**} (\beta,\alpha) \succ^{**} (\delta,\gamma),$$

   violating the fact that $\sim^{**}$ has at most three distinct equivalence classes. If $(\gamma,\delta) \succ^* (\delta,\delta)$ and $(\delta,\delta) \sim^* (\delta,\gamma)$, then URC2 is violated since we have $(\gamma,\delta) \succ^* (\alpha,\beta)$ and $(\delta,\gamma) \succ^* (\beta,\alpha)$. Hence, it must be true that $(\gamma,\delta) \succ^{**} (\delta,\delta)$ implies $(\gamma,\delta) \sim^* (\delta,\delta)$ and $(\delta,\delta) \succ^* (\delta,\gamma)$.

This proves the claim.

We prove that UAC1 $\Rightarrow$ UAC2, the proof of the reverse implication being similar. Suppose UAC2 is violated so that, for some $a, b, c, d \in \mathcal{A}$ and some $\alpha, \beta \in \Gamma$, we have $a \succsim \alpha_i b\ c \succsim \beta_j d$, $Not[a \succsim \beta_i b]$, $Not[c \succsim \alpha_j d]$.

This implies $(\alpha, \beta) \succ^* (\alpha, \delta)$ and $(\gamma, \delta) \succ^* (\gamma, \beta)$, so that $(\alpha, \beta) \succ^{**} (\alpha, \delta)$ and $(\gamma, \delta) \succ^{**} (\gamma, \beta)$.

Because, URC1, URC2, UM1 and UM2 hold, we know that we must be in one of the cases of the above claim.

If either of the last two cases hold, $\succsim^*$ has at most two distinct equivalence classes, so that $(\alpha, \beta) \sim^* (\gamma, \delta)$ and $(\alpha, \delta) \sim^* (\gamma, \beta)$. This implies $(\gamma, \delta) \succ^* (\alpha, \delta)$ and $(\alpha, \beta) \succ^* (\gamma, \beta)$. Since UAC1 implies the right-linearity of $\succsim^*$, $(\gamma, \delta) \succ^* (\alpha, \delta)$ implies $(\gamma, \beta) \succsim^* (\alpha, \beta)$, a contradiction.

Suppose that the first case holds true. We distinguish several subcases.

1. If both (α, β) and (γ, δ) belong to the middle equivalence class of $\succsim^*$, we have $[(\alpha, \beta) \sim^* (\gamma, \delta)] \succ^* [(\alpha, \delta) \sim^* (\gamma, \beta)]$. As shown above, this leads to a contradiction.

2. Suppose that both (α, β) and (γ, δ) belong to the first equivalence class of $\succsim^*$. We therefore have $(\alpha, \beta) \sim^* (\gamma, \delta)$, $(\alpha, \beta) \succ^* (\alpha, \delta)$ and $(\gamma, \delta) \succ^* (\gamma, \beta)$. This implies $(\alpha, \beta) \succ^* (\gamma, \beta)$. Using UAC1, we have $(\alpha, \delta) \succsim^* (\gamma, \delta)$, a contradiction.

3. Suppose that (α, β) belongs to the first equivalence class of $\succsim^*$ and (γ, δ) belongs to the central class of $\succsim^*$. This implies, using the reversibility of $\succsim^{**}$ and the fact that it has at most three equivalence classes, $[(\alpha, \beta) \sim^* (\beta, \gamma)] \succ^* [(\gamma, \delta) \sim^* (\delta, \gamma)] \succ^* [(\gamma, \beta) \sim^* (\beta, \alpha)]$. Hence, we have $(\beta, \gamma) \succ^* (\delta, \gamma)$ and using UAC1, we have $(\beta, \alpha) \succsim^* (\delta, \alpha)$, a contradiction.

Part 2: see examples 11 and 12 in appendix □

This leads to a characterization of LD relations in which $\mathcal{S}$ is a semiorder.

Theorem 4
Let $\succsim$ be a binary relation on $\mathcal{A}$. Then $\succsim$ is a LD relation having a representation $\langle \trianglerighteq, \mathcal{S} \rangle$ in which $\mathcal{S}$ is a semiorder iff it is reflexive and satisfies URC2, UM1, UM2, UAC1 and UAC3.

PROOF
The proof of theorem 4 follows from combining lemmas 9, 10 and 11 with the results in section 5. □

Let us finally mention that in our definition of LD relations in section 3, the only remarkable property imposed on $\trianglerighteq$ is monotonicity w.r.t. inclusion. In most instances, we would expect $\trianglerighteq$ to be transitive as well. It is easy to devise conditions that imply the transitivity of $\trianglerighteq$. We leave the details to the interested reader.

Appendices

A Examples related to model (UM)

Example 2 (URC2, *Not*[URC1])
Let $\Gamma = \{\alpha, \beta, \gamma\}$ and $N = \{1, 2\}$. Let $\succsim$ on $\mathcal{A}$ identical to $\mathcal{A}^2$ except that, using obvious notation, $Not[\alpha_1\gamma_2 \succsim \beta_1\alpha_2]$ and $Not[\gamma_1\alpha_2 \succsim \alpha_1\beta_2]$.

It is easy to see that $\succsim$ is complete (and, hence, reflexive). It violates URC1 since $\alpha_1\alpha_2 \succsim \beta_1\beta_2$ and $\gamma_1\gamma_2 \succsim \alpha_1\alpha_2$ but neither $\alpha_1\gamma_2 \succsim \beta_1\alpha_2$ nor $\gamma_1\alpha_2 \succsim \alpha_1\beta_2$.

It is not difficult to check that we have:

- $[(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\alpha, \gamma), (\beta, \alpha), (\beta, \gamma), (\gamma, \beta)] \succ^* (\alpha, \beta)$ and
- $[(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\alpha, \gamma), (\beta, \alpha), (\beta, \gamma), (\gamma, \beta)] \succ^* (\gamma, \alpha)$,

while (α, β) and (γ, α) are incomparable in terms of $\succsim^*$. Using part 2 of lemma 4, it is easy to check that $\succsim$ satisfies URC2. $\diamond$

Example 3 (URC1, *Not*[URC2])
Let $\Gamma = \{\alpha, \beta\}$ and $N = \{1, 2\}$. Let $\succsim$ on $\mathcal{A}$ be such that:

$$a \succsim b \Leftrightarrow p(a_1, b_1) + p(a_2, b_2) \geq 0,$$

where p is a real valued function on Γ^2 defined by the following table (to be read from line to column):

p	α	β
α	0	-1
β	1	1

It is easy to see that $\succsim$ is complete (and hence, reflexive) and satisfies URC1 (we have: $[(\beta, \beta) \sim^* (\beta, \alpha)] \succ^* (\alpha, \alpha) \succ^* (\alpha, \beta)$). The relation $\succsim$ is not independent since $\beta_1\alpha_2 \succsim \beta_1\beta_2$ but $Not[\alpha_1\alpha_2) \succsim \alpha_1\beta_2]$. Hence, URC2 is violated in view of part 5 of lemma 4. $\diamond$

B Examples related to LD relations

Example 4 (URC1, URC2, UM2, Not[UM1])
Let $\Gamma = \{\alpha, \beta, \gamma\}$ and $N = \{1, 2\}$. Let $\succsim$ on $\mathcal{A}$ be such that:

$$a \succsim b \Leftrightarrow p_1(a_1, b_1) + p_2(a_2, b_2) \geq 0,$$

where p_1 and p_2 are real valued functions on Γ^2 defined by the following table:

p_1	α	β	γ
α	0	4	0
β	0	0	0
γ	0	0	0

p_2	α	β	γ
α	0	0	0
β	-3	0	0
γ	-3	-3	0

The relation $\succsim$ is clearly complete. It is not difficult to see that $\succsim^*$ is such that:

$$(\alpha, \beta) \succ^* [(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\alpha, \gamma), (\beta, \gamma)] \succ^* [(\beta, \alpha), (\gamma, \alpha), (\gamma, \beta)].$$

This shows, in view of lemma 4, that URC1 and URC2 are satisfied. It is easy to check that (7) holds, so that the same is true for UM2. We have $(\alpha, \gamma) \succ^* (\gamma, \alpha)$ but $Not[(\alpha, \gamma) \succsim^* (\alpha, \beta)]$. This shows that (6) is violated. Since URC2 holds, this shows that UM1 is violated in view of part 5 of lemma 6. ◇

Example 5 (URC1, URC2, UM1, Not[UM2])
Let $\Gamma = \{\alpha, \beta, \gamma\}$ and $N = \{1, 2\}$. Let $\succsim$ on $\mathcal{A}$ be such that:

$$a \succsim b \Leftrightarrow g(p_1(a_1, b_1) + p_2(a_2, b_2)) \geq 0,$$

where p_1 and p_2 are real valued functions on Γ^2 defined by the following table:

p_1	α	β	γ
α	0	2	2
β	-2	0	2
γ	-4	-2	0

p_2	α	β	γ
α	0	0	0
β	-2	0	0
γ	-2	-2	0

and g is such that:

$$g(x) = \begin{cases} x \text{ if } |x| > 2, \\ 0 \text{ otherwise.} \end{cases}$$

The relation $\succsim$ is clearly complete. It is not difficult to see that $\succsim^*$ is such that:

$$[(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\alpha, \beta), (\alpha, \gamma), (\beta, \gamma)] \succ^* [(\beta, \alpha), (\gamma, \beta)] \succ^* (\gamma, \alpha).$$

This shows, in view of lemma 4, that URC1 and URC2 are satisfied. It is easy to check that (6) holds, so that the same is true for UM1. We have $(\alpha, \beta) \succ^* (\beta, \alpha)$ but $Not[(\gamma, \alpha) \succsim^* (\beta, \alpha)]$. This shows that (7) is violated. Since URC1 holds, this shows that UM2 is violated in view of part 6 of lemma 6. ◇

Example 6 (URC1, UM1, UM2, Not[URC2])
Let $\Gamma = \{\alpha, \beta\}$ and $N = \{1, 2\}$. Let $\succsim$ on $\mathcal{A}$ be identical $\mathcal{A}^2$ except that $Not[\beta_1\beta_2 \succsim \alpha_1\alpha_2]$ and $Not[\beta_1\beta_2 \succsim \alpha_1\beta_2]$. This relation is clearly complete. It is not independent, so that URC2 is violated in view of lemma 4. We have: $[(\alpha, \alpha), (\alpha, \beta)] \succ^* (\beta, \beta) \succ^* (\beta, \alpha)$. Since $\succsim^*$ is complete, URC1 holds. In view of parts 1 and 2 of lemma 6, we know that (6) and (7) hold. Hence, UM1 and UM2 hold. ◇

Example 7 (URC2, UM1, UM2, Not[URC1])
Let $\Gamma = \{\alpha, \beta, \gamma\}$ and $N = \{1, 2, 3\}$. Let $\succsim$ on $\mathcal{A}$ be identical to $\mathcal{A}^2$ except that the following 25 relations are missing: $\alpha_1\alpha_2\alpha_3 \succsim \gamma_1\alpha_2\gamma_3$, $\alpha_1\alpha_2\alpha_3 \succsim \gamma_1\beta_2\gamma_3$, $\alpha_1\alpha_2\alpha_3 \succsim \gamma_1\gamma_2\gamma_3$, $\alpha_1\beta_2\alpha_3 \succsim \alpha_1\alpha_2\gamma_3$, $\alpha_1\beta_2\alpha_3 \succsim \beta_1\alpha_2\gamma_3$, $\alpha_1\beta_2\alpha_3 \succsim \gamma_1\alpha_2\gamma_3$, $\alpha_1\beta_2\alpha_3 \succsim \gamma_1\beta_2\gamma_3$, $\alpha_1\beta_2\alpha_3 \succsim \gamma_1\gamma_2\gamma_3$, $\alpha_1\gamma_2\alpha_3 \succsim \gamma_1\alpha_2\gamma_3$, $\alpha_1\gamma_2\alpha_3 \succsim \gamma_1\beta_2\gamma_3$, $\alpha_1\gamma_2\alpha_3 \succsim \gamma_1\gamma_2\gamma_3$, $\beta_1\beta_2\alpha_3 \succsim \alpha_1\alpha_2\alpha_3$, $\beta_1\beta_2\alpha_3 \succsim \alpha_1\alpha_2\beta_3$, $\beta_1\beta_2\alpha_3 \succsim \alpha_1\alpha_2\gamma_3$, $\beta_1\beta_2\alpha_3 \succsim \beta_1\alpha_2\gamma_3$, $\beta_1\beta_2\alpha_3 \succsim \gamma_1\alpha_2\gamma_3$, $\beta_1\beta_2\beta_3 \succsim \alpha_1\alpha_2\alpha_3$, $\beta_1\beta_2\beta_3 \succsim \alpha_1\alpha_2\beta_3$, $\beta_1\beta_2\beta_3 \succsim \alpha_1\alpha_2\gamma_3$, $\beta_1\beta_2\gamma_3 \succsim \alpha_1\alpha_2\alpha_3$, $\beta_1\beta_2\gamma_3 \succsim \alpha_1\alpha_2\beta_3$, $\beta_1\beta_2\gamma_3 \succsim \alpha_1\alpha_2\gamma_3$, $\gamma_1\beta_2\alpha_3 \succsim \alpha_1\alpha_2\gamma_3$, $\gamma_1\beta_2\alpha_3 \succsim \beta_1\alpha_2\gamma_3$ and $\gamma_1\beta_2\alpha_3 \succsim \gamma_1\alpha_2\gamma_3$.

It is not difficult to check that $\succsim$ is complete. We have:

$$[(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\beta, \gamma), (\gamma, \alpha), (\gamma, \beta), (\gamma, \beta)] \succ^* (\alpha, \gamma) \text{ and}$$
$$[(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\beta, \gamma), (\gamma, \alpha), (\gamma, \beta), (\gamma, \beta)] \succ^* (\beta, \alpha),$$

while (α, γ) and (β, α) are nor comparable in terms of $\succsim^*$. This shows that URC1 is violated. Using part 2 of lemma 4, it is easy to check that URC2 holds. Using part 1 of lemma 6, it is easy to check that (6) holds. In view of part 3 of lemma 6, this shows that UM1 is satisfied. It remains to check that UM2 holds.

It is not difficult to check that $\beta_2 a \succsim \alpha_2 b$ implies $\tau_2 a \succsim \sigma_2 b$, for all $a, b \in \mathcal{A}$ and all $(\tau, \sigma) \in \Gamma^2$. Furthermore, for all $(\tau, \sigma), (\chi, \psi) \in \Gamma^2 \setminus (\beta, \alpha)$, $\chi_2 a \succsim \psi_2 b \Leftrightarrow \tau_2 a \succsim \sigma_2 b$. Similarly, it is easy to check that $\alpha_3 a \succsim \gamma_3 b$ implies $\tau_3 a \succsim \sigma_3 b$, for all $a, b \in \mathcal{A}$ and all $(\tau, \sigma) \in \Gamma^2$. Furthermore, for all $(\tau, \sigma), (\chi, \psi) \in \Gamma^2 \setminus (\alpha, \gamma)$, $\chi_3 a \succsim \psi_3 b \Leftrightarrow \tau_3 a \succsim \sigma_3 b$.

The two premises of UM2 are that $\tau_i a \succsim \sigma_i b$ and $\sigma_j c \succsim \tau_j d$. The three possible conclusions of UM2 are that $\sigma_i a \succsim \tau_i b$ or $\chi_i a \succsim \psi_i b$ or $\chi_j c \succsim \psi_j d$.

Suppose first that (τ, σ) is distinct from (γ, α) and (α, β). In this case, we know that $(\sigma, \tau) \succsim^* (\tau, \sigma)$, so that $\tau_i a \succsim \sigma_i b$ implies $\sigma_i a \succsim \tau_i b$. Hence, the first conclusion of UM2 will hold.

Suppose henceforth that $(\tau, \sigma) = (\gamma, \alpha)$. If $i = 2$, we know that $\gamma_2 a \succsim \alpha_2 b \Leftrightarrow \alpha_2 a \succsim \gamma_2 b$, so that the first conclusion of UM2 will hold.

Suppose that $i = 3$. If $j = 3$, the second premise of UM2 becomes $\alpha_3 c \succsim \gamma_3 d$. This implies $\gamma_3 c \succsim \alpha_3 d$ so that the last conclusion of UM2 will hold. A similar reasoning shows that the last conclusion of UM2 will hold if $j = 1$. Suppose that $j = 2$. The two premises of UM2 are that $\gamma_3 a \succsim \alpha_3 b$ and $\alpha_2 c \succsim \gamma_2 d$. The three desired conclusions are that either $\gamma_3 a \succsim \alpha_3 b$ or $\chi_3 a \succsim \psi_3 b$ or $\chi_2 c \succsim \psi_2 d$. If (χ, ψ) is distinct from (β, α), we know that $\alpha_2 c \succsim \gamma_2 d \Leftrightarrow \chi_2 c \succsim \psi_2 d$ so that the last conclusion of UM2 will hold. Now if $(\chi, \psi) = (\beta, \alpha)$, we have that $\beta_3 a \succsim \alpha_3 b$ so that the second conclusion of UM2 holds.

Suppose that $i = 1$. If (χ, ψ) is distinct from (β, α), $\gamma_1 a \succsim \alpha_1 b$ will imply $\chi_1 a \succsim \psi_1 b$, so that the second conclusion of UM2 will hold. If $(\chi, \psi) = (\beta, \alpha)$, it is easy to check that there is no $a, b \in \mathcal{A}$ such that $\gamma_1 a \succsim \alpha_1 b$, $Not[\alpha_1 a \succsim \gamma_1 b]$ and $Not[\beta_1 a \succsim \alpha_1 b]$. This shows that UM2 cannot be violated.

Hence, we have shown that UM2 holds if $(\tau, \sigma) = (\gamma, \alpha)$. A similar reasoning shows that UM2 holds if $(\tau, \sigma) = (\alpha, \beta)$. $\diamond$

C Examples related to model (UM*)

Throughout the remaining examples, we use the following notation:

$$\alpha \succsim^{\pm} \beta \Leftrightarrow [(\alpha, \gamma) \succsim^{*} (\beta, \gamma) \text{ and } (\delta, \beta) \succsim^{*} (\delta, \alpha), \forall \gamma, \delta \in \Gamma],$$
$$\alpha \succsim^{+} \beta \Leftrightarrow [(\alpha, \gamma) \succsim^{*} (\beta, \gamma), \forall \gamma \in \Gamma],$$
$$\alpha \succsim^{-} \beta \Leftrightarrow [(\delta, \beta) \succsim^{*} (\delta, \alpha), \forall \delta \in \Gamma].$$

The reader will easily check that:

$$\text{UAC1} \Leftrightarrow \succsim^{+} \text{ is complete},$$
$$\text{UAC2} \Leftrightarrow \succsim^{-} \text{ is complete},$$
$$\text{UAC3} \Leftrightarrow [\alpha \succ^{+} \beta \Rightarrow Not[\beta \succ^{-} \alpha]].$$

It is also interesting to note that:

$$\alpha \succsim^{+} \beta \Leftrightarrow [\beta_i c \succsim d \Rightarrow \alpha_i c \succsim d, \forall c, d \in \mathcal{A}],$$
$$\alpha \succsim^{-} \beta \Leftrightarrow [d \succsim \alpha_i c \Rightarrow d \succsim \beta_i c, \forall c, d \in \mathcal{A}],$$
$$\alpha \succsim^{\pm} \beta \Leftrightarrow [\alpha \succsim^{+} \beta \text{ and } \alpha \succsim^{-} \beta].$$

Example 8 (URC1, URC2, UAC2, UAC3, Not[UAC1])
Let $\Gamma = \{\alpha, \beta, \gamma, \delta\}$ and $N = \{1, 2\}$. Let $\succsim$ on $\mathcal{A}$ be such that:

$$a \succsim b \Leftrightarrow g(p(a_1, b_1) + p(a_2, b_2)) \geq 0,$$

where p is a real valued function on Γ^2 defined by the following table:

p	α	β	γ	δ
α	0	-3	-1	2
β	3	0	1	2
γ	1	-1	0	2
δ	-2	-2	-2	0

and g is such that:

$$g(x) = \begin{cases} x \text{ if } |x| > 2, \\ 0 \text{ otherwise.} \end{cases}$$

The relation $\succsim$ is clearly complete and satisfies URC1 and URC2. It is not difficult to check that we have:

$$\beta \succ^- \gamma \succ^- \alpha \succ^- \delta.$$

We have $\beta \succ^+ \gamma$, $\gamma \succ^+ \alpha$ and $\gamma \succ^+ \delta$ but neither $\alpha \succsim^+ \delta$ (because $\delta_1\alpha_2 \succsim \beta_1\alpha_2$ but $Not[\alpha_1\alpha_2 \succsim \beta_1\alpha_2]$) nor $\delta \succsim^+ \alpha$ (because $\alpha_1\alpha_2 \succsim \alpha_1\gamma_2$ but $Not[\delta_1\alpha_2 \succsim \alpha_1\gamma_2]$). This shows that UAC2 and UAC3 hold but that UAC1 is violated. ◇

Example 9 (URC1, URC2, UAC1, UAC3, Not[UAC2])
Let $\Gamma = \{\alpha, \beta, \gamma, \delta\}$ and $N = \{1, 2\}$. Let $\succsim$ on $\mathcal{A}$ be such that:

$$a \succsim b \Leftrightarrow g(p(a_1, b_1) + p(a_2, b_2)) \geq 0,$$

where p is a real valued function on Γ^2 defined by the following table:

p	α	β	γ	δ
α	0	3	1	-2
β	-3	0	-1	-2
γ	-1	1	0	-2
δ	2	2	2	0

and g is as in example 8.

The relation $\succsim$ is clearly complete and satisfies URC1 and URC2. Observe that p is defined via the transposition of the table used in example 8. This interchanges the roles of UAC1 and UAC2. In fact it is not difficult to see that we have:

$$\delta \succ^+ \alpha \succ^+ \gamma \succ^+ \beta.$$

We have: $\delta \succ^- \gamma$, $\alpha \succ^- \gamma$, $\gamma \succ^- \beta$ but neither $\alpha \succsim^- \delta$ nor $\delta \succsim^- \alpha$. This shows that UAC1 and UAC3 hold but that UAC2 is violated. ◇

Example 10 (URC1, URC2, UAC1, UAC2, Not[UAC3])
Let $\Gamma = \{\alpha, \beta, \gamma, \delta\}$ and $N = \{1, 2\}$. Let $\succsim$ on $\mathcal{A}$ be such that:

$$a \succsim b \Leftrightarrow g(p(a_1, b_1) + p(a_2, b_2)) \geq 0,$$

where p is a real valued function on Γ^2 defined by the following table:

p	α	β	γ	δ
α	0	-5	0	-2
β	5	0	1	2
γ	0	-1	0	0
δ	2	-2	0	0

and g is as in example 8.

The relation $\succsim$ is clearly complete and satisfies URC1 and URC2. We have:

$$\beta \succ^+ \gamma \succ^+ \delta \succ^+ \alpha \text{ and}$$
$$\beta \succ^- \delta \succ^- \gamma \succ^- \alpha.$$

This shows that UAC1 and UAC2 hold but that UAC3 is violated since $\gamma \succ^+ \delta$ but $\delta \succ^- \gamma$. $\diamond$

D Examples related to LD relations in which $\mathcal{S}$ is a semiorder

Example 11 (URC1, URC2, UM1, UM2, UAC1, UAC2, Not[UAC3])
Let $\Gamma = \{\alpha, \beta, \gamma, \delta\}$ and $N = \{1, 2\}$. Let $\succsim$ on $\mathcal{A}$ be such that:

$$a \succsim b \Leftrightarrow g(p(a_1, b_1) + p(a_2, b_2)) \geq 0,$$

where p is a real valued function on Γ^2 defined by the following table:

p	α	β	γ	δ
α	0	-2	0	-2
β	2	0	0	2
γ	0	0	0	0
δ	2	-2	0	0

and g is as in example 8.

The relation $\succsim$ is clearly complete and satisfies URC1 and URC2. Since p takes 3 distinct values, it is easy to see that UM1 and UM2 holds. We have:

$$[\beta, \gamma] \succ^+ \delta \succ^+ \alpha \text{ and}$$
$$\beta \succ^- \delta \succ^- [\gamma, \alpha].$$

This shows that UAC1 and UAC2 hold but that UAC3 is violated since $\gamma \succ^+ \delta$ but $\delta \succ^- \gamma$. ◇

Example 12 (URC1, URC2, UM1, UM2, UAC3, Not[UAC1], Not[UAC2])
Let $\Gamma = \{\alpha, \beta, \gamma, \delta\}$ and $N = \{1, 2\}$. Let $\succsim$ on $\mathcal{A}$ be such that:

$$a \succsim b \Leftrightarrow g(p(a_1, b_1) + p(a_2, b_2)) \geq 0,$$

where p is a real valued function on Γ^2 defined by the following table:

p	α	β	γ	δ
α	0	-2	-2	2
β	2	0	0	0
γ	2	0	0	2
δ	-2	0	-2	0

and g is as in example 8.

The relation $\succsim$ is clearly complete and satisfies URC1 and URC2. Since p takes 3 distinct values, it is easy to see that UM1 and UM2 holds. It is easy to see that: $\beta \sim^+ \gamma$, $\beta \succ^+ \alpha$, $\beta \succ^+ \delta$, $\gamma \succ^+ \alpha$, $\gamma \succ^+ \delta$, but neither $\alpha \succsim^+ \delta$ nor $\delta \succsim^+ \alpha$. Similarly we obtain: $\gamma \succ^- \alpha$, $\gamma \succ^- \beta$, $\gamma \succ^- \delta$, $\alpha \succ^- \delta$, $\beta \succ^- \delta$ but neither $\alpha \succsim^- \beta$ nor $\beta \succsim^- \alpha$. Hence UAC3 holds but UAC1 and UAC2 are violated. ◇

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Multiagent Resource Allocation with k-additive Utility Functions

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Abstract

We briefly review previous work on the *welfare engineering* framework where autonomous software agents negotiate on the allocation of a number of discrete resources, and point out connections to combinatorial optimisation problems, including combinatorial auctions, that shed light on the computational complexity of the framework. We give particular consideration to scenarios where the preferences of agents are modelled in terms of k-additive utility functions, *i.e.* scenarios where synergies between different resources are restricted to bundles of at most k items.

Key words: negotiation, representation of utility functions, social welfare, combinatorial optimisation, bidding languages for combinatorial auctions

1 Introduction

Distributed systems in which autonomous software agents interact with each other, in either cooperative or competitive ways, can often be usefully interpreted as *societies of agents*; and we can employ formal tools from microeconomics to analyse such systems. If we model the interests of individual agents in terms of a notion of *individual welfare*, then the overall performance of the system provides us with a measure of *social welfare*.

Individual welfare may be measured either *quantitatively*, typically by defining a utility function mapping "states of affairs" (outcomes of an election, allocations of resources, agreements on a joint plan of action, etc.) to numeric values; or *qualitatively*, by defining

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a preference relation over alternative states. The concept of social welfare, as studied in welfare economics, is an attempt to characterise the well-being of a society in relation to the welfare enjoyed by its individual members [1, 2, 18, 23]. The best known examples (both relying on quantitative measures of individual welfare) are the *utilitarian* programme, according to which social welfare should be interpreted as the sum of individual utilities, and the *egalitarian* programme, which identifies the welfare of society with the welfare of its "poorest" member.

For instance, in an electronic commerce application where users pay a fee to the provider of the infrastructure depending on the personal benefits incurred by using the system, the increase in utilitarian social welfare correctly reflects the profit generated by the provider. The application discussed by Lemaître *et al.* [17], on the other hand, where agents representing different stake-holders repeatedly negotiate over the access to an earth observation satellite (which has been jointly funded by the stake-holders), requires a *fair* treatment of all agents. Here, the respective values of different access schedules may be better modelled by an egalitarian social welfare ordering.

We are particularly interested in applications where negotiation between autonomous agents serves as a means of addressing a resource allocation problem. Recent results in this framework concern the feasibility of reaching an allocation of resources that is optimal from a social point of view [8, 21], as well as (certain aspects of) the complexity of doing so, in terms of both computational costs and the amount of communication required [5, 6, 7].

Multiagent resource allocation is just one of several recent examples for the successful exploitation of ideas from microeconomics in the context of computer science. Other applications include automatic contracting [21], selfish routing in shared networks [11], distributed reinforcement learning [24], and data mining [16]. This area of activity, which we may term *computational microeconomics*, brings together theoretical computer science and microeconomics in new and fruitful ways, benefiting not only these disciplines themselves but also "hot" research topics such as multiagent systems and electronic commerce.

In previous work, we have put forward the framework of *welfare engineering* [8], which addresses the design of suitable rationality criteria for autonomous software agents participating in negotiations over resources in view of different notions of social welfare, as well as the development of such notions of social welfare themselves. In Section 2, we briefly review the underlying multiagent resource allocation system and recall two previous results on the feasibility of reaching a socially optimal allocation of resources from a utilitarian point of view. As we shall see, in cases where the utility functions used by agents to model their preferences over alternative bundles of resources are *additive*, it is sufficient to use very simple negotiation protocols that only cater for deals involving a single resource at a time.

This result suggests to investigate generalisations of the notion of additivity, and hence we consider the case of k-*additive* functions, as studied, for instance, in the context of fuzzy measure theory [14]. The notion of k-additivity suggests an alternative representation of utility functions, which we introduce in Section 3. We show that this representation is as expressive as the "standard" representation (which involves listing the utility values for all possible bundles) and that it often allows for a more succinct representation of utility functions. Nevertheless, it turns out that the positive result on the complexity of deals obtained for additive functions *cannot* be generalised in the expected manner. Counterexamples are given in Section 4.

In Section 5, we discuss connections between our multiagent resource allocation framework and some well-known *combinatorial optimisation* problems (namely, weighted set packing and the independent set problem). These can be used to prove *NP-hardness* results for the decision problem associated with the task of finding a socially optimal resource allocation. We prove complexity results with respect to both the standard representation of utility functions and the representation based on k-additivity. In this context, we also discuss connections of our optimisation problem to the winner determination problem in *combinatorial auctions*. We are going to point out connections between different ways of representing utility functions and different *bidding languages* for such auctions along the way. Our conclusions are presented in Section 6.

2 Resource Allocation by Negotiation

An instance of our negotiation framework consists of a finite set of (at least two) *agents* $\mathcal{A}$ and a finite set of non-divisible *resources* $\mathcal{R}$. A resource *allocation* A is a partitioning of the set $\mathcal{R}$ amongst the agents in $\mathcal{A}$. For instance, given an allocation A with $A(i) = \{r_3, r_7\}$, agent i would own resources r_3 and r_7. Given a particular allocation of resources, agents may agree on a (multilateral) *deal* to exchange some of the resources they currently hold. In general, a single deal may involve any number of resources and any number of agents. It transforms an allocation of resources A into a new allocation A'; that is, we can define a deal as a pair $\delta = (A, A')$ of allocations (with $A \neq A'$).

Each agent $i \in \mathcal{A}$ is equipped with a *utility function* u_i mapping bundles of resources (subsets of $\mathcal{R}$) to rational numbers. We abbreviate $u_i(A) = u_i(A(i))$ for the utility value assigned by agent i to the set of resources it holds for allocation A. While individual agents may have their own interests, as a system designer, we are interested in the *social welfare* associated with a given allocation. According to the aforementioned *utilitarian* programme, the social welfare of an allocation A is given by the sum of utilities exhibited by all the agents in the system:

$$sw(A) \quad = \quad \sum_{i \in \mathcal{A}} u_i(A)$$

That is, any deal that results in a higher sum of utilities (or equivalently, in higher average utility) would be considered socially beneficial. One of the main questions we are interested in in the welfare engineering framework is under what circumstances negotiation between agents will result in an improvement, and eventually an optimisation, with respect to such a notion of social welfare.

A deal may be coupled with a number of monetary side payments to compensate some of the agents involved for an otherwise disadvantageous deal. We call a deal *rational* iff it results in a gain in utility (or money) that strictly outweighs a possible loss in money (or utility) for each of the agent involved in that deal.

As shown in previous work [9], a deal is rational iff it results in an increase in utilitarian social welfare. Given this connection between the "local" notion of rationality and the "global" notion of social welfare, we can prove the following result on the sufficiency of rational deals to negotiate socially optimal allocations [9, 21]:

Theorem 1 (Maximal social welfare) *Any sequence of rational deals with side payments will eventually result in an allocation of resources with maximal utilitarian social welfare.*

This means that (i) there can be no infinite sequence of deals all of which are rational, and (ii) once no more rational deals are possible the agent society must have reached an allocation that has maximal social welfare. The crucial aspect of this result is that *any* sequence of deals satisfying the rationality condition will cause the system to converge to an optimal allocation. That is, whatever deals are agreed on in the early stages of the negotiation, the system will never get stuck in a local optimum and finding an optimal allocation remains an option throughout.

A drawback of the general framework is that the above result only holds if deals involving any number of resources and agents are admissible [9, 21]. In some cases this problem can be alleviated by putting suitable restrictions on the utility functions agents may use to model their preferences. Interesting special classes of utility functions to consider include, for instance, *non-negative* functions (where an agent may not assign a negative utility to any bundle) and *monotonic* functions (where the utility of a set of resources cannot be lower than the utility assigned to any of its subsets).

A particularly simple class is the class of additive functions. A utility function is called *additive* iff the value ascribed to a set of resources is always the sum of the values of its members. As has been shown in an earlier paper [9], in scenarios where utility functions may be assumed to be additive, it is possible to guarantee optimal outcomes even when agents only negotiate deals involving a single resource and a pair of agents at a time (so-called *one-resource-at-a-time deals*):

Theorem 2 (Additive scenarios) *If all utility functions are additive, then any sequence of rational one-resource-at-a-time deals with side payments will eventually result in an allocation of resources with maximal utilitarian social welfare.*

This result is of great practical relevance, because it shows that it is sufficient to design negotiation protocols for pairs of agents (rather than larger groups) and single resources (rather than sets) for applications in which preferences can be modelled in terms of additive utility functions. In the next section, we are going to introduce a generalisation of this notion of additivity.

3 Representations of Utility Functions

An agent's utility function may be represented in different ways. This situation is similar, for instance, to the case of combinatorial auctions, where one can use different *bidding languages* to express the preferences of the participating agents [19, 22]. Maybe the most intuitive representation of a utility function is the *bundle form*, which amounts to listing all bundles of resources to which the agent assigns a non-zero value. Clearly, this approach can soon become problematic, as there may be up to 2^n such bundles in the worst case.

An alternative representation is based on the notion of k-additive functions, which have been studied in the context of fuzzy measure theory [14]. Given a natural number k, a utility function is called *k-additive* iff the utility assigned to a bundle of resources R can be represented as the sum of basic utilities ascribed to subsets of R with cardinality $\leq k$. More formally, a k-additive utility function can be written as follows:

$$u_i(R) \quad = \sum_{T \subseteq \mathcal{R},\, |T| \leq k} \alpha_i^T \times I_R(T) \quad \text{with } I_R(T) = \begin{cases} 1 & \text{if } T \subseteq R \\ 0 & \text{otherwise} \end{cases}$$

That is, the utility function of agent i is characterised by the coefficients α_i^T for bundles of resources $T \subseteq \mathcal{R}$ with at most k elements. Agent i enjoys an increase in utility of α_i^T when it owns all the items in T *together, i.e.* α_T^i represents the synergetic value of this bundle. An example for a 2-additive utility function would be $u_i(R) = 3 \times I_R(\{r_1\}) - 2 \times I_R(\{r_2, r_3\})$. For the sake of simplicity, we are going to omit the indicator function I_R as well as the explicit mentioning of the bundle variable R when defining concrete k-additive utility functions. Using this simplified notation, the above function becomes $u_i = 3.r_1 - 2.r_2.r_3$.

While the bundle form corresponds to the so-called XOR-language for expressing bids in combinatorial auctions [19, 22], there appears to be no counterpart to the k-additive form in the literature on bidding languages. The connections between our framework and combinatorial auctions will be explored further in Section 5.

Utility functions that are k-additive with $k = 1$ are like the additive functions discussed in the previous section (except that they also allow for a non-zero utility value to be assigned to the empty set). Hence, the notion of k-additivity is a generalisation of the familiar notion of additivity. In fact, as we are going to show next, k-additive utility functions cover a whole range of utility function, from the very simple additive functions to the most general utility functions without any restrictions.

Proposition 1 (Expressive power of k-additive utility functions) *Any utility function can be represented as a k-additive function with $k = |\mathcal{R}|$.*

Proof. Let u_i be any utility function mapping subsets of $\mathcal{R}$ to rational numbers. We recursively define coefficients α_i^T for $T \subseteq \mathcal{R}$ as follows:

$$\begin{aligned} \alpha_i^{\{\}} &= u_i(\{\,\}) \\ \alpha_i^R &= u_i(R) - \sum_{T \subset R} \alpha_i^T \quad \text{for all } R \subseteq \mathcal{R} \text{ with } R \neq \{\,\} \end{aligned}$$

Hence, $u_i(R) = \sum_{T \subseteq R} \alpha_i^T = \sum_{T \subseteq \mathcal{R}} \alpha_i^T \times I_R(T)$. This is a k-additive utility function for $k = |\mathcal{R}|$. □

Clearly, the bundle form is also fully expressive, *i.e.* our two ways of representing utility functions are equivalent in the sense that they can both express any utility function over the set of resources $\mathcal{R}$. Besides expressive power, another important consideration concerns the succinctness of a representation. It turns out that neither of the two representations is more succinct in all cases. In fact, as we are going to see next, there are cases where translating a utility function given in k-additive form into the bundle form results in an exponential blow-up of the representation, and vice versa.[1]

Proposition 2 (Efficiency of the k-additive form) *The bundle form cannot polynomially simulate the k-additive form of representing utility functions.*

Proof. We prove the claim by giving an example for a utility function with a representation that is linear in the size of $\mathcal{R}$ for the k-additive form, but exponential for the bundle form. Consider a utility function that maps a bundle of resources to the number of elements in that bundle. This is a 1-additive function, which requires the specification of exactly $|\mathcal{R}|$ coefficients in the k-additive form (namely $\alpha_1^T = 1$ for all T with $|T| = 1$). For the bundle form, however, the specification of a utility value for each of the $2^{|\mathcal{R}|} - 1$ non-empty bundles is required. □

[1] Nisan [19] proves a number of similar separation results for different types of bidding languages for combinatorial auctions.

Proposition 3 (Efficiency of the bundle form) *The k-additive form cannot polynomially simulate the bundle form of representing utility functions.*

Proof. We give an example for a utility function with a representation that is linear in the size of $\mathcal{R}$ for the bundle form, but exponential for the k-additive form. Consider a utility function u_i that assigns 1 to any bundle consisting of a single resource and 0 to any other bundle. In the bundle form, u_i requires the specification of a utility value for exactly $|\mathcal{R}|$ bundles (namely those with just a single element). In the k-additive form, on the other hand, it requires the specification of $2^{|\mathcal{R}|} - 1$ coefficients: We certainly have $\alpha_i^T = 1$ for any bundle T with $|T| = 1$. To ensure that $u_i(R) = 0$ for any R with $|R| = 2$ we require $\alpha_i^T = -2$ for any T with two elements. For a bundle with three elements, the sum of the coefficients for all its subsets is $3 \times 1 + 3 \times (-2) = -3$, *i.e.* we have to set $\alpha_i^T = 3$ whenever $|T| = 3$, and so on. In general, we have to choose $\alpha_i^T = |T| \times (-1)^{|T|+1}$ (which is different from 0 for any of the $2^{|\mathcal{R}|} - 1$ subsets T of $\mathcal{R}$ with $T \neq \{\,\}$) to be able to represent u_i as a k-additive function. □

The examples given in the proofs of Propositions 2 and 3 are extreme cases, where one form of representation is exponentially more succinct than the other. While the difference is not always going to be this strong, choosing the right representation for a given problem domain is still important. Broadly speaking, the k-additive form will typically be more succinct in cases where there are only limited synergies between different items. This is likely to be the case for many application domains, which makes this a useful language for expressing utilities in practice.

4 Complexity of Deals with k-additive Utilities

Recall that Theorem 2 has shown that it is always possible to negotiate a socially optimal allocation of resources by means of rational deals involving only a single resource at a time whenever the utilities of all the agents involved are additive (*i.e.* 1-additive). Intuitively, we could have expected a similar result for k-additive utilities with $k \geq 2$ (*i.e.* a result that states that rational deals involving at most k resources at a time are sufficient to reach optimal allocations whenever all utility functions are k-additive). However, as we are going to show next, this turns out not to be the case. The deals required to reach allocations with maximal social welfare in the k-additive case are much more complex.

Proposition 4 (Necessity of complete deals) *Even if all utility functions are k-additive for some $k \geq 2$, a deal involving the complete set of resources may be necessary to reach an allocation with maximal utilitarian social welfare by means of a sequence of rational deals with side payments.*

Proof. To prove the claim, we construct an example with 2-additive utility functions in which a deal involving all resources in $\mathcal{R}$ is needed. Consider two agents sharing n resources $\mathcal{R} = \{r_1, r_2, \ldots, r_n\}$, with the following 2-additive utility functions:

$$\begin{aligned} u_1 &= 0 \\ u_2 &= r_1 - r_1.r_2 - r_1.r_3 - r_1.r_4 - \ldots - r_1.r_n \end{aligned}$$

Let A_{init} be the initial allocation of resources describing which agent owns which resource before negotiation commences, and let A_{opt} be the allocation maximising utilitarian social welfare:

	A_{init}	A_{opt}
Agent 1	$\{r_1\}$	$\{r_2, r_3, \ldots, r_n\}$
Agent 2	$\{r_2, r_3, \ldots, r_n\}$	$\{r_1\}$

Here, $sw(A_{init}) = 0$ and $sw(A_{opt}) = 1$. In fact, the *only* allocation which has a social welfare greater than $sw(A_{init})$ is A_{opt}. Recall that a deal increases social welfare iff it is rational with side payments (the proof may be found in [9]). Thus, the only rational deal here is $\delta = (A_{init}, A_{opt})$, which is a bilateral deal involving all n resources at the same time. □

A possible objection to the example used in our proof may be that it is rather artificial. Utility functions that also have some additional properties, such as being monotonic, besides being k-additive may be more relevant in practice. To show that the problem of requiring complex deals persists even when we make such additional assumptions, we give a further, similarly simple, example that demonstrates that also for k-additive functions that are monotonic, rational deals involving no more than k resources do not always suffice to negotiate socially optimal allocations. Consider the case of three agents and four resources with the following utility functions:

$$\begin{aligned} u_1 &= 4.r_1.r_3 \\ u_2 &= 3.r_1.r_2 \\ u_3 &= 2.r_3.r_4 \end{aligned}$$

Let A_{init} be the initial allocation and let A_{opt} be the optimal allocation with maximal utilitarian social welfare:

	A_{init}	A_{opt}
Agent 1	$\{r_1, r_3\}$	$\{\}$
Agent 2	$\{r_2, r_4\}$	$\{r_1, r_2\}$
Agent 3	$\{\}$	$\{r_3, r_4\}$

We have $sw(A_{init}) = 4$ and $sw(A_{opt}) = 5$. Clearly, the only rational deal with side payments (*i.e.* the only deal increasing social welfare) is $\delta = (A_{init}, A_{opt})$, which is a deal involving 3 (rather than just 2) resources at the same time.

In summary, our results show, differently from what one might have expected, that the restriction to utility functions that are k-additive for a given value of k does not, in general, reduce the complexity of deals required to reach a socially optimal allocation of resources in an agent society whose members follow a simple rational negotiation strategy.

5 Connections to Combinatorial Optimisation

In Section 3, we have already mentioned the connection between different representations of utility functions (in our case the bundle form and the k-additive form) in our negotiation framework and different bidding languages in combinatorial auctions. In what follows, we explore a further connection between the two areas.

If we view the problem of finding an allocation with maximal social welfare as an algorithmic problem faced by a central authority (rather than as a problem of designing suitable negotiation mechanisms), then we can observe an immediate relation to the so-called *winner determination problem* in combinatorial auctions [19, 20, 22]. In a combinatorial auction, bidders can put in bids for different *bundles* of items (rather than just single items). After all bids have been received, the auctioneer has to find an allocation for the items on auction amongst the bidders in a way that maximises his revenue. If we interpret the price offered for a particular bundle of items as the utility the agent in question assigns to that set, then maximising revenue (*i.e.* the sum of prices associated with winning bids) is equivalent to finding an allocation with maximal utilitarian social welfare. This equivalence holds, at least, in cases where the optimal allocation of items in an auction is such that *all* of the items on auction are in fact being sold (so-called *free disposal*).

Winner determination in combinatorial auctions is known to be NP-complete [20].[2] The quoted result applies to the case of the "standard" bidding language, which allows bidders to specify prices for particular bundles and makes the implicit assumption that they are prepared to obtain any number of disjoint bundles for which they have submitted a bid (Nisan [19] calls this the "OR language"). Our languages for expressing utilities are more general than this. Hence, the correspondence to combinatorial auctions suggests that the problem of finding an allocation with maximal utilitarian social welfare is at least NP-hard. We can make this observation more precise by showing how our problem relates to well-known NP-complete "reference problems" [3, 13, 15]. One such problem is MAXIMUM WEIGHTED SET PACKING. We use the schema of Ausiello *et al.* [3] to

[2] More precisely, the *decision problem underlying the winner determination problem*, *i.e.* the problem of checking whether it is possible to find an allocation that achieves at least a given minimal revenue K is NP-complete. The concept of NP-completeness applies to decision problems rather than optimisation problems [3]. The winner determination problem is still NP-hard in the sense that solving it is at least as hard as solving any NP-complete decision problem.

define combinatorial optimisation problems:

MAXIMUM WEIGHTED SET PACKING
Instance: Collection $\mathcal{C}$ of finite sets, each associated with a positive weight.
Solution: Collection of disjoint sets $\mathcal{C}' \subseteq \mathcal{C}$.
Measure: Sum of the weights associated with the sets in $\mathcal{C}'$.

The *optimisation problem* known as MAXIMUM WEIGHTED SET PACKING is the problem of finding a solution $\mathcal{C}'$ for which the sum of the weights associated with the sets in $\mathcal{C}'$ is maximal. The underlying *decision problem* is the problem of answering the question whether there exists a solution $\mathcal{C}'$ for which the sum of weights exceeds a given threshold K. This decision problem is known to be NP-complete (in the size of the instance, *i.e.* with respect to the number of sets in $\mathcal{C}$) [3].

Intuitively, we are going to interpret the sets in $\mathcal{C}$ as bundles of resources and the weights associated with them as utility values. To make the correspondence complete, however, we require the following generalisation of MAXIMUM WEIGHTED SET PACKING:

MAXIMUM COLOURED WEIGHTED SET PACKING WITH FULL COVERAGE
Instance: Collection $\mathcal{C}$ of coloured finite sets, each associated with a weight.
Solution: Collection of disjoint sets $\mathcal{C}' \subseteq \mathcal{C}$, including exactly one set of each colour, such that $\{x \in S \mid S \in \mathcal{C}'\} = \{x \in S \mid S \in \mathcal{C}\}$.
Measure: Sum of the weights associated with the sets in $\mathcal{C}'$.

There are three differences between the original weighted set packing problem and our extended problem: (i) we have dropped the restriction to *positive* weights; (ii) every set is associated with a *colour* and every colour is required to be represented exactly once in any valid solution; and (iii) all the items occurring in any of the set in $\mathcal{C}$ need to be *covered* by the set packing $\mathcal{C}'$.

Lemma 1 (Complexity of extended WSP) *The decision problem underlying* MAXIMUM COLOURED WEIGHTED SET PACKING WITH FULL COVERAGE *is NP-complete.*

Proof. NP-membership of our problem follows from the fact that all the conditions imposed on valid solutions can be checked in polynomial time.[3] NP-hardness follows from the known NP-hardness result for the decision problem underlying MAXIMUM WEIGHTED SET PACKING. To see that our extended problem is indeed at least as hard

[3]Recall that a decision problem is in NP iff any proposed proof for a positive answer can be checked (although not necessarily found) in polynomial time.

as the original problem, we need to show how the original problem can be reduced to the extended one. Consider the following mapping: Given an instance $\mathcal{C}$ of MAXIMUM WEIGHTED SET PACKING, first add the set $\{x\}$ (with weight 0) for every $x \in S$ for every $S \in \mathcal{C}$ to the collection (unless that set is already present). Then assign a different colour to each set in the extended collection. Finally, also introduce an empty set (with weight 0) for each of the colours. The additional sets ensure that for any solution of the original problem there is a solution of the extended problem such that all elements as well as colours are covered. □

The following theorem has first been proved by Dunne *et al.* [6] by means of a non-trivial reduction from a variant of 3-SAT where the number of clauses in the input formula is equal to the number of propositional variables occurring in that formula.[4] Having established the complexity of our extended set packing problem, we are in a position to give a much simpler proof.

Theorem 3 (Complexity wrt. bundle form) *The decision problem underlying the problem of finding an allocation with maximal utilitarian social welfare with utilities represented in bundle form is NP-complete.*

Proof. The problem of finding an allocation with maximal utilitarian social welfare is equivalent to MAXIMUM COLOURED WEIGHTED SET PACKING WITH FULL COVERAGE: sets in the collection correspond to bundles, colours correspond to agents, and the weight associated with a coloured set corresponds to the utility assigned to the respective bundle by the respective agent. NP-completeness then follows from Lemma 1. □

Note that we could have proved the same result using a direct reduction from MAXIMUM WEIGHTED SET PACKING, even from the version without weights, but having a combinatorial optimisation problem that is exactly equivalent to our problem of finding a socially optimal allocation of resources in the language familiar from the literature on combinatorial optimisation is interesting in its own right.

Our next aim is to establish the complexity of the same decision problem, but this time with respect to the k-additive form rather than the bundle form of representing utility functions. As the k-additive form may be exponentially more succinct than the bundle form, NP-hardness with respect to the later does not necessarily imply NP-hardness with respect to the former. Nevertheless, as we are going to see, deciding whether there exists an allocation of resources with a utilitarian social welfare that exceeds a given threshold

[4]Fargier *et al.* [10] also prove a very similar result. In their resource allocation framework agents can, by default, share individual resources, but if a particular resource can only be owned by one agent at a time this can be specified by giving additional constraints.

is also NP-complete. This time, we are going use a reduction from another well-known combinatorial optimisation problem:

MAXIMUM INDEPENDENT SET
Instance: Graph $G = (V, E)$.
Solution: Set $V' \subseteq V$ s.t. no two vertices in V' are joined by and edge in E.
Measure: Cardinality $|V'|$.

The problem of finding an independent set whose cardinality exceeds a given threshold is known to be NP-complete [13] (although some special cases, e,g. when all vertices have a degree of at most 2, are solvable in polynomial time).

Theorem 4 (Complexity wrt. k-additive form) *The decision problem underlying the problem of finding an allocation with maximal utilitarian social welfare with utilities represented in k-additive form is NP-complete.*

Proof. Firstly, the problem is certainly in NP, because checking whether the social welfare of a given allocation exceeds a given threshold K can be checked in polynomial time. We show NP-hardness by reducing the decision problem underlying MAXIMUM INDEPENDENT SET to our problem. Given a graph $G = (V, E)$ and a rational number K, we want to establish whether the graph has got an independent set V' with cardinality $|V'| > K$. Without loss of generality, we may assume that no vertex in V is joined with itself by and edge in E, because no solution V' would contain such a vertex. We can map this independent set problem to an instance of our decision problem by introducing an agent for every vertex in V and a resource for every edge in E. We define the utility coefficients in the k-additive form for every agent i as follows: Let T be the set of resources corresponding to edges in E that are adjacent to the vertex corresponding to i. We define $\alpha_i^T = 1$ and there are no other utility coefficients for agent i. Now every allocation A corresponds to an independent set V' and the utilitarian social welfare of A equals the cardinality of V'. Hence, there exists an independent set V' with $|V'| > K$ iff there exists an allocation A with $sw(A) > K$. □

Of course, as with MAXIMUM INDEPENDENT SET, there will be special cases where the above problem is not NP-hard anymore. A very simple example would be the case of $k = 1$: It is easy to devise a polynomial algorithm for finding an allocation with maximal utilitarian social welfare in cases where all agents use 1-additive utility functions (simply assign each resource to the agent that values it the highest).

What about $k = 2$ though? In our proof, k directly corresponds to the maximal degree of vertices in the graph used for the reduction. As pointed out already, the decision problem underlying MAXIMUM INDEPENDENT SET is *not* NP-hard anymore if no vertex has got a degree exceeding 2. Hence, our proof of Theorem 4 would not allow us to conclude that our problem remains NP-hard for $k = 2$. This is the objective of our next theorem. It shows that the problem of finding a socially optimal allocation is still NP-hard for $k = 2$. In this sense, our problem is harder than MAXIMUM INDEPENDENT SET (where the transition to NP-hardness only occurs when we move from 2 to 3).

Theorem 5 (Complexity for $k = 2$) *The decision problem underlying the problem of finding an allocation with maximal utilitarian social welfare with utilities represented in k-additive form remains NP-complete for $k = 2$.*

Proof. NP-membership follows from Theorem 4. To prove NP-hardness for $k = 2$, we show how any problem instance with k-additive utility functions for $k \geq 3$ can be transformed into a problem with 2-additive functions in polynomial time. NP-hardness then follows, again, from Theorem 4.

We will show that a 3-additive resource allocation problem can be reformulated as a 2-additive one. This is an adaptation of an idea by Boros and Hammer [4] to our case. Consider n agents having 3-additive utility functions. We will show here that each 3-additive term appearing in the utility functions can be replaced by a set of five 2-additive ones, in a way that leaves the optimal resource allocation unchanged. Let us suppose u_i contains a 3-additive term $\alpha.r_1.r_2.r_3$ which we want to get rid of. To make it 2-additive, we will have to create a new "pseudo-resource" r_{12} which represents the bundle $\{r_1, r_2\}$. Clearly, the integrity constraint $r_{12} = r_1.r_2$ (with both r_{12} and $r_1.r_2$ being equal to either 0 or 1) has to be fulfilled in order to have $\alpha.r_1.r_2.r_3 = \alpha.r_{12}.r_3$.

For this purpose, let us define the following function with M being a big constant ($M = 1 + 2\sum_{i,T} |\alpha_i^T|$ is sufficient):

$$integrity(r_1, r_2, r_{12}) \quad = \quad -M.r_1.r_2 + 2M.r_1.r_{12} + 2M.r_2.r_{12} - 3M.r_{12}$$

This integrity function, which is 2-additive, will be added to the term $\alpha.r_{12}.r_3$ to penalise it in case the constraint is violated:

$$\begin{array}{ll} \textit{integrity}(r_1, r_2, r_{12}) = 0 & \textit{if } r_{12} = r_1.r_2 \\ \textit{integrity}(r_1, r_2, r_{12}) \leq -M & \textit{otherwise} \end{array}$$

Let us now consider the new utility function equal to u_i in which the term $\alpha.r_1.r_2.r_3$ has been replaced by the 2-additive formula $\alpha.r_{12}.r_3 + \textit{integrity}(r_1, r_2, r_{12})$. This change does not affect social welfare in case the integrity constraint is fulfilled. If not, then the social welfare will have a very low value (far from optimal). Up to now, a single 3-additive term was reduced to five 2-additive terms. By iterating this reduction, a set of 3-additive utilities can be reformulated in 2-additive utilities, without changing the optimal allocation. In addition, note that this can be applied $k-2$ times to transform any k-additive utility function into one that is 2-additive.

It follows that finding a socially optimal resource allocation with 2-additive utility functions is as hard as finding it for k-additive functions with $k > 2$ (modula the polynomial reduction described). Hence, the problem remains NP-hard for $k = 2$. □

As a final complexity result, we are going to show that the problem of *verifying* that a given allocation is socially optimal is co-NP-complete. This holds for both the bundle form and the k-additive form of representing utility functions and is a simple corollary to Theorems 3 and 4.

Corollary 1 (Complexity of verifying optimality) *The problem of verifying that a given allocation has got maximal utilitarian social welfare is co-NP-complete (for both representations of utility functions).*

Proof. Checking that an allocation A is *not* optimal involves firstly computing $sw(A)$, which can be done in polynomial time, and then solving the decision problem "is there an allocation A' with $sw(A') > sw(A)$?". The latter is NP-complete according to Theorem 3 (Theorem 4) for the bundle (k-additive) form. Hence, the complementary problem must be co-NP-complete. □

Related to this result, Dunne *et al.* [6] have shown that the problem of checking whether a given allocation of resources is Pareto optimal is also co-NP-complete.[5]

What is the practical relevance of the connections between our negotiation framework and the combinatorial optimisation problems discussed in this section? In the proof of Theorem 4, for instance, we have reduced MAXIMUM INDEPENDENT SET to a very specific class of instances of the problem of finding a socially optimal allocation of resources, namely those where the utility functions of all agents can be represented as k-additive functions with only a single non-zero coefficient. While this reduction has been useful to establish our NP-hardness result, it does not provide us with much useful information on how to find an optimal allocation in practice. Here, the opposite direction, *i.e.* reductions *from* resource allocation problems *to* standard combinatorial optimisation problems may be more attractive. Such a reduction would allow us to exploit existing algorithms, including highly optimised approximation algorithms [3], to find optimal (or near-optimal) allocations of resources.

[5] An allocation of resources is called *Pareto optimal* iff there is no other allocation that would be better for at least one of the agents without being worse for any of the others. For further results on negotiating Pareto optimal allocations we refer to [9].

In the case of utility functions in k-additive form, the resource allocation problem can be reduced to the weighted variant of MAXIMUM INDEPENDENT SET [3], provided all utility coefficients are positive and all agents value the empty bundle at 0. The mapping firstly involves introducing a vertex for each coefficient (and using the coefficient itself as the weight associated with that vertex). Then we introduce an edge for every possible "conflict": any two vertices α_i^T and $\alpha_j^{T'}$ with $i \neq j$ and $T \cap T' \neq \{\,\}$ are joined together by an edge. The independent set yielding the highest overall weight then corresponds to the optimal allocation.

In the case of the bundle form, we already have established a on-to-one correspondence to MAXIMUM COLOURED WEIGHTED SET PACKING WITH FULL COVERAGE. However, to exploit existing algorithms, we require a reduction to the standard problem of MAXIMUM WEIGHTED SET PACKING. This is possible whenever a resource allocation problem meets the following conditions: (i) all utility functions are non-negative; (ii) all agents value the empty bundle at 0; and (iii) we can assume *free disposal*, *i.e.* for every incomplete allocation (not covering all resources) there is always a complete one that is not worse.[6] The proposed mapping would involve creating a set for every pair of an agent i and a bundle R with $u_i(R) \neq 0$. Here, we consider both the resources and the agent as elements of that set. The weight associated with the set would be $u_i(R)$. It is then not difficult to see that allocations with maximal social welfare correspond to set packings with maximal overall weight. Hence, we can reuse existing algorithms for MAXIMUM WEIGHTED SET PACKING to find optimal allocations of resources.

Finally, we should stress that this would be a methodology for a *centralised* approach to finding optimal resource allocations. It is not immediately applicable to negotiation, which is a *distributed* process. Nevertheless, the techniques used to design optimisation and approximation algorithms may still inspire useful mechanisms for distributed resource allocation. We hope to address this issue in our furture work.

6 Conclusion

In this paper, we have given a brief overview of recent work on multiagent resource allocation in the context of the welfare engineering framework, and we have further analysed the properties of this framework for the case of k-additive utility functions. Our results presented in Section 4 show that, despite the positive expectations raised by the previous result on negotiation in additive domains (Theorem 2), the complexity of the negotiation protocol required to agree on a socially optimal allocation does not necessarily decrease for problems with k-additive utility functions when k gets smaller (as long as $k > 1$).

[6]This may be achieved, for instance, by adding an agent i to the system with $u_i(R) = 0$ for all $R \subseteq \mathcal{R}$, or by having at least one agent with a monotonic utility function.

On the other hand, as we have seen in Section 3, representing utility functions in the k-additive form rather than the bundle form can be significantly more succinct, particularly in cases where a representation with a small value for k is possible.

We have also explored connections to well-known combinatorial optimisation problems, which allowed us to establish complexity results for the problem of finding a socially optimal allocation with respect to different representations of utility functions (Section 5). In this context, we have also briefly discussed the relation of our negotiation framework to combinatorial auctions for different kinds of bidding languages. While our negotiation framework is clearly *not* and auction (it is, for instance, not concerned with the aspect of agreeing on the price for a set of items), the abstract "centralised" problem of finding a socially optimal allocation (which is not itself a problem faced by the agents participating in a negotiation process) directly corresponds to the winner determination problem in combinatorial auctions. Under this view, the languages used to represent utility functions correspond to bidding languages for such auctions. However, it appears that the bidding language corresponding to our k-additive form has not yet been exploited by auction designers.

Finally, we would like to stress that the high complexity of of our negotiation framework does not, at least not necessarily, mean that it cannot be usefully applied in practice. This view is supported by the fact that, in recent years, several algorithms for winner determination in combinatorial auctions (a problem of comparable complexity to the problems arising in the context of welfare engineering) have been proposed and applied successfully [12, 20, 22].

We see the work presented in this paper as part of a wider research trend, which brings together ideas from different areas including microeconomics, game theory, complexity theory, and algorithm design. Some further examples of this kind of interdisciplinary research are cited in the introductory section.

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What can we learn from the transitivity parts of a relation?

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Abstract

A transitivity part of a relation on a set X is any subset of X on which the restriction of the relation is transitive. What can be recovered of a relation from the sole knowledge of its transitivity parts? In general, the relation itself cannot be recovered, because it has the same transitivity parts as its converse. In certain situations, the unordered pair formed by the relation and its converse can be recovered. This is the case for relations known to be indecomposable tournaments. The result first appeared in Boussaïri, Ille, Lopez, and Thomassé [2004]. Our proof is simpler, and at the same time conveys some interesting insight into the structure of tournaments.

Key words : tournament, transitivity, transitively determined

In the applications of combinatorics, the problem of efficiently storing relations in the memory of a computer often arises. In the classic case of a partial order, such a storage can evidently be carried out via its Hasse diagram. The topic of this paper stems from similar concerns and uses related ideas. For instance, consider the transitivity parts of a relation, that is, those subsets on which the restriction of the relation is transitive. As is easily checked, all the transitivity parts of a relation can be recovered from the transitivity parts having two or three elements. Are there situations in which more can be deduced from such small transitivity parts? In the case of symmetric and reflexive relations, the question can be recast as a problem about simple graphs investigated by Hayward [1996]. We will come back on this case at the end of our paper.

We are mostly concerned here with tournaments, and our main result—in Theorem 21— characterizes the tournaments which can be fully recovered (up to their converse)

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from their transitivity parts. Unbeknownst to us, the same result had been established independently by Boussaïri, Ille, Lopez, and Thomassé [2004]. While the latter authors proceed, in several instances, by induction on the number of vertices, our method of proof is different and has the interest of providing some new insights on the general structure of tournaments.

1 Statement of the problem

Definition 1 Let R be a relation defined on a finite set X. We use the abbreviation xy to denote the (ordered) pair (x, y), and we write, as usual, $x\,R\,y$ to mean $xy \in R$ and $R^{-1} = \{xy \mid y\,R\,x\}$ to denote the converse of R. We also sometimes use abbreviations such as $x\,R\,y\,R\,z$ to mean $(x\,R\,y$ and $y\,R\,z)$. All the relations mentioned in this paper are implicitly assumed to be on the same finite vertex set X (if not mentioned otherwise). We often refer to the digraph (X, R), and use the corresponding terminology.

A subset Y of X is called a *transitivity part* of a relation R on X if the restriction of R to Y is transitive. The collection of all the transitivity parts of R is denoted by $\mathcal{T}(R)$. We write $\mathcal{T}_3(R)$ for the subcollection of $\mathcal{T}(R)$ containing all the transitivity parts of size 3. Note in passing that $\mathcal{T}_3(R)$ is considerably smaller than $\mathcal{T}(R)$ in some cases. We denote by $\overline{\mathcal{T}}_3(R)$ the collection of all the 3-subsets of X which do not belong to $\mathcal{T}_3(R)$. Occasionally, when no ambiguity can arise regarding the relation R, we may use abbreviations such as $\mathcal{T}_3$ or $\overline{\mathcal{T}}_3$ to mean $\mathcal{T}_3(R)$ or $\overline{\mathcal{T}}_3(R)$, respectively.

Definition 2 A *tricycle* of a relation R is a cycle of length 3. A *trio* is a 3-subset of X consisting of the vertices of some tricycle.

Definition 3 A relation R is a *tournament* on X if R is complete and asymmetric on X; thus, either $x\,R\,y$ or $y\,R\,x$ for all distinct $x, y \in X$, and $\neg(x\,R\,x)$ for all $x \in X$. It is clear that when R is a tournament, $\mathcal{T}(R)$ can be obtained from $\mathcal{T}_3(R)$; in fact, $Y \in \mathcal{T}(R)$ if and only if each 3-subset of Y is in $\mathcal{T}_3(R)$; moreover, $Y \in \overline{\mathcal{T}}_3(R)$ if and only if Y is a trio.

Notice that distinct tournaments can have the same transitivity parts. Indeed, a tournament R and its converse R^{-1} always have the same trios. Accordingly, our aim in the sequel is the recovery of the unordered pair $\{R, R^{-1}\}$, rather than R itself. As a first step of such a recovery, we can thus arbitrarily fix $a\,R\,b$ for some initial vertices a and b. (Fixing $b\,R\,a$ rather than $a\,R\,b$ amounts to exchanging R for R^{-1}.) For other examples of tournaments with exactly the same transivity parts, take any two strict linear orders on X. Many other examples are easily manufactured.

Definition 4 A tournament R on X is (*transitivity*) *determined* whenever, for any tournament S on X, the equality $\mathcal{T}(R) = \mathcal{T}(S)$ implies $R = S$ or $R = S^{-1}$.

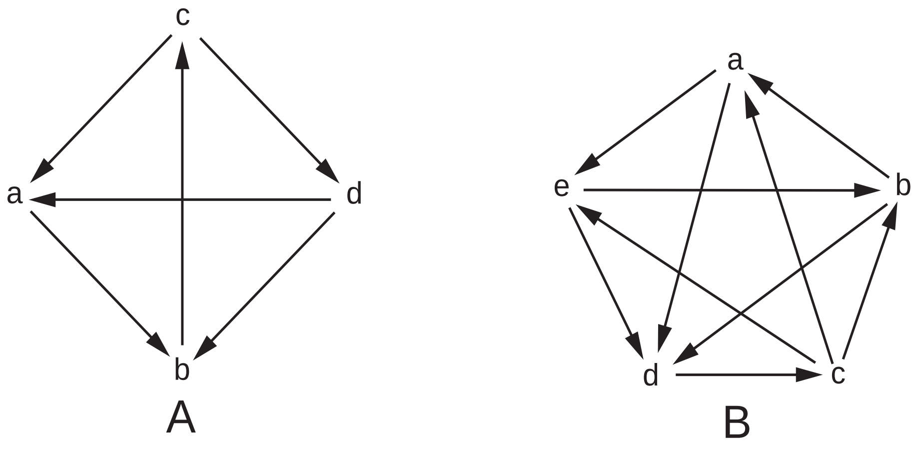

Figure 1: In the tournament represented by the graph of Figure 1A, the pair da can be reversed without altering $\mathcal{T}_3$, so this tournament is not determined. Neither is the tournament of Figure 1B, but the argument is sligthly more involved: this case fails the test derived from Theorem 21.

It is easy to find tournaments which are *not* determined. Two examples are given in Figure 1. In Figure 1A, which defines a tournament on the vertex set $\{a, b, c, d\}$, we can check that the pair da can be transposed without altering $\mathcal{T}_3 = \{\{a, c, d\}, \{a, b, d\}\}$ or $\overline{\mathcal{T}}_3 = \{\{a, b, c\}, \{b, c, d\}\}$. The tournament displayed in Figure 1B is not determined either. (It fails the test derived from Theorem 21.)

There are, however, some tournaments which are determined, such as those represented in Figures 2A and 2B. Indeed, the tournament of Figure 2A (up to its converse) is defined by fixing one initial pair, and then using the information conveyed by

$$\mathcal{T}_3 = \{\{a, b, c\}, \{a, b, e\}, \{a, c, d\}, \{b, d, e\}, \{c, d, e\}\},$$

or equivalently by

$$\overline{\mathcal{T}}_3 = \{\{a, e, d\}, \{a, b, d\}, \{a, e, c\}, \{b, d, c\}, \{b, e, c\}\}.$$

Say we fix the pair ae marked 1 in Figure 2A. The other pairs are then automatically obtained by completing the tricycles, for example in the order 2, 3, …, 10. The tournament of Figure 2B is also determined, but the verification is more complicated. Indeed, two types of inferences can be drawn from $\mathcal{T}_3(R)$ or $\overline{\mathcal{T}}_3(R)$ in the recovery of the pair $\{R, R^{-1}\}$:

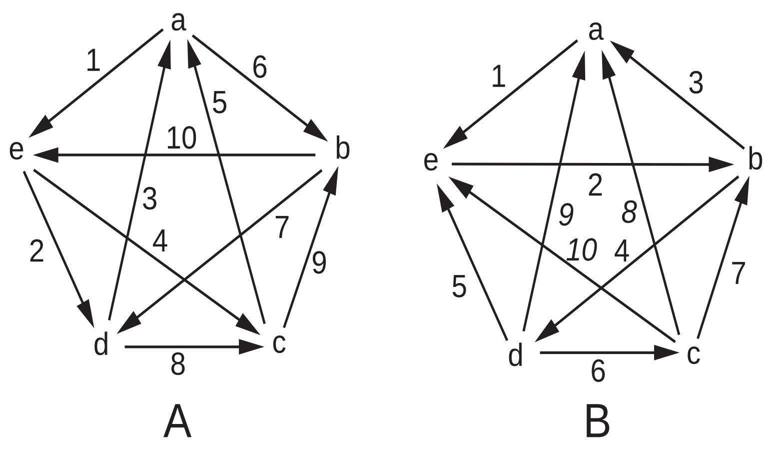

Figure 2: The tournament represented by the graph of Figure 2A is determined by the set of its trios, by applying only Type 1 inferences. By contrast, both types of inferences are required to show that the tournament in Figure 2B is determined (see text).

TYPE 1. If $\{x, y, z\} \in \overline{\mathcal{T}}_3$, then $x\,R\,y$ entails $y\,R\,z$ and $z\,R\,x$.

TYPE 2. If $\{x, y, z\} \in \mathcal{T}_3$, then $x\,R\,y$ and $y\,R\,z$ entails $x\,R\,z$.

In the tournament of Figure 2A, only Type 1 inferences were used to reconstruct the pair $\{R, R^{-1}\}$, while both types of inferences are required in the case of the tournament of Figure 2B. (The numbers 8, 9 and 10 in italics in Figure 2B refer to inferences of Type 2.) These considerations suggest the following problem:

Problem 5 *Characterize the tournaments which are determined.*

Notice an important feature of Definition 4. The quantification “for any tournament S on X” means that we suppose that the unordered pair $\{R, R^{-1}\}$ to be uncovered comes from a tournament R. Thus, the context of tournaments is assumed from the outset. A similar qualification applies to our generalization of Problem 5 in the last section, in which we assume that a family of relations is given (see Problems 23, 24, 25). In the case of Hayward [1996], for instance, the family of reflexive and symmetric relations on X forms the context.

2 Tournament concepts

Definition 6 A tournament R on X is *strongly connected* or *strong* if for any two vertices x and y in X, there is a directed path from x to y.

According to Moon Theorem, a tournament is strong if and only if it is Hamiltonian (cf. Bang-Jensen and Gutin [2001, Theorem 1.5.1], Gross and Yellen [2004], Laslier [1997]). The next characterization is easily checked.

Proposition 7 *A tournament R on X is strong if and only if for any proper subset Y of X, there exist $x, y \in Y$ and $u, v \in X \setminus Y$ such that $x\,R\,u$ and $v\,R\,y$ (we may have $x = y$ or $v = w$, but not both).*

Definition 8 A tournament R on X is *decomposable* if there exists some nontrivial partition $\{C_1, C_2, \ldots, C_k\}$ of X such that, for all distinct indices $i, j \in \{1, \ldots, k\}$,

$$\forall x, y \in C_i, \ \ \forall u, v \in C_j, \qquad x\,R\,u \Rightarrow y\,R\,v.$$

Any indecomposable tournament on at least three vertices is strong, but the converse is false. An example of a strong, decomposable tournament is given in Figure 3.

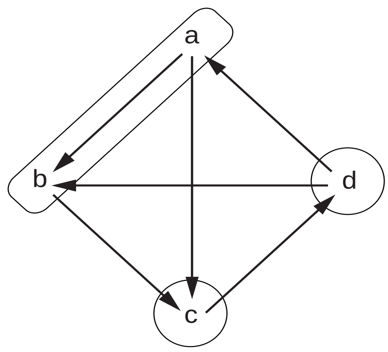

Figure 3: An example of a strong decomposable tournament. The nontrivial partition of the vertex set $\{a, b, c, d\}$ is $\{\{a, b\}, \{c\}, \{d\}\}$.

The partition of Definition 8 being nontrivial, it contains a class that is a proper subset of X with more than one vertex. Thus the following holds:

Proposition 9 *A tournament R on X is indecomposable if and only if for any proper subset Y of X with more than one vertex, there exist $x, y \in Y$ and $z \in X \setminus Y$ such that $x\,R\,z$ and $z\,R\,y$.*

The concepts of a strong and of a decomposable tournament have become classical ones (see e.g. Bang-Jensen and Gutin [2001], Gross and Yellen [2004], Laslier [1997]). We now turn to some further tools and facts that will be instrumental in the proof of our main result.

Definition 10 If V and W are two relations, we write as usual $V\,W = \{xy \mid \exists z,\ \text{with}\ x\,V\,z\,W\,y\}$ for their products. Let R be a tournament on X, and let Q be the set of pairs belonging to some tricycle of R; thus,

$$Q = R \cap R^{-1}R^{-1}. \tag{1}$$

We call Q the *tricyclic relation* of the tournament R. The pairs of R which do not belong to any tricycle of R form the relation

$$P = R \setminus Q = R \setminus (R^{-1}R^{-1}). \tag{2}$$

Notice that P is an irreflexive linear order if and only if $R = P$, that is, $Q = \varnothing$. In general, we refer to P as the *order* of the tournament R, a terminology justified by the next lemma.

Lemma 11 *Suppose that R is a tournament on X. Then the relation $P = R \setminus (R^{-1}R^{-1})$ is an irreflexive partial order on X.*

PROOF. Because R is irreflexive, so is P by definition. To prove the transitivity suppose that $x\,P\,y$ and $y\,P\,z$ with $x \neq z$ for three vertices x, y and z in X. Since $xy \notin R^{-1}R^{-1}$, we cannot have $z\,R\,x$; as R is a tournament, we must have $x\,R\,z$. In fact, $x\,P\,z$ is true because $x\,R^{-1}R^{-1}\,z$ cannot hold. Suppose indeed that $x\,R^{-1}R^{-1}\,z$. There must exists $w \in X$ such that $z\,R\,w$ and $w\,R\,x$. By the completeness of R, we must have either $y\,R\,w$ or $w\,R\,y$. The first case leads to $x\,R^{-1}R^{-1}\,y$ (via $y\,R\,w$ and $w\,R\,x$), and the second one to $y\,R^{-1}R^{-1}\,z$ (via $z\,R\,w$ and $w\,R\,y$), contradicting our hypothesis that $x\,P\,y$ and $y\,P\,z$. □

We now construct a partition of the tricyclic relation Q based on the fact that two pairs in Q can belong to the same tricycle.

Definition 12 Let Q be the tricyclic relation of a tournament R on X (cf. Definition 10). Let then $S \subseteq Q \times Q$ be a relation defined on Q by

$$xy\,S\,zw \quad \Longleftrightarrow \quad \begin{cases} y = z \quad \text{and} \quad x\,Q\,y\,Q\,w\,Q\,x, \\ \text{or} \\ w = x \quad \text{and} \quad x\,Q\,y\,Q\,z\,Q\,x. \end{cases}$$

Because S is symmetric, the reflexive and transitive closure $\widehat{S}$ of S is an equivalence relation on Q. The partition of Q induced by $\widehat{S}$ is the *tricyclic coloring* of Q (or of R); its classes will be called the *tricyclic colors* of Q or of R. Intuitively, pairs xy and zw of Q are in the same tricyclic color C when there is a sequence of pairs in Q, starting from xy and ending in zw, such that two successive pairs belong to some common tricycle. Such a sequence, which lies entirely in C, will be refered to as a *color sequence*. Notice that each tricylic color is a subset of Q, and thus a relation on X.

For an example of tricyclic coloring, take the tournament of Figure 1A; there is only one tricyclic color, namely $\{ab, bc, ca, cd, db\}$. In the tournament of Figure 1B, there are two tricyclic colors, namely $\{ae, eb, ba\}$ and $\{ad, dc, ca, cb, bd, ce, ed\}$.

Definition 13 Given a relation S on X, a vertex x is *covered* by S when there is $x\,S\,y$ or $y\,S\,x$ for some $y \in X$.

Lemma 14 *Let C be a tricyclic color of the tournament R on X, and let Y be the set of vertices covered by C. Then for all $x, y \in Y$ and $z \in X \setminus Y$:*

$$x\,R\,z \implies y\,R\,z.$$

Intuitively, all the vertices covered by a tricyclic color C behave the same way with respect to any vertex not covered by C.

PROOF. Assume first $t\,C\,u\,C\,v\,C\,t$ for some $t, u, v \in X$ (thus $t, u, v \in Y$), and $z \in X \setminus Y$. If $t\,R\,z$, then $v\,R\,z$. Indeed, $z\,R\,v$ would give the tricycle $z\,R\,v\,R\,t\,R\,z$, and by the definition of a color, we would have $tz \in C$, contradicting $z \notin Y$. Similarly, we deduce $u\,R\,z$ from $v\,R\,z$.

Now take x, y, z as in the statement of the lemma, with $x\,R\,z$. Because x is in Y, there must be some $s \in Y$ such that $xs \in C$ or $sx \in C$. Similarly, there must be some $t \in Y$ such that $yt \in C$ or $ty \in C$. Suppose that $xs, ty \in C$. (The argument is the same in the other cases.) By the definition of a color, there is a color sequence from xs to ty. The argument of the previous paragraph, applied to each step of that sequence, yields ultimately $y\,R\,z$. □

Suppose that two tricycles of different colors jointly cover exactly one vertex w. Thus, their union cover in all five vertices (see Figure 4) and four further pairs of those vertices lie in the tournament. The crux of Lemma 15 below is that these four pairs necessarily form two tricycles sharing a pair of one of the original tricycles and so are of the same color as that tricycle.

Lemma 15 *Suppose that two trios of a tournament R on X share a single vertex w. Suppose moreover that the pairs of the two corresponding tricycles belong to distinct tricyclic colors C and D, say $\{wx, xy, yw\} \subseteq C$ and $\{wu, uv, vw\} \subseteq D$. Then, we have*

either (i) $yu, vx, yv, ux \in C$,
or (ii) $yu, vx, vy, xu \in D$.

PROOF. We must have both $y\,R\,u$ and $v\,R\,x$ because otherwise the pairs of the two tricycles $w\,R\,u\,R\,y\,R\,w$ and $w\,R\,x\,R\,v\,R\,w$ would belong to the same tricyclic color. We now have two cases: (i) $y\,R\,v$, forming the tricycle $y\,R\,v\,R\,x\,R\,y$ in C and entailing $u\,R\,x$

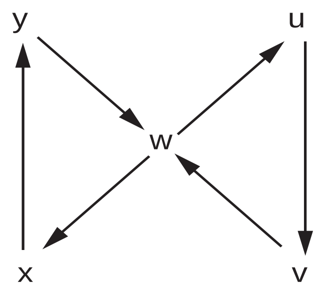

Figure 4: The hypotheses of Lemma 15 and the first step in the proof.

and the tricyle $u\,R\,x\,R\,y\,R\,u$ (because $x\,R\,u$ would yield $vx \in C \cap D$); (ii) $v\,R\,y$, forming the tricycle $v\,R\,y\,R\,u\,R\,v$ in D and entailing $x\,R\,u$ and the tricycle $x\,R\,u\,R\,v\,R\,x$ (because $u\,R\,x$ would yield $yu \in C \cap D$). These are the two cases of the lemma. □

Lemma 16 *Suppose that R is an indecomposable tournament on X, with Q its tricyclic relation and $\mathcal{Q}$ its tricyclic coloring. Then*

(i) *Q covers X;*

(ii) *$\mathcal{Q} = \{Q\}$, that is: there is only one tricyclic color.*

PROOF. Pick arbitrarily some tricyclic color C of $\mathcal{Q}$. Indecomposability of R, as characterized in Proposition 9, together with Lemma 14 imply that C covers X.

It remains to show $C = Q$, that is $C = D$ for any tricyclic color D. Proceeding by contradiction, we suppose $C \neq D$. Take any vertex $w \in X$. Because as just shown both C and D cover X, there must exist a tricycle $\{wx, xy, yw\} \subseteq C$ and another tricycle $\{wu, uv, vw\} \subseteq D$. This is the situation described by the hypotheses of Lemma 15. (A glance at Figure 4 may be helpful.) Thus, either Case (i) or Case (ii) of the Lemma must be true. There is no loss of generality in assuming that Case (i) holds, that is yu, vx, yv, $ux \in C$. Because D covers X, we must have $xk \in D$ or $kx \in D$ for some $k \in X \setminus \{x\}$. Thus according to Definition 12, there is some color sequence starting at wu and ending at xk or kx. Applying Lemma 15 repeatedly, we derive $x\ell \in C$ or $\ell x \in C$ for each ℓ covered by the pairs in the sequence. Then we have $xk \in C \cap D$ or $kx \in C \cap D$, giving $C = D$, a contradiction of our hypothesis $C \neq D$. Thus $C = D$ and therefore $C = Q$. □

Remark 17 The converse of Lemma 16 does not hold, as shown by the decomposable tournament R in Figure 3 (for which $Q = R \setminus \{ab\}$).

We now strengthen the necessary condition in Lemma 16 for a tournament to be indecomposable in order to get a necessary and sufficient condition. In view of later use in the proof of Theorem 21, we formulate the additional requirement in terms of the covering relation or Hasse diagram H of the order $P = R \setminus R^{-1} R^{-1}$ of R. Notice however that reformulating the quantification in Condition (iii) as "for any pair xy in P" would give an equally correct result (as seen from the next proof).

Proposition 18 *A tournament R on X, with $|X| \geq 3$, is indecomposable if and only if the order P and the tricyclic relation Q of R satisfy the following three conditions:*

(i) *Q covers X;*

(ii) *$\mathcal{Q} = \{Q\}$;*

(iii) *for any pair xy in the Hasse diagram H of P, there exists z in $X \setminus \{x, y\}$ satisfying $x\,R\,z\,R\,y$.*

PROOF. If R is indecomposable, Conditions (i) and (ii) hold by Lemma 16. For Condition (iii), the definition of indecomposability implies the existence of z in $X \setminus \{x, y\}$ such that either $x\,R\,z\,R\,y$ or $y\,R\,z\,R\,x$. The second formula cannot be true, because the pair xy, which lies in P, does not belong to any tricycle of R.

If R is decomposable, let us assume that Conditions (i) and (ii) hold, and derive that Condition (iii) fails. By assumption, there exists a proper subset Y of X with more than one element, and such that for $x, y \in Y$ and $z \in X \setminus Y$ we have $x\,R\,z$ implies $y\,R\,z$. Notice that Y cannot contain both vertices of any pair st from Q (otherwise Conditions (i) and (ii) could no be together true: some color sequence must start at st and lead to some pair covering a given vertex outside Y. At some step of the sequence, there appears a tricycle with two vertices in Y and one outside Y, contradicting the choice of Y). Thus all pairs of R formed by two vertices from Y are in P, in other words: P induces on Y a linear order. Let x be the minimum vertex for this order on Y, and let y be the next vertex in Y. Then the pair xy invalidates Condition (iii). Indeed, xy lies in the Hasse diagram H of P, otherwise there would exist $t \in X \setminus \{x, y\}$ satisfying $x\,P\,t\,P\,y$. By the choice of Y, we then have $t \in Y$, and this contradicts the choice of y. Finally, $x\,R\,z\,R\,y$ cannot hold for any $z \in X \setminus Y$ because of the choice of Y. □

Remark 19 The three conditions in Proposition 18 are independent. Three (necessarily decomposable) tournaments failing in turn exactly one of these three conditions are easily built. For instance, Figure 5 gives a counterexample for Condition (i), Figure 1B for Condition (ii) and Figure 3 for Condition (iii).

Corollary 20 *The problem of deciding whether a given tournament R on X is indecomposable is polynomial in the size of X.*

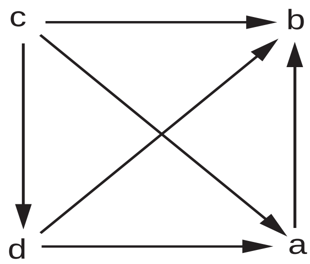

Figure 5: An example showing the independence of Condition (i) in Proposition 18.

PROOF. An algorithm directly based on Proposition 18 is outlined below. (We do not claim that this algorithm is the most efficient one.) Assume R is a tournament given on X, with $|X| \geq 3$.

MAIN STEP 1. Search for a pair uv belonging to some tricycle. If no such pair exists, output that R is decomposable and exit.

MAIN STEP 2. Build the tricyled color C of uv. If C does not cover X or does not contain all pairs belonging to some tricycle, ouptut that R is decomposable and exit.

MAIN STEP 3. Build $P = R \setminus C$. For each pair xy in the Hasse diagram of P, check that Condition (iii) from Proposition 18 holds. If it is the case, output that R is indecomposable, otherwise that R is decomposable.

Each of the three Main Steps above can be performed in time polynomial in $|X|$. □

3 The main result

Theorem 21 *A tournament is determined if and only if it is indecomposable.*

PROOF. Let R be a tournament on X and suppose that R is decomposable. Consider a subset Y invalidating the condition in Proposition 9 (thus $|Y| \geq 2$). Let T be the restriction of R to Y. Then R and $(R \setminus T) \cup T^{-1}$ are tournaments on X which have exactly the same trios. However, $(R \setminus T) \cup T^{-1}$ differs from both R and R^{-1}. Thus a decomposable tournament is not determined.

Conversely, assume R is an indecomposable tournament on X, with $|X| \geq 3$ (for $|X| = 2$, the Theorem holds trivially). By Lemma 16 the tricyclic relation Q of R covers X, and consequently $\overline{\mathcal{T}}_3(R) \neq \varnothing$. Select two vertices a, b in some trio. We may fix

$a\,R\,b$. The goal then is to show that for any other unordered pair $\{x, y\}$ of vertices, we can decide which of $x\,R\,y$ and $y\,R\,x$ holds, that is, we can recover $xy \in R$ or $yx \in R$. Applying inferences of Type 1 starting from $a\,Q\,b$, we are able to recover all pairs xy which belong to Q: this is true because of Lemma 16, which tells us that there is a color sequence from the pair ab to any other pair in Q, and at each step of the sequence we can apply a Type 1 inference. There remains to show that if neither xy nor yx is in Q, it is nevertheless possible to decide for $xy \in R$ or $yx \in R$ solely on the basis of Q and $\overline{\mathcal{T}}_3(R)$. (This is the situation in Figure 2B.) Suppose that $xy \in P$, with P the order of R. (The argument in the case $yx \in P$ is similar). Denote by $\tilde{P}$ the subset of P consisting of the pairs xy of P for which it has been proved that recovering of $xy \in R$ is possible. Thus, at the start, $\tilde{P}$ is empty, and we need to show that after all possible, repeated inferences have been made, $\tilde{P} = P$. We consider three cases for a pair $xy \in P$.

Case (i). There exists some vertex w such that $x\,P\,w$ and y, w are incomparable in P. Then either $y\,Q\,w$ or $w\,Q\,y$, and moreover we have been able to decide which one holds, see paragraph above. Assuming $y\,Q\,w$, we show that we can derive $xy, xw \in \tilde{P}$ by using inferences of Types 1 and 2; the other case, that is $w\,Q\,y$, is similar.

Because R is indecomposable and $|X| \geq 3$, there exists some vertex s with $s\,R\,x$. By Lemma 16, s is covered by the tricyclic color of yw, and this color is equal to the whole of Q. So, there is a color sequence from yw to some pair covering s. The first time a vertex c outside $\{t \in X \mid x\,P\,t\}$ appears in a pair of the sequence, we find vertices a, b, c such that ab is a pair of the sequence and $x\,P\,a$, $x\,P\,b$ and $a\,Q\,b\,Q\,c\,Q\,a$ hold but not $x\,P\,c$. As $c\,R\,x$ cannot hold (because of $x\,P\,b$ and $b\,R\,c$), we have $x\,R\,c$ and thus $x\,Q\,c$. Using $\{x, a, c\} \in \mathcal{T}_3$ together with $x\,Q\,c$ and $c\,Q\,a$, we obtain $x\,R\,a$ by an inference of Type 2, and then $x\,R\,b$ also by such an inference on $x\,R\,a$ and $a\,Q\,b$. Now, following the color sequence backwards from ab to yw and repeatedly applying inferences of Type 2, we deduce $xy, yw \in R$, thus $xy, yw \in \tilde{P}$.

Case (ii). There exists some vertex w such that $w\,P\,y$ and x, w are incomparable in P. This case can be settled in the same way as Case (i) was (in fact, replacing R with R^{-1} transforms Case (ii) into Case (i)).

Case (iii). We still need to establish $xy \in \tilde{P}$ in all situations not covered by Cases (i) or (ii). Let us first assume that xy is moreover in the Hasse diagram H of the order P. Because the tournament R is indecomposable and $|X| \geq 3$, Proposition 18 implies the existence of some vertex z such that $xz, zy \in R$. By the definition of the Hasse diagram H of R, we have $xz \notin P$ or $zy \notin P$. If both of these formulas hold, that is $xz, zy \in Q$, we deduce $xy \in \tilde{P}$ by an inference of Type 2 based on $\{x, z, y\} \in \mathcal{T}_3$. If $x\,P\,z$ and $z\,Q\,y$, we are in Case (i) and so we have $x\,\tilde{P}\,y$. In the last possibility, that is $x\,Q\,z$ and $z\,P\,y$, we are in Case (ii) and $x\,\tilde{P}\,y$ holds then also.

Now if $xy \in P \setminus H$, there exists a sequence $v_1 = x$, v_2, ..., $v_k = y$ of vertices such that $v_i\,H\,v_{i+1}$ for $i = 1, 2, \ldots, k-1$. By the preceding paragraph, $v_iv_{i+1} \in \tilde{P}$. For $i = 1$, $2, \ldots, k-2$, we deduce $v_iv_{i+2} \in \tilde{P}$ from $\{v_i, v_{i+1}, v_{i+2}\} \in \mathcal{T}_3$ by a Type 2 inference. Applying the same argument as many times as required, we will conclude $xy \in \tilde{P}$, which completes the proof. □

4 A generalization

Definition 4 can be generalized to other families of relations than tournaments (we mainly think here of a family of relations defined by first order axioms on a single relation, as for instance reflexiveness). The identity relation I on the set X consists of all loops xx. As in many cases adding or suppressing loops do not alter transitivity, we do not require that the loops of a relation be determined by its transitivity parts.

Definition 22 Let $\mathcal{C}$ be a family of relations on the set X. A relation R from $\mathcal{C}$ is *(transitivity) determined* when for any relation S from $\mathcal{C}$ the following holds: $\mathcal{T}(R) = \mathcal{T}(S)$ only if $R \Delta S \subseteq I$ or $R \Delta S^{-1} \subseteq I$ (here, Δ denotes symmetric difference). The whole family itself is *determined* if any of its relations is determined. In general, let $\mathcal{C}^*$ denote the subfamily of $\mathcal{C}$ consisting of the relations in $\mathcal{C}$ which are determined.

Notice that a relation is determined (or not) only with respect to some given family (changing the family may change the status of the relation determinateness). We now formulate a whole scheme of problems (one problem for each family of relations chosen).

Problem 23 *For a given family $\mathcal{C}$ of relations, find the subfamily $\mathcal{C}^*$ of determined relations, in the sense of Definition 22. Is the problem of deciding whether a relation from $\mathcal{C}$ is determined polynomial in the size of X?*

Particular cases of Problem 23 have been solved already. For instance, Theorem 21 settles the questions for the family $\mathcal{C}$ of tournaments (see also Boussaïri et al. [2004]); it states that $\mathcal{C}^*$ then consists of the indecomposable tournaments. Besides, Corollary 20 asserts that the corresponding decision problem is polynomial.

Next, consider the family $\mathcal{C}$ of all reflexive and symmetric relations on X. Remark that for any $R \in \mathcal{C}$, the subcollection $\mathcal{T}_3(R)$ conveys exactly the same information as $\mathcal{T}(R)$ because the restriction of R to a subset Y of X is transitive if and only if the restriction of R to any 3-element subset of Y is transitive. Moreover, any R in $\mathcal{C}$ corresponds exactly to one (simple) graph $G = (X, E)$, where $\{x, y\} \in E$ if and only if xRy. Under this recasting, finding all the relations R in $\mathcal{C}$ which are determined becomes a question discussed by Hayward [1996] under the following form. Recall that the P_3 structure of a graph G consists of the subsets of 3 vertices on which G induces a P_3 path. Thus, in this case, Problem 23 becomes: Which graphs G are recoverable from their P_3 structure? This question was partially (but elegantly) solved by Hayward [1996, Theorem 4.3 and Corollary 4.4], which in particular builds a polynomial algorithm for recognition.

For the family $\mathcal{C}$ of all relations on X, we are intrigued by the difficulty of the resulting instance of Problem 23. An example of determined relation is the full relation R with no loop (notice $\{a, b\} \notin \mathcal{T}(R)$, for any distinct vertices a, b, which in turn implies $ab, ba \in R$).

Much more ambitious problems are the following general ones.

Problem 24 *Find all families of relations which are determined in the sense of Definition 22.*

Problem 25 *Characterize those families $\mathcal{C}$ for which deciding whether a relation from $\mathcal{C}$ is determined is a polynomial problem.*

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Differential approximation of MIN SAT, MAX SAT and related problems

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Abstract

We present differential approximation results (both positive and negative) for optimal satisfiability, optimal constraint satisfaction, and some of the most popular restrictive versions of them. As an important corollary, we exhibit an interesting structural difference between the landscapes of approximability classes in standard and differential paradigms.

Key words : Satisfiability, Polynomial Approximation, Differential Ratio

1 Introduction and preliminaries

In this paper we deal with the approximation of some of the most famous and classical problems in the domain of the polynomial time approximation theory, the MIN and MAX SAT as well as the MIN and MAX DNF and some of their restricted versions, namely MAX and MIN k and EkSAT and MAX and MIN k and EkDNF. We study their approximability using the so-called *differential approximation ratio* which, informally, for an instance x of a combinatorial optimization problem Π, *measures the relative position of the value of an approximated solution in the interval between the worst-value of x, i.e., the value of a worst feasible solution of x, and optimal-value of x, i.e., the value of a best solution of x.*

Given a set of clauses (i.e., disjunctions) $C_1, \ldots, C_m$ on n variables $x_1, \ldots, x_n$, MAX SAT (resp., MIN SAT) consists of determining a truth assignment to the variables that maximizes (minimizes) the number of clauses satisfied. On the other hand, given a set of cubes (i.e., conjunctions) $C_1, \ldots, C_m$ on n variables $x_1, \ldots, x_n$, MAX DNF (resp., MIN

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DNF) consists of determining a truth assignment to the variables that maximizes (minimizes) the number of conjunctions satisfied. For an integer $k \geqslant 2$, MAX kSAT, MAX kDNF, MIN kSAT, MIN kDNF (resp., MAX EkSAT, MAX EkDNF, MIN EkSAT, MIN EkDNF) are the versions of MAX SAT, MAX DNF, MIN SAT, MIN DNF where each clause or conjunction has size at most (resp., exactly) k. Finally, let us quote two particular weighted satisfiability versions, namely, MAX WSAT and MIN WSAT. In the former, given a set of clauses $C_1, \ldots, C_m$ on n variables $x_1, \ldots, x_n$, with non-negative integer weights $w(x)$ on any variable x, we wish to compute a truth assignment to the variables that both satisfies all the clauses and maximizes the sum of the weights of the variables set to 1. We consider that the assignment setting all the variables to 0 (even if it does not satisfy all the clauses) is feasible and represents the worst-value solution for the problem. The latter problem is similar to the former one, up to the fact that we wish to minimize the sum of the weights of the variables set to 1 and that feasible is now considered the assignment setting all the variables to 1.

A problem Π in **NPO** is a quadruple $(\mathcal{I}_\Pi, \mathrm{Sol}_\Pi, m_\Pi, \mathrm{opt}(\Pi))$ where:

- $\mathcal{I}_\Pi$ is the set of instances (and can be recognized in polynomial time);
- given $x \in \mathcal{I}_\Pi$, $\mathrm{Sol}_\Pi(x)$ is the set of feasible solutions of x; the size of a feasible solution of x is polynomial in the size $|x|$ of the instance; moreover, one can determine in polynomial time if a solution is feasible or not;
- given $x \in \mathcal{I}_\Pi$ and $y \in \mathrm{Sol}_\Pi(x)$, $m_\Pi(x, y)$ denotes the value of the solution y of the instance x; m_Π is called the objective function, and is computable in polynomial time; we suppose here that $m_\Pi(x, y) \in \mathbb{N}$;
- $\mathrm{opt}(\Pi) \in \{\min, \max\}$.

Given an instance x of an optimization problem Π and a feasible solution $y \in \mathrm{Sol}_\Pi(x)$, we denote by $\mathrm{opt}_\Pi(x)$ the value of an optimal solution of x, and by $\omega_\Pi(x)$ the value of a worst solution of x. The *standard approximation ratio* of y is defined as $r_\Pi(x, y) = m_\Pi(x, y)/\mathrm{opt}_\Pi(x)$, while the *differential approximation ratio* of y is defined as $\delta_\Pi(x, y) = |m_\Pi(x, y) - \omega_\Pi(x)|/|\mathrm{opt}_\Pi(x) - \omega_\Pi(x)|$.

For a function f of $|x|$, an algorithm is a *standard f-approximation algorithm* (resp., *differential f-approximation algorithm*) for a problem Π if, for any instance x of Π, it returns a solution y such that $r(x, y) \leqslant f(|x|)$, if $\mathrm{opt}(\Pi) = \min$, or $r(x, y) \geqslant f(|x|)$, if $\mathrm{opt}(\Pi) = \max$ (resp., $\delta(x, y) \geqslant f(|x|)$).

With respect to the best approximation ratios known for them, **NPO** problems can be classified into approximability classes. The most notorious among them are the following:

APX or **DAPX**: the class of problems for which there exists a polynomial algorithm achieving standard or differential approximation ratio $f(|x|)$ where function f is constant (it does not depend on any parameter of the instance);

PTAS or **DPTAS**: the class of problems admitting a polynomial time approximation schema; such a schema is a family of polynomial algorithms $\mathtt{A}_\varepsilon$, $\varepsilon \in]0,1]$, any of them guaranteeing approximation ratio $1-\varepsilon$ (under the differential approximation paradigm and under the standard one in the case where $\mathrm{opt}(\Pi) = \max$), or $1+\varepsilon$ (under the standard approximation paradigm in the case where $\mathrm{opt}(\Pi) = \min$);

FPTAS and **DFPTAS**: the class of problems admitting a fully polynomial time approximation schema; such a schema is a polynomial time approximation schema $(\mathtt{A}_\varepsilon)_{\varepsilon\in]0,1]}$, where the complexity of any $\mathtt{A}_\varepsilon$ is polynomial in both the size of the instance and in $1/\varepsilon$.

We now define a kind of reduction, called *affine reduction* and denoted by AF, which, as we will see, is very natural in the differential approximation paradigm.

Definition 1. Let Π and Π' be two **NPO** problems. Then, Π AF-reduces to Π' ($\Pi \leq_{\mathsf{AF}} \Pi'$), if there exist two functions f and g such that:

1. for any $x \in \mathcal{I}_\Pi$, $f(x) \in \mathcal{I}_{\Pi'}$;
2. for any $y \in \mathrm{Sol}_{\Pi'}(x)$, $g(x,y) \in \mathrm{Sol}_\Pi(x)$; moreover, $\mathrm{Sol}_\Pi(x) = g(x, \mathrm{Sol}_{\Pi'}(f(x)))$;
3. for any $x \in \mathcal{I}_\Pi$, there exist $K \in \mathbb{R}$ and $k \in \mathbb{R}^\star$ ($k > 0$ if $\mathrm{opt}(\Pi) = \mathrm{opt}(\Pi')$, $k < 0$, otherwise) such that, for any $y \in \mathrm{Sol}_{\Pi'}(f(x))$, $m_{\Pi'}(f(x), y) = k m_\Pi(x, g(x,y)) + K$.

If $\Pi \leq_{\mathsf{AF}} \Pi'$ and $\Pi' \leq_{\mathsf{AF}} \Pi$, then Π and Π' are called *affine equivalent*. This equivalence will be denoted by $\Pi \equiv_{\mathsf{AF}} \Pi'$. ∎

It is easy to see that differential approximation ratio is stable under affine reduction. Formally, *if, for* $\Pi, \Pi' \in$ ***NPO****,* $\mathsf{R} = (f,g)$ *is an AF-reduction from* Π *to* Π'*, then for any* $x \in \mathcal{I}_\Pi$ *and for any* $y \in \mathrm{Sol}_{\Pi'}(f(x))$, $\delta_\Pi(x, g(x,y)) = \delta_{\Pi'}(f(x), y)$. Indeed, by Condition 2 of Definition 1, worst and optimal solutions in x and $f(x)$ coincide. Since the value of any feasible solution of Π' is an affine transformation of the same solution seen as a solution of Π, the differential ratios for y and $g(x,y)$ coincide also. Hence, the following holds.

Proposition 1. *If* $\Pi \equiv_{\mathsf{AF}} \Pi'$*, then, for any constant* r*, any* r*-differential approximation algorithm for one of them is an* r*-differential approximation algorithm for the other one.*

Optimization satisfiability problems as MIN SAT and MAX SAT are of great interest from both theoretical and practical points of view. On the one hand, the satisfiability problem (SAT) is the first complete problem for **NP** and MAX SAT, MIN SAT have generalizations or restrictions that are the first problems proved complete for numerous approximation classes under various approximability preserving reductions ([4, 19]). For instance, MAX 3SAT is **APX**-complete under the AP-reduction and **Max-SNP**-complete under the L-reduction ([17]), MAX WSAT and MIN WSAT are **NPO**-complete under the AP-reduction ([8]), etc. In general, many optimal satisfiability problems have for the polynomial approximation theory the same status as SAT for **NP**-completeness theory. On the other hand, many problems in mathematical logic and in artificial intelligence can be expressed in terms of versions of SAT; constraints satisfaction is one such version. Also problems in database integrity constraints, query optimization, or in knowledge bases can be seen as optimization satisfiability problems. Finally, some approaches to inductive inference can be modeled as MAX SAT problems ([13, 14]). The interested reader can be referred to [5] for a survey on standard approximability of optimization satisfiability problems.

Let us note that differential approximability of the problems dealt here, has already been studied in [6]. There, among other results, it was shown that MAX SAT and MIN DNF, as well as MIN SAT and MAX DNF are equivalent for the differential approximation, that all these problems are not solvable by polynomial time differential approximation schemata, unless **P** $=$ **NP**, and, finally, that MIN SAT cannot be approximately solved within differential approximation ratio $1/m^{1-\epsilon}$, for any $\epsilon > 0$ (where m is the number of the clauses in its instance), unless **NP** $=$ **co-RP**. Finally, let us mention here that both MAX WSAT and MIN WSAT belong to **0-DAPX**, the class of the problems for which no algorithm can guarantee differential approximation ratio strictly greater than 0, unless **P** $=$ **NP** ([16]). This class has been also introduced in [6].

	Approximation ratios	Inapproximability bounds
MAX SAT	$4.34/(m+4.34)$	$\notin$ **DAPX**
MAX E2SAT	$17.9/(m+19.3)$	11/12
MAX 3SAT	$4.57/(m+5.73)$	1/2
MAX E3SAT	$8/(m+8)$	1/2
MAX EkSAT	$2^k/(m+2^k)$	$1/p$, p the largest prime s.t. $3(p-1) \leqslant k$
MIN SAT	$2/(m+2)$	
MIN (E)kSAT	$2^k/((2^{k-1}-1)m+2^k)$	$1/p$, p the largest prime s.t. $3(p-1) \leqslant k$
MIN 2SAT	$4/(m+4)$	11/12

Table 1: Summary of the main results of the paper.

In this paper, we further study differential approximability of MAX SAT, MIN SAT, MIN DNF and MAX DNF, and give approximation results and inapproximability bounds for several versions of these problems. A summary of the main results obtained is presented in Table 1. As one can see from the second column of the first line of this table, MAX SAT is not approximable within a constant approximation ratio, unless **P** $=$ **NP**. This result is very interesting since it indicates that **Max-NP** ([17]) is not included in **DAPX**. This is an important difference with the standard approximability classes landscape where **Max-NP** $\subset$ **APX**. Another assessment with respect to our results is that the gap between lower and upper approximation bounds for the problems dealt is still large. However, this paper undertakes a systematic study of satisfiability problems in the differential paradigm, it extends the results of [6] and shows that none of the most classical satisfiability problems is in **0-DAPX**. This approximability class has been introduced in [6] and represents the worst possible configuration for differential approximation since it includes the problems for which no polynomial time approximation algorithm can guarantee differential ratio greater than 0. Inclusion of the problems dealt here in **0-DAPX** or not, was a major question we handled since [6].

2 Affine reductions between optimal satisfiability problems

Let us first note that there does not exist general technique in order to transfer approximation results from differential (resp., standard) paradigm to standard (resp., differential) one, except for the case of maximization problems and for transfers between differential and standard paradigms. Proposition 2 just below deals with this last case.

Proposition 2. *If a maximization problem* Π *can be solved within differential approximation ratio* δ*, then it can be solved within standard approximation ratio* δ*, also.*

Proof. Consider any differential polynomial time approximation algorithm A guaranteeing differential-approximation ratio δ for any instance x of a maximization problem Π. Denote by $\mathtt{A}(x)$, a solution computed by A when running on x. Then,

$$\frac{m(x,\mathtt{A}(x)) - \omega(x)}{\operatorname{opt}(x) - \omega(x)} \geqslant \delta \Longrightarrow m(x,\mathtt{A}(x)) \geqslant \delta \operatorname{opt}(x) + (1-\delta)\omega(x) \overset{\substack{\delta \leqslant 1 \\ \omega(x) \geqslant 0}}{\Longrightarrow} \frac{m(x,\mathtt{A}(x))}{\operatorname{opt}(x)} \geqslant \delta$$

and the claim of the proposition is proved. ∎

Corollary 1. *Any standard inapproximability bound for a maximization problem* Π *is also a differential inapproximability bound for* Π.

We give in this section some affine reductions and equivalences between the problems dealt in the paper. These results will allow us to focus ourselves only in the study of MAX SAT, MIN SAT and their restrictions without studying explicitly MAX and MIN DNF. We first recall a result already proved in [6].

Proposition 3. ***([6])*** MAX SAT $\equiv_{\mathsf{AF}}$ MIN DNF *and* MIN SAT $\equiv_{\mathsf{AF}}$ MAX DNF.

The following proposition shows that one can affinely pass from MAX EkSAT to MAX E$(k+1)$SAT. This, allows us to transfer inapproximability bounds from MAX E3SAT to MAX EkSAT, for any $k \geqslant 4$.

Proposition 4. MAX EkSAT $\leq_{\mathsf{AF}}$ MAX E$(k+1)$SAT.

Proof. Consider an instance φ of MAX EkSAT on n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. Consider also a new variable y and build formula φ', instance of MAX E$(k+1)$SAT as follows: for any clause $C_i = (\ell_{i_1}, \ldots, \ell_{i_k})$ of φ, where, for $j = 1, \ldots, k$, ℓ_{i_j} is a literal associated with x_{i_j}, φ' contains two new clauses $(\ell_{i_1}, \ldots, \ell_{i_k}, y)$ and $(\ell_{i_1}, \ldots, \ell_{i_k}, \bar{y})$. Hence, φ' is the conjunction of $2m$ clauses of size $k+1$ on $n+1$ variables. Assume any truth assignment T on the variables of φ and denote by $(T, 1)$ (resp., $(T, 0)$) the extension of T on φ' by setting $y = 1$ (resp., $y = 0$). Then, it is easy to see that $m(\varphi', (T, 1)) = m(\varphi', (T, 0)) = m + m(\varphi, T)$.

In other words, reduction just described, associating to any assignment T' of φ' its restriction T on variables $x_1, \ldots, x_n$ as assignment for φ, is affine and the proof of the proposition is complete. ∎

We now show that, for k fixed, problems kSAT and kDNF are affine equivalent.

Proposition 5. *For any fixed k,* MAX kSAT, MIN kSAT, MAX kDNF, MIN kDNF, MAX EkSAT, MIN EkSAT, MAX EkDNF *and* MIN EkDNF *are all affine equivalent.*

Proof. We first prove affine equivalence between MAX kSAT and MIN kSAT. Given n variables $x_1, \ldots, x_n$, denote by $\mathcal{C}_k$ the set of clauses of size k and by $\mathcal{C}_{\leqslant k}$ the set of clauses of size at most k on the set $\{x_1, \ldots, x_n\}$. Let us remark that any truth assignment verifies the same number v_k of clauses on $\mathcal{C}_k$ and the same number $v_{\leqslant k}$ of clauses on $\mathcal{C}_{\leqslant k}$. Note also that, since k is assumed fixed, sets $\mathcal{C}_k$ and $\mathcal{C}_{\leqslant k}$ are of polynomial size.

Let φ be an instance of MAX EkSAT on variable-set $\{x_1, \ldots, x_n\}$ and on a set $\mathcal{C} = \{C_1, \ldots, C_m\}$ of m clauses. Consider instance φ' on the clause-set $\mathcal{C}' = \mathcal{C}_k \setminus \mathcal{C}$. Then, for any truth assignment T on $\{x_1, \ldots, x_n\}$: $m(\varphi, S) + m(\varphi', S) = v_k$; in other words, reduction just described is an affine reduction from MAX EkSAT to MIN EkSAT. Considering φ as instance of MIN EkSAT this time, the above describe an affine reduction from MIN EkSAT to MAX EkSAT.

Furthermore, if $\mathcal{C}$ is an instance of MAX kSAT, then we can see the clause-set $\mathcal{C}_{\leqslant k} \setminus \mathcal{C}$ as an instance of MIN kSAT and the same arguments conclude an affine reduction from the former to the latter problem.

We now prove equivalence between versions of SAT and the corresponding versions of DNF. Given a clause $C = (\ell_{i_1} \vee \ldots \vee \ell_{i_k})$ on k literals, we build the cube (conjunction) $D = (\bar{\ell}_{i_1} \wedge \ldots \wedge \bar{\ell}_{i_k})$. Any truth assignment T on ℓ_{i_j} verifies C, if and only if it does not verify D, i.e., $m(C,T) = m - m(D,T)$. This specifies an affine reduction between MAX EkSAT and MIN EkDNF, MIN EkSAT and MAX EkDNF, MAX kSAT and MIN kDNF and between MIN kSAT and MAX kDNF.

We finally show equivalence between MAX kSAT and MAX EkSAT. We first notice that the latter problem being a sub-problem of the former one, direction MAX EkSAT $\leq_{\mathsf{AF}}$ MAX kSAT is immediate. On the other hand, as in Proposition 4, given an instance of MAX kSAT, one can construct, for any clause of size at most k, a set of clauses of size exactly k, in such a way that this reduction is affine.

Combination of equivalences shown above completes the proof of the proposition. ∎

It is shown in [12] (see also [4]), that MAX E3SAT is inapproximable within standard approximation ratio $(7/8) + \epsilon$, for any $\epsilon > 0$, and MAX E2SAT is inapproximable within standard approximation ratio $(21/22) + \epsilon$, for any $\epsilon > 0$ (in what follows for such results we will use, for simplicity, expression "within better than"). Discussion above, together with these bounds leads to the following result.

Proposition 6. MAX 2SAT, MAX E2SAT, MIN 2SAT, MIN E2SAT, MAX 2DNF, MAX E2DNF, MIN 2DNF *and* MIN E2DNF *are inapproximable within differential approximation ratio better than 21/22. Furthermore, for any* $k \geqslant 3$*,* MAX kSAT, MAX EkSAT, MIN kSAT *and* MIN EkSAT, MAX kDNF, MAX EkDNF, MIN kDNF *and* MIN EkDNF*, are inapproximable within differential approximation ratio better than 7/8.*

Proof. Concerning MAX 2SAT and associates, Corollary 1 extends the result of [12] to the differential paradigm. Then, Proposition 5 suffices to conclude the proof.

For MAX kSAT and associates, Corollary 1 extends the result of [15] to the differential paradigm, for MAX 3SAT and Proposition 5 transfers it to MAX E3SAT. Then, Proposition 4 extends it for any $k \geqslant 4$. Finally, Proposition 5 suffices to conclude the proof. ∎

Since the satisfiability problems stated in Proposition 6 are particular cases either of MAX SAT, or of MIN SAT, or of MAX DNF, or, finally, of MIN DNF, application of Proposition 6 and of Proposition 3 concludes the following corollary.

Corollary 2. MAX SAT, MIN SAT, MAX DNF *and* MIN DNF *are inapproximable within differential approximation 7/8.*

Results of Corollary 2 are not the best ones. In Section 4, we strengthen the one for MAX SAT. On the other hand, as it is proved in [6], MIN SAT is inapproximable within differential ratio better than $m^{\epsilon-1}$, for any $\epsilon > 0$. Proposition 5 has to be used with some precautions in order to yield positive or negative approximation results. Indeed, if one of the problem stated in it is approximable within constant differential approximation ratio (i.e., within ratio that does not depend on an instance parameter), then this ratio is naturally transferred to all the other problems. A contrario, one can see in the proof of Proposition 5 that in many cases the number of the clauses for the derived instance can be much larger that the one for the initial instance. In such cases, if we deal with ratios functions of m the form of these ratios is certainly preserved but not their value. For instance, assume that some problem Π among the ones stated in Proposition 5 is approximable within ratio $f(|\varphi|)$, where $|\varphi|$ denotes the number of clauses, or cubes, in φ, and f decreases with $|\varphi|$. Assume also that there exists another problem Π' (among the ones stated in Proposition 5) such that $\Pi' \leq_{\mathsf{AF}} \Pi$ and, furthermore, that this affine reduction transforms a formula φ' of Π' into a formula φ for Π. Then, it transforms an approximation ratio $f(|\varphi|)$ for the latter into an approximation ratio $f(|\varphi|)$ for the former but, if the values $|\varphi|$ and $|\varphi'|$ are very different the one from the other, then $f(|\varphi|) \neq f(|\varphi'|)$.

In fact, one can easily observe that affine reductions of Proposition 5 perform the following differential ratio transformations:

- reduction from MAX EkSAT to MIN EkSAT transforms ratios $f(m,n)$ into $f((2n)^k - m, n)$;
- reduction from MAX kSAT to MIN kSAT transforms ratios $f(m,n)$ into $f((2n+1)^k - m, n)$;
- reductions between SAT and DNF are invariant for approximation ratios;
- reduction from MAX kSAT to MAX EkSAT transforms ratios $f(m,n)$ into $f(2^{k-1}m, n+k-1)$.

In other words, dealing with common approximability of the problems stated in Proposition 5, the following remarks hold:

- if one of these problems is in **DAPX**, then all the other ones are so;
- problems MAX kSAT,MAX EkSAT, MIN k DNF and MIN EkDNF are approximable within differential ratios of $O(f(m))$ for a function f strictly decreasing with m if and only if one of them is $O(f(m))$ differentially approximable for $f(m) = O(m^{\alpha})$, for some $\alpha > 0$, or $f(m) = O(\log m)$; the same holds for the quadruple MIN kSAT,MIN EkSAT, MAX k DNF and MAX EkDNF;

- all problems are in **Log-DAPX** (the class of problems differentially approximable within ratios of $O(1/\log|x|)$) if and only if one of them is so (observe that reductions dealt transform differential ratios of $O(\log m)$ into ratios of the form $O(\log m)$ or of $O(\log n)$, and ratios of $O(\log n)$ into ratios of the same form).

Finally, reduction of Proposition 4 transforms ratios $f(m,n)$ into $f(2m,n+1)$.

3 Positive results

3.1 Maximum satisfiability

Consider an instance φ of an optimal satisfiability problem, defined on n boolean variables $x_1,\ldots,x_n$ and m clauses $C_1,\ldots,C_m$; consider also the very classical algorithm RSAT assigning at any variable value 1 with probability 1/2 and, obviously, value 0 with probability 1/2.Then, denoting by Sol(φ), the set of the 2^n possible truth assignments for φ, and by $E(\texttt{RSAT}(\varphi))$ the expectation of a solution computed by RSAT when running on φ, the following holds: $E(\texttt{RSAT}(\varphi)) = \sum_{T\in\mathrm{Sol}(\varphi)} m(\varphi,T)/2^n$.

Algorithm RSAT can be derandomized by the following technique denoted by DSAT. For $i=1,\ldots,n$:

- compute $E'_i = E(m(\varphi,T)|x_i=1)$ and $E''_i = E(m(\varphi,T)|x_i=0)$, where T is a random assignment and the values of the $i-1$ first variables have already been fixed in iterations $1,\ldots i-1$;
- set $x_i=1$, if $E'_i \geqslant E''_i$; otherwise, set $x_i=0$.

Lemma 1. $m(\varphi,\texttt{DSAT}(\varphi)) \geqslant E(\texttt{RSAT}(\varphi))$.

Proof. It is easy to see that $E(\texttt{RSAT}(\varphi)) = (E'_1/2)+(E''_1/2)$; hence $\max\{E'_1,E''_1\} \geqslant E(\texttt{RSAT}(\varphi))$. Furthermore, at any of the n steps of DSAT, $\max\{E'_i,E''_i\} = (E'_{i+1}/2)+(E''_{i+1}/2) \leqslant \max\{E'_{i+1},E''_{i+1}\}$. Consequently, we have $E(\texttt{RSAT}(\varphi)) \leqslant \max\{E'_1,E''_1\} \leqslant \max\{E'_n,E''_n\} = \texttt{DSAT}(\varphi)$, that concludes the proof of the lemma. ∎

Note finally, that DSAT is polynomial since, for any $i=1,\ldots,n$, computation of E'_i and E''_i is performed in polynomial time. Indeed, for any such computation it suffices to determine with what probability any clause of φ is satisfied and to sum these probabilities over all the clauses of φ.

We are ready now to state and prove positive differential approximation results for the problems dealt here.

Proposition 7. *Algorithm DSAT achieves for* MAX EkSAT *differential approximation ratio* $2^k/(\text{opt}(\varphi) + 2^k)$*. This ratio is bounded below by* $2^k/(m + 2^k)$*.*

Proof. Note first that we can assume that $\text{opt}(\varphi) > \omega(\varphi)$ (otherwise, MAX EkSAT would be polynomial on φ). Then,

$$\omega(\varphi) < E(\texttt{RSAT}(\varphi)) \leqslant m(\varphi, \texttt{DSAT}(\varphi)) \tag{1}$$

From (1) and given that feasible values of MAX EkSAT are integer, we get:

$$m(\varphi, \texttt{DSAT}(\varphi)) - \omega(\varphi) \geqslant 1 \tag{2}$$

Since clauses in φ are of size k, the expectation that any of them is satisfied equals $1-2^{-k}$. Hence,

$$m(\varphi, \texttt{DSAT}(\varphi)) \geqslant E(\texttt{RSAT}(\varphi)) = m\left(1 - \frac{1}{2^k}\right) \geqslant \text{opt}(\varphi)\left(1 - \frac{1}{2^k}\right) \tag{3}$$

Using (2) and (3), we get:

$$\delta(\varphi, \texttt{DSAT}(\varphi)) \geqslant \max\left\{\frac{1}{\text{opt}(\varphi) - \omega(\varphi)}, \frac{\text{opt}(\varphi)\left(1 - \frac{1}{2^k}\right) - \omega(\varphi)}{\text{opt}(\varphi) - \omega(\varphi)}\right\} \tag{4}$$

The first term in (4) is increasing with $\omega(\varphi)$, while the second one is decreasing. Equality holds when $\omega(\varphi) = (\text{opt}(\varphi)(1 - 2^{-k})) - 1$. In this case, (4) gives

$$\delta(\varphi, \texttt{DSAT}(\varphi)) \geqslant \frac{2^k}{\text{opt}(\varphi) + 2^k} \geqslant \frac{2^k}{m + 2^k} \tag{5}$$

Last inequality in (5) holding thanks to the fact that $\text{opt}(\varphi) \leqslant m$, qed. ∎

Notice that the ratio claimed by Proposition 7 increases with k. This is quite natural since for $k > \log m$, MAX kSAT is polynomial. Indeed, using (3) with such a k, we get $m(\varphi, \mathsf{DSAT}(\varphi)) \geqslant m - (m/2^k) > m - 1$, i.e., $m(\varphi, \mathsf{DSAT}(\varphi)) = m$, since the feasible values of MAX kSAT are integer.

We now propose a reduction transferring approximation results for MAX SAT problems from standard to differential paradigm. It will be used in order to achieve differential approximation results for MAX SAT, MAX 3SAT and MAX 2SAT.

Proposition 8. *If a maximum satisfiability problem is approximable on an instance* φ*, within standard approximation ratio* ρ*, then it is approximable in* φ *within differential approximation ratio* $\rho/((1 - \rho)\omega(\varphi) + 1)$*.*

Proof. Fix any maximum satisfiability problem Π, sharing the ones dealt until now, and assume that there exists a polynomial time algorithm achieving standard approximation ratio ρ for Π. Consider an instance φ of Π, run both A and DSAT on φ and retain assignment T satisfying the maximum number of clauses between $\mathtt{A}(\varphi)$ and $\mathtt{DSAT}(\varphi)$. Obviously, $m(\varphi, T) \geqslant \rho\,\mathrm{opt}(\varphi)$. Hence, the differential approximation ratio of T is

$$\delta(\varphi, T) \geqslant \frac{m(\varphi, T) - \omega(\varphi)}{\frac{m(\varphi,T)}{\rho} - \omega(\varphi)} \tag{6}$$

Since, as we have seen in the proof of Proposition 7, $m(\varphi, T) \geqslant \omega(\varphi) + 1$, (6) becomes

$$\delta(\varphi, T) \geqslant \frac{1}{\frac{\omega(\varphi)+1}{\rho} - \omega(\varphi)} = \frac{\rho}{(1-\rho)\omega(\varphi) + 1} \tag{7}$$

The proof of the proposition is now complete. ∎

From the result of Proposition 8, we can deduce several corollaries by specifying values for $\omega(\varphi)$ and ρ. The main such corollaries are stated in the propositions that follow. Before stating and proving them, let us remark that, in the case of MAX kSAT

$$E(\mathtt{RSAT}(\varphi)) \leqslant m\left(1 - \frac{1}{2^k}\right) \tag{8}$$

Then (1) and (8) yield:

$$\omega(\varphi) \leqslant m\left(1 - \frac{1}{2^k}\right) \tag{9}$$

Proposition 9. MAX SAT *is approximable within differential approximation ratio* $4.34/(m+4.34)$.

Proof. We can assume $\omega(\varphi) \leqslant m - 1$, otherwise ($\omega(\varphi) = m$) all feasible solutions of φ have the same value. Since $1 - \rho \geqslant 0$, the differential ratio of (7) decreases with $\omega(I)$. So, it suffices to substitute $m - 1$ for $\omega(\varphi)$, to use the fact that MAX SAT is approximable within standard ratio $1/1.2987$ ([3]), and the proof of the proposition is complete. ∎

Proposition 10. MAX 2SAT *is approximable within differential approximation ratio* $17.9/(m + 19.3)$*, and* MAX 3SAT *within* $4.57/(m + 5.73)$.

Proof. For MAX 2SAT, remark first that, using (3), the expectation of the solution computed by the random algorithm RSAT is, using (9), less than, or equal to, $3m/4$. Consequently, $\omega(\varphi) \leqslant 3m/4$. Next, the fact that MAX SAT is approximable within standard ratio $1/1.0741$ ([10]) suffices to conclude the proof.

For MAX 3SAT, $\omega(\varphi) \leqslant 7m/8$ and $\rho = 1/1.249$ ([18]). ∎

3.2 Minimum satisfiability

We finish this section by studying MIN SAT and some of its versions. Before stating our results, we note that algorithm RSAT can be derandomized in an exactly symmetric way, in order to provide a solution for MIN kSAT with value smaller than expectation's value.

Proposition 11. *If a minimum satisfiability problem is approximable on an instance φ, within standard approximation ratio ρ, then it is approximable in φ within differential approximation ratio*

$$\frac{\rho}{(\rho-1)\left(1-\frac{1}{2^k}\right)m+\rho}$$

Proof. As in the proof of Proposition 7, since we deal with a minimization problem, (1) becomes:

$$\operatorname{opt}(\varphi) \leqslant m(\varphi, \mathtt{DSAT}(\varphi)) \leqslant E(\mathtt{RSAT}(\varphi)) < \omega(\varphi) \tag{10}$$

Consequently, (2) becomes:

$$\omega(\varphi) - m(\varphi, \mathtt{DSAT}(\varphi)) \geqslant 1 \tag{11}$$

Considering the best among the solutions computed by DSAT and A (the ρ-standard approximation algorithm assumed for MIN kSAT in the statement of the theorem), denoting it by T and using (10) and (11), we get:

$$\delta(\varphi, T) \geqslant \max\left\{\frac{1}{\omega(\varphi)-\operatorname{opt}(\varphi)}, \frac{\omega(\varphi)-\rho\operatorname{opt}(\varphi)}{\omega(\varphi)-\operatorname{opt}(\varphi)}, \frac{\omega(\varphi)-m\left(1-\frac{1}{2^k}\right)}{\omega(\varphi)-\operatorname{opt}(\varphi)}\right\} \tag{12}$$

where the third term in (12) is due to the fact that T has a better value than the value of algorithm RSAT.

The first term in (12) is decreasing with $\omega(\varphi)$, while the second and third ones are increasing. We distinguish two cases depending on the relation between these terms.

If the second term is greater than the third one, i.e., if $\rho\operatorname{opt}(\varphi) \leqslant m(1-2^{-k})$, then equality of the first two terms of (12) is achieved when $\omega(\varphi) = 1 + \rho\operatorname{opt}(\varphi)$. In this case, (12) gives:

$$\delta(\varphi, T) \geqslant \frac{\rho}{(\rho-1)m\left(1-\frac{1}{2^k}\right)+\rho} \tag{13}$$

If, on the other hand, second term is smaller than the third one, i.e., if $\rho\operatorname{opt}(\varphi) \geqslant m(1-2^{-k})$, then equality of the first and the third term in (12) is achieved when $\omega(\varphi) = m(1-2^{-k})+1$. In this case also, $\delta(\varphi, T)$ verifies (13). The proof of the proposition is now complete. ∎

The best standard approximation ratios known for MIN kSAT and MIN SAT are $2(1-2^{-k})$ and 2, respectively ([7]). With the ratio just mentioned for MIN kSAT, the result of Proposition 11 can be simplified as indicated in the following corollary.

Corollary 3. MIN kSAT *is approximable within differential ratio* $2^k/((2^{k-1}-1)m+2^k)$.

Proposition 12. MIN SAT *is approximable within differential ratio* $2/(m+2)$.

Proof. Use Proposition 11 with $\rho = 2$ ([7]). ∎

Also, using Corollary 3 with $k = 2$ and $k = 3$, the following corollary holds and concludes the section.

Corollary 4. MIN 2SAT *and* MIN 3SAT *are approximable within differential ratios* $4/(m+4)$ *and* $8/(3m+8)$, *respectively.*

4 Inapproximability

We first recall some basics about MAX E3LINp that will be used for deriving our results. In this problem, we are given a positive prime p, n variables $x_1, \ldots, x_n$ in $\mathbb{Z}/p\mathbb{Z}$, m linear equations of type $\alpha_{i_\ell}x_{i_\ell} + \alpha_{j_\ell}x_{j_\ell} + \alpha_{k_\ell}x_{k_\ell} = \beta_\ell$ and our objective is to determine an assignment on $x_1, \ldots, x_n$, in such a way that a maximum number among the m equations is satisfied.

As it is proved in [12] (see also [9] for the case where all the coefficients equal 1), for any $p \geqslant 2$ and for any $\epsilon > 0$, MAX E3LINp cannot be approximated within standard approximation ratio $(1/p) + \epsilon$, even if coefficients in the left-hand sides of the equations are all equal to 1. Note that, due to Corollary 1, this bound is immediately transferred to the differential paradigm.

Finally, let us quote the following GAP-reduction (see [2] for more about this kind of reductions), proved in [12], that will be used in order to yield our results.

Proposition 13. *([12]) Given a problem* $\Pi \in$ ***NP*** *and a real* $\delta > 0$, *there exists a polynomial transformation* g *from any instance* I *of* Π *into an instance of* MAX E3LIN2 *such that:*

- *if* I *is a yes-instance of* Π *(we use here classical terminology from [11]), then* $\mathrm{opt}(g(I)) \geqslant (1-\delta)m$;
- *if* I *is a no-instance of* Π, *then* $\mathrm{opt}(g(I)) \leqslant (1+\delta)m/2$.

Proposition 13 shows, in fact, that MAX E3LIN2 is not approximable within standard ratio $1/2 + \epsilon$, for any $\epsilon > 0$, because an algorithm achieving it would allow us to distinguish in polynomial time the *yes*-instances of any problem $\Pi \in$ **NP** from the *no*-ones. Devising of such reductions is one of the most common strategies for proving inapproximability results in standard approximation.

4.1 Bounds for MAX E3SAT

We first prove a GAP-reduction analogous to the one of Proposition 13 from any problem $\Pi \in$ **NP** to MAX E3SAT. Note that this is the first time that a GAP-reduction is used in the differential approximation paradigm.

Proposition 14. *Given a problem* $\Pi \in$ ***NP*** *and a real* $\delta > 0$*, there exists a polynomial transformation* f *from any instance* I *of* Π *into an instance of* MAX E3SAT *such that:*

- *if* I *is a yes-instance of* Π*, then* $\text{opt}(f(I)) - \omega(f(I)) \geqslant (1-2\delta)m/4$*;*
- *if* I *is a no-instance of* Π*, then* $\text{opt}(f(I)) - \omega(f(I)) \leqslant \delta m/4$*.*

Proof. We first prove that the reduction of Proposition 13 can be translated into the differential paradigm also. Consider an instance $I' = g(I)$ of MAX E3LIN2 and a feasible solution $\vec{x} = (x_1, x_2, \ldots, x_n)$ for I (we will use the same notation for both variables and their assignment) verifying k among the m equations of I'. Then, vector $\bar{\vec{x}} = (1 - x_1, \ldots, 1 - x_n)$, verifies the $m - k$ equations not verified by $\vec{x}$. In other words, $\text{opt}(I) + \omega(I) = m$; hence, function g claimed by Proposition 13 is such that:

- if I is a *yes*-instance of Π, then $\text{opt}(I') - \omega(I') \geqslant (1-2\delta)m$;
- if I is a *no*-instance of Π, then $\text{opt}(I') - \omega(I') \leqslant \delta m$.

We are ready now to continue the proof of the proposition. Consider an instance I of MAX E3LIN2 on n variables x_i, $i = 1, \ldots, n$ and m equations of type $x_i + x_j + x_k = \beta$ in $\mathbb{Z}/2\mathbb{Z}$, i.e., where variables and second members equal 0, or 1. In the same spirit as in [12], we transform I into an instance $\varphi = h(I)$ of MAX E3SAT in the following way:

- for any equation $x_i + x_j + x_k = 0$, we add in $h(I)$ the following four clauses: $(\bar{x}_i \vee x_j \vee x_k)$, $(x_i \vee \bar{x}_j \vee x_k)$, $(x_i \vee x_j \vee \bar{x}_k)$ and $(\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k)$;
- for any equation $x_i + x_j + x_k = 1$, we add in $h(I)$ the following four clauses: $(x_i \vee x_j \vee x_k)$, $(\bar{x}_i \vee \bar{x}_j \vee x_k)$, $(\bar{x}_i \vee x_j \vee \bar{x}_k)$ and $(x_i \vee \bar{x}_j \vee \bar{x}_k)$.

It can immediately be seen that $h(I)$ has n variables and $4m$ (distinct) clauses.

Given a solution y for MAX E3SAT on $h(I)$, we construct a solution y' for I by setting $x_i = 1$ if $x_i = 1$ in $h(I)$ also; otherwise, we set $x_i = 0$.

For instance, consider equation $x_i + x_j + x_k = 0$ in I. It is verified if either 0 or 2 of the variables are equal to 1. The several satisfaction possibilities for the clauses derived in $h(I)$ for this equation are the following:

- if zero, or two variables are set to 1 (true), then all the four clauses are satisfied;
- if one, or three variables are set to 1, then 3 clauses are satisfied.

As a consequence, iterating this argument for any clause set built from an equation, we conclude that solution y for MAX E3SAT on $h(I)$ verifies $m(h(I), y) = 3m + m(I, y')$. Since transformation between y' and y is bijective, we get $\omega(h(I)) = 3m + \omega(I)$ and $\text{opt}(h(I)) = 3m + \text{opt}(I)$. In other words, the reduction just described is an affine reduction from MAX E3LIN2 to MAX E3SAT.

It suffices now to remark that the composition $f = h \circ g$ verifies the statement of the proposition and its proof is concluded. ∎

Proposition 14 has a very interesting corollary, expressed in the Proposition 15 just below, that exhibits another point of dissymmetry between standard and differential paradigms.

Proposition 15. *Unless **P** $=$ **NP**, no polynomial algorithm can compute, on an instance φ of* MAX E3SAT *a value that is a constant approximation of the quantity* $\text{opt}(\varphi) - \omega(\varphi)$.

In view of Proposition 15, what is different between standard and differential paradigms with respect to the GAP-reduction is that in the former such a reduction immediately concludes the impossibility for a problem (assume that it is a maximization one) to be approximable within some ratio, by showing the impossibility for the optimal value to be approximated within this ratio. For that, it suffices that one reads the value of the solution returned by the approximation algorithm. In the latter paradigm such a conclusion is not always immediate. In fact, a reasoning similar to the one of the standard approximation is possible when computation of the worst solution can be done in polynomial time (this is, for instance, the case of maximum independent set and of many other **NP**-hard problems). In this case a simple reading of the value of the approximate solution is sufficient to give an approximation of $\text{opt}(x)-\omega(x)$. A contrario, when it is **NP**-hard to compute $\omega(x)$ (this is the case of the problems dealt here –simply think that the worst solution for MAX SAT is the optimal one for MIN SAT and that both of them are **NP**-hard –, of traveling salesman, etc.), then reading the value $m(x, y)$ of the approximate solution does not provide us with knowledge about $m(x, y) - \omega(x)$ and, consequently no approximation of $\text{opt}(x) - \omega(x)$ can be immediately estimated. So, use of GAP-reduction for achieving inapproximability results is different from the one paradigm to the other.

However, for the case we deal with, we will take advantage of a combination of Propositions 5 and 15 in order to achieve the inapproximability bound for MAX E3SAT given in Proposition 16 that follows.

Proposition 16. *Unless **P** $=$ **NP**,* MAX E3SAT *is inapproximable within differential approximation ratio greater than 1/2.*

Proof. Assume that an approximation achieves differential ratio $\delta > 1/2$, for MAX E3SAT. Then, by Proposition 5, there exists an algorithm achieving the same differential ratio for MIN E3SAT. Denote by T_1 and T_2, respectively, the solutions computed by these algorithms on an instance φ of these problems. We have:

$$m\left(\varphi, T_1\right) - \omega(\varphi) \geqslant \delta(\mathrm{opt}(\varphi) - \omega(\varphi)) \tag{14}$$

where $\mathrm{opt}(\cdot)$ and $\omega(\cdot)$ are referred to MAX E3SAT. By the relations between all these parameters for the two problems specified in the proof of Proposition 5, we get:

$$\mathrm{opt}(\varphi) - m\left(\varphi, T_2\right) \geqslant \delta(\mathrm{opt}(\varphi) - \omega(\varphi)) \tag{15}$$

Adding (14) and (15) member-by-member, we get $m(\varphi, T_1) - m(\varphi, T_2) \geqslant (2\delta - 1)(\mathrm{opt}(\varphi) - \omega(\varphi))$. So, simple reading of the values of T_1 and T_2, can provide us a constant approximation (since δ has been assumed to be a fixed constant greater than 1/2) of the quantity $\mathrm{opt}(\varphi) - \omega(\varphi)$, impossible by Proposition 15. ∎

Proposition 16 together with Proposition 5 conclude the following corollary.

Corollary 5. *For any $k \geqslant 3$, MAX EkSAT, MIN EkSAT, MAX kSAT and MIN kSAT are differentially inapproximable within ratios better than 1/2.*

4.2 MAX EkSAT, $k \geqslant 3$

In this section, we will generalize the GAP-reduction of Proposition 14 in order to further strengthen inapproximability results of Corollary 5.

Proposition 17. *For any prime $p > 0$,* MAX E3LINp $\leq_{\mathsf{AF}}$ MAS E3$(p-1)$SAT.

Proof. Consider a positive prime p and an instance I of MAX E3LINp on n variables and m equations. Consider an equation $x_1 + x_2 + x_3 = \beta$ (in $\mathbb{Z}/p\mathbb{Z}$) of I and, for any $i = 1, 2, 3$, $p - 1$ new variables $x_i^1, \ldots, x_i^{p-1} \in \{0, 1\}$. Consider, finally, equation

$$\sum_{j=1}^{p-1} x_1^j + \sum_{j=1}^{p-1} x_2^j + \sum_{j=1}^{p-1} x_3^j = \beta \tag{16}$$

It is easy to see that (16) is verified if and only if the number of variables set to 1 is either β or $\beta + p$, or, finally, $\beta + 2p$.

Consider now the set of all the possible clauses on $3(p-1)$ literals issued from variables $x_i^1, \ldots, x_i^{p-1}$, $i = 1, 2, 3$. Any truth assignment will satisfy all but one clause. For example, if any variable is assigned with 1, the only unsatisfied clause is the one where all variables appear negative.

What is of interest for us is to specify when the number of variables set to 1 is either β or $\beta + p$, or, $\beta + 2p$. For this, denote by $\mathcal{C}_k$ the set of clauses on $3(p-1)$ literals issued from variables $x_i^1, \ldots, x_i^{p-1}$, $i = 1, 2, 3$ with exactly k negative literals. Then, a truth assignment setting k variables to 1, verifies $|\mathcal{C}_k| - 1$ clauses of $\mathcal{C}_k$, while any other truth assignment on the variables of $\mathcal{C}_k$ verifies all the $|\mathcal{C}_k|$ clauses. So, for an equation $x_1 + x_2 + x_3 = \beta$, we will add in the instance of MAX E3$(p-1)$SAT the set $\mathcal{C}_k$, for $k \in \{0, \ldots, 3(p-1)\}$ and $k \notin \{\beta, \beta + p, \beta + 2p\}$. Hence, if a truth assignment for these clauses has β, or $\beta + p$, or $\beta + 2p$ variables set to 1, it will verify all the clauses constructed, otherwise it will verify all but one of these clauses.

In all, for any of the variables $x_i^1, \ldots, x_i^{p-1}$ we will build one new variable and we will transform any of the m equations of I into an equation as in (16). Then, for any of these new equations we add in the instance of MAX E3$(p-1)$SAT the set of clauses as built just above. The instance φ of MAS E3$(p-1)$SAT so constructed has $n(p-1)$ variables and, since the number of clauses issued from any equation is no more than $2^{3(p-1)}$, φ will have at most $m_\varphi \leqslant m2^{3(p-1)}$ clauses.

Given a truth assignment T on the variables of φ, we set $x_i = |\{x_i^k : x_i^k = 1 \text{ in } T\}|$. Discussion above leads to $m(\varphi, T) = m_\varphi - m + m(I, S)$. On the other hand, it is easy to see that our reduction implies that any solution S of I is transformed into a truth assignment T on the variables of φ such that the relation between the values of S and T given just above is always satisfied. This relation confirms that the reduction specified is an affine one from MAX E3LINp to MAX E3$(p-1)$SAT.

Finally, let us remark that it is possible that formula φ contains many times the same clause. This, for instance, is the case if I simultaneously contains equations say $x_1 + x_2 + x_3 = \beta_1$ and $x_1 + x_2 + x_3 = \beta_2$, for $\beta_1 \neq \beta_2$. In this case, we can modify the construction described, by building the subset of $\mathcal{C}_k$ or $k \in \{0, \ldots, 3(p-1)\}$ and $k \notin \{\beta_1, \beta_1 + p, \beta_1 + 2p, \beta_2, \beta_2 + p, \beta_2 + 2p\}$. This concludes the proof of the proposition. ∎

The result of Proposition 17 together with the result of [12] stated in the beginning of the section and Proposition 1, lead to the following corollary.

Corollary 6. *For any prime* p, MAX E3$(p-1)$SAT *is inapproximable within differential ratio greater than* $1/p$.

Furthermore, Propositions 4 and 5 allow us to rewrite Proposition 17 as follows.

Proposition 18. *For any* $k \geqslant 3$, *neither* MAX EkSAT, *nor* MIN EkSAT *can be approximately solved within differential ratio greater than* $1/p$, *where* p *is the largest positive prime such that* $3(p-1) \leqslant k$.

Easy consequences of Proposition 18 are the following differential inapproximability bounds for several instantiations of maximum and minimum k-satisfiability:

- MAX and MIN 3SAT 4SAT and 5SAT are differentially inapproximable within ratio better than 1/2;
- MAX and MIN 6SAT, ..., 11SAT are differentially inapproximable within ratio better than 1/3;
- MAX and MIN 12SAT, ..., 17SAT are differentially inapproximable within ratio than 1/5, ...

Finally, MAX SAT being harder to approximate than any MAX kSAT problem, the following result holds and concludes the section.

Proposition 19. MAX SAT $\notin$ ***DAPX***.

In [17] is defined a logical class of **NPO** maximization problems called **MAX-NP**. A maximization problem $\Pi \in$ **NPO** belongs to **Max-NP** if and only if there exist two finite structures $(U, \mathcal{I})$ and $(U, \mathcal{S})$, a quantifier-free first order formula φ and two constants k and ℓ such that, the optima of Π can be logically expressed as:

$$\max_{S \in \mathcal{S}} \left| \left\{ x \in U^k : \exists y \in U^\ell, \varphi(\mathcal{I}, S, x, y) \right\} \right| \tag{17}$$

The predicate-set $\mathcal{I}$ draws the set of instances of Π, set $\mathcal{S}$ the solutions on $\mathcal{I}$ and φ the feasibility conditions for the solutions of Π. In the same article is proved that MAX SAT $\in$ **Max-NP** and that **MAX-NP** $\subset$ **APX**.

It is easy to see that (17) can be identically used in both standard and differential paradigms. So, Proposition 19 draws an important structural difference in the landscape of approximation classes in the two paradigms, since an immediate corollary of this proposition is that **MAX-NP** $\not\subset$ **DAPX**. We conjecture that the same holds for the other one of the celebrated logical classes of [17], the class **MAX-SNP**, i.e., we conjecture that **MAX-SNP** $\not\subset$ **DAPX**

4.3 MAX E2SAT

We have already seen in Proposition 6 that MAX E2SAT is differentially inapproximable within ratio 21/22. In this section, we improve this result by operating an affine reduction from MAX E2LIN2 to MAX E2SAT.

Indeed, consider an instance I of the former problem (on n variables and m equations) and an equation $x_1 + x_2 = 0$ in I. Add in φ (the instance of MAX E2SAT under construction) clauses $\bar{x}_1 \vee x_2$ and $x_1 \vee \bar{x}_2$. On the other hand, for an equation $x_1 + x_2 = 1$, add in φ clauses $x_1 \vee x_2$ and $\bar{x}_1 \vee \bar{x}_2$. Performing this transformation for any equation in I, we finally build a formula φ of MAX E2SAT on n variables and $2m$ clauses. Moreover, for any truth assignment T on the variables of φ, one gets a solution S for I such that $m(\varphi, T) = m + m(I, S)$, qed.

It is shown in [12] that MAX E2LIN2 is inapproximable within standard approximation ratio better than 11/12. By Proposition 2, this bound is transferred to the differential paradigm. Then, the affine reduction just described concludes the following result.

Proposition 20. MAX E2LIN2 $\leq_{\mathsf{AF}}$ MAX E2SAT. *Consequently,* MAX E2SAT *is differentially inapproximable within ratio greater than 11/12.*

5 Ideas for further research

We give in this concluding section a few ideas about possible ways for further improving results of the paper or for yielding new ones.

Consider a graph $G(V, E)$ of order n and with maximum degree Δ. We construct an instance φ of MAX DNF on n variable $x_1, \ldots, x_n$ and n cubes $C_1, \ldots, C_n$ as follows: for any vertex $v_i \in V$, with neighbors $v_{i_1}, \ldots, v_{i_{\delta_i}}$, we add in φ clause $x_i \wedge \bar{x}_{i_1} \wedge \ldots \wedge \bar{x}_{i_{\delta_i}}$. Let T be a truth assignment satisfying k cubes, say $C_{j_1}, \ldots, C_{j_k}$. Then, obviously, the vertex-set $V' = \{v_{j_1}, \ldots, v_{j_k}\}$ is an independent set for G (of size k). Conversely, given an independent set of G of size k consisting of vertices $v_{j_1}, \ldots, v_{j_k}$, the truth assignment setting variables $x_{j_1}, \ldots, x_{j_k}$ to 1 and any other variable of φ in 0 satisfies k cubes. Observe finally that the size of the cubes built for φ is bounded by $\Delta + 1$. In all we have just exhibited an affine reduction from MAX INDEPENDENT SET-Δ (i.e., MAX INDEPENDENT SET on graphs with maximum degree bounded by Δ) to MAX $\Delta + 1$DNF.

On the other hand, there exists an $\epsilon > 0$ such that, for any $\Delta \geqslant 3$, MAX INDEPENDENT SET-Δ is not approximable within approximation ratio $1/\Delta^{\epsilon}$ ([1]). Since standard and differential approximation ratios coincide for MAX INDEPENDENT SET (the worst independent set in a graph is the empty set), the result of [1] holds immediately for differential paradigm and can be used in order to conclude that there exists an $\epsilon > 0$ such that, for any $k \geqslant 4$, MAX kDNF is not differentially approximable within ratio greater than $1/k^{\epsilon}$. This recovers the result of Proposition 19, namely, that MAX SAT $\notin$ **DAPX**.

If one wishes to improve this result, a possible issue is the following. Recall that transformation of MAX kDNF to MAX kSAT of Proposition 5, consists of substituting any cube of size ℓ by $2^{\ell} - 1$ clauses of size ℓ. We so can affinely (but not polynomially) reduce MAX INDEPENDENT SET to MAX SAT by building an instance φ of the latter on n variables and at most $n2^{\Delta+1}$ clauses. But, if Δ is bounded by $\log n$, then this reduction is polynomial. In other words, if one obtains an inapproximability bound for MAX INDEPENDENT SET-$\log n$ (for example a bound of the type $1/\log^{\epsilon} n$, for some positive ϵ), then one can extend it immediately to MAX SAT improving so the bound of the paper.

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Conciliation and Consensus in Iterated Belief Merging

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Abstract

Two conciliation processes for intelligent agents based on an iterated merge-then-revise change function for belief profiles are introduced and studied. The first approach is skeptical in the sense that at any revision step, each agent considers that her current beliefs are more important than the current beliefs of the group, while the other case is considered in the second, credulous approach. Some key features of such conciliation processes are pointed out for several merging operators; especially, the "convergence" issue, the existence of consensus and the properties of the induced iterated merging operators are investigated.

Key words : Belief Merging

1 Introduction

Belief merging is about the following question: given a set of agents associated to belief bases which are (typically) mutually inconsistent, how to define a belief base reflecting the beliefs of the group of agents?

The belief merging issue is central in many applications. For example, when a distributed database is to be queried, conflicting answers coming from different bases must be handled. The same difficulty occurs when one wants to define the beliefs of a group of experts, or the global beliefs within a multi-agent system.

There are many different ways to address the belief merging issue in a propositional setting (see e.g.[11, 19, 17, 16, 2, 3, 13, 14]). The variety of approaches just reflects the various ways to deal with inconsistent beliefs.

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The belief merging issue is not concerned with the way the beliefs of the group are exploited. One possibility is to suppose that all the belief bases are replaced by the (agreed) merged base. This scenario is sensible with low-level agents that are used for distributed computation, or for applications with distributed information sources (like distributed databases). Once the merged base has been computed, all the agents participating to the merging process are equivalent in the sense that they share the same belief base. Such a drastic approach, when repeated, clearly leads to impoverish the beliefs of the system. Contrastingly, when high-level intelligent agents are considered, the previous scenario looks rather unlikely: it is not reasonable to assume that the agents are ready to completely discard their current beliefs and inconditionnally accept the merged base as a new belief base. It seems more adequate for them to incorporate the result of the merging into their current belief base. Such an incorporation of new beliefs calls for what is known as belief revision [1, 7, 8], which can be considered as a specific case of IC belief merging.

In this perspective, two revision strategies can be considered. The first one consists in giving more priority to the previous beliefs; this is the strategy at work for skeptical agents. The second one, used by credulous agents, views the current beliefs of the group as more important than their own, current beliefs. Thus, given a revision strategy, every IC merging operator Δ induces what we called a conciliation operator which maps every belief profile E (i.e., the beliefs associated to each agent at start) to a new belief profile where the new beliefs of an agent are obtained by revising its previous beliefs with the merged base given by E and Δ, or vice-versa.

Obviously enough, it makes sense to iterate such a merge-then-revise process when the objective of agents is to reach an agreement (if possible): after a first merge-then-revise round, each agent has possibly new beliefs, defined from her previous ones and the beliefs of the group; this may easily give rise to new beliefs for the group, which must be incorporated into the previous beliefs of agents, and so on. The objective of this paper is to study the two conciliation processes induced by the two revision strategies for various IC merging operators under two simplifying assumptions: homogeneity (the same strategy and the same revision operators are used by all the agents) and compatibility (the revision operator used is the one induced by the IC merging operator under consideration). Some key issues are considered, including the “convergence” of the processes, i.e., the existence of a round from which no further evolution is possible, the existence of consensus (i.e., the joint consistency of all belief bases at some stage), and the logical properties of the iterated merging operator defined by the last merged base once a fixed point has been reached.

The rest of the paper is organized as follows. In the next section, some formal preliminaries are provided. Section 3 presents the main results of the paper: in Section 3.1 the conciliation processes are defined, in Section 3.2 the focus is laid on the skeptical ones and in Section 3.3 on the credulous ones. In Section 4 we investigate the connections between the conciliation processes and the merging operators they induce. Especially, we give some properties of the corresponding iterative merging operators. Section 5 is devoted to related work. Finally, Section 6 gives some conclusions and perspectives of this work.

2 Preliminaries

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{P}$ of propositional symbols. An interpretation is a function from $\mathcal{P}$ to $\{0, 1\}$. The set of all the interpretations is denoted $\mathcal{W}$. An interpretation ω is a model of a formula K, noted $\omega \models K$, if and only if it makes it true in the usual classical truth functional way. Let K be a formula, $mod(K)$ denotes the set of models of K, i.e., $mod(K) = \{\omega \in \mathcal{W} \mid \omega \models K\}$.

A *belief base* K is a consistent propositional formula (or, equivalently, a finite consistent set of propositional formulas considered conjunctively). Let us note $\mathcal{K}$ the set of all belief bases.

Let $K_1, \ldots, K_n$ be n belief bases (not necessarily pairwise different). We call *belief profile* the vector E consisting of those n belief bases in a specific order, $E = (K_1, \ldots, K_n)$, so that the n^{th} base gathers the beliefs of agent n. When belief merging is considered only, every belief profile can typically be viewed as the multi-set composed of its coordinates; this just comes from the fact that usual belief merging frameworks make an anonymity assumption about agents (roughly, no agent is considered more important than another one): the merged base associated to a given belief profile is invariant under any permutation of the agents. In the following, we need nevertheless to keep track of the origins of beliefs, so as to be able to associate to each agent the right beliefs after each evolution step. This is why belief profiles are represented as vectors of belief bases, and not just multi-sets of belief bases; clearly enough, this is without any loss of generality since more information is preserved by the vector representation. We note $\bigwedge E$ the conjunction of the belief bases of E, i.e., $\bigwedge E = K_1 \wedge \cdots \wedge K_n$. We say that a belief profile is consistent if $\bigwedge E$ is consistent. The union of belief profiles (actually, of the associated multi-sets) will be noted $\sqcup$. The cardinal of a (multi-)set or vector E is noted $\#(E)$ (the cardinal of a finite multi-set is the sum of the numbers of occurrences of each of its elements).

Let $\mathcal{E}$ be the set of all finite non-empty belief profiles. Two belief profiles E_1 and E_2 from $\mathcal{E}$ are said to be equivalent (noted $E_1 \equiv E_2$) if and only if there is a bijection between the multi-set associated to E_1 and the multi-set associated to E_2 s.t. each belief base of E_1 is logically equivalent to its image in E_2.

For every belief revision operator $*$, every profile $E = (K_1, \ldots, K_n)$ and every belief base K, we define the revision of E by K (resp. the revision of K by E) as the belief profile given by $(K_1, \ldots, K_n) * K = (K_1 * K, \ldots, K_n * K)$ (resp. $K * (K_1, \ldots, K_n) =$

$(K * K_1, \ldots, K * K_n)$). Since sequences of belief profiles will be considered, we use superscripts to denote belief profiles obtained at some stage, while subscripts are used (as before) to denote belief bases within a profile. For instance, E^i denotes the belief profile obtained after i elementary evolution steps (in our framework, i merge-then-revise steps), and K_j^i the belief base associated the the j^{th} coordinate of vector E^i.

2.1 IC merging operators

Some basic work in belief merging aims at determining sets of axiomatic properties valuable operators should exhibit [18, 19, 16, 12, 13, 15]. We focus here on the characterization of Integrity Constraints (IC) merging operators [13, 14].

Definition 1 (IC merging operators) $\triangle$ *is an* IC merging operator *if and only if it satisfies the following properties:*

(IC0) $\triangle_\mu(E) \models \mu$

(IC1) *If* μ *is consistent, then* $\triangle_\mu(E)$ *is consistent*

(IC2) *If* $\bigwedge E$ *is consistent with* μ, *then* $\triangle_\mu(E) \equiv \bigwedge E \wedge \mu$

(IC3) *If* $E_1 \equiv E_2$ *and* $\mu_1 \equiv \mu_2$, *then* $\triangle_{\mu_1}(E_1) \equiv \triangle_{\mu_2}(E_2)$

(IC4) *If* $K_1 \models \mu$ *and* $K_2 \models \mu$, *then* $\triangle_\mu(\{K_1, K_2\}) \wedge K_1$ *is consistent if and only if* $\triangle_\mu(\{K_1, K_2\}) \wedge K_2$ *is consistent*

(IC5) $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2) \models \triangle_\mu(E_1 \sqcup E_2)$

(IC6) *If* $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ *is consistent, then* $\triangle_\mu(E_1 \sqcup E_2) \models \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$

(IC7) $\triangle_{\mu_1}(E) \wedge \mu_2 \models \triangle_{\mu_1 \wedge \mu_2}(E)$

(IC8) *If* $\triangle_{\mu_1}(E) \wedge \mu_2$ *is consistent, then* $\triangle_{\mu_1 \wedge \mu_2}(E) \models \triangle_{\mu_1}(E)$

The intuitive meaning of the properties is the following: *(IC0)* ensures that the result of merging satisfies the integrity constraints. *(IC1)* states that, if the integrity constraints are consistent, then the result of merging will be consistent. *(IC2)* states that if possible, the result of merging is simply the conjunction of the belief bases with the integrity constraints. *(IC3)* is the principle of irrelevance of syntax: the result of merging has to depend only on the expressed opinions and not on their syntactical presentation. *(IC4)* is a fairness postulate meaning that the result of merging of *two* belief bases should not give preference to one of them (in the sense that if it is consistent with one of them, it has to be consistent with the other one.) It is a symmetry condition, that aims at ruling out operators which give priority to one of the bases. *(IC5)* expresses the following idea: if belief profiles are viewed as expressing the beliefs of the members of a group, then if E_1 (corresponding to a first group) compromises on a set of alternatives A belongs to, and E_2 (corresponding to a second group) compromises on another set of alternatives which contains A too, then A has to be in the chosen alternatives if we join the two groups. *(IC5)* and *(IC6)* together state that if one could find two subgroups which agree on at least one alternative, then the result of the global merging has to be exactly those alternatives the two groups agree on. *(IC7)* and *(IC8)* state that the notion of closeness is well-behaved, i.e., that an alternative that is preferred among the possible alternatives (μ_1), remains preferred if one restricts the possible choices ($\mu_1 \wedge \mu_2$). For more explanations on those properties see [14].

Two sub-classes of IC merging operators have been defined. *IC Majority operators* aim at resolving conflicts by adhering to the majority wishes, while *IC arbitration operators* exhibit a more consensual behaviour:

Definition 2 (majority and arbitration) *An* IC majority operator *is an IC merging operator which satisfies the following majority postulate:*

(Maj) $\exists n \ \triangle_{\mu}(E_1 \sqcup {E_2}^n) \models \triangle_{\mu}(E_2)$.

An IC arbitration operator *is an IC merging operator which satisfies the following arbitration postulate:*

(Arb) $$\left.\begin{array}{l}\triangle_{\mu_1}(K_1) \equiv \triangle_{\mu_2}(K_2)\\ \triangle_{\mu_1 \Leftrightarrow \neg\mu_2}(\{K_1, K_2\}) \equiv (\mu_1 \Leftrightarrow \neg\mu_2)\\ \mu_1 \not\models \mu_2\\ \mu_2 \not\models \mu_1\end{array}\right\} \Rightarrow \triangle_{\mu_1 \vee \mu_2}(\{K_1, K_2\}) \equiv \triangle_{\mu_1}(K_1).$$

See [13, 15] for explanations about those two postulates and the behaviour of the two corresponding classes of merging operators.

Let us now give some examples of IC merging operators.

Definition 3 *A* pseudo-distance *between interpretations is a total function* $d : \mathcal{W} \times \mathcal{W} \mapsto \mathbb{R}^+$ *such that for any* $\omega, \omega', \omega'' \in \mathcal{W}$*:*

- $d(\omega, \omega') = d(\omega', \omega)$*, and*
- $d(\omega, \omega') = 0$ *if and only if* $\omega = \omega'$*.*

Two widely used distances between interpretations are Dalal distance [6], denoted d_H, which is the Hamming distance between interpretations (i.e., the number of propositional variables on which the two interpretations differ); and the drastic distance, denoted d_D, which is the simplest pseudo-distance one can define: it gives 0 if the two interpretations are the same one, and 1 otherwise.

Definition 4 *An* aggregation function f *is a total function associating a nonnegative real number to every finite tuple of nonnegative real numbers and s.t. for any* $x_1, \ldots, x_n, x, y \in \mathbb{R}^+$*:*

- *if* $x \leq y$*, then* $f(x_1, \ldots, x, \ldots, x_n) \leq f(x_1, \ldots, y, \ldots, x_n)$*.* *(non-decreasingness)*
- $f(x_1, \ldots, x_n) = 0$ *if and only if* $x_1 = \ldots = x_n = 0$*.* *(minimality)*
- $f(x) = x$*.* *(identity)*

Widely used functions are the $\max$ [19, 15], the sum Σ [19, 17, 13], or the leximax $GMax$ [13, 15].

The chosen distance between interpretations induces a "distance" between an interpretation and a base, which in turn gives a "distance" between an interpretation and a profile, using the aggregation function. This latter distance gives the needed notion of closeness $\leq_E$ (a pre-order induced by E):

Definition 5 *Let* d *be a pseudo-distance between interpretations and* f *be an aggregation function. The result* $\triangle_\mu^{d,f}(E)$ *of the (model-based) merging of* E *given the integrity constraints* μ *is defined by:*

- $d(\omega, K) = min_{\omega' \models K} d(\omega, \omega')$*.*
- $d(\omega, E) = f_{\{K_i \in E\}}(d(\omega, K_i))$*.*
- $\omega \leq_E \omega'$ *if and only if* $d(\omega, E) \leq d(\omega', E)$*.*
- $[\triangle_\mu^{d,f}(E)] = \min([\mu], \leq_E)$*.*

Let us illustrate now the behaviour of merging operators on an example. This example shows the result of a merging for the IC arbitration operator $\triangle^{d_H,GMax}$, using the Hamming distance and the leximax aggregation function, the IC majority operator $\triangle^{d_H,\Sigma}$, and the operator $\triangle^{d_H,Max}$ which is not an IC merging operator, but satisfies all IC properties (and (Arb)), except (IC6).

Example 1 *Let us consider a belief profile* $E = (K_1, K_2, K_3, K_4)$ *and an integrity constraint* μ *defined on a propositional language built over four symbols, as follows:*

$$mod(\mu) = \mathcal{W} \setminus \{ (0,1,1,0), (1,0,1,0), (1,1,0,0), (1,1,1,0)\}$$
$$mod(K_1) = \{(1,1,1,1), (1,1,1,0)\}$$
$$mod(K_2) = \{(1,1,1,1), (1,1,1,0)\}$$

	$\mathbf{K_1}$	$\mathbf{K_2}$	$\mathbf{K_3}$	$\mathbf{K_4}$	$\mathbf{d_{d_H},}$*Max*	$\mathbf{d_{d_H},\Sigma}$	$\mathbf{d_{d_H},}$*GMax*
$(0,0,0,0)$	3	3	0	2	3	8	(3,3,2,0)
$(0,0,0,1)$	3	3	1	3	3	10	(3,3,3,1)
$(0,0,1,0)$	2	2	1	1	**2**	6	**(2,2,1,1)**
$(0,0,1,1)$	2	2	2	2	**2**	8	(2,2,2,2)
$(0,1,0,0)$	2	2	1	1	**2**	6	**(2,2,1,1)**
$(0,1,0,1)$	2	2	2	2	**2**	8	(2,2,2,2)
$(0,1,1,0)$	1	1	2	0	2	4	(2,1,1,0)
$(0,1,1,1)$	1	1	3	1	3	6	(3,1,1,1)
$(1,0,0,0)$	2	2	1	2	**2**	7	(2,2,2,1)
$(1,0,0,1)$	2	2	2	3	3	9	(3,2,2,2)
$(1,0,1,0)$	1	1	2	1	2	5	(2,1,1,1)
$(1,0,1,1)$	1	1	3	2	3	7	(3,2,1,1)
$(1,1,0,0)$	1	1	2	1	2	5	(2,1,1,1)
$(1,1,0,1)$	1	1	3	2	3	7	(3,2,1,1)
$(1,1,1,0)$	0	0	3	0	3	3	(3,0,0,0)
$(1,1,1,1)$	0	0	4	1	4	**5**	(4,1,0,0)

Table 1: Distances

$mod(K_3) = \{(0,0,0,0)\}$
$mod(K_4) = \{(1,1,1,0),(0,1,1,0)\}$

The computations are reported in Table 1. The shadowed lines correspond to the interpretations rejected by the integrity constraints. Thus the result has to be taken among the interpretations that are not shadowed. The first four columns show the Hamming distance between each interpretation and the corresponding source. The last three columns show the distance between each interpretation and the profile according to the different aggregation functions. So the selected interpretations for the corresponding operators are the ones with minimal aggregated distance.

With the $\triangle^{d_H,Max}$ *operator, the minimum distance is* 2 *and the chosen interpretations are* $mod(\triangle_\mu^{d_H,Max}(E)) = \{(0,0,1,0), (0,0,1,1), (0,1,0,0), (0,1,0,1), (1,0,0,0)\}$.

We can see on that example why $\triangle^{d,Max}$ *operators are not IC merging operators. For example, the two interpretations* $(0,0,1,0)$ *and* $(0,0,1,1)$ *are chosen by* $\triangle^{d_H,Max}$*, although* $(0,0,1,0)$ *is better for* K_3 *and* K_4 *than* $(0,0,1,1)$*, whereas these two interpretations are equally preferred by* K_1 *and* K_2*. It seems then natural to globally prefer* $(0,0,1,0)$ *to* $(0,0,1,1)$*. It is in fact what (IC6) requires.*

The $\triangle^{d,GMax}$ *family has been built with the purpose of being more selective than the* $\triangle^{d,Max}$ *family. With the* $\triangle^{d_H,GMax}$ *operator, the result is* $mod(\triangle_\mu^{d_H,GMax}(E)) = \{(0,0,1,0),$

$(0, 1, 0, 0)\}$.

Finally, if one chooses $\triangle^{d_H,\Sigma}$ *for solving the conflict according to majority wishes, the result is* $mod(\triangle_{\mu}^{d_H,\Sigma}(E)) = \{(1, 1, 1, 1)\}$.

2.2 Merging vs. revision

Belief revision operators can be viewed as special cases of belief merging operators when applied to singleton profiles, as stated below.

Proposition 1 *If* $\triangle$ *is an IC merging operator (it satisfies (IC0-IC8)), then the operator* $*$*, defined as* $K * \mu = \triangle_{\mu}(K)$*, is an AGM revision operator (it satisfies (R1-R6)).*

So to each belief merging operator $\triangle$, one can associate a corresponding revision operator $*_{\triangle}$, which is called the revision operator *associated to* the merging operator $\triangle$.

3 Conciliation Operators

3.1 Definitions

Conciliation operators aim at reflecting the evolution of belief profiles, typically towards the achievement of some agreements between agents. It can be viewed as a simple form of negotiation, where the way beliefs may evolve is uniform.

Let us first give the following, very general, definition of conciliation operators:

Definition 6 *A conciliation operator is a function from the set of belief profiles to the set of belief profiles.*

This definition does not impose any strong constraints on the result, except that each resulting belief profile is solely defined from the previous one (i.e., no additional information, like a further observation, are taken into account). Clearly, pointing out the desirable properties for such conciliation operators is an interesting issue. We let this for future work, but one can note that the *s*ocial contraction functions introduced by Booth [5] are very close to this idea.

In this paper we focus on a particular familly of conciliation operators: conciliation operators induced by an iterated merge-then-revise process. The idea is to compute the belief merging from the profile, to revise the beliefs of each source by the result of the merging, and to repeat this process until a fixed point is reached. When such a fixed point exists, the conciliation operator is defined and the resulting profile is the image of the original profile by this operator.

When a fixed point has been reached, incorporating the beliefs of the group has no further impact on the own beliefs of each agent; in some sense, each agent did its best w.r.t. the group, given its revision function. Then there are two possibilities: either a consensus has been obtained, or no consensus can be obtained that way:

Definition 7 *There is a consensus for a belief profile E if and only if E is consistent (with the integrity constraints).*

The existence of a consensus for a belief profile just means that the associated agents agree on at least one possible world. When this is the case, the models of the corresponding merged base w.r.t. any IC merging operator reduce to such possible worlds ((IC2) ensures it). Interestingly, it can be shown that the existence of a consensus at some stage of the merge-then-revise process is sufficient to ensure the existence of a fixed point, hence the termination of the process.

Let us now consider two additional properties on conciliation operators in order to keep the framework simple enough: homogeneity and compatibility.

Definition 8 *An iterated merging conciliation operator is a function from the set of belief profiles to the set of belief profiles, where the evolution of a profile is characterized by a merge-then-revise approach. It is:*

- *homogeneous if all the agents use the same revision operator,*
- *compatible if the revision operator is associated to the merging operator.*

In this work, we focus on compatible homogeneous iterated merging conciliation operators (CHIMC in short). Under the compatibility and homogeneity assumptions, defining a CHIMC operator just requires to make precise the belief merging operator under use and the revision strategy (skeptical or credulous):

Definition 9 (skeptical CHIMC operators) *Let $\triangle$ be an IC merging operator, and $*$ its associated revision operator (i.e., $\varphi * \mu = \triangle_\mu(\{\varphi\})$). Let E be any belief profile. We define the sequence $(E_s^i)_i$ (depending on both $\triangle$ and E) by:*

- $E_s^0 = E$,
- $E_s^{i+1} = \Delta_\mu(E_s^i) * E_s^i$

The skeptical CHIMC operator induced by $\triangle$ is defined by $\triangle^*_\mu(E) = E^k_s$, where k is the lowest rank i such that $E^i_s = E^{i+1}_s$, and $\triangle^*_\mu(E)$ is undefined otherwise. We note $E^*_s = E^k_s$ the "resulting" profile.

Definition 10 (credulous CHIMC operators) *Let $\triangle$ be an IC merging operator, and $*$ its associated revision operator Let E be any belief profile. We define the sequence $(E^i_c)_i$ (depending on both $\triangle$ and E) by:*

- $E^0_c = E$,
- $E^{i+1}_c = E^i_c * \Delta_\mu(E^i_c)$

The credulous CHIMC operator induced by $\triangle$ is defined by ${}^* \triangle_\mu (E) = E^k_c$, where k is the lowest rank i such that $E^i_c = E^{i+1}_c$, and ${}^* \triangle_\mu (E)$ is undefined otherwise. We note $E^*_c = E^k_c$ the "resulting" profile.

Clearly enough, each sequence induces a corresponding merged base when a fixed point is reached: the merged base of the "last" profile in the sequence (i.e., at the rank from which the sequence is stationary). Formally:

Definition 11 (CHIM operators) *Let $\triangle$ be an IC merging operator, and $*$ its associated revision operator.*

- *The skeptical CHIM operator induced by $\triangle$ is the function that maps every profile E to $\triangle_\mu(\triangle^*_\mu(E))$ whenever $\triangle^*_\mu(E)$ exists and is undefined otherwise.*
- *The credulous CHIM operator induced by $\triangle$ is the function that maps every profile E to $\triangle_\mu({}^* \triangle_\mu (E))$ whenever ${}^* \triangle_\mu (E)$ exists and is undefined otherwise.*

Let us now study the key features of the two sequences $(E^i_s)_i$ and $(E^i_c)_i$ and the properties of the corresponding iterated merging operators, based on various IC merging operators.

3.2 Skeptical operators

We start with skeptical CHIMC operators. Let us first give an important monotony property, which states that the conciliation process given by any IC merging operator $\triangle$ may only lead to strengthen the beliefs of each agent:

Proposition 2 *Let K_j^i denote the belief base corresponding to agent j in the belief profile E_s^i characterized by the initial belief profile E and the IC merging operator $\triangle$. For every i, j, we have $K_j^{i+1} \models K_j^i$.*

On this ground, it is easy to prove that the sequence $(E_s^i)_i$ is stationary at some stage[1], for every profile E and every IC merging operator $\triangle$. Accordingly, the induced skeptical conciliation operator and the induced skeptical iterated merging operator are defined for every E:

Proposition 3 *For every belief profile E and every IC merging operator $\triangle$, the stationarity of $(E_s^i)_i$ is reached at a rank bounded by $(\sum_{K \in E} \#(mod(K))) - \#(E)$. Therefore, the CHIMC operator $\triangle^*$ and the CHIM operator $\triangle(\triangle^*)$ are total functions.*

The bound on the number of iterations is easily obtained from the monotony property.

Another interesting property is that the sequence of profiles and the corresponding sequence of merged bases are equivalent with respect to stationarity:

Proposition 4 *Let E be a belief profile and $\triangle$ be an IC merging operator. Let μ be any integrity constraint. The sequence $(E_s^i)_i$ is stationary from some stage if and only if the sequence $(\triangle_\mu(E_s^i))_i$ is stationary from some stage.*

The number of iterations needed to reach the fixed point of $(E_s^i)_i$ is one for the IC merging operators defined from the drastic distance. More precisely, the skeptical CHIM operator induced by any IC merging operator $\triangle$ defined from the drastic distance coincides with $\triangle$.

Proposition 5 *Let $E = (K_1, \ldots, K_n)$ be a profile. If the IC merging operator $\triangle$ is among $\triangle^{d_D,Max}$,$\triangle^{d_D,\Sigma}$,$\triangle^{d_D,GMax}$, then for every j, the base K_j^* from the resulting profile $E^* = \triangle_\mu^*(E)$ can be characterized by:*

$$K_j^* = \begin{cases} \mu \wedge \triangle_\mu(E) & \textit{if consistent, else} \\ \triangle_\mu(E) & \textit{otherwise.} \end{cases}$$

Furthermore, the resulting profile is obtained after at most one iteration (i.e., for every $i > 0$, $E^i = E^{i+1}$).

We have no direct (i.e., non-iterative) definition for any skeptical CHIM operator based on an IC merging operator defined from the Hamming distance. Let us see an example of such an operator:

[1] Abusing words, we sometimes say that the sequence is “convergent” to express that there exists a rank k s.t. the sequence is stationary from k.

Example 2 *Let us consider the profile $E = (K_1, K_2, K_3)$ with $mod(K_1) = \{(0,0,0),$ $(0,0,1),(0,1,0)\}$, $mod(K_2) = \{(0,1,1),(1,1,0),(1,1,1)\}$, $mod(K_3) = \{(0,0,0),(1,0,$ $0),(1,0,1),(1,1,1)\}$, no integrity constraints ($\mu \equiv \top$), and the skeptical CHIMC operator defined from the $\triangle^{d_H,GMax}$ operator. The complete process is represented in Table 2. The columns have the same meanings than in table 1, but here, as there are several (three in that case) iterations, we sum up the three tables (corresponding to the three merging steps) in the same one. So, for example in column $d(\omega, K_1^i)$, the first number denotes the distance of the interpretation with respect to K_1^1, the second one the distance with respect to K_1^2, etc.*

Let us explain the full process in details. The first profile is $E^0 = E$. The first merging iteration gives as result $mod(\triangle^{d_H,GMax}(E^0)) = \{(0,0,1),(0,1,0),(0,1,1),(1,0,0),$ $(1,0,1),(1,1,0)\}$. Then, every source revises the result of the merging with its old beliefs, i.e., $K_1^1 = \triangle^{d_H,GMax}(E^0) * K_1^0$, so $mod(K_1^1) = \{(0,0,1),(0,1,0\}$. Similarly $mod(K_2^1) =$ $\{(0,1,1),(1,1,0\}$ and $mod(K_3^1) = \{(1,0,0),(1,0,1\}$. Since each of the three bases is consistent with the merged base, the new base of each agent is just the conjunction of her previous base with the merged base (in accordance to revision postulates). Then, the second merging iteration gives $mod(\triangle^{d_H,GMax}(E^1)) = \{(0,0,1),(1,1,0)\}$, and the revision of each base gives $mod(K_1^2) = \{(0,0,1)\}$, $mod(K_2^2) = \{(1,1,0)\}$, and $mod(K_3^2) =$ $\{(1,0,0),(1,0,1)\}$. Then the third iteration step gives $mod(\triangle^{d_H,GMax}(E^2)) = \{(1,0,0),$ $(1,0,1)\}$, and the revision step does not change any belief base, i.e., $E^2 \equiv E^3$, so the stationary point is reached and the process stops on this profile.

ω	$d(\omega, K_1^i)$	$d(\omega, K_2^i)$	$d(\omega, K_3^i)$	$d_{GMax}(\omega, E^i)_{d(\omega,\Delta_\mu(E^i))}$
(0,0,0)	0,1,1	2,2,2	0,1,1	$(2,0,0)_1$,$(2,1,1)_1$,$(2,1,1)_1$
(0,0,1)	0,0,0	1,1,3	1,1,1	$(1,1,0)_0$,$(1,1,0)_0$,$(3,1,0)_1$
(0,1,0)	0,0,2	1,1,1	1,2,2	$(1,1,0)_0$,$(2,1,0)_1$,$(2,2,1)_2$
(0,1,1)	1,1,1	0,0,2	1,2,2	$(1,1,0)_0$,$(2,1,0)_1$,$(2,2,1)_2$
(1,0,0)	1,2,2	1,1,1	0,0,0	$(1,1,0)_0$,$(2,1,0)_1$,$(2,1,0)_0$
(1,0,1)	1,1,1	1,2,2	0,0,0	$(1,1,0)_0$,$(2,1,0)_1$,$(2,1,0)_0$
(1,1,0)	1,1,3	0,0,0	1,1,1	$(1,1,0)_0$,$(1,1,0)_0$,$(3,1,0)_1$
(1,1,1)	2,2,2	0,1,1	0,1,1	$(2,0,0)_1$,$(2,1,1)_1$,$(2,1,1)_1$

Table 2: $\Delta_\mu^{*\,d_H,GMax}$

We have also proven that a skeptical conciliation process cannot lead to a consensus, unless a consensus already exists at start:

Proposition 6 *Let E be a belief profile and $\triangle$ be an IC merging operator. There exists a rank i s.t. a consensus exists for E_s^i if and only if $i = 0$ and there is a consensus for E.*

3.3 Credulous operators

Let us now turn to credulous CHIMC operators. Let us first give some general properties about credulous operators.

Proposition 7 *Let K_j^i now denote the belief base corresponding to agent j in the belief profile E_c^i characterized by the initial belief profile E and the IC merging operator $\triangle$.*

- $\forall i, j\ K_j^{i+1} \models \triangle_\mu(E_c^i)$,
- $\forall i > 0 \forall j\ K_j^i \models \mu$,
- $\forall i, j$, *if* $K_j^i \wedge \triangle_\mu(E_c^i)$ *is consistent, then* $K_j^{i+1} \equiv K_j^i \wedge \triangle_\mu(E_c^i)$.

The first item states that, during the evolution process, each base implies the previous merged base. The second item states that from the first iteration, all the bases implies the integrity constraints. The last one is simply a consequence of a revision property: if, at a given step, a base is consistent with the result of the merging, then the base at the next step will be that conjunction.

Unfortunately, no monotony property can be derived from this proposition. At that point, we can just conjecture that our credulous CHIMC operators (and the corresponding iterated merging operators) are defined for every profile:

Conjecture 1 *For every belief profile E and every merging operator $\triangle$ using the aggregation function Max, $GMax$ or Σ, the sequence $(E_c^i)_i$ is stationary from some rank.*

This claim is supported by some empirical evidence. We have conducted exhaustive tests for profiles containing up to three bases, when the set of propositional symbols contains up to three variables. The following IC merging operators have been considered: $\triangle^{d_H,\mathit{Max}}$, $\triangle^{d_H,\mathit{GMax}}$ and $\triangle^{d_H,\Sigma}$. We have also conducted non-exhaustive tests when four propositional symbols are considered in the language (this leads to billions of tests). All the tested instances support the claim (stationarity is reached in less than five iterations when up to three symbols are considered, and less than ten iterations when four symbols are used).

We can nevertheless prove the stationarity of $(E_c^i)_i$ for every belief profile E when some specific IC merging operators $\triangle$ are considered. In particular, for each IC merging operator defined from the drastic distance, it is possible to find out a non-iterative definition of the corresponding CHIMC operator, and to prove that it is defined for every profile.

Proposition 8 *Let $E = (K_1, \ldots, K_n)$ be a profile. If the IC merging operator is $\triangle^{d_D,Max}$, then for every j, the base K_j^* from the resulting profile $E^* =^* \triangle_\mu^{d_D,Max}(E)$ can be characterized by:*

$$K_j^* = \begin{cases} \mu \wedge \bigwedge_{K_i : K_i \wedge \mu \nvdash \perp} K_i & \text{if consistent, else} \\ \mu \wedge K_j & \text{if consistent, else} \\ \mu & \text{otherwise.} \end{cases}$$

Furthermore, the resulting profile is obtained after at most two iterations (i.e., for every $i > 1$, $E^i = E^{i+1}$).

Proposition 9 *Let $E = (K_1, \ldots, K_n)$ be a profile. If the IC merging operator is $\triangle^{d_D,GMax}$ of $\triangle^{d_D,\Sigma}$, then for every j, the base K_j^* from the resulting profile $E^* =^* \triangle_\mu^{d_D,GMax}(E) =^* \triangle_\mu^{d_D,\Sigma}(E)$ can be characterized by:*

$$K_j^* = \begin{cases} K_j \wedge \triangle_\mu^{d_D,GMax}(E) & \text{if consistent, else} \\ \triangle_\mu^{d_D,GMax}(E) & \text{otherwise.} \end{cases}$$

Furthermore, the resulting profile is obtained after at most one iteration (i.e., for every $i > 0$, $E^i = E^{i+1}$).

Finally, we have proven that, like for the skeptical case, the sequence of profiles and the corresponding sequence of merged bases are equivalent w.r.t. stationarity in the credulous case:

Definition 12 *Let E be a belief profile and $\triangle$ be an IC merging operator. Let μ be any integrity constraint. The sequence $(E_s^i)_i$ is stationary from some stage if and only if the sequence $(\triangle_\mu(E_s^i))_i$ is stationary from some stage.*

Let us see an example of credulous operator at work.

Example 3 *Consider the profile $E = (K_1, K_2, K_3, K_4)$, with $mod(K_1) = \{(0,0,0), (0,0,1), (0,1,0)\}$, $mod(K_2) = \{(1,0,0), (1,0,1), (1,1,1)\}$, $mod(K_3) = \{(0,0,1), (0,1,0), (0,1,1), (1,1,0)\}$ and $mod(K_4) = \{(0,1,1), (1,0,0), (1,1,0), (1,1,1)\}$. There is no integrity constraint $\mu \equiv \top$, and let us consider the credulous CHIMC operator defined from the merging operator $\triangle^{d_H,\Sigma}$. The computations are summed up in table 3. The resulting profile is $mod(K_1^2) = \{(0,0,1)\}$, $mod(K_2^2) = \{(1,0,0)\}$, $mod(K_3^2) = \{(0,0,1)\}$ and $mod(K_4^2) = \{(1,0,0)\}$. And the corresponding CHIM operator gives as result a base whose models are $\{(0,0,0), (0,0,1), (1,0,0), (1,0,1)\}$, that is different from the result of the merging of E by the IC merging operator $mod(\triangle^{d_H,\Sigma}(E)) = \{(0,0,1), (0,1,1), (1,0,0), (1,1,0)\}$.*

ω	$d(\omega, K_1^i)$	$d(\omega, K_2^i)$	$d(\omega, K_3^i)$	$d(\omega, K_4^i)$	$d_\Sigma(\omega, E^i)$
(0,0,0)	0,1,1	1,1,1	1,1,1	1,1,1	3,4,4
(0,0,1)	0,0,0	1,2,2	0,0,0	1,1,2	2,3,4
(0,1,0)	0,2,2	2,2,2	0,1,2	1,1,2	3,6,8
(0,1,1)	1,1,1	1,3,3	0,0,1	0,0,3	2,4,8
(1,0,0)	1,2,2	0,0,0	1,1,2	0,0,0	2,3,4
(1,0,1)	1,1,1	0,1,1	1,1,1	1,1,1	3,4,4
(1,1,0)	1,3,3	1,1,1	0,0,3	0,0,1	2,4,8
(1,1,1)	2,2,2	0,2,2	1,1,2	0,1,2	3,6,8

Table 3: ${}^*\Delta_\mu^{d_H,\Sigma}$

4 Iterated Merging Operators

We have also investigated the properties of the iterated merging operators induced by the conciliation processes.

A first important question is whether such operators are IC merging operators. The answer is negative in general: only six basic postulates over the nine characterizing IC merging operators are guaranteed to hold:

Proposition 10 *Credulous and Skeptical CHIM operators satisfy (IC0)-(IC3), (IC7) and (IC8).*

Thus, some important properties of IC merging operators are usually lost through the merge-then-revise process. We claim that this is not so dramatic since the main purpose of conciliation processes is not exactly the one of belief merging. Furthermore, specific iterated merging operators (i.e., those induced by some specific merging operators $\triangle$) may easily satisfy additional postulates:

Proposition 11 *The credulous iterated merging operator associated to* ${}^*\triangle_\mu^{d_D,Max}$ *satisfies (IC0)-(IC5), (IC7)-(IC8) and (Arb). It satisfies neither (IC6) nor (Maj).*

In fact, the CHIM operator defined from ${}^*\triangle_\mu^{d_D,Max}$ can be defined as follows (this is a straightforward consequence of proposition 8):

$$\triangle_\mu^{d_D,Max}({}^*\triangle_\mu^{d_D,Max}(E)) = \begin{cases} \mu \wedge \bigwedge_{K_i : K_i \wedge \mu \nvdash \perp} K_i & \text{if consistent, else} \\ \mu & \text{otherwise.} \end{cases}$$

Proposition 12 *The credulous iterated operator associated to* ${}^*\triangle_\mu^{d_D,GMax} = {}^* \triangle_\mu^{d_D,\Sigma}$ *satisfies (IC0)-(IC8), (Arb) and (Maj).*

This result easily comes from the fact that this credulous CHIM operator actually coincides with the IC merging operator $\triangle_\mu^{d_D,GMax} = \triangle_\mu^{d_D,\Sigma}$ it is based on.

Thus, as for skeptical operators (see Proposition 5), each CHIM operator based on the Drastic distance coincides with the underlying IC merging operator, so it satisfies exactly the same properties (see [14]).

As to the operators based on the Hamming distance, things are less easy. Up to now, we did not find an equivalent, non-iterative, definition for any of them. Furthermore, since stationarity is only conjectured for credulous operators (cf. Conjecture 1), we do not have a proof that the corresponding CHIM operators are total functions. So the two following results operators are guaranteed under the conjecture of stationarity, only.

Proposition 13 *The credulous CHIM operator associated to* ${}^*\triangle_\mu^{d_H,\Sigma}$ *satisfies (IC0)-(IC3), (IC7)-(IC8) and (Maj), but does not satisfy (IC5)-(IC6) and (Arb). The satisfaction of (IC4) is an open issue.*

Proposition 14 *The credulous CHIM operators associated to* ${}^*\triangle_\mu^{d_H,Max}$ *and* ${}^*\triangle_\mu^{d_H,GMax}$ *satisfy (IC0)-(IC3), (IC7)-(IC8), but satisfy none of (IC5)-(IC6), (Maj) and (Arb). The satisfaction of (IC4) is an open issue.*

We have similar results for skeptical operators, though the proofs are different:

Proposition 15 *The skeptical CHIM operator associated to* $\triangle_\mu^{*\,d_H,\Sigma}$ *satisfies (IC0)-(IC3), (IC7)-(IC8) and (Maj), but does not satisfy (IC5)-(IC6) and (Arb). The satisfaction of (IC4) is an open issue.*

Proposition 16 *The skeptical CHIM operators associated to* $\triangle_\mu^{*\,d_H,Max}$ *and* $\triangle_\mu^{*\,d_H,GMax}$ *satisfy (IC0)-(IC3), (IC7)-(IC8), but satisfy none of (IC5)-(IC6), (Maj) and (Arb). The satisfaction of (IC4) is an open issue.*

5 Related Work

In [5, 4] Richard Booth presents what he calls *Belief Negociation Models.* Such negociation models can be formalized as games between sources: until a coherent set of sources is reached, at each round a contest is organized to find out the weakest sources, then those sources have to be logically weakened. This idea leads to numerous new interesting operators (depending of the exact meaning of *"weakest"* and *"weaken"*, which correspond to the two parameters for this family). Booth is interested at the same time in the evolution of the profile (in connection to what he calls "Social Contraction"), and to the resulting merged base (the result of the Belief Negociation Model).

In [10, 9] a systematic study of a sub-class of those operators, called *Belief Game Models*, is achieved. This sub-class contains operators closer to merging ones than the general class which also allows negotiation-like operators.

All those operators are close in spirit to the CHIMC/CHIM operators defined in this work. A main difference is that in the work presented in this paper, the evolution of a profile does not always lead to a consensus. Scenarios where agents disagree at a final stage are allowed. Whereas in the former work, the evolution process leads to consensus (in fact consensus is the halting condition of the iterative definition). So CHIMC operators seem more adequate to formalize interaction between agents' beliefs. Thus, they are closer to negotiation processes, since the agents' beliefs change due to the interaction with other agents' beliefs, but this interaction can be stopped when the agents have achieve the best possible compromise.

6 Conclusion

In this paper, we have introduced two conciliation processes based on an iterated merge-the-revise change function for the beliefs of agents. On this ground, a family of conciliation operators and an associated family of iterated merging operators have been defined and studied.

This work calls for several perspectives. One of them concerns the stationarity conjecture related to credulous CHIMC operators (it would clearly be nice to have a formal proof of it, or to disprove it). A second perspective is about rationality postulates for conciliation operators; such postulates should reflect the fact that at the end of the conciliation process, the disagreement between the agents participating to the conciliation process is expected not to be more important than before; a difficulty is that it does not necessarily mean that this must be the case at each step of a conciliation process. A last perspective is to enrich our framework in several directions; one of them consists in relaxing the homogeneity assumption; in some situations, it can prove sensible to consider that an agent is free to reject a negociation step, would it lead her to a belief state "too far" from its original one; it would be interesting to incorporate as well such features in our approach.

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Graphical Models for Utility Elicitation under Risk[1]

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Abstract

This paper deals with preference representation and elicitation in the context of multiattribute utility theory under risk. Assuming the decision maker behaves according to the EU model, we investigate the elicitation of generalized additively decomposable utility functions on a product set (GAI-decomposable utilities). We propose a general elicitation procedure based on a new graphical model called a GAI-network. The latter is used to represent and manage independences between attributes, as junction graphs model independences between random variables in Bayesian networks. It is used to design an elicitation questionnaire based on simple lotteries involving completely specified outcomes. Our elicitation procedure is convenient for any GAI-decomposable utility function, thus enhancing the possibilities offered by UCP-networks.

Key words : Decision theory, graphical representations, preference elicitation, multiattribute expected utility, GAI-decomposable utilities

1 Introduction

Over the last few years the growing interest in decision systems has stressed the need for compact representations of individual's beliefs and preferences, both for user-friendliness of elicitation and reduction of memory consumption. In Decision under Uncertainty, the

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diversity of individuals behaviors and application contexts have led to different mathematical models including Expected Utility (EU) [27, 23], Choquet EU [24], Qualitative EU [10], Generalized EU [12, 13]. The concern in compact numerical representations of preferences being rather recent, studies have mainly focused on EU and emphasized the potential of graphical models such as UCP-nets [3] or influence diagrams [15, 25].

Using EU requires both a numerical representation of the Decision Maker's (DM) preferences over all the possible outcomes (a *utility function*) and a family of probability distributions over these outcomes. In this paper we focus on the assessment of utility, which is usually performed through an interactive process. The DM is asked to answer "simple" questions such as "do you prefer a to b?" and a numerical representation follows.

Theoretically, the assessment of preferences over every pair of outcomes may be needed to elicit completely the DM's utility, but in practice the large size of the outcome set prevents such a procedure to be feasible. Fortunately, preferences often have an underlying structure that can be exploited to drastically reduce the elicitation burden. Several structures described in terms of different independence concepts have emerged from the multiattribute utility theory community [18, 11, 20] and led to different forms of utilities, the most popular of which being the additive and the multilinear decompositions. The particular independences both of these decompositions assume significantly simplify the elicitation procedures, yet as they compel the DM's preferences to satisfy very stringent constraints they are inadequate in many practical situations.

A "good" trade-off between easiness of elicitation and generality of the model can certainly be achieved by *Generalized Additive Independence* (GAI) [11]. This "weak" form of independence is sufficiently flexible to apply to most situations and as such deserves the elaboration of elicitation procedures. Although introduced in the sixties, GAI has not received many contributions yet. In particular, elicitation procedures suggested in the literature for GAI-decomposable utilities are not general purpose. They assume either that the utilities satisfy constraints imposed by CP-net structure (see UCP-nets [3]) or that utilities are random variables (the prior distribution of which is known) and that the elicitation consists in finding an *a posteriori* utility distribution [6, 7]. We feel that these additional assumptions might not be suitable in a significant number of practical decision problems. For instance, as we shall see later in this paper, there exist "simple" GAI-decomposable preferences that cannot be compacted by UCP-nets. Similarly, the existence of prior utility distributions is not always natural, for instance there is not much chance that a company manager facing a given decision problem may have a prior distribution of other managers utilities at hand. Hence an elicitation procedure applicable to any GAI decomposition should prove useful. The purpose of this paper is to propose such a procedure in the context of Decision Making under Risk. More precisely, we assume uncertainties are handled through probabilities and DM's preferences are consistent with EU.

The key idea in our elicitation procedure is to take advantage of a new graphical representation of GAI decompositions we call a GAI network. It is essentially similar to the junction graphs used for Bayesian networks [26, 16, 16, 8]. As such, it keep tracks of all the dependences between the different components of the utilities and the sequence of questions to be asked to the DM can be retrieved directly from this graph.

The paper is organized as follows: the first section provides necessary background in multiattribute utility theory. Then, a typical example showing how a GAI-decomposable utility can be elicited is presented. The third section introduces GAI networks, a graphical tool for representing GAI-decompositions. It also describes a general elicitation procedure relying on this network which applies to any GAI-decomposable utility, as well as a generic scheme for constructing the GAI network. We finally conclude by emphasizing some significant advantages of our elicitation procedure.

2 Utility Decompositions

In this paper, we address problems of decision making under risk [27] (or under uncertainty [23]), that is the DM has a preference relation $\succsim_d$ over a set of decisions $\mathcal{D}$, "$d_1 \succsim_d d_2$" meaning the DM either prefers decision d_1 to d_2 or feels indifferent between both decisions. The consequence or *outcome* resulting from making a particular decision is uncertain and only known through a probability distribution over the set of all possible outcomes. Decisions can thus be described in terms of these distributions, i.e., to each decision is attached a *lottery*, that is a finite tuple of pairs (outcome, probability of the outcome), and to $\succsim_d$ is associated a preference relation $\succsim$ over the set of lotteries such that $d_1 \succsim_d d_2 \Leftrightarrow \text{lottery}(d_1) \succsim \text{lottery}(d_2)$. Taking advantage of this equivalence, we will use lotteries instead of decisions in the remainder of the paper.

Let $\mathcal{X}$ be the finite set of outcomes and let $\mathcal{L}$ be the set of lotteries. $\langle p^1, x^1; p^2, x^2; \ldots; p^q, x^q \rangle$ denotes the lottery such that each outcome $x^i \in \mathcal{X}$ obtains with a probability $p^i > 0$ and $\sum_{i=1}^q p^q = 1$. Moreover, for convenience of notation, when unambiguous, we will note x instead of lottery $\langle 1, x \rangle$. Under some axioms expressing the "rational" behavior of the DM, [23] and [27] have shown that there exist some functions $U : \mathcal{L} \mapsto \mathbb{R}$ and $u : \mathcal{X} \mapsto \mathbb{R}$, unique up to strictly positive affine transforms, such that $L_1 \succsim L_2 \Leftrightarrow U(L_1) \succsim U(L_2)$ for all $L_1, L_2 \in \mathcal{L}$ and $U(\langle p^1, x^1; \ldots; p^q, x^q \rangle) = \sum_{i=1}^q p^i u(x^i)$. Such functions assigning higher numbers to the preferred outcomes are called *utility functions*. As $U(\cdot)$ is the expected value of $u(\cdot)$, we say that the DM is an *expected utility maximizer*.

Eliciting $U(\cdot)$ consists in both assessing the probability distribution over the outcomes for each decision and eliciting function $u(\cdot)$. The former has been extensively addressed in the UAI community [5, 14]. Now eliciting $u(\cdot)$ is in general a complex task as the size of $\mathcal{X}$ is usually very large. The first step to circumvent this problem is to remark that usually the set of outcomes can be described as a Cartesian product of *attributes*

$\mathcal{X} = \prod_{i=1}^n X_i$, where each X_i is a finite set. For instance, a mayor facing the Decision Making problem of selecting one policy for the industrial development of his city can assimilate each policy to a lottery over outcomes defined as tuples of type (investment cost supported by the city, environmental consequence, impact on employment, etc). This particular structure can be exploited by observing that some independences hold between attributes. For instance, preferences over environment consequences should not depend on preferences over employment. Several types of independence have been suggested in the literature, taking into account different preference structures and leading to different functional forms of the utilities. The most usual is the following:

Definition 1 (Additive Independence) *Let L_1 and L_2 be any pair of lotteries and let p and q be their respective probability distributions over the outcome set. Then $X_1, \ldots, X_n$ are additively independent for $\succsim$ if p and q having the same marginals on every X_i implies that both lotteries are indifferent, i.e. $L_1 \succsim L_2$ and $L_2 \succsim L_1$ (or $L_1 \sim L_2$ for short).*

[1] illustrates additive independence on the following example: let $\mathcal{X} = X_1 \times X_2$ where $X_1 = \{a_1, b_1\}$ and $X_2 = \{a_2, b_2\}$. Let L_1 and L_2 be lotteries whose respective probability distributions on $\mathcal{X}$ are p and q. Assume $p(a_1, a_2) = p(a_1, b_2) = p(b_1, a_2) = p(b_1, b_2) = 1/4$, $q(a_1, a_2) = q(b_1, b_2) = 1/2$ and $q(a_1, b_2) = q(b_1, a_2) = 0$. Then p and q have the same marginals on X_1 and X_2 since $p(a_1) = q(a_1) = 1/2$, $p(b_1) = q(b_1) = 1/2$, $p(a_2) = q(a_2) = 1/2$ and $p(b_2) = q(b_2) = 1/2$. So under additive independence, lotteries L_1 and L_2 should be indifferent.

As additive independence captures the fact that preferences only depend on the marginal probabilities on each attribute, it rules out interactions between attributes and thus results in the following simple form of utility [1]:

Proposition 1 *$X_1, \ldots, X_n$ are additively independent for $\succsim$ iff there exist some functions $u_i : X_i \mapsto \mathbb{R}$ such that $u(x) = \sum_{i=1}^n u_i(x_i)$ for any $x = (x_1, \ldots, x_n)$.*

Additive decomposition allows all u_i's to be elicited independently, thus considerably reducing the amount of questions required to determine $u(\cdot)$. However, as no interaction is possible among attributes, such functional form cannot be applied in many practical situations. Hence other types of independence have been introduced that capture more or less dependences. For instance *utility independence* of every attribute [1] leads to a more general form of utility called *multilinear utility*:

$$u(x_1, \ldots, x_n) = \sum_{\emptyset \neq Y \subseteq \{1, \ldots, n\}} k_Y \prod_{i \in Y} u_i(x_i),$$

where the u_i's are scaled from 0 to 1. Multilinear utilities are more general than additive utilities but many interactions between attributes still cannot be taken into account by such functionals. Consider for instance the following example:

Example 1 Let $\mathcal{X} = X_1 \times X_2$, where $X_1 = \{$lamb, vegetable, beef$\}$ and $X_2 = \{$red wine, white wine$\}$. Assume a DM has the following preferences over meals:

$$\begin{aligned}&\text{(lamb, red wine)} \succ \text{(vegetable, red wine)}\\ \sim\ &\text{(lamb, white wine)} \sim \text{(vegetable, white wine)}\\ \succ\ &\text{(beef, red wine)} \succ \text{(beef, white wine)},\end{aligned}$$

that is the DM has some kind of lexicographic preference over food, and then some preference over wine. Then, if a multilinear utility $u(\text{food}, \text{wine}) = k_1 u_1(\text{food}) + k_2 u_2(\text{wine}) + k_3 u_1(\text{food}) u_2(\text{wine})$ existed, since utilities are scaled from 0 to 1, the above preference relations would imply that $u_1(\text{lamb}) = 1 \geq u_1(\text{vegetable}) = x \geq u_1(\text{beef}) = 0$ and that $u_2(\text{red wine}) = 1$ and $u_2(\text{white wine}) = 0$. But then the preference relations could be translated into a system of inequalities $k_1 + k_2 + k_3 > k_1 x + k_2 + k_3 x = k_1 = k_1 x > k_2 > 0$ having no solution, a contradiction. Consequently no multilinear utility can represent these DM preferences, although they are not irrational. ♦

Within multilinear utilities, interactions between attributes are taken into account using the products of subutilities on every attribute. The advantage is that the elicitation task remains reasonably tractable since only the assessments of the u_i's and of constants k_Y's are needed. But the price to pay is that many preference relations cannot be represented by such functions. One way out would be to keep the types of interactions between attributes unspecified, that is, separating the utility function into a sum of subutilities on sets of interacting attributes: this leads to the GAI decompositions. Those result from a generalization of additive utilities:

Definition 2 (Generalized Additive Independence) *Let L_1 and L_2 be any pair of lotteries and let p and q be their probability distributions over the outcome set. Let $Z_1, \ldots, Z_k$ be some subsets of $N = \{1, \ldots, n\}$ such that $N = \cup_{i=1}^k Z_i$ and let $X_{Z_i} = \{X_j : j \in Z_i\}$. Then $X_{Z_1}, \ldots, X_{Z_k}$ are generalized additively independent for $\succsim$ if the equality of the marginals of p and q on all X_{Z_i}'s implies that $L_1 \sim L_2$.*

As proved in [1, 11] the following functional form of the utility called a GAI decomposition can be derived from generalized additive independence:

Proposition 2 *Let $Z_1, \ldots, Z_k$ be some subsets of $N = \{1, \ldots, n\}$ such that $N = \cup_{i=1}^k Z_i$. $X_{Z_1}, \ldots, X_{Z_k}$ are generalized additively independent (GAI) for $\succsim$ iff there exist some real functions $u_i : \prod_{j \in Z_i} X_j \mapsto \mathbb{R}$ such that*

$$u(x) = \sum_{i=1}^{k} u_i(x_{Z_i}), \text{ for all } x = (x_1, \ldots, x_n) \in \mathcal{X},$$

where x_{Z_i} denotes the tuple of components of x having their index in Z_i.

Example 1 (continued) GAI decompositions allow great flexibility because they do not make any assumption on the kind of relations between attributes. Thus, if besides main course and wine, the DM wants to eat a dessert and a starter, her choice for the starter will certainly be dependent on that of the main course, but her preferences for desserts may not depend on the rest of the meal. This naturally leads to decomposing the utility over meals as $u_1(\text{starter}, \text{main course}) + u_2(\text{main course}, \text{wine}) + u_3(\text{dessert})$ and this utility corresponds precisely to a GAI decomposition. ♦

Note that the undecomposed utility $u(\cdot)$ and the additively decomposed utility $\sum_{i=1}^{n} u_i(\cdot)$ are special cases of GAI-decomposable utilities. The amount of questions required by the elicitation is thus closely related to the GAI decomposition itself. In practice, it is unreasonable to consider eliciting subutilities with more than 3 parameters. But GAI decompositions involving "small" X_{Z_i}'s can be exploited to keep the number of questions to a reasonable amount as shown in the next two sections.

3 Elicitation of a GAI-decomposable Utility

In this section, we will first present the general type of questions to be asked to the DM during the elicitation process and, then, we will specialize them to the GAI-decomposable model case.

Let $\succsim$ be a preference relation on the set $\mathcal{L}$ of all possible lotteries over an outcome set $\mathcal{X}$. Let x, y and z be three arbitrary outcomes such that the DM prefers making any decision the result of which is always outcome y (resp. x) to any decision resulting in x (resp. z), i.e., $y \succsim x \succsim z$. In terms of utilities, $u(y) \geq u(x) \geq u(z)$. Consequently, there exists a real number $p \in [0, 1]$ such that $u(x) = pu(y) + (1-p)u(z)$, or equivalently, there exists a probability p such that $x \sim \langle p, y; 1-p, z \rangle$. This gamble is illustrated on Figure 1. Knowing the values of p, $u(y)$ and $u(z)$ thus completely determines that of $u(x)$. This is

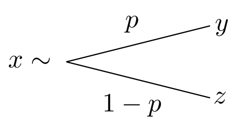


Figure 1: Gamble $x \sim \langle p, y; 1-p, z \rangle$.

the very principle of utility elicitation under risk. In the remainder, to avoid testing which of the outcomes x, y or z are preferred to the others, for any three outcomes x^1, x^2, x^3, we will denote by $G(x^1, x^2, x^3)$ the gamble $x^{\sigma(2)} \sim \langle p, x^{\sigma(1)}; 1-p, x^{\sigma(3)} \rangle$ where σ is a permutation of $\{1, 2, 3\}$ such that $x^{\sigma(1)} \succsim x^{\sigma(2)} \succsim x^{\sigma(3)}$.

Assume that y and z correspond to the most and least preferred outcomes in $\mathcal{X}$ respectively, then all the x's in $\mathcal{X}$ are such that $y \succsim x \succsim z$, and the utility assigned to every outcome in $\mathcal{X}$ can be determined from the knowledge of p, $u(y)$ and $u(z)$. Moreover, as under von Neumann-Morgenstern's axioms utilities are unique up to strictly positive affine transforms, we can assume that $u(y) = 1$ and $u(z) = 0$. Hence there just remains to assess probabilities p. Different interactive procedures exist but they all share the same key idea: the DM is asked which of the following options she prefers: x or $\langle p, y; 1-p, z\rangle$ for a given value of p. If she prefers the first option, another similar question is asked with an increased value of p, else the value of p is decreased. When the DM feels indifferent between both options, p has been assessed.

Of course, as in practice $\mathcal{X}$ is a Cartesian product, $\mathcal{X}$'s size tends to increase exponentially with the number of attributes so that, as such, the above procedure cannot be completed using a reasonable number of questions. Fortunately, GAI decomposition helps reducing drastically the number of questions to be asked. The key idea can be illustrated with the following example:

Example 2 Consider an outcome set $\mathcal{X} = X_1 \times X_2 \times X_3$ and assume that $u(x_1, x_2, x_3) = u_1(x_1) + u_2(x_2, x_3)$. Then it is easily seen that gamble

$$(x_1, a_2, a_3) \sim \langle p, (y_1, a_2, a_3); 1-p, (z_1, a_2, a_3)\rangle$$

is equivalent to gamble

$$(x_1, b_2, b_3) \sim \langle p, (y_1, b_2, b_3); 1-p, (z_1, b_2, b_3)\rangle$$

as they both assert that $u_1(x_1) = pu_1(y_1) + (1-p)u_1(z_1)$. Hence, assuming preferences are stable over time, there is no need to ask the DM questions to determine the value of p in the second gamble: it is equal to that of p in the first one. Thus many questions can be avoided during the elicitation process. Note that in essence this property is closely related to a Ceteris Paribus statement [4]. ♦

Now let us introduce our elicitation procedure with the following example:

Example 3 Let $\mathcal{X} = \prod_{i=1}^{4} X_i$ and assume that utility $u : \mathcal{X} \mapsto \mathbb{R}$ over the outcomes is decomposable as $u(x_1, \ldots, x_4) = u_1(x_1, x_2) + u_2(x_2, x_3) + u_3(x_3, x_4)$. The elicitation algorithm consists in asking questions to determine successively the value of $u_1(\cdot)$, then that of $u_2(\cdot)$ and finally that of $u_3(\cdot)$.

Let (a_1, a_2, a_3, a_4) be an arbitrary outcome that will be used as a reference point. In the sequel, for notational convenience, instead of writing $x_{\{1,2\}}$ for (x_1, x_2) we shall write x_{12}. Let us show that we may assume without loss of generality that:

$$\begin{array}{ll} u_1(b_1, a_2) = 1, & u_1(a_1, x_2) = 0 \text{ for all } x_2 \in X_2, \\ u_3(a_3, a_4) = 0, & u_2(a_2, x_3) = 0 \text{ for all } x_3 \in X_3. \end{array} \quad (1)$$

Assume the DM's preferences are representable by a utility

$$v(x_1, \ldots, x_4) = v_1(x_1, x_2) + v_2(x_2, x_3) + v_3(x_3, x_4)$$

on the outcome set such that $v(\cdot)$ does not necessarily satisfy Eq. (1). Let

$$u_1(x_1, x_2) = v_1(x_1, x_2) - v_1(a_1, x_2).$$

Then $v(x_1, \ldots, x_4) = u_1(x_1, x_2) + [v_2(x_2, x_3) + v_1(a_1, x_2)] + v_3(x_3, x_4)$ and $v_2(x_2, x_3) + v_1(a_1, x_2)$ is a function on $X_2 \times X_3$ and $u_1(a_1, x_2) = 0$ for all x_2's. It can thus be said that $v_2(\cdot)$ has "*absorbed*" a part of $v_1(\cdot)$. Similarly, some part of $v_2(\cdot)$ may be absorbed by $v_3(\cdot)$ in such a way that the resulting $u_2(a_2, x_3) = 0$ for all x_3's: it is sufficient to define

$$\begin{aligned} u_2(x_2, x_3) = \; & v_2(x_2, x_3) + v_1(a_1, x_2) \\ & - v_2(a_2, x_3) - v_1(a_1, a_2). \end{aligned}$$

$v(x_1, \ldots, x_4)$ thus equals to $u_1(x_1, x_2) + u_2(x_2, x_3) + v_3(x_3, x_4) + v_2(a_2, x_3) + v_1(a_1, a_2)$. Note that $u_3(x_3, x_4) = v_3(x_3, x_4) + v_2(a_2, x_3) + v_1(a_1, a_2)$ is a function over $X_3 \times X_4$ as $v_1(a_1, a_2)$ is a constant.

Von Neumann-Morgenstern's utilities being unique up to positive affine transforms, it can be assumed without loss of generality that $u(a_1, a_2, a_3, a_4) = 0$ and that $u(b_1, a_2, a_3, a_4) = 1$ for some arbitrary $b_1 \in X_1$ such that outcome $(b_1, a_{234}) \succsim (a_1, a_{234})$, hence resluting in $u_3(a_3, a_4) = 0$ and $u_1(b_1, a_2) = 1$. Consequently, hypotheses (1) may be assumed without loss of generality.

Thus, the assessment of $u_1(x_1, a_2)$ for all x_1's can be derived directly from gambles such as:

$$\begin{gathered} (x_1, a_{234}) \sim \langle p, (b_1, a_{234}); 1-p, (a_1, a_{234}) \rangle \\ \text{also denoted as } G((b_1, a_{234}), (x_1, a_{234}), (a_1, a_{234})), \end{gathered} \tag{2}$$

as they are equivalent to $u_1(x_1, a_2) = p$. Note that in the above gambles lotteries only differ by the first attribute value, hence the questions asked to the DM should not be cognitively too complicated and the DM should not have difficulties answering them. Then

$$G((b_1, a_2, a_{34}), (a_1, x_2, a_{34}), (a_1, a_2, a_{34})) \tag{3}$$

determines the value of $u_2(x_2, a_3)$. For instance, if $(b_1, a_2, a_{34}) \succ (a_1, x_2, a_{34})$, then the above gamble is equivalent to:

$$(a_1, x_2, a_{34}) \sim \langle q, (b_1, a_2, a_{34}); 1-q, (a_1, a_2, a_{34}) \rangle,$$

which implies that $u_2(x_2, a_3) = q$. Combining Eq. (3) with

$$G((b_1, a_2, a_{34}), (x_1', x_2, a_{34}), (a_1, a_2, a_{34})), \tag{4}$$

where x'_1 is an arbitrary value of X_1, the determination of $u_1(x'_1, x_2)$ follows. Note that until now all calls to function $G(\cdot)$, and especially in equations (3) and (4), shared the same first and third outcomes, i.e., (b_1, a_2, a_{34}) and (a_1, a_2, a_{34}). Note also that the gambles remain cognitively "simple" as most of the attributes are the same for all outcomes. Now, the value of $u_1(x'_1, x_2)$ is sufficient to induce from $G((x'_1, x_2), (x_1, x_2), (a_1, x_2))$ the values of all the $u_1(x_1, x_2)$'s and the determination of $u_1(\cdot)$ is completed.

The same process applies to assess $u_2(\cdot)$. First, using gambles similar to that of Eq. (3), i.e., $G((b_1, a_2, a_{34}), (a_1, b_2, a_{34}), (a_1, a_2, a_{34}))$, $u_2(b_2, a_3)$ can be assessed for arbitrary values b_2 of X_2. Then $G((a_1, b_2, a_{34}), (a_1, x_2, a_{34}), (a_1, a_2, a_{34}))$ will enable the determination of the $u_2(x_2, a_3)$'s for all x_2's (in fact, they will involve terms in $u_1(\cdot)$ and $u_2(\cdot)$ but as $u_1(\cdot)$ has been elicited, only the $u_2(\cdot)$'s remain unknown). Once the $u_2(x_2, a_3)$'s are known, gambles similar to those of Eq. (3) and Eq. (4) but applied to X_2, X_3 instead of X_1, X_2 lead to the complete determination of $u_2(\cdot)$.

Finally as function $u_3(\cdot)$ is the only remaining unknown, $u_3(x_3, x_4)$ can be elicited directly using any gamble involving two "elicited" outcomes. For instance $G((b_1, a_{23}, a_4), (a_1, a_{23}, x_4), (a_1, a_{23}, a_4))$ will determine the $u_3(a_3, x_4)$'s for all values of x_4 and, then, $G((a_{12}, a_3, b_4), (a_{12}, x_3, x_4), (a_{12}, a_3, a_4))$ will complete the assessment of $u_3(\cdot)$. ♦

Note that only a few attributes differed in the outcomes of each of the above gambles, hence resulting in cognitively simple questions. At first sight, this elicitation scheme seems to be a *ad hoc* procedure but, as we shall see in the next section, it proves to be in fact quite general.

4 GAI Networks

To derive a general scheme from the above example, we introduce a graphical structure we call a *GAI network*, which is essentially similar to the junction graphs used in Bayesian networks [17, 8]:

Definition 3 (GAI network) *Let $Z_1, \ldots, Z_k$ be some subsets of $N = \{1, \ldots, n\}$ such that $\bigcup_{i=1}^k Z_i = N$. Assume that $\succsim$ is representable by a GAI-decomposable utility $u(x) = \sum_{i=1}^k u_i(x_{Z_i})$ for all $x \in \mathcal{X}$. Then a GAI network representing $u(\cdot)$ is an undirected graph $G = (V, E)$, satisfying the following properties:*

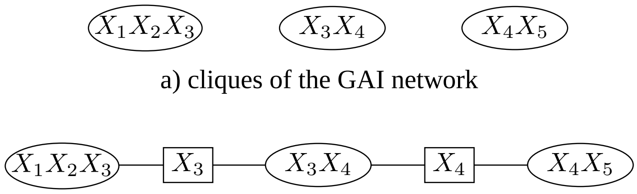
1. *$V = \{X_{Z_1}, \ldots, X_{Z_k}\}$;*
2. *For every $(X_{Z_i}, X_{Z_j}) \in E$, $Z_i \cap Z_j \neq \emptyset$. Moreover, for every pair of nodes X_{Z_i}, X_{Z_j} such that $Z_i \cap Z_j = T_{ij} \neq \emptyset$, there exists a path in G linking X_{Z_i} and X_{Z_j} such that all of its nodes contain all the indices of T_{ij} (Running intersection property).*

Nodes of V are called cliques. *Moreover, every edge $(X_{Z_i}, X_{Z_j}) \in E$ is labeled by $X_{T_{ij}} = X_{Z_i \cap Z_j}$, which is called a* separator.

Throughout this paper, cliques will be drawn as ellipses and separators as rectangles. The rest of this section will be devoted to the construction of GAI networks, and especially GAI trees, from GAI decompositions of utilities, and an elicitation procedure applicable to any GAI tree will be inferred from the example of the preceding section.

4.1 From GAI Decompositions to GAI Networks

For any GAI decomposition, Definition 3 is explicit as to which cliques should be created: these are simply the sets of variables of each subutility. For instance, if $u(x_1, \ldots, x_5) = u_1(x_1, x_2, x_3) + u_2(x_3, x_4) + u_3(x_4, x_5)$ then, as shown in Figure 2.a, cliques are $\{X_1, X_2, X_3\}$, $\{X_3, X_4\}$ and $\{X_4, X_5\}$.



a) cliques of the GAI network

b) edges of the GAI network

Figure 2: The construction of a GAI network.

Property 2 of Definition 3 gives us a clue for determining the set of edges of a GAI network: the algorithm constructing this set should always preserve the running intersection property. A simple — although not always efficient — way to construct the edges thus simply consists in linking cliques that have some nodes in common. Hence the following algorithm:

Algorithm 1 (Construction of a GAI network)
construct set $V = \{X_{Z_1}, \ldots, X_{Z_k}\}$;
for $i \in \{1 \ldots, k-1\}$ **do**
  for $j \in \{i+1 \ldots, k\}$ **do**
    if $Z_i \cap Z_j \neq \emptyset$ **then**
      add edge (X_{Z_i}, X_{Z_j}) *to* E
    fi
  done
done

Applying this algorithm on set $V = \{\{X_1, X_2, X_3\}, \{X_3, X_4\}, \{X_4, X_5\}\}$, sets $\{X_1, X_2, X_3\}$ and $\{X_3, X_4\}$ having a nonempty intersection, an edge should be created between these two cliques. Similarly, edge $(\{X_3, X_4\}, \{X_4, X_5\})$ should also be added as X_4 belongs to both cliques. Consequently the network of Figure 2.b is a GAI network representing $u(x_1, \ldots, x_5) = u_1(x_1, x_2, x_3) + u_2(x_3, x_4) + u_3(x_4, x_5)$.

As we shall see in the next subsection, GAI trees are more suitable than multiply-connected networks for conducting the elicitation process. Unfortunately, GAI networks representing utility decompositions often contain cycles. For instance, consider the following decomposition: $u(x_1, x_2, x_3, x_4) = u_1(x_1, x_2)+u_2(x_2, x_3)+u_3(x_3, x_4)+u_4(x_4, x_1)$. Then the only possible GAI network is that of Figure 3.

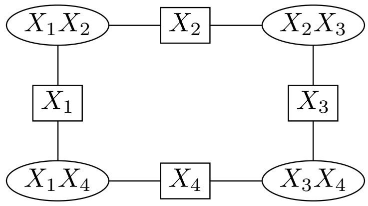


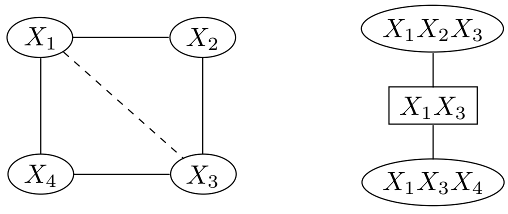
Figure 3: A GAI network containing a cycle.

Unlike GAI trees where a sequence of questions revealing the DM's utility function naturally arises, GAI multiply-connected networks do not seem to be appropriate to easily infer the sequence of questions to ask to the DM. Fortunately, they can be converted into GAI trees using the same triangulation techniques as in Bayesian networks [19, 9]:

Algorithm 2 (Construction of a GAI tree)
1/ create a graph $G' = (V', E')$ such that
a/ $V' = \{X_1, \ldots, X_n\}$;
b/ edge (X_i, X_j) belongs to E' iff there exists a subutility containing both X_i and X_j
2/ triangulate G'
3/ derive from the triangulated graph a junction tree: the GAI tree

For instance, consider again the GAI network of Figure 3 representing utility $u(x_1, x_2, x_3, x_4) = u_1(x_1, x_2) + u_2(x_2, x_3) + u_3(x_3, x_4) + u_4(x_4, x_1)$. Graph G' constructed on step 1 of the above algorithm is depicted on Figure 4.a: the nodes of this graph are X_1, X_2, X_3, X_4, i.e., they correspond to the attributes of the utility. As function $u_1(\cdot)$ is defined over $X_1 \times X_2$, G' contains edge (X_1, X_2). Similarly, functions $u_2(\cdot)$, $u_3(\cdot)$ and $u_4(\cdot)$ imply that E' contains edges (X_2, X_3), (X_3, X_4) and (X_4, X_1), hence resulting in the solid edges in Figure 4.a. Note that graph G' corresponds to a CA-independence map of [1].

On step 2, G' is triangulated using any triangulation algorithm [2, 19, 21], for instance using the following one:



a. attribute dependences b. final GAI tree

Figure 4: From a GAI network to a GAI tree.

Algorithm 3 (triangulation) *Let $G' = (V', E')$ be an undirected graph, where $V' = \{X_1, \ldots, X_n\}$. Let $adj(X_i)$ denote the set of nodes adjacent to X_i in G'. A node $X_i \in V'$ is said to be eliminated from graph G' when*

i) the edges $(adj(X_i) \times adj(X_i)) \backslash E'$ are added to E' so that $adj(X_i) \cup \{X_i\}$ becomes a clique;

ii) the edges between X_i and its neighbors are removed from E', as well as X_i from V'.

Let σ be any permutation of $\{1, \ldots, n\}$. Let us eliminate $X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}$ successively and call E'_T the set of edges added to graph G' by these eliminations. Then graph $G'_T = (V', E' \cup E'_T)$ is triangulated.

This triangulation algorithm, when applied with elimination sequence X_2, X_3, X_1, X_4, precisely produces the graph of Figure 4.a, in which edges in E'_T are drawn with dashed lines.

Step 3 consists in constructing a new graph the nodes of which are the cliques of G' (i.e., maximal complete subgraphs of G'): here, $\{X_1, X_2, X_3\}$ and $\{X_1, X_3, X_4\}$ (see Figure 4.a). The edges between these cliques derive from the triangulation [8, 19, 22]: each time a node X_i is eliminated, it will either create a new clique C_i or a subclique of an already existing clique C_i. In both cases, associate C_i to each X_i. Once a node X_i is eliminated, it cannot appear in the cliques created afterward. However, just after X_i's elimination, all the nodes in $C_i \backslash \{X_i\}$ still form a clique, hence the clique associated to the first eliminated node in $C_i \backslash \{X_i\}$ contains $C_i \backslash \{X_i\}$. Thus linking C_i to this clique ensures the running intersection property. In our example, clique $\{X_1, X_2, X_3\}$ is associated to node X_2 while clique $\{X_1, X_3, X_4\}$ is associated to the other nodes. As X_2 is the first eliminated node, we shall examine clique $\{X_1, X_2, X_3\}$. $C_i \backslash \{X_i\}$ is thus equal to $\{X_1, X_3\}$. Among these nodes, X_3 is the first to be eliminated and clique $\{X_1, X_3, X_4\}$ is associated to this node. Hence, there should exist an edge between cliques $\{X_1, X_2, X_3\}$ and $\{X_1, X_3, X_4\}$. As each clique is linked to at most one other clique, the process ensures that the resulting graph is actually a tree (see Figure 4.b).

Note that the GAI tree simply corresponds to a coarser GAI decomposition of the DM's utility function, i.e., it simply occults some known local independences, but this is the price to pay to make the elicitation process easy to perform.

4.2 Utility Elicitation in GAI Trees

This subsection first translates into a GAI tree-conducted algorithm the elicitation process of the preceding section and, then, a general algorithm is derived.

Example 3 (continued) The GAI network related to Example 3 is shown on Figure 5: ellipses represent the attributes of each subutility and rectangles the intersections between pairs of ellipses. Separators are essential for elicitation because they capture all the dependencies between sets of attributes. For instance separator X_2 reveals that clique X_1X_2 is independent of the rest of the graph for any fixed value of X_2. Hence answers to questions involving gambles on outcomes of type $(\cdot, a_2, a_3, a_4)$ do not depend on a_3, a_4, thus simplifying the elicitation of $u_1(\cdot, a_2)$.

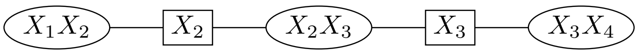


Figure 5: The GAI tree of Example 3.

The elicitation process described in Example 3 can be reformulated using the GAI tree as follows: we started with an outer clique, i.e., a clique connected to at most one separator. The clique we chose was X_1X_2. Function $u_1(\cdot)$ was assessed for every value of the attributes in the clique except those in the separator (here X_2) that were kept to the reference point a_2. This led to assessing $u_1(x_1, a_2)$ for all x_1's using Eq. (2)'s gamble:

$$G((b_1, a_{234}), (x_1, a_{234}), (a_1, a_{234})).$$

Then the values of the attributes in the separator were changed to, say x_2, and $u_1(\cdot)$ was elicited for every value of the attributes in clique X_1X_2 except those in the separator that were kept to x_2. This was performed using the gambles of Eq. (3) and (4), as well as gambles similar to the one above, i.e.,

$$G((b_1, a_2, a_{34}), (a_1, x_2, a_{34}), (a_1, a_2, a_{34})),$$
$$G((b_1, a_2, a_{34}), (x'_1, x_2, a_{34}), (a_1, a_2, a_{34})),$$
$$G((x'_1, x_2, a_{34}), (x_1, x_2, a_{34}), (a_1, x_2, a_{34})).$$

After $u_1(\cdot)$ was completely determined, clique X_1X_2 and its adjacent separator were removed from the network and we applied the same process with another outer clique, namely clique X_2X_3: using gamble $G((b_1, a_2, a_{34}), (a_1, b_2, a_{34}), (a_1, a_2, a_{34}))$, $u_2(b_2, a_3)$ could be determined. Then gamble

$$G((a_1, b_2, a_3, a_4), (a_1, x_2, a_3, a_4), (a_1, a_2, a_3, a_4))$$

was used to assess the value of $u_2(x_2, a_3)$ for any x_2 in X_2. In other words, we assessed the value of $u_2(\cdot)$ for every value of the attributes in the clique except those in the separator (X_3) that were kept to the reference point a_3. Once the $u_2(x_2, a_3)$'s were known, $u_2(\cdot)$ was determined for different values of x_3 using gambles

$$G((b_1, a_2, a_3, a_4), (a_1, a_2, x_3, a_4), (a_1, a_2, a_3, a_4)),$$
$$G((b_1, a_2, a_3, a_4), (a_1, b_2, x_3, a_4), (a_1, a_2, a_3, a_4)),$$
$$G((a_1, b_2, x_3, a_4), (a_1, x_2, x_3, a_4), (a_1, a_2, x_3, a_4)),$$

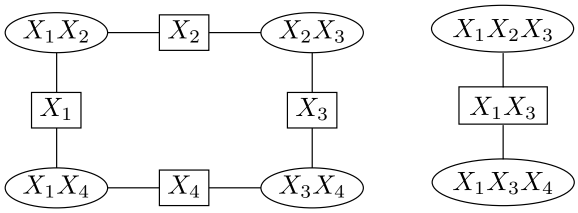
i.e., the values of the attributes in the separator were changed to x_3 and $u_2(\cdot)$ was elicited for every value of the attributes in clique X_2X_3 except those in the separator that were kept to x_3, and so on.

All cliques can thus be removed by induction until there remains only one clique. This one deserves a special treatment as the hypotheses of Eq. (1) specifying that, when we elicit a subutility $u_i(\cdot)$, $u_i(\cdot) = 0$ whenever the value of the attributes not in the separator equal those of the reference point, apply to every clique except the last one. When determining the value of the utility of the last clique, all the other subutilities are known and a direct elicitation can thus be applied. ♦

The above example suggests the following general elicitation procedure, which is applicable to any GAI tree: let $\succsim$ be a preference relation on lotteries over the outcome set $\mathcal{X}$. Let $Z_1, \ldots, Z_k$ be some subsets of $N = \{1, \ldots, n\}$ such that $N = \cup_{i=1}^k Z_i$ and such that $u(x) = \sum_{i=1}^k u_i(x_{Z_i})$ is a GAI-decomposable utility. Assume that the X_{Z_i}'s are such that they form a GAI tree $G = (V, E)$ and that for every i, once all X_{Z_j}'s, $j < i$, have been removed from G as well as their adjacent edges and separators, X_{Z_i} has only one adjacent separator left we will denote by X_{S_i}. In other words, the X_{Z_i}'s are ordered giving priorities to outer nodes. Call $C_i = Z_i \backslash S_i$, and let $C_k = Z_k \backslash S_{k-1}$. Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be arbitrary outcomes of $\mathcal{X}$ such that $(b_{C_i}, a_{N \backslash C_i}) \succ (a_{C_i}, a_{N \backslash C_i})$ for all i's. Then algorithm 4 completely determines the value of each subutility which can then be stored in cliques, thus turning the GAI network into a compact representation of $u(\cdot)$.

Algorithm 4
$u_1(b_{C_1}, a_{N\setminus C_1}) \leftarrow 1$; $u_1(a_N) \leftarrow 0$
for all i *in* $\{1, \ldots, k\}$ **and all** x_{S_i} **do**
  $u_i(a_{C_i}, x_{N\setminus C_i}) \leftarrow 0$
done
for all i *in* $\{1, \ldots, k-1\}$ **do**
  if $i \neq 1$ **then**
    compute $u_i(b_{C_i}, a_{N\setminus C_i})$ *using*
      $G((b_{C_1}, a_{N\setminus C_1}), (b_{C_i}, a_{N\setminus C_i}), (a_N))$
  endif
  for all x_{S_i} **do**
    if $x_{S_i} \neq a_{S_i}$ **then**
      compute $u_i(b_{C_i}, x_{S_i}, a_{N\setminus Z_i})$ *using*
        $G((b_{C_1}, a_{N\setminus C_1}), (x_{S_i}, a_{N\setminus S_i}), (a_N))$
        and $G((b_{C_1}, a_{N\setminus C_1}), (b_{C_i}, x_{S_i}, a_{N\setminus Z_i}), (a_N))$
    endif
    for all x_{Z_i} **do**
      compute $u_i(x_{Z_i})$ *using* $G((b_{C_i}, x_{S_i}, a_{N\setminus Z_i}),$
        $(x_{Z_i}, a_{N\setminus Z_i}), (a_{C_i}, x_{S_i}, a_{N\setminus Z_i}))$
    done
  done
done
/* *computation of the final clique* */
compute $u_k(b_{C_k}, a_{S_{k-1}})$ *using*
  $G((b_{C_1}, a_{N\setminus C_1}), (b_{C_k}, a_{N\setminus C_k}), (a_N))$
for all x_{Z_k} **do**
  compute $u_k(x_{Z_k})$ *using*
    $G((b_{C_k}, a_{N\setminus C_k}), (x_{Z_k}, a_{N\setminus Z_k}), (a_N))$
done

Of course, algorithm 4 can be applied whichever way the GAI tree is obtained. In particular, it can be applied on GAI trees resulting from triangulations. For the latter, the algorithm may be improved taking into account the knowledge of the GAI decomposition before triangulation. Consider for instance the following GAI decomposition: $u(x_1, x_2, x_3, x_4) = u_1(x_1, x_2) + u_2(x_2, x_3) + u_3(x_3, x_4) + u_4(x_4, x_1)$, representable by the GAI network of Figure 6.a and inducing the GAI tree of Figure 6.b, or equivalently the GAI decomposition $u(x_1, \ldots, x_4) = v_1(x_1, x_2, x_3) + v_2(x_1, x_3, x_4)$. Applying directly the elicitation process in the graph of Figure 6.b would be quite inefficient as many questions would be asked to the DM although their answers could be computed from the answers given to previous questions. For instance, assume that X_1 (resp. X_2; X_3) can take values



a. original GAI network　　b. final GAI tree

Figure 6: A GAI tree resulting from a triangulation.

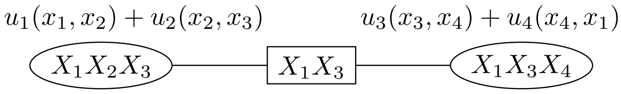


Figure 7: Subutilities in a GAI tree.

a_1, b_1 (resp. a_2, b_2; a_3, b_3). Then, as obviously $v_1(x_1, x_2, x_3) = u_1(x_1, x_2) + u_2(x_2, x_3)$, the above elicitation algorithm ensures that

$$v_1(a_1, a_2, a_3) = u_1(a_1, a_2) + u_2(a_2, a_3) = 0.$$

But, then,

$$\begin{aligned} v_1(b_1, a_2, b_3) &= u_1(b_1, a_2) + u_2(a_2, b_3) \\ &= u_1(b_1, a_2) + u_2(a_2, a_3) + \\ &\quad u_1(a_1, a_2) + u_2(a_2, b_3) \\ &= v_1(b_1, a_2, a_3) + v_1(a_1, a_2, b_3). \end{aligned}$$

Hence, after the elicitation of both $v_1(b_1, a_2, a_3)$ and $v_1(a_1, a_2, b_3)$, that of $v_1(b_1, a_2, b_3)$ can be dispensed with. Intuitively, such questions can be found simply by setting down the system of equations linking the v_i's to the u_i's and identifying colinear vectors.

5 Conclusion

In this paper, we provided a general algorithm for eliciting GAI-decomposable utilities. Unlike UCP-nets, GAI networks do not assume some CP-net structure and thus extend the range of application of GAI-decomposable utilities. For instance, consider a DM having some preferences over some meals constituted by a main course (either a stew or some fish), some wine (red or white) and a dessert (pudding or an ice cream), in particular

$$\begin{aligned} &\text{(stew,red wine,dessert)} \succ \text{(fish,white wine,dessert)} \\ \succ\ &\text{(stew,white wine,dessert)} \succ \text{(fish,red wine,dessert)} \end{aligned}$$

for any dessert. Moreover, assume that the DM would like to suit the wine to the main course and she prefers having ice cream when she eats a stew. Then such preferences can be represented efficiently by $u(\text{meal}) = u_1(\text{course, wine}) + u_2(\text{course, dessert})$ and thus be compacted by the associated GAI network. Nevertheless, since preferences over courses depend on wine and conversely, and since there exists some dependence between courses and desserts, UCP-nets do not help in compacting utility function $u(\cdot)$ despite its GAI decomposability.

Another specificity of our procedure is that we always consider gambles over completely specified outcomes, i.e., including all the attributes. This is an advantage because answers to questions involving only a subset of attributes are not easily interpretable. Consider for instance a multi-attribute decision problem where the multi-attribute space is $\mathcal{X} = X_1 \times X_2 \times X_3 \times X_4$, with $X_1 = \{a_1, c_1, b_1\}$, $X_2 = \{a_2, c_2, b_2\}$, $X_3 = \{a_3, c_3\}$, and $X_4 = \{a_4, c_4\}$. Assume the preferences of the DM can be represented by the following utility function:

$$u(x) = u_1(x_1) + u_2(x_1, x_2) + u_3(x_2, x_3) + u_4(x_3, x_4),$$

where the u_i's are given by the tables below:

x_1	a_1	c_1	b_1
$u_1(x_1)$	0	500	1000

$u_2(x_1, x_2)$	a_2	c_2	b_1
a_1	0	10	70
c_1	50	10	90
b_1	60	80	100

$u_3(x_2, x_3)$	a_3	c_3
a_2	0	7
c_2	5	2
b_2	9	10

$u_4(x_3, x_4)$	a_4	c_4
a_3	0	0.6
c_3	0.4	1

Note that the big-stepped structure of utilities in the above tables is consistent with the Ceteris Paribus assumption about preferences, hence $u(\cdot)$ can be characterized by the UCP-net of Figure 8. Asking the DM to provide probability p such that $c_1 \sim \langle p, b_1; 1-p, a_1 \rangle$ would, at first sight, be meaningful and, assuming $u_1(a_1) = 0$ and $u_1(b_1) = 1000$, it would certainly imply that $u_1(c_1) = 1000p$. However, a careful examination highlights that it is not so obvious. Indeed, such gamble, involving only attribute X_1 would be meaningful only if the DM had a preference relation $\succsim_1$ over X_1 that could be exploited to extract informations about $\succsim$, the DM's preference relation over $\mathcal{X}$. In the classical framework of additive conjoint measurement [11, 20, 28], this property holds because $c_1 \sim$

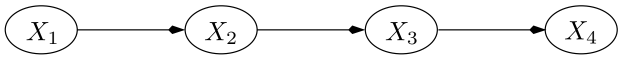
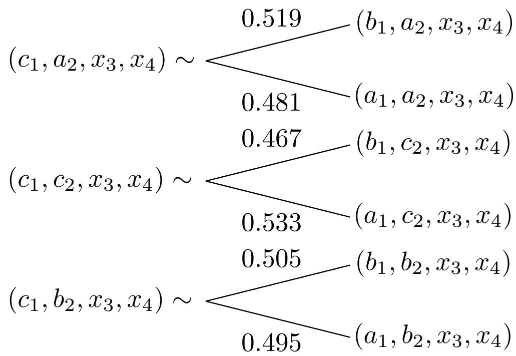


Figure 8: A simple UCP-net.

$\langle p, b_1; 1-p, a_1\rangle$ is equivalent to $(c_1, x_2, x_3, x_4) \sim \langle p, (b_1, x_2, x_3, x_4); 1-p, (a_1, x_2, x_3, x_4)\rangle$ for any $(x_2, x_3, x_4) \in X_2 \times X_3 \times X_4$, but this does not hold for GAI decompositions involving intersecting factors. For instance, using the above tables, it is easily seen that, whatever values for X_3 and X_4:



The explanation of this unfortunate property lies in the misleading interpretation we may have of Ceteris Paribus statements: in the above UCP-net, Ceteris Paribus implies that preferences over X_1 do not depend on the values of the other attributes. The observation of the subutility tables confirm this fact: b_1 is preferred to c_1, that is also preferred to a_1. However, the CP property does not take into account the strength of these preferences while the probabilities involved in the lotteries do: whatever the value of X_2, (b_1, x_2) is always preferred to (c_1, x_2), but the DM prefers more (b_1, c_2) to (c_1, c_2) than (b_1, a_2) to (c_1, a_2) and this results in different values of p in gambles. This explains the discrepancy between $c_1 \sim \langle p, b_1; 1-p, a_1\rangle$ and the same gamble taking into account the other attributes. This discrepancy is not restricted to UCP-net root nodes, it is easily seen that it also occurs for other nodes such as X_2 or X_3.

To conclude, the GAI networks introduced in this paper allow taking advantage of any GAI decomposition of a multiattribute utility function to construct a compact representation of preferences. The efficiency of the proposed elicitation procedure lies both in the relative simplicity of the questions posed and in the careful exploitation of independences between attributes to reduce the number of questions. This approach of preference elicitation is a good compromise between two conflicting aspects: the need for sufficiently flexible models to capture sophisticated decision behaviors under uncertainty and the practical necessity of keeping the elicitation effort at an admissible level. A similar approach might be worth investigating for the elicitation of multiattribute utility functions under certainty. Resorting to GAI networks in this context might also be efficient to elicit subutility functions under some solvability assumptions on the product set.

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Computation of median orders: complexity results

Olivier Hudry[*]

Abstract

Given a set of individual preferences defined on a same finite set of candidates, we consider the problem of aggregating them into a collective preference minimizing the number of disagreements with respect to the given set and verifying some structural properties like transitivity. We study the complexity of this problem when the individual preferences as well as the collective one must verify different properties, and we show that the aggregation problem is NP-hard for different types of collective preferences, even when the individual preferences are linear orders.

Key words: Complexity, partially ordered relations, median relations, aggregation of preferences.

1 Introduction

The problem that we deal with in this paper can be stated as follows: given a set (called a *profile*) $\Pi = (R_1, R_2, \ldots, R_m)$ of m binary relations R_i ($1 \le i \le m$) defined on the same finite set X, find a binary relation R^* defined on X verifying certain properties like transitivity and summarizing Π as accurately as possible. This problem occurs in different fields, for instance in the social sciences, in electrical engineering, in agronomy or in mathematics (see for example L. Hubert (1976), J.-P. Barthélemy *et alii* (1981, 1986, 1988, 1989, 1995), M. Jünger (1985), G. Reinelt (1985), A. Guénoche *et*

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alii (1994)). For example, in voting theory, X can be considered as a set of candidates, Π as a profile of individual preferences expressed by voters and R^* as the collective preference that we look for. Though the problem occurs in different fields, as said above, we shall keep this illustration from voting theory in the following.

The aim of this paper is to study the complexity of finding R^* (for the theory of complexity, see for instance M.R. Garey and D.S. Johnson (1979) or J.-P. Barthélemy *et alii* (1996)). We consider different types of ordered relations for the individual preferences of Π as well as for R^* and we show that for most cases, the computation of R^* is an NP-hard problem. This problem has been already studied in some special cases, namely for the aggregation of a profile of linear orders into a linear order by J.B. Orlin (1988) and by J.J. Bartholdi III, C.A. Tovey and M.A. Trick (1989), and for the aggregation of a profile of binary relations into a linear order, a partial order, a complete preorder or a preorder (see below for the definitions of these structures) by Y. Wakabashi (1986 and 1998). The results displayed in this paper generalize the previous ones by extending them to other cases. They slightly strengthen and sometimes generalize the ones presented in O. Hudry (1989).

In the following, the relations to aggregate are assumed to represent preferences, and thus will not be symmetric. Anyway, the aggregation of symmetric relations has also been studied: M. Krivanek and J. Moravek (1986) showed that the approximation of a symmetric relation by an equivalence relation (a reflexive, symmetric, and transitive relation) is NP-hard. This case corresponds with the aggregation of a profile reduced to only one symmetric relation while R^* is assumed to be an equivalence relation. From this, we may derive that the aggregation of several symmetric relations or of equivalence relations into one equivalence relation is also NP-hard (see J.-P. Barthélemy and B. Leclerc (1995)). On contrary, the aggregation of symmetric relations or of equivalence relations into a symmetric relations is trivially polynomial.

The paper is organized as follows. Section 2 recalls the definitions of the ordered relations that we take into account. In Section 3, we show how the aggregation problems can be formulated in graph theoretical terms. Then we prove our complexity results upon these aggregation problems in Section 4. The conclusions take place in Section 5 and summarize the main results got in Section 4.

2 The ordered relations

Given a finite set X, a binary relation R defined on X is a subset of $X \times X = \{(x, y): x \in X \text{ and } y \in X\}$. We note n the number of elements of X and we suppose that n is great enough (typically, at least equal to 4). We note xRy instead of

$(x, y) \in R$ and $x\overline{R}y$ instead of $(x, y) \notin R$. The following properties that a binary relation R can satisfy are basic:

• *reflexive*: $\forall\ x \in X$, xRx;

• *irreflexive*: $\forall\ x \in X$, $x\overline{R}x$;

• *antisymmetric*: $\forall\ (x, y) \in X^2$, $(xRy$ and $x \neq y) \Rightarrow y\overline{R}x$;

• *asymmetric*: $\forall\ (x, y) \in X^2$, $xRy \Rightarrow y\overline{R}x$;

• *transitive*: $\forall\ (x, y, z) \in X^3$, $(xRy$ and $yRz) \Rightarrow xRz$;

• *complete*: $\forall\ (x, y) \in X^2$ with $x \neq y$, xRy or (inclusively) yRx.

From a binary relation R, we may define an asymmetric relation R^a (called the *asymmetric part* of R) by: $xR^ay \Leftrightarrow (xRy$ and $y\overline{R}x)$.

By combining the above properties, we may define different types of binary relations (see for instance J.-P. Barthélemy and B. Monjardet (1981) or P.C. Fishburn (1985)). As a binary relation R defined on X is the same as the oriented graph $G = (X, R)$ (*i.e.* (x, y) is an arc of G if and only if we have xRy), we illustrate these types with graph theoretic examples:

• a *partial order* is an asymmetric and transitive binary relation; $\mathcal{O}$ will denote the set of the partial orders defined on X;

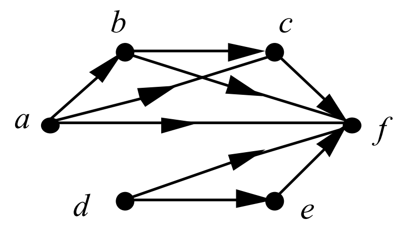

Figure 1. A partial order.

• a *linear order* is a complete partial order; $\mathcal{L}$ will denote the set of the linear orders defined on X;

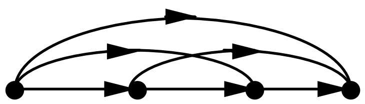

Figure 2. A linear order. The partial order of Figure 1 is not a linear order, for instance because the vertices a and d are not compared.

• a *tournament* is a complete and asymmetric binary relation; $\mathcal{T}$ will denote the set of the tournaments defined on *X*; notice that a transitive tournament is a linear order and conversely;

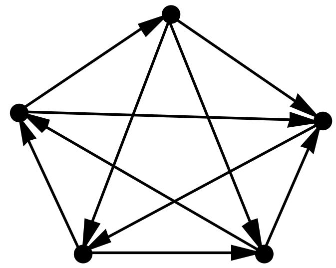

Figure 3. A tournament.

• a *preorder* is a reflexive and transitive binary relation; $\mathcal{P}$ will denote the set of the preorders defined on *X*;

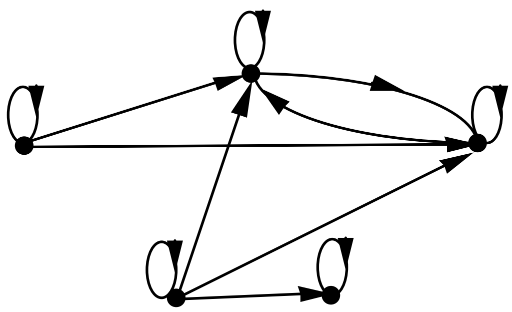

Figure 4. A preorder.

• a *complete preorder* is a reflexive, transitive and complete binary relation; $\mathcal{C}$ will denote the set of the complete preorders defined on *X*;

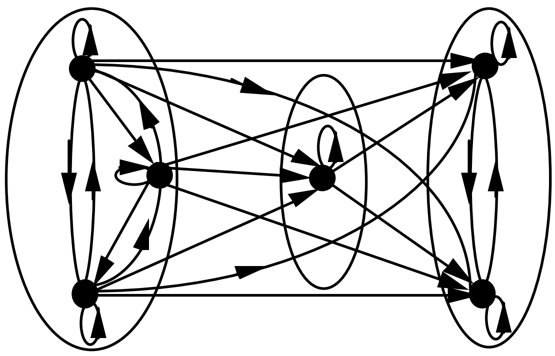

Figure 5. A complete preorder.

• a *weak order* is the asymmetric part of a complete preorder; $\mathcal{W}$ will denote the set of the weak orders defined on X;

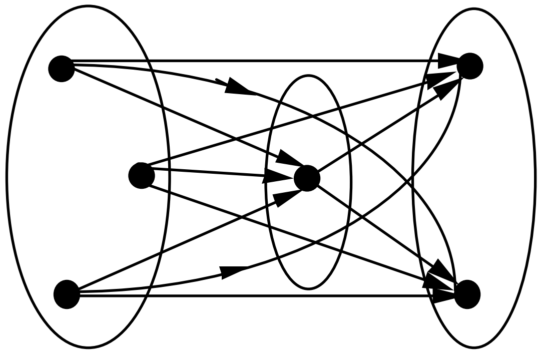

Figure 6. A weak order (namely, the asymmetric part of the complete preorder of Figure 5).

• an *interval order* is a partial order R satisfying: $\forall$ $(x, y, z, t) \in X^4$, $(xRy$ and $zRt) \Rightarrow \{xRt$ or (inclusively) $zRy\}$; $\mathcal{I}$ will denote the set of interval orders defined on X (the name *interval order* comes from the fact that we may represent such an order by intervals spread on the real axis and associated with each element x of X: then xRy means that the interval associated with x is completely on the left of the one associated with y, while $x\overline{R}y$ and $y\overline{R}x$ mean that the intervals associated with x and y overlap; the above condition means that if x is on the left of y and z on the left of t, then the intervals associated with x and t on one hand and the ones associated with z and y on the other hand cannot overlap simultaneously).

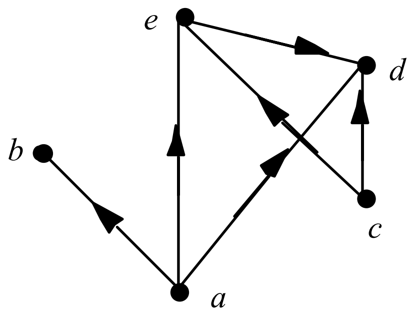

Figure 7. An interval order .The partial order of Figure 1 is not an interval order for instance because of the vertices b, c, d, e: bRc and dRe but we have not bRe nor dRc.

• a *semiorder* is an interval order R satisfying: $\forall$ $(x, y, z, t) \in X^4$, (xRy and yRz) $\Rightarrow$ {xRt or (inclusive) tRz}; $\mathcal{S}$ will denote the set of interval orders defined on X (with respect to the representation as intervals, an interval orders is a semiorder if we may associate intervals with the same length to all the elements of X).

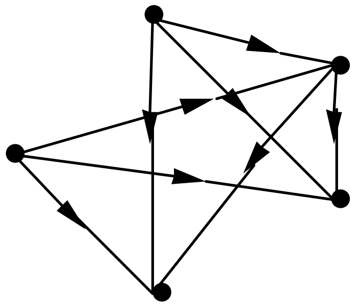

Figure 8. A semiorder .The interval order of Figure 7 is not a semiorder for instance because of the vertices b, c, d, e: cRe and eRd but we have not cRb nor bRd.

• a *quasi-order* is a reflexive and complete relation of which the asymmetric part is a semiorder; $\mathcal{Q}$ will denote the set of the quasi-orders defined on X;

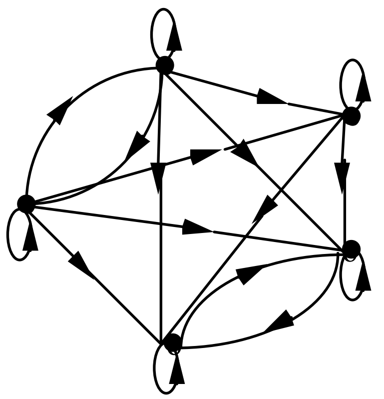

Figure 9. A quasi-order. Its asymmetric part is the semiorder of Figure 8.

• an *acyclic relation* is a relation R without directed cycle (*circuit*), *i.e.* verifying: $\forall$ $1 \leq k \leq n$, ($x_i R x_{i+1}$ for $1 \leq i \leq k - 1$) $\Rightarrow$ $x_k \overline{R} x_1$; $\mathcal{A}$ will denote the set of acyclic relations defined on X.

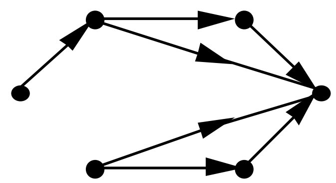

Figure 9. An acyclic relation. Its transitive closure is the partial order of Figure 1.

Checking that a given relation (or a given graph) fulfils the requirements of these structures can be done in polynomial time with respect to n. From this remark, it will follow that the problems considered below all belong to NP.

It is possible to get other structures by adding or by removing reflexivity or irreflexivity from the above definition (and by changing asymmetry by antisymmetry when necessary). In fact, the distinction between reflexive and irreflexive relations is not relevant for our study, as we shall see below: the complexity results will remain the same. Thus, in the following, we do not take reflexivity or irreflexivity into account (for instance, we will consider that a linear order is also a preorder).

These types include the most studied and used partially ordered relations. We will also consider generic binary relations, without any particular property. The set of the binary relations will be noted $\mathcal{R}$. We may notice several inclusions between these sets, especially the following one: $\forall\ \mathcal{Z} \in \{\mathcal{A}, \mathcal{C}, \mathcal{I}, \mathcal{L}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{W}\}$, $\mathcal{L} \subseteq \mathcal{Z}$; in other words, a linear order can be considered as a special case of any one of the other types.

3 Formulations of the aggregation problem

In order to get an optimization problem to deal with, it is necessary to explicit what we mean when we say that R^* must summarize Π "as accurately as possible". To do so, we consider the symmetric difference distance δ: given two binary relations R and S defined on the same set X, we have

$$\delta(R,S) = \left|\left\{(x,y) \in X^2 : \left[xRy \text{ and } x\bar{S}y\right] \text{ or } \left[x\bar{R}y \text{ and } xSy\right]\right\}\right|$$

This quantity $\delta(R,S)$ measures the number of disagreements between R and S. Though some authors consider sometimes another distance, δ is used widely and is appropriate for many applications. J.-P. Barthélemy (1979) shows that δ satisfies a number of naturally desirable properties and J.-P. Barthélemy and B. Monjardet (1981) recall that $\delta(R,S)$ is the Hamming distance between the characteristic vectors of R and S and point out the links between δ and the L_1 metric or the square of the Euclidean distance between these vectors (see also K.P. Bogart (1973 and 1975) and B. Monjardet (1979 and 1990)).

Then, for a profile $\Pi = (R_1, R_2, \ldots, R_m)$ of m relations, we can define the *remoteness* $\Delta(\Pi, R)$ (J.-P. Barthélemy and B. Monjardet (1981)) between a relation R and the profile Π by:

$$\Delta(\Pi, R) = \sum_{i=1}^{m} \delta(R, R_i).$$

The remoteness $\Delta(\Pi, R)$ measures the total number of disagreements between Π and R.

Our aggregation problem can be seen now as a combinatorial problem: given a profile Π, determine a binary relation R^* minimizing Δ over one of the sets $\mathcal{A}$, $\mathcal{C}$, $\mathcal{I}$, $\mathcal{L}$, $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{W}$. Such a relation R^* will be called a *median relation* of Π (J.-P. Barthélemy and B. Monjardet (1981)). According to the number m of relations of the profile and to the properties assumed for the relations belonging to Π or required from the median relation, we get many combinatorial problems. They are too numerous to state them explicitly; so we note them as follows:

Problems $P_f(\mathcal{Y}, \mathcal{Z})$. For $\mathcal{Y}$ belonging to $\{\mathcal{A}, \mathcal{C}, \mathcal{I}, \mathcal{L}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{W}\}$ and $\mathcal{Z}$ belonging also to $\{\mathcal{A}, \mathcal{C}, \mathcal{I}, \mathcal{L}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{W}\}$, for a function f defined from the set $\mathbf{N}$ of integers to $\mathbf{N}$, $P_f(\mathcal{Y}, \mathcal{Z})$ denotes the following problem: given a finite set X of n elements, given a profile Π of $m = f(n)$ binary relations all belonging to $\mathcal{Y}$, find a relation R^* belonging to $\mathcal{Z}$ with: $\Delta(\Pi, R^*) = \underset{R \in Z}{\text{Min}}\, \Delta(\Pi, R)$.

An interesting case is the one for which f is a constant m, i.e. the particular case for which the number m of relations is fixed outside the instance. We will denote this problem by $P_m(\mathcal{Y}, \mathcal{Z})$. For instance, $P_2(\mathcal{R}, \mathcal{L})$ will denote the aggregation of 2 binary relations into a linear order. We will see that the parity of m will play a role in the following results. Anyway, it will be easy to see from the following computations that, if $P_f(\mathcal{Y}, \mathcal{Z})$ is NP-hard for some function f and some sets $\mathcal{Y}$ and $\mathcal{Z}$, then $P_{f+2}(\mathcal{Y}, \mathcal{Z})$ will also be NP-hard: it will be sufficient to add any linear order and its reverse order to the considered profile to get this result (since the linear orders are special cases of any one of the other types).

We do not explicit the statements of the decision problems associated with the problems $P_f(\mathcal{Y}, \mathcal{Z})$, because they are obvious. Similarly, it is obvious to show that these decision problems belong to NP. Thus, we deal with the NP-hardness of $P_f(\mathcal{Y}, \mathcal{Z})$, but we could deal with the NP-completeness of the decision problems associated with $P_f(\Upsilon, \mathcal{Z})$.

To study the complexity of $P_f(\Upsilon, \mathcal{Z})$, we develop the expression of Δ. For this, consider the characteristic vectors $r^i = \left(r^i_{xy}\right)_{(x,y)\in X^2}$ of the relations R_i $(1 \le i \le m)$ defined by $r^i_{xy} = 1$ if xR_iy and $r^i_{xy} = 0$ otherwise, and similarly the characteristic vector $r = \left(r_{xy}\right)_{(x,y)\in X^2}$ of any binary relation R. Then, it is easy to get a linear expression of $\Delta(\Pi, R)$:

$$\delta(R, R_i) = \sum_{(x,y)\in X^2}\left|r_{xy} - r^i_{xy}\right| = \sum_{(x,y)\in X^2}\left|r_{xy} - r^i_{xy}\right|^2 = \sum_{(x,y)\in X^2}\left[r_{xy}\left(1 - 2r^i_{xy}\right) + r^i_{xy}\right]$$

hence $$\Delta(\Pi, R) = \sum_{i=1}^{m}\sum_{(x,y)\in X^2}\left|r_{xy} - r^i_{xy}\right|$$

and, after simplifications: $$\Delta(\Pi, R) = C - \sum_{(x,y)\in X^2} m_{xy} \cdot r_{xy}$$

with $C = \sum_{i=1}^{m}\sum_{(x,y)\in X^2} r^i_{xy}$ and $m_{xy} = \sum_{i=1}^{m}\left(2r^i_{xy} - 1\right) = 2\sum_{i=1}^{m} r^i_{xy} - m$.

Notice that the quantities m_{xy} can be non-positive or non-negative, and that they all have the same parity (the one of m). Notice also that, from this expression of $\Delta(\Pi, R)$, it is easy to get a 0-1 linear programming formulation of the problems $P_f(\Upsilon, \mathcal{Z})$ by adding the 0-1 linear constraints associated with each type of median relation (but it will not be the way that we are going to follow in the sequel). For example, the transitivity of R can be written: $\forall (x, y, z) \in X^3$, $r_{xy} + r_{yz} - r_{xz} \le 1$ (see for instance Y. Wakabayashi (1986) or O. Hudry (1989) for details). Such a 0-1 linear programming formulation was applied as soon as 1960 (A.W. Tucker (1960); see also D.H. Younger (1963), J.S. de Cani (1969), D. Arditti (1984), and more generally J.-P. Barthélemy and B. Monjardet (1981) for references).

Before going further, the following lemma shows that reflexivity or irreflexivity of the median relation do not change the complexity of the problems $P_f(\Upsilon, \mathcal{Z})$.

Lemma 1. For any set $\mathcal{Z}$ of median relations, let $\mathcal{Z}_r$ (resp. $\mathcal{Z}_i$) be the set of median relations got from the elements of $\mathcal{Z}$ by adding the reflexivity (resp. irreflexivity) property. Then, for any set Υ and any function f, $P_f(\Upsilon, \mathcal{Z})$, $P_f(\Upsilon, \mathcal{Z}_r)$, and $P_f(\Upsilon, \mathcal{Z}_i)$ have the same complexity.

Proof. To show this result, consider any profile Π of m (= $f(n)$) relations belonging to $\mathcal{Y}$ and any relation Z belonging to $\mathcal{Z}$. Let Z_r (resp. Z_i) be the reflexive (resp. irreflexive) relation got from Z by adding the reflexivity (resp. irreflexivity) property. Then it is easy to state the following relations:

$$\Delta(\Pi, Z_r) = \Delta(\Pi, Z) + \sum_{x:(x,x)\notin Z} m_{xx} \text{ and } \Delta(\Pi, Z_i) = \Delta(\Pi, Z) - \sum_{x:(x,x)\in Z} m_{xx}.$$

Hence the result, since the computation of $\sum_{x:(x,x)\notin Z} m_{xx}$ and of $\sum_{x:(x,x)\in Z} m_{xx}$ can trivially be done in polynomial time w.r.t. the size of the considered instance. ❑

Because of Lemma 1, we shall not pay attention from now on to reflexivity or irreflexivity: all the complexity results remain the same if we add or remove reflexivity or irreflexivity.

In the following, we will not consider the previous 0-1 linear programming formulation to study the complexity of the problems $P_f(\mathcal{Y}, \mathcal{Z})$, but a graph theoretic representation. Indeed, we may associate a complete, symmetric, weighted, oriented graph $G = (X, U)$ to any profile Π: the vertex set of G is X and G owns all the arcs (i.e. oriented edges) that a simple graph can own; in other words, we have: $U = X \times X - \{(x,x) \text{ for } x \in X\}$ (remember that reflexivity does not matter now on). In the following, we will write U_X to denote the set $X \times X - \{(x,x) \text{ for } x \in X\}$ and the graph associated with Π is thus $G = (X, U_X)$. The arcs (x, y) of G (with $x \in X$, $y \in X$ and $x \neq y$) are weighted by m_{xy}. Then minimizing $\Delta(\Pi, Z)$ for Z belonging to one of the sets $\mathcal{A}$, $\mathcal{C}$, $\mathcal{I}$, $\mathcal{L}$, $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{W}$ is exactly the same as extracting a partial graph $H = (X, Z)$ from G in order to maximize $\sum_{(x,y)\in Z} m_{xy}$ while the kept arcs describe the structure that Z must respect (H must belong to $\mathcal{A}$, $\mathcal{C}$, $\mathcal{I}$, $\mathcal{L}$, $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{W}$, where these sets are seen as sets of graphs).

Then the question arises: which weighted graphs $G = (X, U_X)$ can be associated to a profile Π? By combining results by P. Erdös, L. Moser (1964) and by B. Debord (1987) (see also D. McGarvey (1953) and R. Stearns (1959)), we get such characterisations, which depend on the nature of the relations of Π. For the statements of the following theorems, let M denote the highest absolute value of the weights of G: $M = \underset{(x,y) \in U_X}{\text{Max}} \left| m_{xy} \right|$.

Theorem 2. The graph $G = (X, U_X)$ weighted by the (non-positive or non-negative) integers m_{xy} represents a profile Π of m binary relations if the following conditions are fulfilled:

1. all the weights m_{xy} have the same parity;
2. m has the same parity as the weights m_{xy};
3. $m \geq M$.

Theorem 3. The graph $G = (X, U_X)$ weighted by the (non-positive or non-negative) integers m_{xy} represents a profile Π of m tournaments if the following conditions are fulfilled:

1. all the weights m_{xy} have the same parity;
2. m has the same parity as the weights m_{xy};
3. $m \geq M$;
4. $\forall\ (x, y) \in U_X,\ m_{xy} = -m_{yx}$.

Theorem 4. The graph $G = (X, U_X)$ weighted by the (non-positive or non-negative) integers m_{xy} represents a profile Π of m linear orders if the following conditions are fulfilled:

1. all the weights m_{xy} have the same parity;
2. m has the same parity as the weights m_{xy};
3. $m \geq \dfrac{c.n.M}{\log n}$ where c is a constant;
4. $\forall\ (x, y) \in U_X,\ m_{xy} = -m_{yx}$.

Notice that, for Theorems 2 and 3, M is the lowest possible value of m. For Theorem 4, it is sometimes possible to find a profile of m linear orders associated with

G with m less than $\frac{c.n.M}{\log n}$. Anyway, in all these cases, there exists a profile Π with $m = M$ binary relations, or $m = M$ tournaments, or $m = \left|\frac{c.n.M}{\log n}\right|$ or $m = \left|\frac{c.n.M}{\log n}\right| + 1$ (depending on the parity of $\left|\frac{c.n.M}{\log n}\right|$ and of the weights of G) linear orders. Moreover, if we assume that M is upper-bounded by a polynomial in n (as it will be the case further), then the construction of Π can be done in polynomial time with respect to the size of G. Indeed, as any binary relation R defined on X can be described by $O(n^2)$ bits, it is possible to code $\Pi = (R_1, R_2, \ldots, R_M)$ with $O(M.n^2)$ bits, while the size of G is at least n^2 (at least 1 bit for the weight of each arc of G); hence the result. Notice also that, if M is upper-bounded by a constant, it is possible to fix the number m of relations of Π in Theorems 2 and 3; in this case, we may associate to G an instance of the problems $P_m(\mathcal{R}, \mathcal{Z})$ (Theorem 2) or the problems $P_m(\mathcal{T}, \mathcal{Z})$ (Theorem 3) for an appropriate set $\mathcal{Z}$.

From this polynomial link between the problems $P_f(\mathcal{Y}, \mathcal{Z})$ and their graph theoretic representations, it appears that we may study the complexity of the problems $P_f(\mathcal{Y}, \mathcal{Z})$ with the help of weighted graphs. It is what we do below. More precisely, we are going to study the following decision problems, stated as graph theoretic problems:

Problems $Q_0(\mathcal{Y}, \mathcal{Z})$ with $\mathcal{Y} \in \{\mathcal{L}, \mathcal{R}, \mathcal{T}\}$ and $\mathcal{Z} \in \{\mathcal{A}, \mathcal{C}, \mathcal{I}, \mathcal{L}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{W}\}$

Instance: a graph $G = (X, U_X)$ weighted by (non-positive or non-negative) even integers m_{xy} and which represents a profile of relations belonging to $\mathcal{Y}$; an integer K;

Question: does there exist a partial graph (X, U) of G belonging to $\mathcal{Z}$ with $\sum_{(x,y)\in U} m_{xy} \geq K$?

Problems $Q_1(\mathcal{Y}, \mathcal{Z})$ with $\mathcal{Y} \in \{\mathcal{L}, \mathcal{R}, \mathcal{T}\}$ and $\mathcal{Z} \in \{\mathcal{A}, \mathcal{C}, \mathcal{I}, \mathcal{L}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{W}\}$

Instance: a graph $G = (X, U_X)$ weighted by (non-positive or non-negative) odd integers m_{xy} and which represents a profile of relations belonging to $\mathcal{Y}$; an integer K;

Question: does there exist a partial graph (X, U) of G belonging to $\mathcal{Z}$ with $\sum_{(x,y)\in U} m_{xy} \geq K$?

4 The complexity results

As for the problems $P_f(\mathcal{Y}, \mathcal{Z})$, the problems $Q_0(\mathcal{Y}, \mathcal{Z})$ and $Q_1(\mathcal{Y}, \mathcal{Z})$ obviously belong to NP for $\mathcal{Y} \in \{\mathcal{L}, \mathcal{R}, \mathcal{T}\}$ and $\mathcal{Z} \in \{\mathcal{A}, \mathcal{C}, \mathcal{I}, \mathcal{L}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{W}\}$. To show that they are NP-complete, we use the well-known Feedback Arcset Problem (see M.R. Garey, D.S. Johnson (1979)):

Instance: a directed, asymmetric graph $H = (X, W)$; an integer h;

Question: does there exist $W' \subset W$ with $|W'| \leq h$ and such that W' contains at least one arc of each circuit (directed cycle) of H ?

R. Karp (1972) showed that this problem is NP-complete. We may also state it as follows:

Problem FAS

Instance: a directed, asymmetric graph $H = (X, W)$; an integer h;

Question: does there exist $W' \subset W$ with $|W'| \leq h$ and such that removing the elements of W' from H leaves a graph $(X, W - W')$ without any circuit (such a set W' is called a *feedback arc set* of H of cardinality at most h) ?

In the following, we use this latter formulation. We use also the following (obvious) lemma:

Lemma 5.

a. Any partial graph of a graph without circuit is itself without circuit.

b. Any graph without circuit can be completed into a linear order by adding appropriate arcs.

We now pay attention to the complexity of $Q_0(\mathcal{R}, \mathcal{Z})$ and $Q_1(\mathcal{R}, \mathcal{Z})$ (i.e. when the graph represents a profile of any binary relations) for $\mathcal{Z} \in \{\mathcal{A}, \mathcal{I}, \mathcal{L}, \mathcal{O}, \mathcal{S}, \mathcal{W}\}$ (i.e. when we look for an acyclic relation, an interval order, a linear order, a partial order, a semiorder or a weak order).

Theorem 6. The problems $Q_0(\mathcal{R}, \mathcal{Z})$ are NP-complete for $\mathcal{Z} \in \{\mathcal{A}, \mathcal{I}, \mathcal{L}, \mathcal{O}, \mathcal{S}, \mathcal{W}\}$.

Proof. We polynomially transform any instance $H = (X, W)$ and h of FAS into an instance G and K of $Q_0(\mathcal{R}, \mathcal{Z})$. For this, we set the vertex set of G as being X; hence the set of arcs of G: U_X. We define the weights m_{xy} of the arcs (x, y) of G and K as follows:

* if $(x, y) \in W$, $m_{xy} = 2$

* if $(x, y) \notin W$, $m_{xy} = 0$.

Then we set $K = 2(|W| - h)$.

This transformation is obviously polynomial. Let us show that it keeps the answer « yes » or « no ».

Indeed, assume that there exists a subset W' of W with $|W'| \leq h$ and such that removing the elements of W' from H leaves a graph $(X, W - W')$ without any circuit. If we consider it as a partial graph of G, its weight is $2(|W| - |W'|)$, which is greater than or equal to $2(|W| - h) = K$. If $\mathcal{Z} = \mathcal{A}$, we are done. Otherwise, thanks to Lemma 5 b, it is possible to complete $(X, W - W')$ into a linear order by adding extra arcs. As all the weights are non-negative, we get a linear order (X, L), that we may consider as a partial order if $\mathcal{Z} = \mathcal{O}$, or as an interval order if $\mathcal{Z} = \mathcal{I}$, or as a semiorder if $\mathcal{Z} = \mathcal{S}$, or as a weak order if $\mathcal{Z} = \mathcal{W}$, with $\sum_{(x,y) \in L} m_{xy} \geq K$.

Conversely, assume that the instance (G, K) of $Q_0(\mathcal{R}, \mathcal{Z})$ admits the answer « yes »: there exists a partial graph (X, U) of $G = (X, U_X)$ which is without circuit if $\mathcal{Z} = \mathcal{A}$, or a linear order if $\mathcal{Z} = \mathcal{L}$, or a partial order if $\mathcal{Z} = \mathcal{O}$, or an interval order if $\mathcal{Z} = \mathcal{I}$, or a semiorder if $\mathcal{Z} = \mathcal{S}$, or a weak order if $\mathcal{Z} = \mathcal{W}$, with $\sum_{(x,y) \in U} m_{xy} \geq K$. In every case, (X, U) is without circuit. By definition of G, we have $\sum_{(x,y) \in U_X} m_{xy} = 2|W|$. Let W' be the subset of W defined by $W' = W - W \cap U$. Then we have $|W'| = |W| - |W \cap U|$. Since the elements of U which do not belong to W have a weight equal to 0, and since the other arcs have a weight equal to 2, we have then $|W'| = |W| - \frac{1}{2} \sum_{(x,y) \in U} m_{xy} \leq |W| - \frac{1}{2} K = h$.

Moreover, the graph (X, $W - W'$) is equal to (X, $W \cap U$), which is without circuit by Lemma 5 a and because the graph (X, U) is without circuit.

In conclusion, the answer is kept by the transformation, and hence the problems $Q_0(\mathcal{R}, \mathcal{Z})$ are NP-complete for $\mathcal{Z} \in \{\mathcal{A}, \mathcal{I}, \mathcal{L}, \mathcal{O}, \mathcal{S}, \mathcal{W}\}$. ❑

Corollary 7. For any even integer $m \geq 2$, the problems $P_m(\mathcal{R}, \mathcal{Z})$ are NP-hard for $\mathcal{Z} \in \{\mathcal{A}, \mathcal{I}, \mathcal{L}, \mathcal{O}, \mathcal{S}, \mathcal{W}\}$.

Proof. It follows from the fact that, in the proof of Theorem 6, it is possible to upper bound the weights of the graph by 2. ❑

Theorem 8. The problem $Q_1(\mathcal{R}, \mathcal{L})$ is NP-complete.

Proof. We apply the same transformation as for Theorem 6 (and thus we keep the same notations), but with the following weights:

* if $(x, y) \in W$, $m_{xy} = 1$ and $m_{yx} = -1$

* if $(x, y) \notin W$ and $(y, x) \notin W$, $m_{xy} = 1$ and $m_{yx} = 1$

(notice that the weights m_{xy} are well-defined, because H is assumed to be asymmetric),

and with $K = \frac{n(n-1)}{2} - 2h$.

This transformation is obviously polynomial. Let us show that it keeps the answer « yes » or « no ». The proof is quite similar as the one of Theorem 6.

Indeed, consider a minimum-sized subset W' of W such that removing the elements of W' from H gives a graph (X, $W - W'$) without any circuit, and assume that we have $|W'| \leq h$. If we consider (X, $W - W'$) as a partial graph of G, its weight is $|W| - |W'|$, which is greater than or equal to $|W| - h$. Thanks to Lemma 5 b, it is possible to complete (X, $W - W'$) into a linear order (X, L) by adding extra arcs. As we are looking for a linear order, it is necessary to add the arcs (x, y) such that (y, x) belongs to W' (because of the completeness of a linear order); there are $|W'|$ such arcs, and their weights are equal to –1. Because of the asymmetry of a linear order, the other extra arcs

(x, y) cannot belong to W' and are such that (y, x) neither belong to W'; there are $\frac{n(n-1)}{2} - |W|$ such arcs, and their weights are equal to 1. Thus we get:

$$\sum_{(x,y)\in L} m_{xy} = \sum_{(x,y)\in W-W'} m_{xy} + \sum_{\substack{(x,y)\in L \\ (y,x)\in W'}} m_{xy} + \sum_{\substack{(x,y)\in L-(W-W') \\ (y,x)\notin W'}} m_{xy}$$

$$= |W| - |W'| - |W'| + \frac{n(n-1)}{2} - |W|$$

$$= \frac{n(n-1)}{2} - 2|W'|.$$

From $|W'| \le h$, we get $\sum_{(x,y)\in L} m_{xy} \ge \frac{n(n-1)}{2} - 2h = K$: the answer admitted by the instance (G, K) of $Q_1(\mathcal{R}, \mathcal{L})$ is also « yes ».

Conversely, assume that the instance (G, K) of $Q_1(\mathcal{R}, \mathcal{L})$ admits the answer « yes »: there exists a partial graph (X, L) of $G = (X, U_X)$ which represents a linear order, with $\sum_{(x,y)\in L} m_{xy} \ge K$. Notice that (X, L) is without circuit. Let W' be the subset of W defined by $W' = W - W \cap L$. If an arc (x, y) belongs to W' (and thus to W), it does not belong to L; then (y, x) belongs to L (completeness of L) but not to W (asymmetry of H), and so its weight is equal to –1. Conversely, let (x, y) be an arc of L with a weight equal to –1; then it does not belong to W but is such that (y, x) does belong to W and not to L: (y, x) belongs to W'. So, the number of arcs of L with a weight equal to –1 is equal to $|W'|$. The other elements of L (there are $\frac{n(n-1)}{2} - |W'|$ such arcs) have a weight equal to 1. Hence the relation: $\sum_{(x,y)\in L} m_{xy} = \frac{n(n-1)}{2} - 2|W'|$. From the inequality $\sum_{(x,y)\in L} m_{xy} \ge K = \frac{n(n-1)}{2} - 2h$, we draw $|W'| \le h$. Moreover, the graph $(X, W - W')$ is equal to $(X, W \cap L)$, which is without circuit by Lemma 5 a and because the graph (X, L) is without circuit.

In conclusion, the answer is kept by the transformation, and hence the problem $Q_1(\mathcal{R}, \mathcal{L})$ is NP-complete. ❑

Corollary 9. For any odd integer $m \ge 1$, the problems $P_m(\mathcal{R}, \mathcal{L})$ are NP-hard.

Proof. It follows from the fact that, in the proof of Theorem 8, it is possible to upper bound the weights of the graph by 1. ❑

Theorem 10. $Q_1(\mathcal{R}, \mathcal{A})$ is NP-complete.

Proof. The proof is similar to the one of Theorem 8, and we do not detail it here. The construction is the following, with the same notations as above:

* if $(x, y) \in W$, $m_{xy} = 1$ and $m_{yx} = -1$

* if $(x, y) \notin W$ $m_{xy} = 1$

* $K = \frac{n(n-1)}{2} - h$.

With respect to the proof of Theorem 8, instead of considering the linear order called (X, L) above, we consider the same set of arcs L without the arcs with a weight equal to –1. Details are left to the reader. ❑

Corollary 11. For any odd integer $m \geq 1$, the problems $P_m(\mathcal{R}, \mathcal{A})$ are NP-hard.

Proof. It follows from the fact that, in the proof of Theorem 10, it is possible to upper bound the weights of the graph by 1. ❑

Theorem 12. The problems $Q_1(\mathcal{R}, \mathcal{Z})$ are NP-complete for $\mathcal{Z} \in \{\mathcal{I}, \mathcal{O}, \mathcal{S}, \mathcal{W}\}$.

Proof. We apply the same transformation as for Theorem 8 (and thus we keep the same notations), but with the following weights:

* if $(x, y) \in W$, $m_{xy} = 3$

* if $(x, y) \notin W$ $m_{xy} = 1$ and $m_{yx} = 1$

and with $K = \frac{n(n-1)}{2} + 2|W| - 2h$.

This transformation is obviously polynomial. Let us show that it keeps the answer « yes » or « no ». The proof is quite similar as those of Theorems 6 and 8.

Indeed, assume that there exists a subset W' of W with $|W'| \leq h$ and such that removing the elements of W' from H leaves a graph (X, $W - W'$) without any circuit. If we consider it as a partial graph of G, its weight is $3(|W| - |W'|)$. Thanks to Lemma 5 b, it is possible to complete (X, $W - W'$) into a linear order (X, L) by adding $\frac{n(n-1)}{2} - (|W| - |W'|)$ extra arcs. As the weights are all greater than or equal to 1, we get a linear order (X, L), that we may consider as a partial order if $\mathcal{Z} = \mathcal{O}$, or as an interval order if $\mathcal{Z} = \mathcal{I}$, or as a semiorder if $\mathcal{Z} = \mathcal{S}$, or as a weak order if $\mathcal{Z} = \mathcal{W}$, with:

$$\sum_{(x,y)\in L} m_{xy} \geq 3(|W| - |W'|) + \frac{n(n-1)}{2} - (|W| - |W'|) = \frac{n(n-1)}{2} + 2|W| - 2|W'| \geq K,$$

which shows that the answer of the instance (G, K) of $Q_1(\mathcal{R}, \mathcal{Z})$ is « yes ».

Conversely, assume that the instance (G, K) of $Q_1(\mathcal{R}, \mathcal{Z})$ admits the answer « yes »: there exists a partial graph (X, U) of $G = (X, U_X)$ which represents an element of $\mathcal{Z}$, with $\sum_{(x,y)\in U} m_{xy} \geq K$. Notice that ($X$, U) is without circuit. It is then possible, by Lemma 5 b, to complete U into a linear order L by adding extra arcs. As all the weights are positive, we get a linear order (X, L) with $\sum_{(x,y)\in L} m_{xy} \geq K$. Let W' be the subset of W defined by $W' = W - W \cap L$. The graph (X, $W - W'$) is equal to (X, $W \cap L$), which is without circuit by Lemma 5 a. Let us now compute $\sum_{(x,y)\in L} m_{xy}$:

$$\begin{aligned}\sum_{(x,y)\in L} m_{xy} &= \sum_{(x,y)\in L\cap W} m_{xy} + \sum_{(x,y)\in L-W} m_{xy} \\ &= 3|L \cap W| + |L - W| \\ &= 2|L \cap W| + |L| \\ &= 2|W - W'| + \frac{n(n-1)}{2} \\ &= 2|W| - 2|W'| + \frac{n(n-1)}{2}.\end{aligned}$$

Hence, from $\sum_{(x,y)\in L} m_{xy} \geq K = \frac{n(n-1)}{2} + 2|W| - 2h$, we get $|W'| \leq h$. The set W' shows that the instance (H, h) of FAS admits the answer « yes ».

In conclusion, the answer is kept by the transformation, and hence the problems $Q_1(\mathcal{R}, \mathcal{Z})$ is NP-complete for $\mathcal{Z} \in \{\mathcal{I}, \mathcal{O}, \mathcal{S}, \mathcal{W}\}$. ❑

Corollary 13. For any odd integer $m \geq 3$ and for $\mathcal{Z} \in \{\mathcal{I}, \mathcal{O}, \mathcal{S}, \mathcal{W}\}$, the problems $P_m(\mathcal{R}, \mathcal{Z})$ are NP-hard.

Proof. It follows from the fact that, in the proof of Theorem 12, it is possible to upper bound the weights of the graph by 3. ❑

Remarks.

In fact, the above proof can more generally be applied to any set $\mathcal{Z}$ with $\mathcal{L} \subseteq \mathcal{Z} \subseteq \mathcal{A}$, what is the case for the above sets.

We may notice that the weights of the graph G of Theorem 12 are chosen to be positive, so that an optimal solution is in fact a linear order. The « price » of this trick is that we need some weights to be greater than 1. Because of this, the complexities of the problems $P_1(\mathcal{R}, \mathcal{Z})$ for $\mathcal{Z} \in \{\mathcal{I}, \mathcal{O}, \mathcal{S}, \mathcal{W}\}$ remain open.

We now consider a profile Π of linear orders, i.e. the problems $Q_0(\mathcal{L}, \mathcal{Z})$ and $Q_1(\mathcal{L}, \mathcal{Z})$ for $\mathcal{Z} \in \{\mathcal{A}, \mathcal{C}, \mathcal{I}, \mathcal{L}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}$, or $\mathcal{W}\}$. The study of the complexity is more difficult because the graphs associated with Π are more constrained. Another consequence is that we cannot fix the number m of relations of Π any longer (because of the reconstruction of Π from the graph; see above) though it will be possible to upper bound m by a polynomial of n. To study the complexities of $Q_0(\mathcal{L}, \mathcal{Z})$ and $Q_1(\mathcal{L}, \mathcal{Z})$, we use the NP-completeness of two more constrained versions of FAS, that we call BFAS and BFAS′ because they deal with bipartite graphs.

Problem BFAS

Instance: a directed, asymmetric, and bipartite graph $H = (Y \cup Z, W_1 \cup W_2)$ where $Y = \{y_i\text{: } 1 \leq i \leq |Y|\}$ and $Z = \{z_i\text{: } 1 \leq i \leq |Z| = |Y|\}$ give the two classes of H and with $W_1 = \{(z_i, y_i)$ for $1 \leq i \leq |Y|\}$ and $W_2 \subseteq \{(y_i, z_j)$ for $1 \leq i \leq |Y|$ and $1 \leq j \leq |Y|\}$; an integer h;

Question: does there exist $W' \subset W_1 \cup W_2$ with $|W'| \leq h$ and such that removing the elements of W' from H leaves a graph $(Y \cup Z, (W_1 \cup W_2) - W')$ without any circuit ?

Problem BFAS′

Instance: the same as for BFAS;

Question: does there exist $W' \subset W_1$ with $|W'| \leq h$ and such that removing the elements of W' from H leaves a graph $(Y \cup Z, (W_1 \cup W_2) - W')$ without any circuit ?

Figure 10 shows how such a graph looks like. So the only difference between BFAS and BFAS′ is that, in BFAS′ and with respect to the drawing of Figure 10, W' is only made of horizontal arcs.

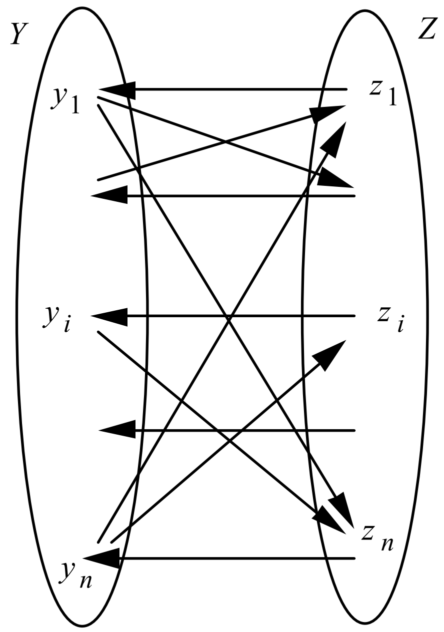

Figure 10: an instance of BFAS.

Theorem 14. BFAS and BFAS′ are NP-complete.

Proof. It is easy to show that BFAS and BFAS′ belong to NP (details are left to the reader). To prove that they are NP-complete, we transform the following problem, called Vertex Cover, into BFAS or BFAS′:

Problem Vertex Cover (VC)

Instance: an undirected graph $G = (X, U)$; an integer g;

Question: does there exist $X' \subset X$ with $|X'| \leq g$ and verifying the following property: $\forall \ \{x, \ y\} \in U$, $x \in X'$ or $y \in X'$ (X' is then a *vertex cover* of G of cardinality at most g) ?

It is known that VC is NP-complete (see R. Karp [1972]). Let (G, g) be any instance of VC. We define an instance (H, h) of BFAS or of BFAS′ as follows:

* for any vertex $x_i \in X$ ($1 \leq i \leq |X| = n$), we create two vertices of H: y_i and z_i ($1 \leq i \leq n$) and we set $Y = \{y_i, 1 \leq i \leq n\}$ and $Z = \{z_i, 1 \leq i \leq n\}$;

* for any edge $\{x_i, x_j\}$ of H, we create two arcs of H: (y_i, z_j) and (y_j, z_i); W_2 will denote the set of these arcs: $W_2 = \{(y_i, z_j), (y_j, z_i)$, for i and j such that $\{x_i, x_j\}$ belongs to $U\}$;

* we complete H by adding all the arcs of the form (z_i, y_i) for $1 \leq i \leq n$; they constitute the set W_1: $W_1 = \{(z_i, y_i)$ for $1 \leq i \leq n\}$;

* we set $g = h$.

Then we claim that G admits a vertex cover of cardinality at most g if and only if H admits a feedback arc set included into W_1 of cardinality at most h.

Indeed, assume that there exists a vertex cover X' of G with $|X'| \leq g$. Then let W' be defined by: $W' = \{(z_i, y_i)$ for $x_i \in X'\}$. We clearly have $|W'| = |X'| \leq g = h$. Moreover, assume that there exists a circuit in the graph $(Y \cup Z, (W_1 \cup W_2) - W')$. Then this circuit necessarily goes through an arc (z_i, y_i) for some i such that x_i does not belong to X' and then goes through an arc (y_i, z_j) for an appropriate j ($1 \leq j \leq n$); the only way to go on the circuit is to follow the arc (z_j, y_j) (it is the only arc with z_j as its tail),which involves that x_j does not belong to X'. But, as the arc (y_i, z_j) exists in H, $\{x_i, x_j\}$ must be an edge of G, and this edge is not covered by X', a contradiction.

Conversely, assume that (H, h) admits a subset W' of $W_1 \cup W_2$ which is a feedback arc set of cardinality at most h. Then there exists a subset W'' of W_1 which is a feedback arc set of H of cardinality at most h (for BFAS′, W' is necessarily such a set; so the following is useful only for BFAS). Indeed, for any arc (y_i, z_j) of $W' \cap W_2$, remove (y_i, z_j) from W' and replace it in W'' by the arc (z_i, y_i). We get thus a subset of W_1 with at most h elements. To see that W'' is a feedback arc set of H, it is enough to notice that any circuit of H going through (y_i, z_j) goes also through (z_i, y_i), since (z_i, y_i) is the only arc with y_i as its head. So define X' as the set of vertices of G associated with the elements of W'': $X' = \{x_i \text{ for } (z_i, y_i) \in W''\}$. Then obviously: $|X'| \leq g$. Moreover, assume that X' is not a VC of G. It means that there exists an edge $\{x_i, x_j\}$ of G with $x_i \notin X'$ and $x_j \notin X'$. So, similarly, we have in H: $(z_i, y_i) \notin W''$ and $(z_j, y_j) \notin W''$. But in these conditions, the arcs (z_i, y_i), (y_i, z_j), (z_j, y_j), and (y_j, z_i) (these four arcs do exist in H) define a circuit in the graph $(Y \cup Z, W_1 \cup W_2 - W'')$, and W'' is not a feedback arc set, a contradiction.

So the proposed transformation keeps the answer. As it is trivially polynomial with respect to the size of the transformed instance (G, g), BFAS and BFAS′ are NP-complete. ❑

Now we study the complexity of the problem $Q_0(\mathcal{L}, \mathcal{L})$, and then the one of $Q_1(\mathcal{L}, \mathcal{L})$:

Theorem 15. $Q_0(\mathcal{L}, \mathcal{L})$ is NP-complete.

Proof. As noticed above, $Q_0(\mathcal{L}, \mathcal{L})$ belongs to NP. We transform BFAS into $Q_0(\mathcal{L}, \mathcal{L})$. Let $(H = (Y \cup Z, W_1 \cup W_2), h)$ be any instance of BFAS, with the same notations as above. Let (G, K) be the instance of $Q_0(\mathcal{L}, \mathcal{L})$ defined by:

- the vertex set of G is $X = Y \cup Z$;

- the arc set of G is U_X;

- for any arc (x, y) of G, the weight m_{xy} of (x, y) is equal to: 2 if $(x, y) \in W_1 \cup W_2$, –2 if $(y, x) \in W_1 \cup W_2$, 0 otherwise;

- $K = 2|W_1 \cup W_2| - 4h$.

Notice that G is well defined since H is asymmetric. Moreover, the transformation is clearly polynomial with respect to the size of the instance (H, h) of BFAS.

Now, assume that the instance (H, h) of BFAS admits the answer « yes »: there exists $W' \subset W_1 \cup W_2$ with $|W'| \leq h$ and such that the graph $(X, (W_1 \cup W_2) - W')$ is without any circuit. If we consider $(X, (W_1 \cup W_2) - W')$ as a partial graph of G, its weight is $2|(W_1 \cup W_2) - W'|$, i.e. $2|W_1 \cup W_2| - 2|W'|$. Then, by Lemma 5 b, we may complete $(X, (W_1 \cup W_2) - W')$ into a linear order (X, L) by adding appropriate arcs. Among these extra arcs, there are at most $|W'|$ arcs (x, y) such that (y, x) belongs to $W_1 \cup W_2$, i.e. at most $|W'|$ arcs with a weight equal to –2. More precisely, the weights of the arcs of L belonging to $W_1 \cup W_2$ are equal to 2, and there are at least $|(W_1 \cup W_2) - W'|$ such arcs;

the weights of the arcs (x, y) of L such that (y, x) belongs to $W_1 \cup W_2$ are equal to –2, and there are at most $|W'|$ such arcs; the other arcs of L have a weight equal to 0. Hence:

$$\sum_{(x,y)\in L} m_{xy} \geq 2|W_1 \cup W_2| - 2|W'| - 2|W'| = 2|W_1 \cup W_2| - 4|W'| \geq 2|W_1 \cup W_2| - 4h = K.$$

So (G, K) admits also the answer « yes ».

Conversely, assume that (G, K) admits the answer « yes »: there exists $L \subset U_X$ with $\sum_{(x,y)\in L} m_{xy} \geq K$ and such that (X, L) is a linear order. Thus consider the set $W' = (W_1 \cup W_2) - (W_1 \cup W_2) \cap L$. As (X, L) is a linear order, (X, L) is without any circuit. Thus, by Lemma 5 a, $(W_1 \cup W_2) \cap L = (W_1 \cup W_2) - W'$ is also without any circuit. Let (x, y) be an element of W' (and thus of $W_1 \cup W_2$); then the arc (y, x) belongs to L (because L is complete and (x, y) does not belong to L) and its weight is –2. So, suppose that we have $|W'| > h$. Then there are at most $|(W_1 \cup W_2) - W'|$ arcs which belong to $W_1 \cup W_2$ and to L. So we get:

$$\sum_{(x,y)\in L} m_{xy} \leq 2|(W_1 \cup W_2) - W'| - 2|W'| = 2|W_1 \cup W_2| - 4|W'| < 2|W_1 \cup W_2| - 4h = K,$$

a contradiction. It means that W' satisfies all the conditions and the answer admitted by (X, L) is « yes ».

All these considerations show that $Q_0(\mathcal{L}, \mathcal{L})$ is NP-complete. ❑

Corollary 16. For $f = \Omega(n/\log n)$, with f taking even values, $P_f(\mathcal{L}, \mathcal{L})$ is NP-hard and, for any even integer $m \geq 2$, $P_m(\mathcal{T}, \mathcal{L})$ is NP-hard.

Theorem 17. $Q_1(\mathcal{L}, \mathcal{L})$ is NP-complete.

Proof. It is the same proof as for $Q_0(\mathcal{L}, \mathcal{L})$, but with the weights 1, n, and $-n$ instead of 0, 2, and –2 respectively, and with $K = n|W_1 \cup W_2| - 2n.h$ instead of $2|W_1 \cup W_2| - 4h$. Details are left to the reader. ❑

Corollary 18. For $f = \Omega(n^2/\log n)$, with f taking odd values, $P_f(\mathcal{L}, \mathcal{L})$ is NP-hard and, for $f = \Omega(n)$ with f taking odd values, $P_f(\mathcal{T}, \mathcal{L})$ is NP-hard.

Remark. An interesting case is the problem $P_1(\mathcal{T}, \mathcal{L})$, set by P. Slater (1961) (also known under other names; see for instance I. Charon *et alii* (1997) or O. Hudry *et alii* (2005) for references), i.e. the approximation of a tournament by a linear order. Unfortunately, the proof of Corollary 18 does not allow to know the complexity of Slater's problem, which remains an open problem.

Proofs similar to the previous ones (and not given here) lead to the following results:

Theorem 19. $Q_0(\mathcal{L}, \mathcal{A})$, $Q_1(\mathcal{L}, \mathcal{A})$, $Q_0(\mathcal{L}, \mathcal{O})$, and $Q_1(\mathcal{L}, \mathcal{O})$ are NP-complete. For $f = \Omega(n/\log n)$, with f taking even values, $P_f(\mathcal{L}, \mathcal{A})$ and $P_f(\mathcal{L}, \mathcal{O})$ are NP-hard. For any even integer $m \geq 2$, $P_m(\mathcal{T}, \mathcal{A})$ and $P_m(\mathcal{T}, \mathcal{O})$ are NP-hard. For $f = \Omega(n^2/\log n)$, with f taking odd values, $P_f(\mathcal{L}, \mathcal{A})$ and $P_f(\mathcal{L}, \mathcal{O})$ are NP-hard. For $f = \Omega(n)$ with f taking odd values, $P_f(\mathcal{T}, \mathcal{A})$ and $P_f(\mathcal{T}, \mathcal{A})$ are NP-hard.

Now we study the complexity of the problems $Q_0(\mathcal{L}, \mathcal{Z})$ for $\mathcal{Z} \in \{\mathcal{C}, \mathcal{I}, \mathcal{S}\}$.

Theorem 20. For $\mathcal{Z} \in \{\mathcal{C}, \mathcal{I}, \mathcal{S}\}$, $Q_0(\mathcal{L}, \mathcal{Z})$ is NP-complete.

Proof. Let $\mathcal{Z}$ belong to $\{\mathcal{C}, \mathcal{I}, \mathcal{S}\}$. As for the other problems above, $Q_0(\mathcal{L}, \mathcal{Z})$ obviously belongs to NP. We transform BFAS′ into $Q_0(\mathcal{L}, \mathcal{Z})$. Let $(H = (Y \cup Z, W_1 \cup W_2), h)$ be any instance of BFAS′, with the same notations as above. Let (G, K) be the instance of $Q_0(\mathcal{L}, \mathcal{Z})$ defined by:

- the vertex set of G is $X = Y \cup Z$;
- the arc set of G is U_X;
- for any arc (x, y) of G, the weight m_{xy} of (x, y) is equal to: 2 if $(x, y) \in W_1$, –2 if $(y, x) \in W_1$, $4n - 2$ if $(x, y) \in W_2$, $-(4n - 2)$ if $(y, x) \in W_2$, 0 otherwise;
- $K = (4n-2)|W_2| + 2n - 4h$.

Notice that G is well defined since H is asymmetric. Moreover, the transformation is clearly polynomial with respect to the size of the instance (H, h) of BFAS′.

Now, assume that the instance (H, h) of BFAS′ admits the answer « yes »: there exists $W' \subset W_1$ with $|W'| \leq h$ and such that the graph $(X, (W_1 \cup W_2) - W')$ is without any circuit. We prove that the instance (G, K) admits also the answer « yes » as in

Theorem 15. If we consider (X, $(W_1 \cup W_2) - W'$) as a partial graph of G, its weight is $2|W_1 - W'| + (4n+2)|W_2|$. By Lemma 5 b, we may complete (X, $(W_1 \cup W_2) - W'$) into a linear order (X, L) (that we shall consider as an interval order if Z is I, or as a semiorder if Z is S, or as a complete preorder if Z is C) by adding appropriate arcs. Among these extra arcs, there are at most $|W'|$ arcs (x, y) such that (y, x) belongs to W_1, i.e. at most $|W'|$ arcs with a weight equal to –2, while the other extra arcs all belong to $U_X - (W_1 \cup W_2)$ and have a weight equal to 0. More precisely, the weights of the arcs of L belonging to W_1 are equal to 2; there are at least $|W_1 - W'|$ such arcs. The weights of the $|W_2|$ arcs of L belonging to W_2 are equal to $4n - 2$. The weights of the arcs (x, y) of L such that (y, x) belongs to W_1 are equal to –2; there are at most $|W'|$ such arcs. The other arcs of L have a weight equal to 0. Hence, since $|W_1| = n$ and $W' \subset W_1$:

$$\sum_{(x,y)\in L} m_{xy} \geq 2|W_1 - W'| + (4n-2)|W_2| - 2|W'| = 2n + (4n-2)|W_2| - 4|W'|$$

and so, since $|W'| \leq h$: $\sum_{(x,y)\in L} m_{xy} \geq 2n + (4n-2)|W_2| - 4h = K.$

So (G, K) admits also the answer « yes ».

Conversely, assume that (G, K) admits the answer « yes ». We consider two main subcases: $Z \in \{I, S\}$ or $Z = C$.

• 1rst subcase: $Z \in \{I, S\}$

Since a semiorder is an interval order, there exists $I \subset U_X$ with $\sum_{(x,y)\in I} m_{xy} \geq K$ and such that (X, I) is an interval order. We want to show that then the instance (H, h) of BFAS′ also admits the answer « yes ». Notice that if h is greater than n, the answer of (H, h) is trivially « yes »; so, assume that we have $h \leq n - 1$. Let us show that I contains all the arcs of W_2. Assume the contrary. The arcs of I with a non-negative weight would be at most the n elements of W_1 (with a weight equal to 2) and at most $|W_2| - 1$ arcs of W_2 (with a weight equal to $4n - 2$). So we would get: $\sum_{(x,y)\in I} m_{xy} \leq 2n + (4n-2)(|W_2| - 1)$. On the other hand, we are supposed to have: $\sum_{(x,y)\in I} m_{xy} \geq K = (4n-2)|W_2| + 2n - 4h$, from which we draw $4h \geq 4n - 2$, which is incompatible with $h \leq n - 1$. Hence: $W_2 \subset I$ and, because of the antisymmetry of I, there is no arc in I of the form (z_i, y_j) with $i \neq j$. We prove now that we may construct a linear order L with $\sum_{(x,y)\in L} m_{xy} \geq K$ from I. For this, set $J = I - \{(x, y) \in I \text{ with } m_{xy} = 0\}$, and gather the vertices of X into the following three sets:

* $X_1 = \{y_k \in Y, z_k \in Z$ such that $(z_k, y_k) \in J\}$
* $X_2 = \{y_k \in Y, z_k \in Z$ such that $(y_k, z_k) \notin J$ and $(z_k, y_k) \notin J\}$
* $X_3 = \{y_k \in Y, z_k \in Z$ such that $(y_k, z_k) \in J\}$.

The situation is illustrated by Figure 11. We are going to show that the dashed arcs of Figure 11 do not exist in fact. Notice that, as a subset of I which contains no circuit, J contains no circuit.

The dashed arcs with their two extremities inside X_1 cannot exist, otherwise there would exist a circuit in J. Now consider an arc (y_j, z_i) with $y_j \in Y$, $z_i \in Z$ (thus $i \neq j$) and with an extremity inside X_1 and the other inside X_2. As one extremity belongs to X_1, the arc (z_i, y_i) or the arc (z_j, y_j) exists in I, and thus in J since its weight is not equal to 0. Also, by construction of G, (y_i, z_j) is an arc of G, and thus of I ($W_2 \subseteq I$). Assume that (z_i, y_i) belongs to I (and thus to J); then y_i and z_i belong to X_1, while y_j and z_j belong to X_2. In this case, as I is transitive, the arcs (y_j, z_i), (z_i, y_i), (y_i, z_j) involve the existence of the arc (y_j, z_j), a contradiction with the belonging of y_j and z_j to X_2. Similarly, the dashed arcs with their two extremities inside X_2 cannot exist in I. Indeed, assume that such a pair of arcs (y_j, z_i) and (y_i, z_j) exist with $y_i \in Y \cap X_2$, $y_j \in Y \cap X_2$, $z_i \in Z \cap X_2$, and $z_j \in Z \cap X_2$ ($i \neq j$) exist (notice that if one of these two arcs exists, the other one must exist too). As y_i, y_j, z_i, and z_j belong to X_2, the arcs (y_i, z_i) and (y_j, z_j) do not belong to I. But then the arcs (y_j, z_i) and (y_i, z_j) do not respect the definition of an interval order. So the look of J is as the one shown by Figure 11 without the dashed arcs.

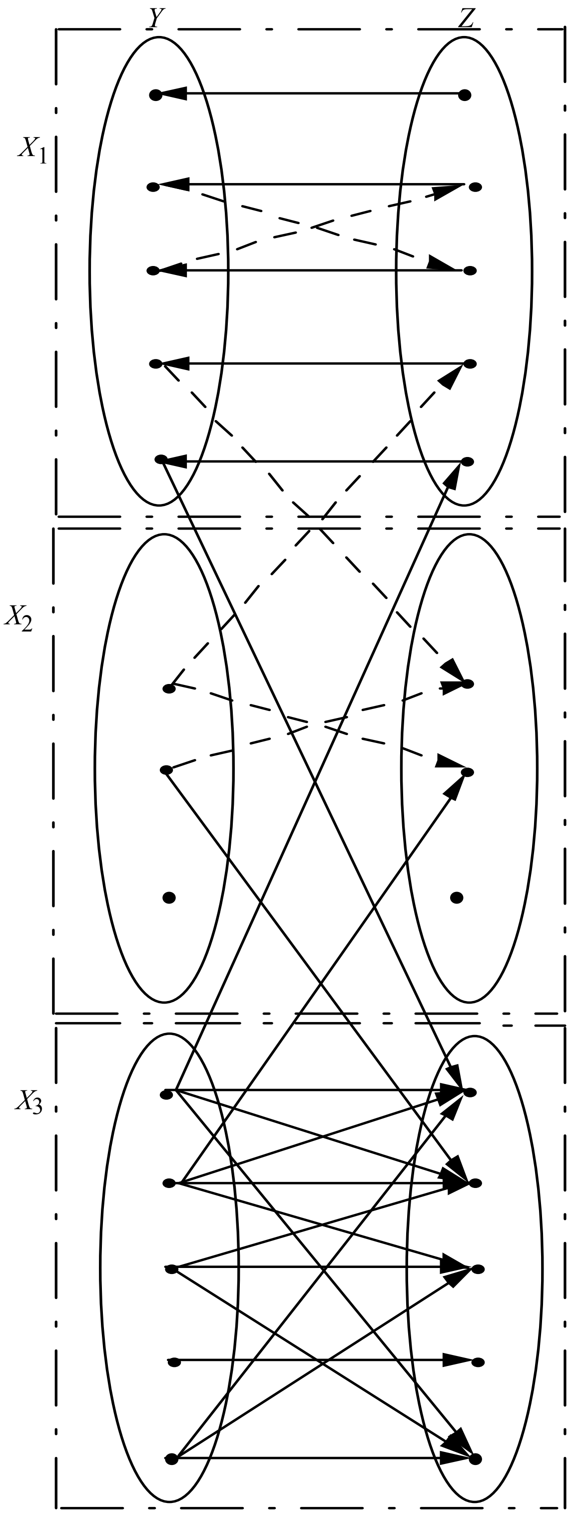

Figure 11. The graph induced by J for $Q_0(\mathcal{L}, \mathcal{Z})$ when $\mathcal{Z}$ is equal to $\mathcal{I}$ or $\mathcal{S}$.

Now, set $J' = J \cup \{(z, y) \in W_1$ for $y \in X_2$ and $z \in X_2\}$ (with respect to Figure 11, we add all the horizontal arcs from right to left with their two extremities in X_2). As the vertices of X_2 are linked only with vertices of X_3, it is easy to see that J' is still without any circuit. As the weights of these arcs are positive, we get: $\sum_{(x,y) \in J'} m_{xy} \geq \sum_{(x,y) \in J} m_{xy} = \sum_{(x,y) \in I} m_{xy} \geq K$. As J' is without circuit, by Lemma 5 b, we may extend J' into a linear order L. As we had $W_2 \subseteq I$ and as, for any index k with $1 \leq k \leq n$, y_i and z_i are already linked by an arc belonging to J', all the arcs that we add in order to define L from J' have a weight equal to 0. Hence $\sum_{(x,y) \in L} m_{xy} = \sum_{(x,y) \in J'} m_{xy} \geq K$. The end of the proof is exactly the same as in Theorem 15, and we do not duplicate it here: from L we define a subset W' which shows that the instance (H, h) of BFAS′ admits the answer « yes », which completes the proof for the subcase $\mathcal{Z} \in \{\mathcal{I}, \mathcal{S}\}$.

• 2nd subcase: $\mathcal{Z} = \mathcal{C}$

As for the previous case, we are going to prove that, if the answer admitted by the instance (G, K) is « yes », then we can build a linear order which gives this answer « yes ». Then the conclusion will be the same as above.

So, assume that there exists a subset C of U_X such that (X, C) is a complete preorder with $\sum_{(x,y) \in C} m_{xy} \geq K$. As above, this inequality involves that C contains all the elements of W_2 and no arc (x, y) such that (y, x) would belong to W_2 (details are left to the reader). Let D be the set made of the arcs of C with a non-zero weight: $D = C - \{(x, y) \in C \text{ with } m_{xy} = 0\}$. Moreover, gather the vertices of X into the following three sets:

* $X_1 = \{y_k \in Y, z_k \in Z$ such that $(z_k, y_k) \in D$ and $(y_k, z_k) \notin D\}$

* $X_2 = \{y_k \in Y, z_k \in Z$ such that $(y_k, z_k) \in D$ and $(z_k, y_k) \notin D\}$

* $X_3 = \{y_k \in Y, z_k \in Z$ such that $(y_k, z_k) \in D$ and $(z_k, y_k) \in D\}$.

The look of the graph induced by D is given by Figure 12. We are going to show that the dashed arcs of Figure 12 do not exist in fact.

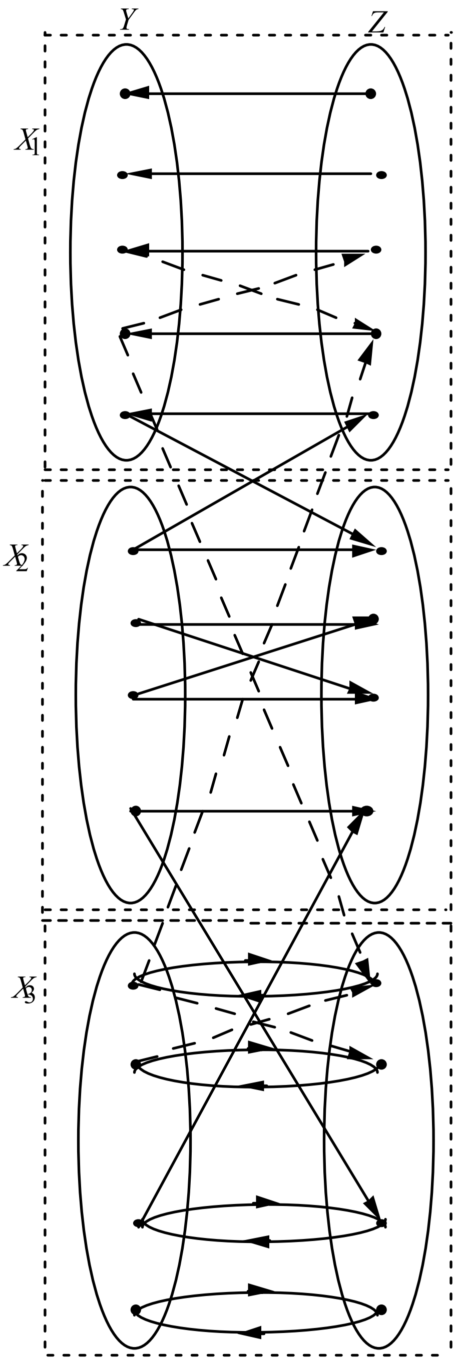

Figure 12. The graph induced by D for $Q_0(\mathcal{L}, \mathcal{C})$.

Indeed, let (y_j, z_i) be such an arc with $z_i \in Z$ and $y_j \in Y$. Then its weight is not equal to 0, and it is the same for (y_i, z_j), which thus belongs to D. If we assume that the four vertices y_i, y_j, z_i, and z_j belong to $Z_1 \cup Z_3$, then the arcs (z_i, y_i) and (z_j, y_j) belong to D and, by transitivity, the arcs (z_j, y_i) and (z_i, y_j) also belong to D, what is impossible (see above). So, the look of the graph induced by D is the one depicted by Figure 12 without the dashed arcs.

The next step consists in showing that we may extract a set D' of arcs from D such that D' is without circuit while its weight $\sum_{(x,y)\in D'} m_{xy}$ is still greater than or equal to K. For this, let D' be defined by $D' = D - \{(y_i, z_i) \text{ for } y_i \in X_3, z_i \in X_3\}$ (in other words, with respect to Figure 12, we remove the – almost – horizontal arcs inside X_3 and oriented from left to right). As the removed arcs have a negative weight, we get: $\sum_{(x,y)\in D'} m_{xy} \geq \sum_{(x,y)\in D} m_{xy} \geq K$. Moreover, D' is without circuit. Indeed, consider any circuit in D, which is transitive. Such a circuit must contain an arc of the form (z_i, y_i) with $y_i \in Y$ and $z_i \in Z$ (since it is the only way to go from Z to Y in the graph induced by D). Because of the transitivity of D applied to the considered circuit, (y_i, z_i) must also be an arc of D, and so y_i and z_i must belong to X_3. So the removal from D of the arcs (y_i, z_i) with $y_i \in X_3$, $z_i \in X_3$ leaves a graph (induced by D') without any circuit.

We may now conclude. As D' is without any circuit and by Lemma 5 b, we may complete it into a linear order L by adding appropriate arcs. As D' already contains W_2 and, for $1 \leq i \leq n$, exactly one of the two arcs (y_i, z_i) or (z_i, y_i), the extra arcs have a weight equal to 0. So, we get: $\sum_{(x,y)\in L} m_{xy} \geq \sum_{(x,y)\in D'} m_{xy} \geq K$. Then it is sufficient to apply the same argument as in Theorem 15 to show the existence of a subset W' of $W_1 \cup W_2$ which gives the answer « yes » to the instance (H, h) of BFAS′, which completes the proof for the subcase $\mathcal{Z} = \mathcal{C}$. ❑

Corollary 21. For $\mathcal{Z} \in \{\mathcal{C}, \mathcal{I}, \mathcal{S}\}$ and for $f = \Omega(n^2/\log n)$, with f taking even values, $P_f(\mathcal{L}, \mathcal{Z})$ is NP-hard. For $\mathcal{Z} \in \{\mathcal{C}, \mathcal{I}, \mathcal{S}\}$ and for $f = \Omega(n)$ with f taking even values, $P_f(\mathcal{T}, \mathcal{Z})$ is NP-hard.

Remark. If we transpose the proof of Theorem 20 in terms of preferences, we build a profile of linear orders such that there exists an optimal interval order, or an optimal semiorder, or a complete order which is in fact a linear order. An interesting question would be to know whether it is always the case, for any profile of linear orders.

Proofs similar to the previous ones (and not given here) lead to the following results:

Theorem 22. For $\mathcal{Z} \in \{\mathcal{C}, \mathcal{I}, \mathcal{S}\}$, the problems $Q_1(\mathcal{L}, \mathcal{Z})$ are NP-complete. For $\mathcal{Z} \in \{\mathcal{C}, \mathcal{I}, \mathcal{S}\}$ and for $f = \Omega(n^3/\log n)$, with f taking odd values, $P_f(\mathcal{L}, \mathcal{Z})$ is NP-hard. For $\mathcal{Z} \in \{\mathcal{C}, \mathcal{I}, \mathcal{S}\}$ and for $f = \Omega(n^2)$, with f taking odd values, $P_f(\mathcal{T}, \mathcal{Z})$ is NP-hard.

To study the complexity of the problems $P_f(\mathcal{Y}, \mathcal{Q})$ and $P_f(\mathcal{Y}, \mathcal{W})$, we first prove a lemma. In order to state it, we recall a previous notation. For any set $\mathcal{Z}$ of binary relations defined by some properties, we define $\mathcal{Z}^a$ as the set of preferences which are the asymmetric part of a preference belonging to $\mathcal{Z}$. In particular, we have $\mathcal{C}^a = \mathcal{W}$, $\mathcal{Q}^a = \mathcal{S}$, and $\mathcal{P}^a = \mathcal{O}$.

Lemma 23. For $\mathcal{Y} \in \{\mathcal{L}, \mathcal{T}\}$ and for any set $\mathcal{Z}$ and any function f, $P_f(\mathcal{Y}, \mathcal{Z})$ and $P_f(\mathcal{Y}, \mathcal{Z}^a)$ have the same complexity.

Proof. The result comes from the fact that we have $\Delta(\Pi, Z) = \Delta(\Pi, Z^a)$, for any profile Π of linear orders or of tournaments and any element Z of $\mathcal{Z}$. ❑

Corollary 24.

- For $f = \Omega(n/\log n)$, with f taking even values, $P_f(\mathcal{L}, \mathcal{P})$ is NP-hard. For $f = \Omega(n^2/\log n)$, with f taking even values, $P_f(\mathcal{L}, \mathcal{Q})$ and $P_f(\mathcal{L}, \mathcal{W})$ are NP-hard.
- For $f = \Omega(n^2/\log n)$, with f taking odd values, $P_f(\mathcal{L}, \mathcal{P})$ is NP-hard. For $f = \Omega(n^3/\log n)$, with f taking odd values, $P_f(\mathcal{L}, \mathcal{Q})$ and $P_f(\mathcal{L}, \mathcal{W})$ are NP-hard.
- For $m \geq 2$ with m even, $P_f(\mathcal{T}, \mathcal{P})$ is NP-hard. For $f = \Omega(n)$ with f taking even values, $P_f(\mathcal{T}, \mathcal{Q})$ and $P_f(\mathcal{T}, \mathcal{W})$ are NP-hard.
- For $f = \Omega(n)$ with f taking odd values, $P_f(\mathcal{T}, \mathcal{P})$ is NP-hard. For $f = \Omega(n^2)$ with f taking odd values, $P_f(\mathcal{T}, \mathcal{Q})$ and $P_f(\mathcal{T}, \mathcal{W})$ are NP-hard.
- For any even $m \geq 2$, the problems $P_m(\mathcal{R}, \mathcal{P})$ and $P_m(\mathcal{R}, \mathcal{Q})$ are NP-hard.
- For $f = \Omega(n)$ with f taking odd values, $P_f(\mathcal{R}, \mathcal{P})$ is NP-hard. For any odd $m \geq 3$, $P_m(\mathcal{R}, \mathcal{Q})$ is NP-hard.

Proof. For $\mathcal{Y} = \mathcal{L}$ or $\mathcal{Y} = \mathcal{T}$, these results come as a consequence of Theorem 19, Corollary 21 and Theorem 22 and from the application of Lemma 23 to $\mathcal{Z} = \mathcal{O}$, $\mathcal{Z} = \mathcal{Q}$ and to $\mathcal{Z} = \mathcal{C}$. For $\mathcal{Y} = \mathcal{R}$, this comes from Lemma 23, Corollary 7 and Corollary 13 for $\mathcal{Z} = \mathcal{Q}$ or $\mathcal{Z} = \mathcal{C}$, or from the complexity of $P_f(\mathcal{T}, \mathcal{P})$ for $\mathcal{Z} = \mathcal{Q}$ (by considering $\mathcal{T}$ as included into $\mathcal{R}$). ❑

The last result of this section deals with any set $\mathcal{Y}$ containing $\mathcal{L}$.

Theorem 25. For $f = \Omega(n/\log n)$, with f taking even values, for any set $\mathcal{Y}$ with $\mathcal{L} \subseteq \mathcal{Y}$, for $\mathcal{Z} \in \{\mathcal{A}, \mathcal{L}, \mathcal{O}, \mathcal{P}\}$, $P_f(\mathcal{Y}, \mathcal{Z})$ is NP-hard. For $f = \Omega(n^2/\log n)$, with f taking odd values, for any set $\mathcal{Y}$ with $\mathcal{L} \subseteq \mathcal{Y}$, for any set $\mathcal{Z} \in \{\mathcal{A}, \mathcal{L}, \mathcal{O}, \mathcal{P}\}$, $P_f(\mathcal{Y}, \mathcal{Z})$ is NP-hard. For $f = \Omega(n^2/\log n)$, with f taking even values, for any set $\mathcal{Y}$ with $\mathcal{L} \subseteq \mathcal{Y}$, for any set $\mathcal{Z} \in \{\mathcal{C}, \mathcal{I}, \mathcal{Q}, \mathcal{S}, \mathcal{W}\}$, $P_f(\mathcal{Y}, \mathcal{Z})$ is NP-hard. For $f = \Omega(n^3/\log n)$, with f taking odd values, for any set $\mathcal{Y}$ with $\mathcal{L} \subseteq \mathcal{Y}$, for any set $\mathcal{Z} \in \{\mathcal{C}, \mathcal{I}, \mathcal{Q}, \mathcal{S}, \mathcal{W}\}$, $P_f(\mathcal{Y}, \mathcal{Z})$ is NP-hard.

Proof. The previous results give the statement of Theorem 25 for $\mathcal{Y} = \mathcal{L}$. For $\mathcal{L} \subset \mathcal{Y}$, it is sufficient to consider any instance of the NP-hard problem $P_f(\mathcal{L}, \mathcal{Z})$ as an instance of $P_f(\mathcal{Y}, \mathcal{Z})$. This transformation (the identity !) is obviously polynomial and keeps the answer. Hence the result. ❑

In particular, we may apply Theorem 25 when $\mathcal{Y}$ is any one of the sets $\mathcal{A}$, $\mathcal{C}$, $\mathcal{I}$, $\mathcal{L}$, $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, or $\mathcal{W}$, but also to « mixed » profiles belonging to any union of two or more sets $\mathcal{A}$, $\mathcal{C}$, $\mathcal{I}$, $\mathcal{L}$, $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, or $\mathcal{W}$, for instance to profiles which may contain tournaments, preorders, and interval orders simultaneously...

5 Conclusion

The previous section was devoted to NP-hard problems. There are also some problems $P_f(\mathcal{Y}, \mathcal{Z})$ which are polynomial. It is trivially the case for $P_f(\mathcal{Y}, \mathcal{R})$ and for $P_f(\mathcal{Y}, \mathcal{T})$, for any set $\mathcal{Y}$ and any function f. Indeed, if we consider the associated problems $Q_0(\mathcal{Y}, \mathcal{R})$, $Q_1(\mathcal{Y}, \mathcal{R})$, $Q_0(\mathcal{Y}, \mathcal{T})$, or $Q_1(\mathcal{Y}, \mathcal{T})$, it is easy to see that an optimal solution consists in keeping all the arcs of G with a positive weight for $\mathcal{Z} = \mathcal{R}$, or in keeping, for each pair of arcs (x, x') and (x', x), the arc with the greatest weight for $\mathcal{Z} = \mathcal{T}$. Another interesting polynomial case is the one of unimodular orders; in this case, the aggregation of unimodular orders into a unimodular order is polynomial (see D. Black (1948)).

Anyway, it seems that properties like the lack of circuits or transitivity usually lead to NP-hard problems. The previous complexity results illustrate this trend. We may summarize them by the tables of Figure 13 (m even) and Figure 14 (m odd). In these tables, « NPH » means that the considered problem $P_f(\mathcal{Y}, \mathcal{Z})$ is NP-hard. In such a case, we indicate the range of a lower bound of the number m of relations inside the profile which ensures that $P_f(\mathcal{Y}, \mathcal{Z})$ is NP-hard; for instance, $m = \Omega(n)$ with m odd for $P_f(\mathcal{T}, \mathcal{L})$ means that $P_f(\mathcal{T}, \mathcal{L})$ is NP-hard if the range of the odd number m of tournaments of the profile is at least n. As a general result, remember that the NP-hardness of $P_f(\mathcal{Y}, \mathcal{Z})$ involves the one of $P_{f+2}(\mathcal{Y}, \mathcal{Z})$. To my knowledge, when not trivial, the complexity for lower values of m is not known. The letter « P » means that $P_f(\mathcal{Y}, \mathcal{Z})$ is (trivially) polynomial. Remember also that all the results displayed in the tables of Figures 13 and 14 remain the same if we add the reflexivity or the irreflexivity to the considered types of relations.

From this table, it appears that some cases are still unsolved, when m is low. One such interesting case is the problem stated by P. Slater (1961), i.e. $P_1(\mathcal{T}, \mathcal{L})$ for which Π is reduced to one tournament while the median relation must be a linear order. In spite of repeated efforts, its complexity remains open...

Median relation ($\mathcal{Z}$)	$\Pi \in \mathcal{R}^m$ ($\mathcal{Y} = \mathcal{R}$)	$\Pi \in \mathcal{T}^m$ ($\mathcal{Y} = \mathcal{T}$)	$\Pi \in \mathcal{Y}^m$ with $\mathcal{L} \subseteq \mathcal{Y}$
binary relation ($\mathcal{R}$)	P	P	P
tournament ($\mathcal{T}$)	P	P	P
acyclic relation ($\mathcal{A}$)	NPH, $m \geq 2$	NPH, $m \geq 2$	NPH, $m = \Omega(n / \log n)$
complete preorder ($\mathcal{C}$)	NPH, $m = \Omega(n)$	NPH, $m = \Omega(n)$	NPH, $m = \Omega(n^2 / \log n)$
interval order ($\mathcal{I}$)	NPH, $m \geq 2$	NPH, $m = \Omega(n)$	NPH, $m = \Omega(n^2 / \log n)$
linear order ($\mathcal{L}$)	NPH, $m \geq 2$	NPH, $m \geq 2$	NPH, $m = \Omega(n / \log n)$
partial order ($\mathcal{O}$)	NPH, $m \geq 2$	NPH, $m \geq 2$	NPH, $m = \Omega(n / \log n)$
preorder ($\mathcal{P}$)	NPH, $m \geq 2$	NPH, $m \geq 2$	NPH, $m = \Omega(n / \log n)$
quasi-order ($\mathcal{Q}$)	NPH, $m \geq 2$	NPH, $m = \Omega(n)$	NPH, $m = \Omega(n^2 / \log n)$
semiorders ($\mathcal{S}$)	NPH, $m \geq 2$	NPH, $m = \Omega(n)$	NPH, $m = \Omega(n^2 / \log n)$
weak order ($\mathcal{W}$)	NPH, $m \geq 2$	NPH, $m = \Omega(n)$	NPH, $m = \Omega(n^2 / \log n)$

Figure 13. The complexity results for m even.

Median relation ($\mathcal{Z}$)	$\Pi \in \mathcal{R}^m$ ($\Upsilon = \mathcal{R}$)	$\Pi \in \mathcal{T}^m$ ($\Upsilon = \mathcal{T}$)	$\Pi \in \Upsilon^m$ with $\mathcal{L} \subseteq \Upsilon$
binary relation ($\mathcal{R}$)	P	P	P
tournament ($\mathcal{T}$)	P	P	P
acyclic relation ($\mathcal{A}$)	NPH, $m \geq 1$	NPH, $m = \Omega(n)$	NPH, $m = \Omega(n^2 / \log n)$
complete preorder ($\mathcal{C}$)	NPH, $m = \Omega(n^2)$	NPH, $m = \Omega(n^2)$	NPH, $m = \Omega(n^3 / \log n)$
interval order ($\mathcal{I}$)	NPH, $m \geq 3$	NPH, $m = \Omega(n^2)$	NPH, $m = \Omega(n^3 / \log n)$
linear order ($\mathcal{L}$)	NPH, $m \geq 1$	NPH, $m = \Omega(n)$	NPH, $m = \Omega(n^2 / \log n)$
partial order ($\mathcal{O}$)	NPH, $m \geq 3$	NPH, $m = \Omega(n)$	NPH, $m = \Omega(n^2 / \log n)$
preorder ($\mathcal{P}$)	NPH, $m = \Omega(n)$	NPH, $m = \Omega(n)$	NPH, $m = \Omega(n^2 / \log n)$
quasi-order ($\mathcal{Q}$)	NPH, $m \geq 3$	NPH, $m = \Omega(n^2)$	NPH, $m = \Omega(n^3 / \log n)$
semiorders ($\mathcal{S}$)	NPH, $m \geq 3$	NPH, $m = \Omega(n^2)$	NPH, $m = \Omega(n^3 / \log n)$
weak order ($\mathcal{W}$)	NPH, $m \geq 3$	NPH, $m = \Omega(n^2)$	NPH, $m = \Omega(n^3 / \log n)$

Figure 14. The complexity results for *m* odd.

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Continuous Ordinal Clustering: A Mystery Story[1]

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Abstract

Cluster analysis may be considered as an aid to decision theory because of its ability to group the various alternatives. There are often errors in the data that lead one to wish to use algorithms that are in some sense continuous or at least robust with respect to these errors. Known characterizations of continuity are order theoretic in nature even for data that has numerical significance. Reasons for this are given and arguments presented for considering an ordinal form of robustness with respect to errors in the input data. The work is preliminary and some open questions are posed.

1 The Background

In their book "Mathematical Taxonomy" [1], N. Jardine and R. Sibson presented a model for clustering algorithms that only allowed one feasible algorithm that produced an ultrametric output: single-linkage clustering. Among other things they assumed two axioms:

1. Clustering algorithms should be continuous.
2. Clustering algorithms should not be concerned with values of dissimilarities – only whether one value is larger or smaller than another.

But how can this be? The first condition involves the consideration of what happens when objects are close together. The second condition tells us to ignore closeness. This is a puzzle to be unravelled.

[1]Note: The present work has different goals and was done independently of the paper by O. Gascuel and A. McKenzie, *Performance Analysis of Hierarchical Clustering*, Journal of Classification, **11**, 2004, pp. 3-18, though there is some overlap of ideas.

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2 Definitions

The terminology in the area is not universal, so let's clarify the terms.

Input Data This is a finite nonempty set P of objects to classify. Each object has associated with it a set of numerical, binary, or nominal attributes.

Output Data A partition of P or an indexed nested sequence of partitions, the top one having a single class.

Intermediate step Convert the attribute data into a dissimilarity coefficient (DC). A DC on P is a mapping $d : P \times P \mapsto \Re_0^+$ (the non-negative reals) such that

(1) $d(a, b) = d(b, a) \geq 0$

(2) $d(a, a) = 0$ for all $a \in P$.

d is *definite* if also

(3) $d(a, b) = 0$ implies $a = b$ in the sense that they are identical.

d is an *ultrametric* if it satisfies (1), (2) and the ultrametric inequality

(4) $d(a, b) \leq \max\{d(a, c), d(b, c)\}$ for all $c \in P$.

The DCs are ordered by the rule $d_1 \leq d_2 \iff d_1(a, b) \leq d_2(a, b)$ for all $a, b \in P$. The smallest DC is then given by $\underline{0}$ which is defined by $\underline{0}(a, b) = 0$ for all $a, b \in P$.

The T–transform For the DC d, define Td by the rule

$$Td(h) = \{(a, b) : d(a, b) \leq h\},$$

noting that $Td(h)$ is a reflexive symmetric relation. $Td(h)$ is an equivalence relation for all h if and only if d is an ultrametric. When ordered by set inclusion, the smallest reflexive symmetric relation is denoted $R_\emptyset$, and is defined by $R_\emptyset = \{(a, a) : a \in P\}$, and the largest one is given by $R_{PP} = \{(a, b) : a, b \in P\}$. It is easy to show that the reflexive symmetric relations then form a Boolean algebra isomorphic to the power set of the two element subsets of P.

Relations of the form $Td(h)$ are called *threshold relations* of d, and the *proper* threshold relations are those other than $R_\emptyset$.

There is a natural well known bijection between ultrametrics and indexed nested sequences of equivalence relations, the top one being R_{PP}.

A cluster method is then a mapping $d \mapsto F(d)$ where d and F(d) are DCs. The usual algorithm takes $F(d)$ to be an ultrametric.

If $|P| = p$, and $k = p(p-1)/2$, then DCs may be viewed as vectors in the positive cone of a k-dimensional Euclidean vector space, and cluster methods may be viewed as mappings on the positive cone of this space. Any of the usual metrics for Euclidean spaces may then be used. In particular, we use Δ_0 which is defined by

$$\Delta_0(d_1, d_2) = \max\{d_1(a,b) - d_2(a,b)| : a, b \in P\},$$

and is based on the L_∞-norm. Continuity, left continuity, and right continuity of a cluster method then all have their expected meanings.

It is easy to justify continuity as a desirable condition for a cluster method. The input data may very well have small errors, and it would be nice if a small error for the input would translate to a small error for the output. But in their book [1], N. Jardine and R. Sibson showed that in the presence of continuity and certain other properties, the only acceptable cluster method is single-linkage clustering. This is defined by taking $[TF(d)](h) = \gamma \circ Td(h)$, where $\gamma(R)$ is the equivalence relation generated by the reflexive symmetric relation R.

3 Properties of Cluster Methods

We rephrase here some of the axioms that were introduced by Jardine and Sibson [1] for a cluster method F.

(JS1) *Idempotent* $F = F \circ F$.

(JS3) *Scale invariance*. $F(\alpha d) = \alpha F(d)$ for all $\alpha > 0$.

(JS3a) *Monotone equivariance* $F(\theta d) = \theta F(d)$ for every order automorphism θ of the nonnegative reals.

(JS5) *Isotone* $d_1 \leq d_2$ implies that $F(d_1) \leq F(d_2)$.

(JS5a) *0-isotone* If $Td_1(0) = Td_2(0)$, then $d_1 \leq d_2$ implies that $F(d_1) \leq F(d_2)$.

Theorem: For a monotone equivariant cluster method F, the following conditions are equivalent:

1. There exists a mapping η on the reflexive symmetric relations such that for every DC d, $TF(d) = \eta \circ Td$.

2. F is continuous.

3. F is right continuous.

Theorem Let F be monotone equivariant.

- Then F is left continuous if and only if there is a family $(\eta_R)_{R\in\Sigma(P)}$ of mappings on $\Sigma(P)$ such that $TF(d) = \eta_{Td(0)} \circ Td$.

- F is continuous if and only if there is a mapping η on $\Sigma(P)$ such that $TF(d) = \eta \circ Td$.

F being isotone has unexpected consequences.

Theorem If the image of F contains all ultrametrics, and if F satisfies JS1 and JS5, then $F(d) \leq d$ for every DC d.

Lemma: Let F satisfy JS3 and JS5a, and d a DC. There then exist positive constants $\delta(d), M(d)$ such that $0 < \Delta_0(d, d') < \delta(d)$ with $Td(0) = Td'(0) \Longrightarrow \Delta_0(F(d), F(d')) < M(d)\delta(d)$. If F is isotone, the implication holds with $Td(0) = Td'(0)$ replaced by $Td(0) \subseteq Td'(0)$.

Theorem: If F satisfies JS3 and JS5a, then F is left continuous. If it also satisfies JS5, it is in fact continuous at all definite DCs. **Question:** What does it take to make F continuous at all DCs?

Here is an example illustrating this Theorem. Take $F(d) = \underline{0}$ if d is not definite, and $F(d)$ to be single linkage clustering on the definite DCs. But this example is in fact monotone equivariant. **Question:** Is there a cluster method satisfying JS3 and JS5 that is not monotone equivariant?

Theorem Let F be monotone equivariant. Then JS5a is equivalent to left continuity.

Thus continuity plus monotone eqivariance rules out almost all cluster algorithms that are commonly used by investigators. We will argue that the important property of continuity may be ordinal in nature rather than metric.

4 Clustering Data Having Ordinal Significance

A DC d has ordinal significance if the numerical values of d have no meaning, only whether one of $d(a, b) < d(x, y)$, $d(a, b) > d(x, y)$ or $d(a, b) = d(x, y)$ is true. But Jardine and Sibson [1] argue that one should use a *monotone equivariant* cluster method. Recall that this is a cluster method F having the property that $F(\theta d) = \theta F(d)$ for every DC d, and every order automorphism θ of $\Re_0^+$. This is a rather strong assumption, and in a later paper Sibson [2] argues that it suffices to use a cluster algorithm that preserves *global order equivalence*, which is denoted $\sim_g$, and defined by the rule that $d_1 \sim_g d_2$ if and only if there is an order automorphism θ of $\Re_0^+$ such that $d_1 = \theta \circ d_2$. Thus one wants $d_1 \sim_g d_2$ to imply that $F(d_1) \sim_g F(d_2)$. Two cluster methods F, G are globally order equivalent if $F(d) \sim_g G(d)$ for every DC d defined on P. It turns out that every cluster method F that

> preserves global order equivalence and has the property that
> the image of $F(d)$ cannot have more members than the image of d

is globally order equivalent to a monotone equivariant cluster method, so we have not moved far from monotone equivariance.

But let $P = \{a, b, c\}$ with $d_1(a, b) = 0, d_1(a, c) = 1$ and $d_1(b, c) = 3$. If $d_2 = d_1 + 1$, then d_1 and d_2 are not globally order equivalent; yet they are equivalent in a way that we need to preserve. The proper definition is to say that d_1 and d_2 are weakly order equivalent (denoted $d_1 \sim_w d_2$) in case $d_1(a, b) < d_1(x, y) \iff d_2(a, b) < d_2(x, y)$. But now things are not so nice. A monotone equivariant cluster method need not preserve weak order equivalence. One can characterize when a cluster method that preserves weak order equivalence is weakly order equivalent (obvious definition) to a monotone equivariant cluster method.

The big question now is this. What in the world does any of this have to do with continuity in the Δ_0 metric? Hang on. A clue is coming.

5 The Connection with Continuity

If continuity is a desirable condition, it would be very nice to find a continuous cluster method that is not monotone equivariant. Where does one look? Let's start by seeing if there is any property that all continuous cluster methods might have in common.

For any DC d, define the *mesh width* of d by

$$\mu(d) = \frac{1}{2} \min\{|h_i - h_{i-1}| : 1 \leq i \leq t\},$$

where the image of d is $0 = h_0 < h_1 < \cdots < h_t$.

Fundamental Result: If $\Delta_0(d, d') < \mu(d)$, then $d \preceq d'$ in the sense that

$$d(a, b) < d(x, y) \Longrightarrow d'(a, b) < d'(x, y).$$

Note that $d \sim_w d' \iff d \preceq d'$ and $d' \preceq d$. So suddenly there is a connection between metric properties of Δ_0 and ordinal considerations. Indeed, if $d_n \to d$, there must exist a positive integer N such that $n \geq N \Longrightarrow d_n \preceq d$. There is a weak converse connection given by the fact that $d \preceq d'$ implies the existence of d'' such that $d' \sim_w d''$ and $\Delta_0(d, d'') < \mu(d)$. In fact $d \preceq d'$ is equivalent to d being arbitrarily close to some d'' with d'' weakly order equivalent to d'.

Theorem: $d \preceq d'$ if and only if there is a sequence (d_n) of DCs all weakly order equivalent to d' such that $d_n \to d$,

Theorem: $d \preceq d'$ if and only if every proper threshold relation of d is a threshold relation of d'.

Definition. A cluster method F is called *ordinally continuous* if

$$d \preceq d' \Longrightarrow F(d) \preceq F(d').$$

Theorem: Let F be continuous.
There exists $\delta > 0$ such that $\Delta_0(d, d') < \delta \Longrightarrow d \preceq d'$ and $F(d) \preceq F(d')$.
$d \preceq d' \Longrightarrow \exists$ a DC d'' such that $d' \sim_w d''$ and $F(d) \preceq F(d'')$.

Corollary: If F is continuous and preserves weak order equivalence, then F is ordinally continuous.

It is natural to conjecture that monotone eqivariance together with ordinal continuity might imply continuity. Here is an example showing this to be false. Let $R_1, R_2, \ldots, R_n$ denote the proper threshold relations of d. Take as the threshold relations for $F(d)$ those R_i that happen to be equivalence relations. Assign each such equivalence relation the level at which it came into being for d. This cluster method is monotone equivariant, order continuous, but not continuous. We illustrate this concretely.

Let $P = \{a, b, c\}$, and define $d(a, b) = d(a, c) = 1$, with $d(b, c) = 2$. d' is defined by $d'(a, b) = 1, d'(a, c) = 1 + \varepsilon, d'(b, c) = 2$, where $0 < \varepsilon < 1/4$. Note that $\mu(d) = 1/2$, and $\Delta_0(d, d') < 1/4$. The reader can verify that R_{PP} is the only proper threshold relation of $F(d)$, while $F(d')$ has R_{PP}, as well as $R_\emptyset \cup \{(a,b),(b,a)\}$. It follows that $Fd(a,b) = Fd(a,c) = Fd(b,c) = 2$, while $Fd'(a,b) = 1$ with $Fd'(a,c) = Fd'(b,c) = 2$. Thus $\Delta_0(d,d') = \varepsilon$, while $\Delta_0(Fd, Fd') = 2$. Letting $\varepsilon \to 0$, it follows that F is not continuous.

If we take the view that it is only the partitions that $F(d)$ produces that are of interest, and not the levels at which they occur, then if we define a cluster method G to be single linkage clustering with the levels of the output rank ordered, then G is just as good as single linkage as a cluster algorithm. Thus we want conditions of a cluster method that tell us when the method is weakly order equivalent to a continuous cluster method. The only clear fact for such a cluster method is that if it preserves weak order equivalence, it must be order continuous. Such a cluster method need not be isotone, nor need it preserve multiplication by a positive scalar α.

Is Continuity the Issue? The motivation usually given for continuous cluster methods is that small errors in the input should translate to small errors in the output. But small errors in the input d produce a DC d' such that $d \preceq d'$, so this is really an argument for ordinal continuity.

Fundamental Question: Find necessary and sufficient conditions on a cluster method F so that F is weakly order equivalent to a continuous cluster method.

Examples are wanted (if there are any) of useful continuous cluster methods that are not monotone equivariant.

Complete Linkage Clustering Complete linkage clustering is not continuous, but does have the property that $d \sim_w d' \implies F(d) \sim_w F(d')$. Is this the key property that needs to be preserved? Is there a version of complete linkage clustering that is ordinally continuous?

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Compact preference representation and combinatorial vote

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Abstract

In many real-world social choice problems, the set of alternatives is defined as the Cartesian product of (finite) domain values for each of a given set of variables, and these variables cannot be asusmed to be prefentially independent (to take an example, if X is the main dish of a dinner and Y the wine, preferences over Y depends on the value taken for X). Such combinatorial domains are much too large to allow for representing preference relations or utility functions explicitly (that is, by listing alternatives together with their rank or utility); for this reason, artificial intelligence researchers have been developing languages for specifying preference relations or utility functions as compactly as possible. This paper first gives a brief survey of compact representation languages, and then discusses its role for representing and solving social choice problems, especially from the point of view of computational complexity.

1 Introduction

Voting procedures have been extensively studied by researchers in social choice theory who have studied extensively all properties of various families of voting rules, up to an important detail: candidates are supposed to be listed explicitly (typically, they are individual or lists of individuals, as in political elections), which assumes that they should not be too numerous. In this paper, we focus on the case where the set of candidates has a *combinatorial structure*, i.e., is a Cartesian product of finite value domains for each one of a set of variables: this problem will be referred to as *combinatorial vote*. In this case, the space of possible alternatives has a size being exponential in the number of variables

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and it is therefore not reasonable asking the voters to rank or evaluate on a utility scale all alternatives.

Consider for example that the voters have to agree on a common menu to be composed if a first course dish, a main course dish, a dessert and a wine, with a choice of 6 items for each. This makes 6^4 candidates. This would not be a problem if the four items to choose were independent from the other ones: in this case, this vote problem over a set of 6^4 candidates would come down to four independent problems over sets of 6 candidates each, and any standard voting rule could be applied without difficulty. Things become more complicated if voters express dependencies between items, such as "I would like to have risotto ai funghi as first course, except if the main course is a vegetable curry, in which case I would prefer smoked salmon as first course", "I prefer white wine if one of the courses is fish and none is meat, red wine if one of the courses is meat and none is fish, and in the remaining cases I would like equally red or white wine", etc.

As soon as variables are not preferentially independent, it is generally a bad idea to decompose a combinatorial vote problem with p variables into a set of p smaller problems, each one bearing on a single variable: "multiple election paradoxes" [9] show that such a decomposition leads to suboptimal choices, and give real-life examples of such paradoxes, including simultaneous referenda on related issues. They argue that the only way of avoiding the paradox would consist in "voting for combinations [of values]", but they stress its practical difficulty: "To be sure, if there are more than eight or so combinations to rank, the voter's task could become burdensome. How to package combinations (e.g., of different propositions on a referendum, different amendments to a bill) so as not to swamp the voter with inordinately many choices – some perhaps inconsistent – is a practical problem that will not be easy to solve."

In this paper we address this issue. Since the preference structure of each voter cannot be reasonably expressed explicitly by listing all candidates, what is needed is a compact *preference representation language*. Such preference representation languages have been developed within the KR community; they are often build up on propositional logic, but not always (see for instance utility networks [1] [14] or valued constraint satisfaction [18] – however in this paper we restrict the study to logical approaches); they enable a much more concise representation of the preference structure, while preserving a good readability (and hence a proximity with the way agents express their preferences in natural language). *Therefore, the first parameter to be fixed, for a combinatorial vote problem, is the language for representing the preferences of the voters.*

Now, two other problems arise:

1. *How are these compactly represented preferences pratcically specified by the voters?* Assuming that voters can easily express by themselves (without any kind of help) their preferences over combination of values using complex logical objects

is often not reasonable; even if they do, it is highly possible that the preference relation induced by the specification is incomplete or inconsistent. So as to help agents expressing their preferences, *interactive elicitation procedures* work by finding relevant questions to ask, until the agent's preference relation is consistent and complete. Preference elicitation in combinatorial domains has been investigated in several recent works [2, 3, 13] and will not be considered here.

2. *Once preference have been elicited, how is the outcome of the voting rule computed?* Obviously, the prohibitive number of candidates makes it hard, or even practically impossible, to apply voting rules in a straightforward way, sicne all but the simplest voting procedures need a number of operations at least linear (sometimes quadratic, sometimes even exponential) in the number of candidates, which is not reasonable when the set of candidates has a strong combinatorial structure. Computational complexity of some voting procedures when applied on combinatorial domains has been investigated in [16], but this does not really address the question of *how* these procedures should be applied in practice so as to get their outcome (or an approximation of it) in a reasonable amount of time.

This article addresses the latter point.

2 Logical languages for compact preference representation

In this Section we are concerned with the preferences of a *single voter* over a finite set of candidates $\mathcal{X}$. We assume that $\mathcal{X}$ has a combinatorial nature, namely, $\mathcal{X}$ is a set of possible assignments of each of a certain number of variables to a value of its (finite) domain: $\mathcal{X} = D_1 \times ... \times D_n$, where D_i is the set of possible values for variable v_i; the size of $\mathcal{X}$ is exponentially large in n. Because specifying a preference structure explicitly in such a case is unreasonable, the AI community has developed several preference representation languages that escape this combinatorial blow up. Such languages are said to be *factorized*, or *succinct*, because they enable a much more concise representation of preference structures than explicit representations. For the sake of brevity, following we focus on *logical* languages, which means that domains are assumed to be binary. This does not imply a real loss of generality, since a variable over a finite domain with k possible values can be expressed using $\lceil \log k \rceil$ binary variables.

A *preference relation* $\succeq$ is a preorder, i.e., a reflexive and transitive binary relation on $\mathcal{A}$. $M \succeq M'$ means that alternative M is at least as good (to the agent) as alternative M'. Such a relation $\succeq$ is not necessarily complete, that is, it may be that neither $M \succeq M'$ nor $M' \succeq M$ holds for a pair of alternatives M and M' in $\mathcal{A}$. We note $M \succ M'$ for $M \succeq M'$

and not ($M \succeq M'$) (strict preference of M over M'), and $M \sim M'$ for $M \succeq M'$ and $M' \succeq M$ (indifference). It is important to note that $M \sim M'$ means that the agent takes M and M' to be equally preferred, while the incomparability between M and M' ($M \not\succeq M'$ and $M' \not\succeq M$) simply means that no preference between them is expressed.

These definitions are about preferences over an arbitrary set of alternatives $\mathcal{A}$. In this paper, we consider propositional languages expressing preferences: such languages express preferences over the set of possible interpretations W over a given alphabet VAR. A refinement of this definition is that of assuming that the set of possible alternatives excludes some interpretations of W. In this case, we assume that a formula K is given: this formula represents “integrity constraints” on the set of *feasible* alternatives, i.e., the only interpretations we accept as possible alternatives are those of $Mod(K)$, i.e., $\mathcal{A} = Mod(K)$. For instance, in a decision making problem consisting of recruiting at least one and at most two of three candidates a, b and c, the feasible alternatives are the models of $K = (a \vee b \vee c) \wedge (\neg a \vee \neg b \vee \neg c)$.

We now briefly recall the propositional languages for preference representation we study. In the following, the formulas G_i are propositional formulas representing elementary *goals*. The input of a logically-represented preference relation is a pair $\Delta = \langle K, GB \rangle$ where K is the propositional formula restricting the possible alternatives (the integrity constraints) and GB (the *goal base*) is a set of elementary goals, generally associated with extra data such as weights, priorities, contexts or distances. $\succeq_{K,GB}$ (or simply $\succeq_{GB}$ when there is no risk of ambiguity) denotes the preference relation induced by GB over $Mod(K)$.

2.1 A brief overview of languages

2.1.1 Penalties

In this natural and frequently used preference representation language, the agent expresses her preferences in terms of a set of propositional formulas that she wants to be satisfied. In order to compare alternatives (models), formulas are associated with weights (usually, numbers), which tell how important the satisfaction of the formula is considered. Formally, the preferences of an agent are expressed as a finite set of goals, where each goal is a propositional formula with an associated weight. The complete preference is given by a set of these goals: $GB = \{\langle \alpha_1, G_1 \rangle, \ldots, \langle \alpha_n, G_n \rangle\}$, where each α_i is an integer and each G_i is a propositional formula. The degree of preference of a model is measured as follows: for any $M \in Mod(K)$, we define $p_{GB}(M) = \sum\{\alpha_i | M \not\models G_j\}$ to be the penalty of M. The preference relation $\succeq^{pen}_{GB}$ is defined by $M \succeq^{pen}_{GB} M'$ if and only if

$p_{GB}(M) \leq p_{GB}(M')$ (with the convention $\sum(\emptyset) = 0$).[1]

2.1.2 Distance to goals

The preference relation based on penalties only makes a distinction between models satisfying a formula and models violating it. On the other hand, if an agent prefers a formula G_i to be satisfied, we could infer that she also prefers models "close" to this formula than models "far". Let d be a pseudo-distance between models, that is, a symmetric function from $\mathcal{X}^2 \rightarrow I\!R$ such that $d(M, M') = 0$ if and only of $M = M'$. For instance, the Hamming distance $d_H(M, M')$ is the number of variables that are assigned different values in M and M'.) The "distance" between a model M and a formula G is defined by $d(M, G) = \min_{M' \models G} d(M, M')$. A goal base is a finite set of pairs $\langle \alpha_i, G_i \rangle$; the distance of a model to a goal base is defined by $d(M, GB) = \sum_i \{\alpha_i . d(M, G_i)\}$. and finally, $\succeq^H_{GB}$ is defined by

$$M \succeq^H_{GB} M' \text{ if and only if } d(M, GB) \leq d(M', GB)$$

2.1.3 Prioritized Goals

The languages defined above allow for compensations among goals (the violation of a goal may be compensated by the satisfaction of a sufficient number of goals of lower importance). Prioritization is used when such a compensation should not be possible, and does not need any numerical data. In this case, a goal base is a pair $GB = \langle \{G_1, \ldots, G_n\}, r \rangle$ where each G_i is a propositional formula and r is a rank function from $\{1, \ldots, n\}$ to $I\!N$: if $r(i) = j$, then j is called the rank of the formula G_i. By convention, a lower rank means a higher priority. The question is now how to extend the priority on goals to a preference relation on alternatives. The following three choices are the most frequent ones:

best-out ordering Let $r_{GB}(M) = \min\{r(i) \mid M \not\models G_i\}$ Then $M \succeq^{bo}_{GB} M'$ iff $r_{GB}(M) \geq r_{GB}(M')$

discrimin ordering Let $discr^+_{GB}(M, M') = \{i \mid M \models G_i \text{ and } M' \not\models G_i\}$ and $discr_{GB}(M, M') = discr^+_{GB}(M, M') \cup discr^+_{GB}(M', M)$ Then:

$$\left| \begin{array}{l} M \succ^{discrimin}_{GB} M' \text{iff} \min_{i \in discr^+_{GB}(M,M')} r(i) < \min_{j \in discr^+_{GB}(M',M)} r(j) \\ M \succeq^{discrimin}_{GB} M' \text{iff} M \succ^{discrimin}_{GB} M' \text{ or } discr_{GB}(M, M') = \emptyset. \end{array} \right.$$

[1]Many other operators can be used, in place of the sum, for aggregating weights of violated (or symmetrically, satisfied) formulas (see [15] for a general discussion).

leximin ordering Let $d_k(M)$ be the cardinal of $\{i \mid M \models G_i \text{ and } r(i) = k\}$.

$$\left| \begin{array}{l} M \succ_{GB}^{leximin} M' \text{iff} \exists k \leq n \text{ s. t.} d_k(M) > d_k(M') \text{ and } \forall j < k, d_k(M) = d_k(M'); \\ M \succeq_{GB}^{leximin} M' \text{iff} M \succ_{GB}^{leximin} M' \text{ or } d_i(M) = d_i(M') \text{ for any } i. \end{array} \right.$$

Note that $\succeq_{GB}^{leximin}$ and $\succeq_{GB}^{bo}$ are complete preference relations while $\succeq_{GB}^{discrimin}$ is generally not. We moreover have the following chain of implications: $M \succ_{GB}^{bo} M' \Rightarrow M \succ_{GB}^{discrimin} M' \Rightarrow M \succ_{GB}^{leximin} M'$.

More discussion, references and examples can be found in [16, 10].

2.1.4 Ceteris Paribus preferences

In this language, preferences are expressed in terms of statements like: "all other things being equal, I prefer these alternatives over these other ones." Formally, let C, G, and G' be three propositional formulas and V being a subset of VAR such that $Var(G) \cup Var(G') \subseteq V$. The *ceteris paribus desire* $C : G > G'[V]$ means: "all irrelevant things being equal, I prefer $G \wedge \neg G'$ to $\neg G \wedge G'$", where the "irrelevant things" are the variables that are not in V. The definitions proposed in various places differ somehow – we take here the definition of [10]. For natural reasons, and to remain consistent with the original definitions, we impose that $Var(G) \cup Var(G') \subseteq V$.

Furthermore, we add to the original definition the ability to express *indifference statements* – without them, $M \sim M'$ could not be expressed.

Let $GB = \mathcal{D}_P \cup \mathcal{D}_I$, where $\mathcal{D}_P$ and $\mathcal{D}_I$ are defined as follows.

$$\begin{array}{rcl} \mathcal{D}_P & = & \{C_1 : G_1 > G'_1[V_1], \ldots, C_m : G_m > G'_m[V_m]\} \\ \mathcal{D}_I & = & \{C_n : G_n \sim G'_n[V_n], \ldots, C_p : G_p \sim G'_p[V_p]\} \end{array}$$

We call the elements of $\mathcal{D}_P$ as "preference desires" while elements of $\mathcal{D}_I$ are "indifference desires". For all i, C_i, G_i and G'_i are propositional formulas and $Var(G_i) \cup Var(G'_i) \subseteq V_i \subseteq VAR$. We define the preference induced by a single desire $D_i = C_i : G_i > G'_i[V_i]$, denoted by $M >_{D_i} M'$, by the following three conditions:

1. $M \models C_i \wedge G_i \wedge \neg G'_i$;
2. $M' \models C_i \wedge \neg G_i \wedge G'_i$;
3. M and M' coincide on all variables in $VAR \backslash V_i$.

If the above conditions 1-3 are satisfied for an indifference desire $D_i = C_i : G_i \sim G'_i[V_i]$ in $\mathcal{D}_I$, then we say that M and M' are *indifferent* with respect to D_i, denoted by $M \sim_{D_i} M'$. Now, the preference order $\succeq^{cp}_{GB}$ is defined from the above dominance relations by transitive closure of their union: $M \succeq^{cp}_{GB} M'$ holds if and only if there exists a finite chain $M_0 = M, M_1, \ldots, M_{q-1}, M_q = M'$ of alternatives such that for all $j \in \{0, \ldots, q-1\}$ there is a $D_i \in GB$ such that $M_j >_{D_i} M_{j+1}$ or such that $M_j \sim_{D_i} M_{j+1}$.

An important sublanguage of CP-preferences is the language of (binary) **CP-nets**, which is obtained by imposing the following syntactical restriction:

- goals G and G' are *literals*, that is, CP-statements express preference of a value over its opposite for a given single variable, given some context (in other words, G and G' are of the form $(\mathbf{x}_i = v_i)$, where $\mathbf{x}_i \in VAR$ and $v_i \in \{T, F\}$.
- the variables mentioned in the context C of a preference statement about variable $\mathbf{x}_i$ must be contained in a fixed set of variables, called the *parents of* $\mathbf{x}_i$, denoted by $Parents(\mathbf{x}_i)$.
- for each variable $\mathbf{x}_i$ and each possible assignment π of the parents of $\mathbf{x}_i$, there is *one and only one* CP-preference $C : x_i > \neg x_i$ or $C : \neg x_i > x_i$ such that $\pi \models C$.

The more expressive language of **TCP-nets** [7] can also be obtained by syntactical restrictions. See [19] for a discussion about the expressivity of these various languages.

For the sake of brevity, we omitted the family of preference representation languages based on *conditional logics*. See [16, 10].

2.2 Issues in preference representation

At least four very important problems must be addressed when investigating the relevance and complexity of preference representation languages.

Elicitation We already discussed this issue in Introduction and we do not want to come back on this, since since is left outside the scope of this paper.

Expressive power R being a representation language, a relevant quesiton is whether R can express all preorders and/or all utility functions, or only complete preorders, or only a strict subclass of them, etc. This issue is investigated in [10].

Computational complexity Let R being representation language. What is the computational complexity of comparing two alternatives given an input GB of R, of deciding whether a given alternative is optimal, of finding an optimal alternative? This issue is investigated in [16].

Comparative succinctness Given R, R' two representation languages, R' is said to be at least as succinct as R iff is there a function F from R to R' such that

a. for each $GB \in R$, GB and $F(GB)$ induce the same preference relation (or utility function);

b. F is polysize, i.e., there exists a polynomial function p such that for all $GB \in R$, $size(F(GB)) \leq p(size(GB))$.

This issue is investigated in [10].

3 Combinatorial vote

Let $\mathcal{A} = \{1, ..., N\}$ be a finite set of *voters*; $\mathcal{X}$ is a *finite* set of *alternatives* (or candidates); a *individual preference profile* P is a complete weak order $\succeq_i$ (reflexive and transitive relation) on $\mathcal{X}$. A *preference profile* w.r.t. $\mathcal{A}$ and $\mathcal{X}$ is a collection of N individual preference profiles: P = ($\succeq_1$, ..., $\succeq_N$). Lastly, let $\mathcal{P}_{A,X}$ set of all preference profiles.

A *voting correspondance* $C : P_{A,X} \rightarrow 2^{\mathcal{X}} \setminus \{\emptyset\}$ maps each preference profile P of $P_{A,X}$ into a nonempty subset $C(P)$ of $\mathcal{X}$. A *voting (deterministic) rule* $r : P \rightarrow \mathcal{X}$ maps each preference profile P of $P_{A,X}$ into a single candidate $r(P)$. A deterministic rule can be obtained from a correspondance by prioritization over candidates (for more details see [8]). In the rest of the paper we focus on deterministic rules.

A combinatorial vote problem consists in applying voting rules when the set of alternatives has a combinatorial structure and the voters' preferences are expressed in a compact preference representation language. Practically, a combinatorial vote problem is composed of two steps: first, the agents express their preferences within a fixed (and common) representation language R, and second, one or several optimal (i.e., non-dominated) candidate(s) is (are) determined automatically, using a fixed voting rule.

For any representation language R, one defines a *R-profile for p voters* as a collection $B = \langle GB_1, ..., GB_p \rangle$ of goal bases (one for each of the p voters), expressed in the language R, generating a profile $P = Induce_R(B) = \{\succeq_{GB_1}, ..., \succeq_{GB_p}\}$.

3.1 Combinatorial vote: direct approach

The "direct" approach to solving a combinatorial vote consists in applying these tasks in sequence:

- elicit the preference relation for each voter, using a compact representation language;

- generate the whole preference relations on $D_1 \times \ldots \times D_n$ from the input;
- apply the voting rule r.

The good point with this direct approach is that it leads to finding an optimal outcome, more precisely, it allows for determining the exact winners according to the chosen voting rule and the true preference of the agents. The (very) bad point is its very high computational complexity in the general case. Here are examples, in the simplest compact representation language, that is, the basic propositional representation (where each agent specifies a unique propositional formula as his/her goal):

- computing a winner for the plurality rule needs $O(\log N)$ satisfiability tests (N = number of agents);
- determining whether there exists a Condorcet winner is both NP-hard and coNP-hard, and in in Θ_2^p (the exact complexity is an open problem).

Further results, including for instance the complexity of determining whether there exists a Condorcet winner for a given profile specified in a compact preference representation language, can be found in [16].

3.2 Combinatorial vote: sequential approach

The principle of the sequential appraoch is to exploit preferential independence of the preference profiles. It is well-known that preferences relations (or utility functions) over combinatorial "real-life" domains most often enjoy structural properties such as *(conditional) preferential independence* between sets of variables. This assumption was central to the development of several preference representation languages, especially *graphical languages* such as CP-nets of weighted constraint satisfaction. In these languages, the input consists of two distinct part: a structural part (an hypergraph in the CSP case, a directed acyclic graph in the CP-net case) over the variables, and a "internal" part consisting of the local preference relations over the subsets of variables identified by the structural part.

For instance, let $V = \{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}\}$, all three being Boolean variables, and assume that preference of a given agent over 2^V can be defined by a CP-net whose structural part is the directed acyclic graph $G = \{(x, y), (y, z), (y, t), (z, t)\}$; this means that, for the agent considered, preference over the values of **x** is unconditional, preference over the values of **y** is fully defined given the value of **x**, and so on.

Now comes the central assumption to the sequential approach to combinatorial vote: *the preferential independence structure is common to all agents*. Therefore, for instance, if preference obver 2^V for agent 1 can be described by a CP-net with the structure as above, then all other agents are assumed to be able to express their preferences within a CP-net using the same structure. This is a strong assumption; however, in many real-life domains it can be considered as reasonable.

Let us first consider an example. Let $N = 7$, $V = \{\mathbf{x}, \mathbf{y}\}$ with $Dom(\mathbf{x}) = \{x, \bar{x}\}$ and $Dom(\mathbf{y}) = \{y, \bar{y}\}$, and let us consider the following preference relations, where each agent expresses his preference relation by a CP-net corresponding to the following fixed preferential structure: preference on **x** is unconditional (but preference on **y** may depend on the value given to **x**).

3 agents	2 agents	2 agents
$\bar{x} \succ x$	$x \succ \bar{x}$	$x \succ \bar{x}$
$x : \bar{y} \succ y$	$x : y \succ \bar{y}$	$x : \bar{y} \succ y$
$\bar{x} : y \succ \bar{y}$	$\bar{x} : \bar{y} \succ y$	$\bar{x} : y \succ \bar{y}$

This corresponds to the following preference relations:

3 agents	2 agents	2 agents
$\bar{x}y$	xy	$x\bar{y}$
$\bar{x}\bar{y}$	$x\bar{y}$	xy
$x\bar{y}$	$\bar{x}\bar{y}$	$\bar{x}y$
xy	$\bar{x}y$	$\bar{x}\bar{y}$

Let r be a deterministic rule r. Since for all 7 voters, preference on **x** is unconditional, we may consider first the projections of the 7 preference relations on $Dom(\mathbf{x})$, namely $\langle P_1^{\mathbf{x}}, \ldots, P_n^{\mathbf{x}} \rangle$, and start by applying r to these, which results in a value of **x**, denoted by x^*, called the **x**-winner[2]. The value of **x** is now fixed to x^*; then, let us consider the projections of the 7 preference relations on $Dom(\mathbf{y})$, given **x** $= x^*$; denote these by $\langle P_1^{\mathbf{y}|\mathbf{x}=x^*}, \ldots, P_n^{\mathbf{y}|\mathbf{x}=x^*} \rangle$; we then apply r to these, which results in a value of **y**, denoted by y^*, called the conditional $\mathbf{y}$-winner given $\mathbf{x} = x^*$. The *sequential winner* is now obtained by combining the **x**-winner and the conditional $\mathbf{y}$-winner given $\mathbf{x} = x^*$, namely (x^*, y^*).

Example: let r be the plurality rule (where the plurailty score of a candidate is the number of voters ranking this candidate in the highest position, the plurality winners then being those maximizing the plurality score). Because 4 agents out of 7 unconditionally prefer x over $\bar{x}$, we get $x^* = x$; then, given **x** = x, 5 agents out of 7 prefer $\bar{y}$ to y, which leads to $y^* = x$. Therefore, the sequential plurality winner is (x, y). However, the direct plurality winner is $(\bar{x}, y)$.

[2]In case of ties, we therefore need a deterministic tiebreaking mechanism, for instance using a predeterminate order over the possible values of **x**).

The above example shows that when r is the plurality rule, sequential winners (obtained by sequential applications of r) and direct winners (obtained by a direct application of r) do not always coincide, which is an argument against the use of the sequential approach for such a voting rule. Note that more generally, this failure of sequential winners to coincide with direct winners holds for any scoring rule.

A more general question is the following: are there deterministic rules r, does the sequential winner (obtained by sequential applications of r) and the direct winner (obtained by a direct application of r) coincide? We do not know any positive answer to this question in the general case. We first show a second negative result, and lastly we give a restriction on preferences under which the answer to the above question turns out to be positive.

Here comes the second negative result. A Condorcet winner is a candidate preferred to any other candidate by a majority of voters. The notion of Condorcet winner naturally leads to the determination of *sequential Condorcet winners*: let X and Y being two subsets of the set of variables; then

- if preference on X is unconditional, then $\vec{x} \in D_X$ is a X-Condorcet winner if and only if
$$(\forall \vec{y} \in D_{\bar{X}}) \forall \vec{x}' \in D_X \; \#\{i, \vec{x}\vec{y} \succ_i \vec{x}'\vec{y}\} > \frac{N}{2}$$
- if and preference on Y given X is unconditional, then $\vec{y} \in D_Y$ is a Y-Condorcet winner given $X = \vec{x}$ if and only if
$$(\forall \vec{z} \in D_{\overline{X \cup Y}}) \forall \vec{y}' \in D_X \; \#\{i, \vec{x}\vec{y}\vec{z} \succ_i \vec{x}\vec{y}'\vec{z}\} > \frac{N}{2}$$

The sequential Condorcet winner is then the sequential combination of "local" Condorcet winners. The question is now, is a sequential Condorcet winner a direct Condorcet winner and vice versa? The following example shows that this fails.

2 voters	1 voter	2 voters
$x\bar{y}$	xy	$\bar{x}y$
$\bar{x}\bar{y}$	$x\bar{y}$	$\bar{x}\bar{y}$
xy	$\bar{x}y$	xy
$\bar{x}y$	$\bar{x}\bar{y}$	$x\bar{y}$

x and **y** are preferentially independent, therefore the sequential Condorcet winner is the mere combination of the local Condorcet winner for $\{$**x**$\}$ and the local Condorcet winner for $\{$**y**$\}$, provided that both exist. Since 3 voters unconditionally prefer x to $\bar{x}$, x is the $\{\mathbf{x}\}$-Condorcet winner; similarly, 3 voters unconditionally prefer y to $\bar{y}$ and is the $\{\mathbf{y}\}$-Condorcet winner. Therefore, xy is the sequential Condorcet winner – *but* xy is not a direct Condorcet winner, because four voters out of seven prefer $\bar{x}\bar{y}$ to xy.

We now give a condition on the preference relations such that direct and sequential Condorcet winners coincide. We say that a preference relation on $Dom(\mathbf{x}_1 \times \ldots \times Dom(\mathbf{x}_p\}$ is *lexicographic* if and only if there is a total ordering of the variables, say without loss of generality $\mathbf{x}_1 \rhd \mathbf{x}_2 \rhd \ldots \rhd \mathbf{x}_p$, and p local preference relations on $Dom(\mathbf{x}_1)$, ..., $Dom(\mathbf{x}_p)$, such that $x = (x_1, \ldots, x_p)$ is preferred to $y = (y_1, \ldots, y_p)$ iff there is an index $j \leq p$ such that (a) for every $k \leq j$, $x_k \sim y_k$ and (b) $x_j \succ y_j$. Now, assume that all agents have lexicographic preference relations (with the same variable ordering). $(v_1, \ldots, v_p) \in D_1 \times \ldots D_p$ is a sequential Condorcet winner iff

- $v_1 \in D_1$: $\{\mathbf{x}_1\}$-Condorcet winner;
- $v_2 \in D_2$: $\{\mathbf{x}_2\}$-Condorcet winner given $\mathbf{x}_1 = v_1$;
- ...
- $v_p \in D_p$: $\{\mathbf{x}_p\}$-CW given $\mathbf{x}_1 = v_1, \ldots, \mathbf{x}_{p-1} = v_{p-1}$

Thenwe have the following positive result: if there exists a sequential Condorcet winner $(v_1, \ldots, v_p)$ then $(v_1, \ldots, v_p)$ is also the (direct) Condorcet winner for the given profile, and *vice versa*.

Now, the restriction on lexicographic preference relation is a strong one. Thie leads to the following questions and problems:

Question 1 are there voting rule such that sequential winners and direct winners always coincide?

Problem 2 find reasonable restrictions on the preference relations so that the answer to Question 1 becomes positive;

Problem 3 find good algorithms (using the preferential structure) for determining winners of a combinatorial vote problem.

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On the consensus of closure systems

Bruno Leclerc [*]

Abstract

The problem of aggregating a profile of closure systems into a consensus system has applications in domains such as social choice, data analysis, or clustering. We first briefly recall the results obtained by a lattice approach and observe that there is some need for a finer approach. Then, we develop some considerations based on implications and related notions, and present a uniqueness result. It appears to be a generalization of a previous result relevant from cluster analysis.

Key words: Closure, Implication, Lattice, Consensus

1 Introduction

We consider the problem of aggregating a profile (k-tuple) $\mathtt{F}^* = (\mathtt{F}_1, \mathtt{F}_2, \ldots, \mathtt{F}_k)$ of closure systems on a given finite set S into a consensus closure systems $\mathtt{F} = c(\mathtt{F}^*)$. The aim is, for instance, to find a structure on a set S described by variables of different types. Structural information (order, tree structure) provided by these variables may be totally or partially retained by a derived closure system (see examples in Section 2). Moreover, several consensus problems already studied in the literature are particular cases of the consensus of closure systems. A basic example is provided by hierarchical classification, where many works have followed those of Adams [Ada72] and Margush and McMorris [MM81] (see the survey [Lec98]).

Closure systems and their uses are presented in Section 2.1. Several equivalent structures are recalled in Section 2.2. Section 2.3 give elements about the involved lattice structures. Section 3 presents results provided by the particularization of general results on the consensus problem in lattices. An original approach based on implications is initiated in Section 4.

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2 Closure systems

2.1 Definitions and uses

A *closure system* (abbreviated as CS) on a finite given set S is a set $\mathcal{F} \subseteq \mathcal{P}(S)$ of subsets of S satisfying the following two conditions:

(C1) $S \in \mathcal{F}$;

(C2) $C, C' \in \mathcal{F} \Rightarrow C \cap C' \in \mathcal{F}$.

When considering classical types of preference or classification data describing a given set S of objects, one observes that, frequently, they naturally correspond to closure systems. A list, of course not limitative, of such situations is given in Table 1. A CS $\mathcal{F}$ is *nested* if it is linearly ordered by set inclusion: for all $F, F' \in \mathcal{F} \Rightarrow F \cap F' \in \{F, F'\}$; it is a *tree of subsets* if, for all $F, F' \in \mathcal{F} \Rightarrow F \cap F' \in \{\emptyset, F, F'\}$ (and *hierarchical* if, moreover, $\{s\} \in \mathcal{F}$ for all $s \in S$); it is *distributive* if, for all $F, F' \in \mathcal{F} \Rightarrow F \cap F' \in \mathcal{F}$ and $F \cup F' \in \mathcal{F}$.

Type of data	***S* endowed with a**	**Subsets of *S***	**Type of closure system**
Numerical, ordinal variable	Weak order W	Down-sets of W	Nested
Transitive preference relation	Preorder P	Down-sets of P	Distributive
Nominal variable	Partition Π	S, $\emptyset$, and classes of Π	Tree of subsets of length 2
Taxonomy	Hierarchy $\mathcal{H}$	Classes of $\mathcal{H}$, and $\emptyset$	Hierarchical

Table 1. Types of data and related closure systems

2.2 Equivalent structures

Three notions are defined in this section. Together with CS's, they turn to be equivalent to each other. A *closure operator* φ is a mapping onto $\mathtt{P}(S)$ satisfying the properties of isotony (for all A, $B \subseteq S$, $A \subseteq B$ implies $\varphi(A) \subseteq \varphi(B)$, extensivity ((for all $A \subseteq S$, $A \subseteq \varphi(A)$) and idempotence (for all $A \subseteq S$, $\varphi(\varphi(A)) = \varphi(A)$. Then, the elements of the image $\mathtt{F}_\varphi = \varphi(\mathtt{P}(S))$ of $\mathtt{P}(S)$ by φ are the *closed* (by φ) sets, and $\mathtt{F}_\varphi$ is a closure system on S. Conversely, the closure operator $\varphi_{\mathtt{F}}$ on $\mathtt{P}(S)$ is given by $\varphi_{\mathtt{F}}(A) = \cap\{F \in \mathtt{F}: A \subseteq F\}$.

A *full implicational system*, denoted hereafter by I, $\rightarrow_I$ or simply $\rightarrow$, is a binary relation on $\mathtt{P}(S)$ satisfying the following conditions:

(I1) $B \subseteq A$ implies $A \rightarrow B$;

(I2) for any A, B, $C \subseteq S$, $A \rightarrow B$ and $B \rightarrow C$ imply $A \rightarrow B$;

(I3) for any A, B, C, $D \subseteq S$, $A \rightarrow B$ and $C \rightarrow D$ imply $A \cup C \rightarrow B \cup D$.

An *overhanging order* on S is also a binary relation Œ on $\mathtt{P}(S)$, now satisfying:

(O1) A Œ $B \Rightarrow A \subset B$;

(O2) $A \subset B \subset C \Rightarrow [A$ Œ $C \Longleftrightarrow A$ Œ B ou B Œ $C]$;

(O3) A Œ $A \cup B \Rightarrow A \cap B$ Œ B.

It follows from (01) and (02) that the relation Œ is a strict order on $\mathtt{P}(S)$ (whereas $\rightarrow$ is a preorder). The sets of, respectively, closure systems, closure operators, full implicational systems and overhanging orders on S are denoted, respectively, as **M**, **C**, **I** and **O**. They are related to each other by one-to-one correspondences. The equivalence between closure systems and operators has been recalled above. For a closure operator φ and its associated full implicational system $\rightarrow$ and overhanging order Œ, the first of the equivalences below is due to Armstrong [Arm74], and the second is given in [DL04a]:

$$A \rightarrow B \Longleftrightarrow B \subseteq \varphi(A)$$

$$A \text{ Œ } B \Longleftrightarrow A \subset B \text{ and } \varphi(A) \subset \varphi(B)$$

There is an important literature, with meaningful results, on implications, due to their importance in domains such as logic, lattice theory, relational databases, knowledge representation, or latticial data analysis (see the survey [CM03]). Overhanging orders take their origin in Adams ([Ada86]), where, named *nestings*, they were characterized in the particular case of hierarchies. Their generalization to all closure systems [DL04a] make them a further tool for the study of closure systems.

2.3. Lattices

The results of this section may be found in [CM03] and, for overhangings, in [DL03] and [DL04a]. First, each of the sets **M**, **C**, **I** and **O** is naturally ordered: **M**, **I** and **O** by inclusion, and **C** by the pointwise order: for $\varphi, \varphi' \in$ **C**, $\varphi \leq \varphi'$ means that $\varphi(A) \subseteq \varphi'(A)$ for any $A \subseteq S$. These orders are isomorphic or dually isomorphic:

$$\mathtt{F} \subseteq \mathtt{F}' \Longleftrightarrow \varphi' \leq \varphi \Longleftrightarrow I' \subseteq I \Longleftrightarrow Œ \subseteq Œ',$$

where φ, I and Œ (resp. φ', I' and Œ') are the closure operator, full implication system and overhanging relation associated to F (resp. to F ').

The sets **M** and **I** preserve set intersection, while **O** preserves set union and **C** pointwise intersection ($\varphi \wedge \varphi'(A) = \varphi(A) \cap \varphi'(A)$, for any $A \subseteq S$). The main correspondences between elements or operations in **M**, **C**, **I** and **O** are given in Table 2. Here, $(A] = \{B \subseteq S: B \subseteq A\}$ (prime ideal in P(S))and $[A) = \{B \subseteq S: A \subseteq B\}$ (prime filter).

From these observations, **M** and **I** are closure systems, respectively on P(S) and (P(S))^2. The closure operator associated to **M** is denoted as **Φ**. It is well-known that, with the inclusion order, any closure system F on S is a lattice (F, $\vee$, $\cap$) with the meet $F \cap F'$ and the join $F \vee F' = \varphi(F \cup F')$. If $F \subseteq F'$, F' *covers* F (denoted as $F \prec F'$) if $F \subseteq G \subseteq F'$ implies $G = F$ or $G = F'$.

An element J of F is *join irreducible* if G $\subseteq$ F and $J = \vee$G imply $J \in$ G; an equivalent property is that J covers exactly one element, denoted J^-, of F. The set of all the join irreducibles is denoted by J. Setting J(F) = $\{J \in$ J: $J \subseteq F\}$ for any $F \in$ F, one has $F = \vee$J(F) for all $F \in$ F. A join irreducible is an *atom* if it covers the minimum element of F, and the lattice F is *atomistic* if all its join irreducibles are atoms. Similarly, an element M of F is *meet irreducible* if G $\subseteq$ F and $M = \cap$G imply $M \in$ G; equivalently, M is covered by exactly one element M^+ of F. For any $F \in$ F, we have $F = \cap$M(F), where M is the set of all the meet irreducibles of F and M(F) = $\{M \in$ M: $F \subseteq M\}$.

M	C	I	O
P(S) (maximum)	$\varphi_{min} = id_{P(S)}$ (minimum)	{(X, Y) ∈ P(S)^2: $Y \subseteq X$} (minimum)	{(X, Y) ∈ P(S)^2: $X \subset Y$} (maximum)
{S} (minimum)	$\varphi_{max}(A) = S$, all $A \subseteq S$ (maximum)	P(S)^2 (maximum)	∅ (minimum)
join M∨M'	meet (pointwise intersection)	meet I∩I'	join Œ∪Œ'
meet M∩M'	join	join I∨I'	meet Œ∧Œ'
{S, A}, $A \subset S$ (join irreducible)	$\varphi(X) = A$ if $X \subseteq A$; $\varphi(X) = S$ otherwise (meet irreducible)	P(A)^2∪{(X, Y) ∈ P(S)^2: $A \not\subseteq X$} (meet irreducible)	{(X, Y) ∈ (A]× (P(S)–(A]): $X \subset Y$} (join irreducible)
{$X \subseteq S$: $A \subseteq X \Rightarrow s \in X$}, $A \subset S$, $s \in S$-A (meet irreducible)	$\varphi(X) = X$+s if $A \subseteq X$; $\varphi(X) = X$ otherwise (join irreducible)	{(X, Y) ∈ P(S)^2: $X \subseteq Y$ or $A \subseteq X$, $Y = X$+s} (join irreducible)	{(X, Y) ∈ P(S)^2: $X \subset Y$} - {(X, Y) ∈ P(S)^2: $A \subseteq X$, $Y = X$+s} (meet irreducible)

Table 2. Correspondences between M, C, I and O

The lattice F is *lower semimodular* if, for every F, F' ∈ F, $F \prec F \vee F'$ and $F' \prec F \vee F'$ imply $F \cap F' \prec F$ and $F \cap F' \prec F$. The lattice F is *ranked* if it admits a numerical *rank function* r such that $F \prec F'$ implies $r(F') = r(F) + 1$. Lower semimodular lattices are ranked.

The lattice F is a *convex geometry* if it satisfies one of the following equivalent conditions (among many other characterizations [Mon90b]:

(CG1) For any F ∈ F, there is a unique minimal subset R of J such that $F = \vee$R;

(CG2) F is ranked with rank function $r(F) = |$J$(F)|$;

(CG3) F is lower semimodular with a rank function as in (CG2) above;

Since it is a closure system on $\mathrm{P}(S)$, the ordered set $\mathbf{M}$ is itself a lattice. This lattice is an atomistic convex geometry. For $\mathrm{F} \in \mathbf{M}$, we have $\mathbf{J}(\mathrm{F}) = \{\{A, S\}: A \neq S, A \in \mathrm{F}\}$ and, so, $|\mathbf{J}(\mathrm{F})| = |\mathrm{F}|-1$.

3 Lattice consensus for closure systems

In this section, we consider the main consequences of the lattice structure of $\mathbf{M}$ for the consensus problem on closure systems, that is the aggregation of a profile $\mathrm{F}^* = (\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_k)$ (of length k) of closure systems into a closure system $\mathrm{F} = c(\mathrm{F}^*)$. General results on the consensus problem in lattices may be found, among others, in [BM90b], [BJ91] and [Lec94]. Concerning closure systems, the results obtained in an axiomatic approach by Raderanirina [Rad01] (see also [MR04] about the related case of choice functions) are described in another contribution and not recalled here.

3.1. A property of quota rules

A *federation* on K is a family K of subsets of $K = \{1,\ldots, k\}$ satisfying the monotonicity property: $[L \in \mathrm{K}, L' \supseteq L] \Rightarrow [L' \in \mathrm{K}]$. Then, the federation consensus function c_{K} on $\mathbf{M}$ is associated to K by $c_{\mathrm{K}}(\mathrm{F}^*) = \vee_{L\in \mathrm{K}} (\cap_{i\in L} \mathrm{F}_i)$. Such consensus function includes the *quota rules*, where $\mathrm{K} = \{L \subseteq K: |L| \geq q\}$, for a fixed number q, $0 \leq q \leq k$. The quota rule c_q is equivalently defined as:

$$c_q(\mathrm{F}^*) = \Phi(\mathrm{A}_q),$$

where $\mathrm{A}_q = \{A \subset S: |\{i \in K: A \in \mathrm{F}_i\}| \geq q\}$, the set of all proper subsets of S appearing in at least q elements of F^*, and Φ is the operator mentioned in Section 2.3: $\Phi(\mathrm{A}_q)$ is the smallest closure system including A_q. For $q = k/2$, $c_q(\mathrm{F}^*) = m(\mathrm{F}^*)$ is the so-called (weak) *majority rule* and, for $q = k$, it is the *unanimity rule* $u(\mathrm{F}^*)$.

Quota rules have good properties in any lattice structure, for instance:

- *Unanimity* : for any $F \in \mathbf{M}$, $c_q(F, F,\ldots, F) = F$;
- *Isotony* : for any $\mathrm{F}^* = (\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_k)$,$\mathrm{F}'^* = (\mathrm{F}'_1, \mathrm{F}'_2, \ldots, \mathrm{F}'_k)$, profiles of $\mathbf{M}$, $\mathrm{F}_i, \subseteq \mathrm{F}'_i$: for all $i = 1, \ldots, k$ implies $c_q(\mathrm{F}^*) \subseteq c_q(\mathrm{F}'^*)$.

The next property of consistency type (see Section 3.2) is not general (for instance it is not true in partition lattices [BL95]) but holds in the so-called LLD lattices [Lec03], which include convex geometries; it implies unanimity. In what follows, the profile F*F'* is just the concatenation of profiles F* and F'*, which are not required to have the same length.

Proposition 3.1. *Let* F* *and* F'* *be two profiles of* **M**. *If* c_q(F*) = c_q(F'*) = F, *then* c_q(F*F'*) = F.

3.2. Bounds on medians

For a metric approach of the consensus in **M**, we first have to define metrics. For that, we just follow [BM81] and [Lec94]. A real function v on **M** such as F $\subseteq$ F' implies v(F) < v(F') is a *lower valuation* if it satisfies one the following two equivalent properties:

(LV1) For all s, $t \in L$ such that $s \vee t$ exists, $v(s) + v(t) \leq v(s \vee t) + v(s \wedge t)$;

(LV2) The real function d_v defined on $\mathbf{M}^2$ by the following formula is a metric on **M**:

$$d_v(\mathrm{F}, \mathrm{F}') = v(\mathrm{F}) + v(\mathrm{F}') - 2v(\mathrm{F} \cap \mathrm{F}').$$

A characteristic property of lower semimodular semilattices is that their rank functions are lower valuations. So, taking property (CG2) into account, the *rank metric* is obtained taking $v(\mathrm{F}) = |\mathrm{F}|$ and, so, $d_v(\mathrm{F}, \mathrm{F}') = \partial(\mathrm{F}, \mathrm{F}') = |\mathrm{F} \Delta \mathrm{F}'|$, where Δ is the symmetric difference on subsets. The equality between the rank and the symmetric difference metric is characteristic of convex geometries or of close structures [Lec03] and is a reason to focuse on that metric.

Given the metric ∂, the *median* consensus procedure consists of searching for the *medians* of the profile F*, that is the elements F^{μ} of **M** minimizing the *remoteness* $\rho(\mathrm{F}^{\mu}, \mathrm{F}^*) = \sum_{1 \leq i \leq k} \partial(\mathrm{F}^{\mu}, \mathrm{F}_i)$ (see [BM81]). If $\mu(\mathrm{F}^*)$ is the set of all the medians of the profile F*, the median procedure has a *consistency* type property (YL78), described as follows.

Let F* and F'* be two profiles of **M**. If $\mu(\mathrm{F}^*) \cap \mu(\mathrm{F}'^*) \neq \emptyset$, then $\mu(\mathrm{F}^*\mathrm{F}'^*) = \mu(\mathrm{F}^*) \cap \mu(\mathrm{F}'^*)$.Set $\mathbf{J}_m = \{\{A, S\}: A \in \mathrm{A}_{k/2}\}$ (the set of majority atoms). From results in [Lec94], every median M of F* is the join of some subset of $\mathbf{J}_m$. It then follows that every median CS is included into the majority rule one:

Theorem 3.2. *For any profile* F* *and for any median* F^{μ} *of* F*, *the inclusion* $\mathrm{F}^{\mu} \subseteq m(\mathrm{F}^*)$ *holds.*

4 The fitting of overhangings

The results of the previous section (and those of the same type mentioned there) only take into account the presence or absence of the same closed set in enough (oligarchies, majorities) or all (unanimity) elements of the profile. It was observed in the literature that this strong limitation may prevent one to recognize actual common features. This criticism, essentially pointed out in the classification context, remains valid for general closure systems [DL04b]. Moreover, consensus systems based on closet sets may frequently be trivial. For instance, if there does not exist any majority non-trivial closed set, then, the majority rule (and unique median) is the trivial closure system reduced to $\{S\}$. Adams [Ada86] presented a consensus method (for hierarchies) able to retain common features even in such cases. It is based on overhanging orders (and, then, on implications). Here we initiate the same approach for closure systems of any type.

We state here a very general uniqueness result. Let Ξ be a binary relation on $\mathcal{P}(S)$, with the only assumption that $(A, B) \in \Xi$ implies $A \subset B$. Consider the following two properties for a closure system $\mathcal{F}$, with associated closure operator φ and overhanging relation Œ:

(AΞ1) $\Xi \subseteq$ Œ; (preservation of Ξ)

(AΞ2) for all $M \in \mathcal{M}_{\mathcal{F}}$, $(M, M^{+}) \in \Xi$. (qualified overhangings)

Note that (AΞ2) is a weakening of the converse of (AΞ1) since, by definition, any pair (M, M^{+}) belongs to Œ. Only those special pairs are required to already belong to the relation Ξ.

Theorem 4.1. *If both* $\mathcal{F}$ *and* $\mathcal{F}'$ *satisfy Conditions* (AΞ1) *and* (AΞ2)*, then* $\mathcal{F} = \mathcal{F}'$.

Proof. Observe first that the maximum set S is in both $\mathcal{F}$ and $\mathcal{F}'$. If $\mathcal{F} \neq \mathcal{F}'$, The symmetric difference $\mathcal{F}\Delta\mathcal{F}'$ is not empty. Let F be a maximal element of $\mathcal{F}\Delta\mathcal{F}'$. This subset F is not equal to S and it may be assumed without loss of generality that F belongs to $\mathcal{F}$ (and, so, not to $\mathcal{F}'$). If F was not a meet-irreducible element of $\mathcal{F}$, it would be an intersection of meet-irreducibles, all belonging to both $\mathcal{F}$ and $\mathcal{F}'$ and, so, F would belong to $\mathcal{F}'$.

Thus, F is a meet-irreducible, covered by a unique element F^{+} of $\mathcal{F}$, with $F^{+} \in \mathcal{F}'$. By (AΞ2), $(F, F^{+}) \in \Xi$ and, by (AΞ1), F Œ' F^{+}. Set $F' = \varphi'(F)$. We have $F \subset F'$, since $F \notin \mathcal{F}'$, and F' Œ' F^{+}, since $F' = \varphi'(F) = \varphi'(F') \subset \varphi'(F^{+}) = F^{+}$. But, according to the hypotheses, $F \subset F'$ implies $F' \in \mathcal{F}$, with $F \subset F' \subset F^{+}$, a contradiction with the

hypothesis that F^+ covers F in $\mathbb{F}$.
❑

The following question then arises: given a binary relation Ξ on $\mathbb{P}(S)$ (implying strict inclusion), does it exist an overhanging relation Œ satisfying conditions (AΞ1) and (AΞ2). Adams provides a positive answer in the case of a profile of hierarchies, and with $\Xi = \cap_{i\in K}$ Œ$_i$ where Œ$_1$, Œ$_2$, …, Œ$_k$ are the overhanging orders associated to the elements of the profile. In the general case, one can consider any convenient combination of Œ$_1$, Œ$_2$, …, Œ$_k$. For instances, $\Xi = \cap_{i\in K}$ Œ$_i$ corresponds to a kind of unanimity rule on overhangings, and $\Xi = \cup_{L\subseteq K,\ 2|L|>k} \cap_{i\in L}$ Œ$_i$ to a majority rule.

5 Conclusion

The last section provides a framework for the consensus of closure systems. One of the main questions is to recognize the binary relations Ξ on $\mathbb{P}(S)$ for which an overhanging order Œ satisfying (AΞ1) and (AΞ2) exists. For instance, setting $\Xi = \cup_{L\subseteq K,\ 2|L|>k} \cap_{i\in L}$ Œ$_i$ accounts for the fact that a CS appears several times in a profile, contrary to intersection rules. Algorithmic issues are very important, since overhanging relations are very big objects.

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A Complete Description of Comparison Meaningful Functions

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Abstract

Comparison meaningful functions acting on some real interval E are completely described as transformed coordinate projections on minimal invariant subsets. The case of monotone comparison meaningful functions is further specified. Several already known results for comparison meaningful functions and invariant functions are obtained as consequences of our description.

Key words : comparison meaningful function, invariant function, ordinal scale.

1 Introduction

Measurement theory (see e.g. [6, 14]) studies, among others, the assignments to each measured object of a real number so that the ordinal structure of discussed objects is preserved. When aggregating several observed objects, their aggregation is often also characterized by a real number, which can be understood as a function of numerical characterizations of fused objects. A sound approach to such aggregation cannot lead to contradictory results depending on the actual scale (numerical evaluation of objects) we are dealing with. This fact was a key motivation for Orlov [11] when introducing comparison meaningful functions. Their strengthening to invariant functions (scale independent functions) was proposed by Marichal and Roubens [9]. The general structure of invariant functions

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(and of monotone invariant functions) is now completely known from recent works of Ovchinnikov [12], Ovchinnikov and Dukhovny [13], Marichal [7], Bartlomiejczyk and Drewniak [2], and Mesiar and Rückschlossová [10]. Moreover, comparison meaningful functions were already characterized in some special cases, e.g., when they are continuous; see Yanovskaya [16] and Marichal [7]. However, a complete description of all comparison meaningful functions was still missing. This gap is now filled by the present paper, which is organized as follows. In the next section, we give some preliminaries and recall some known results. In Section 3, a complete description of comparison meaningful functions is given, while in Section 4 we describe all monotone comparison meaningful functions.

2 Preliminaries

Let $E \subseteq \mathbb{R}$ be a nontrivial convex set and set $e_0 := \inf E$, $e_1 := \sup E$, and $E^\circ := E \setminus \{e_0, e_1\}$. Let $n \in \mathbb{N}$ be fixed and set $[n] := \{1, \ldots, n\}$. Denote also by $\Phi(E)$ the class of all automorphisms (nondecreasing bijections) $\phi : E \to E$, and for $x = (x_1, \ldots, x_n) \in E^n$ put $\phi(x) := (\phi(x_1), \ldots, \phi(x_n))$.

Following the earlier literature, we introduce the next notions and recall a few results.

Definition 2.1 ([9]). A function $f : E^n \to E$ is invariant if, for any $\phi \in \Phi(E)$ and any $x \in E^n$, we have $f(\phi(x)) = \phi(f(x))$.

Definition 2.2 ([1, 11, 16]). A function $f : E^n \to \mathbb{R}$ is comparison meaningful if, for any $\phi \in \Phi(E)$ and any $x, y \in E^n$, we have

$$f(x) \begin{Bmatrix} < \\ = \end{Bmatrix} f(y) \quad \Rightarrow \quad f(\phi(x)) \begin{Bmatrix} < \\ = \end{Bmatrix} f(\phi(y)). \tag{1}$$

Definition 2.3 ([1, 5]). A function $f : E^n \to \mathbb{R}$ is strongly comparison meaningful if, for any $\phi_1, \ldots, \phi_n \in \Phi(E)$ and any $x, y \in E^n$, we have

$$f(x) \begin{Bmatrix} < \\ = \end{Bmatrix} f(y) \quad \Rightarrow \quad f(\phi(x)) \begin{Bmatrix} < \\ = \end{Bmatrix} f(\phi(y)),$$

where here the notation $\phi(x)$ means $(\phi_1(x_1), \ldots, \phi_n(x_n))$.

Definition 2.4 ([2]). A nonempty subset B of E^n is called invariant if $\phi(B) \subseteq B$ for any $\phi \in \Phi(E)$, where $\phi(B) = \{\phi(x) \mid x \in B\}$. Moreover, an invariant subset B of E^n is called minimal invariant if it does not contain any proper invariant subset.

It can be easily proved that $B \subseteq E^n$ is invariant if and only if its characteristic function $\mathbf{1}_B : E^n \to \mathbb{R}$ is comparison meaningful (or invariant if $E = [0, 1]$).

Let $\mathcal{B}(E^n)$ be the class of all minimal invariant subsets of E^n, and define

$$B_x(E) := \{\phi(x) \mid \phi \in \Phi(E)\}$$

for all $x \in E^n$. Then, we have

$$\mathcal{B}(E^n) = \{B_x(E) \mid x \in E^n\},$$

which clearly shows that the elements of $\mathcal{B}(E^n)$ partition E^n into equivalence classes, where $x, y \in E^n$ are equivalent if there exists $\phi \in \Phi(E)$ such that $y = \phi(x)$. A complete description of elements of $\mathcal{B}(E^n)$ is given in the following proposition:

Proposition 2.1 ([2, 10]). *We have $B \in \mathcal{B}(E^n)$ if and only if there exists a permutation π on $[n]$ and a sequence $\{\lhd_i\}_{i=0}^n$ of symbols $\lhd_i \in \{<, =\}$, containing at least one symbol $<$ if $e_0 \in E$ and $e_1 \in E$, such that*

$$B = \{x \in E^n \mid e_0 \lhd_0 x_{\pi(1)} \lhd_1 \cdots \lhd_{n-1} x_{\pi(n)} \lhd_n e_1\},$$

where $\lhd_0$ is $<$ if $e_0 \notin E$ and $\lhd_n$ is $<$ if $e_1 \notin E$.

Example 2.1. The unit square $[0, 1]^2$ contains exactly eleven minimal invariant subsets, namely the open triangles $\{(x_1, x_2) \mid 0 < x_1 < x_2 < 1\}$ and $\{(x_1, x_2) \mid 0 < x_2 < x_1 < 1\}$, the open diagonal $\{(x_1, x_2) \mid 0 < x_1 = x_2 < 1\}$, the four square vertices, and the four open line segments joining neighboring vertices.

We also have the following important result:

Proposition 2.2 ([2, 7, 10]). *Consider a function $f : E^n \to E$.*

$i)$ *If f is idempotent (i.e., $f(x, \ldots, x) = x$ for all $x \in E$) and comparison meaningful, then it is invariant.*

$ii)$ *If f is invariant, then it is comparison meaningful.*

$iii)$ *If E is open, then f is idempotent and comparison meaningful if and only if it is invariant.*

$iv)$ *f is invariant if and only if, for any $B \in \mathcal{B}(E^n)$, either $f|_B \equiv c$ is a constant $c \in \{e_0, e_1\} \cap E$ (if this constant exists) or there is $i \in [n]$ so that $f|_B = P_i|_B$ is the projection on the ith coordinate.*

For nondecreasing invariant functions, a crucial role in their characterization is played by an equivalence relation $\sim$ acting on $\mathcal{B}(E^n)$, namely $B \sim C$ if and only if $P_i(B) =$

$P_i(C)$ for all $i \in [n]$. Note that projections $P_i(B)$ of minimal invariant subsets are necessarily either $\{e_0\} \cap E$ or $\{e_1\} \cap E$ or E°. Further, for any $B \in \mathcal{B}(E^n)$, the set

$$B^* = \bigcup_{\substack{C \in \mathcal{B}(E^n) \\ C \sim B}} C = P_1(B) \times \cdots \times P_n(B)$$

is an invariant subset of E^n, and

$$\mathcal{B}^*(E^n) = \{B^* \mid B \in \mathcal{B}(E^n)\}$$

is a partition of E^n coarsening $\mathcal{B}(E^n)$. We also have $\operatorname{card}(\mathcal{B}^*(E^n)) = k^n$, where $k = 1 + \operatorname{card}(E \cap \{e_0, e_1\})$.

Notice that any subset B^* can also be regarded as a minimal "strongly" invariant subset of E^n in the sense that

$$\{(\phi_1(x_1), \ldots, \phi_n(x_n)) \mid x \in B^*\} \subseteq B^* \qquad (\phi_1, \ldots, \phi_n \in \Phi(E)).$$

Equivalently, the characteristic function $\mathbf{1}_{B^*} : E^n \to \mathbb{R}$ is strongly comparison meaningful.

From the natural order

$$\{e_0\} \prec E^\circ \prec \{e_1\}$$

we can straightforwardly derive a partial order $\preceq$ on $\mathcal{B}(E^n)$, namely $B \preceq C$ if and only if $P_i(B) \preceq P_i(C)$ for all $i \in [n]$. A partial order on $\mathcal{B}^*(E^n)$ can be defined similarly.

Denote by $\mathcal{M}_n$ the system of all nondecreasing functions $\mu : \{0,1\}^n \to \{0,1\}$, and let

$$\mathcal{M}_n(E) := \mathcal{M}_n \setminus \{\mu_j \mid j \in \{0,1\},\, e_j \notin E\},$$

where $\mu_j \in \mathcal{M}_n$ is the constant set function $\mu_j \equiv j$. Clearly $\mathcal{M}_n(E)$ is partially ordered through the order defined as

$$\mu \preceq \mu' \quad \Leftrightarrow \quad \mu(x) \leq \mu'(x) \;\; \forall x \in \{0,1\}^n.$$

For $\mu \in \mathcal{M}_n(E)$, we define a function $L_\mu : E^n \to E$ by

$$L_\mu(x_1, \ldots, x_n) = \bigvee_{\substack{t \in \{0,1\}^n \\ \mu(t)=1}} \bigwedge_{t_i = 1} x_i$$

with obvious conventions

$$\bigvee_{\varnothing} = e_0 \quad \text{and} \quad \bigwedge_{\varnothing} = e_1.$$

Observe that for any $\mu \in \mathcal{M}_n(E)$, L_μ is a continuous invariant function which is also idempotent whenever $\mu(0, \ldots, 0) < \mu(1, \ldots, 1)$, that is, whenever $\mu(0, \ldots, 0) = 0$ and $\mu(1, \ldots, 1) = 1$.

Remark. Functions $\mu \in \mathcal{M}_n(E)$ with $\mu(0,\ldots,0) < \mu(1,\ldots,1)$ are called also $\{0,1\}$-valued fuzzy measures (when an element $t \in \{0,1\}^n$ is taken as the characteristic vector of a subset of $[n]$). For any such μ, the corresponding function L_μ is exactly the Choquet integral with respect to μ [4, 13], but also the Sugeno integral with respect to μ [15, 13]. These functions are called also lattice polynomials [3] or Boolean max-min functions [8].

We also have the following result:

Proposition 2.3 ([7, 10]). *Consider a function $f : E^n \to E$. Then we have*

$i)$ *f is continuous and invariant if and only if $f = L_\mu$ for some $\mu \in \mathcal{M}_n(E)$.*

$ii)$ *f is nondecreasing and invariant if and only if there exists a nondecreasing mapping $\xi : \mathcal{B}^*(E^n) \to \mathcal{M}_n(E)$ so that*

$$f(x) = L_{\xi(B^*)}(x) \qquad (x \in B^* \in \mathcal{B}^*(E^n)).$$

3 Comparison meaningful functions

Following Definition 2.1, the invariance of a function $f : E^n \to E$ can be reduced to the invariance of $f|_B$ for all minimal invariant subsets $B \in \mathcal{B}(E^n)$. This observation is a key point in the description of invariant functions as given in Proposition 2.2, $iv)$. However, in the case of comparison meaningful functions, the situation is more complicated. In fact, we have to examine property (1) for $x \in B$, $y \in C$, with $B, C \in \mathcal{B}(E^n)$, to be able to describe comparison meaningful functions. We start first with the case when $B = C$, i.e., when $y = \phi(x)$ for some $\phi \in \Phi(E)$.

Proposition 3.1. *Let $f : E^n \to \mathbb{R}$ be a comparison meaningful function. Then, for any $B \in \mathcal{B}(E^n)$, there is an index $i_B \in [n]$ and a strictly monotone or constant function $g_B : P_{i_B}(B) \to \mathbb{R}$ such that*

$$f(x) = g_B(x_{i_B}) \qquad (x = (x_1,\ldots,x_n) \in B).$$

As an easy corollary of Proposition 3.1 we obtain the characterization of invariant functions stated in Proposition 2.2, $iv)$; see also [2]. Indeed, for a fixed $B \in \mathcal{B}(E^n)$, we should have $f(x) = g(x_i)$ and hence, for all $\phi \in \Phi(E)$ with fixed point x_i, we have

$$\phi(g(x_i)) = \phi(f(x)) = f(\phi(x)) = g(x_i),$$

which implies that $g(x_i)$ is a fixed point of all such ϕ's, that is,

$$g(x_i) = x_i \text{ or } e_0 \text{ or } e_1.$$

As we have already observed, the structure of invariant functions on a given minimal invariant subset is completely independent of their structure on any other minimal invariant subset. This fact is due to the invariance property: $\phi(x) \in B$ for all $x \in B$, $\phi \in \Phi(E)$ and $B \in \mathcal{B}(E^n)$. However, in the case of comparison meaningful functions we are faced a quite different situation, in which we should take into account all minimal invariant subsets.

Observe first that for a given comparison meaningful function $f : E^n \to \mathbb{R}$ and a given $B \in \mathcal{B}(E^n)$, the corresponding index i_B need not be determined univocally. This happens for instance when g_B is constant or when B is defined with equalities on coordinates (see Proposition 2.1). On the other hand, given i_B, the function g_B is necessarily unique.

Now, we are ready to give a complete description of all comparison meaningful functions.

Theorem 3.1. *The function $f : E^n \to \mathbb{R}$ is comparison meaningful if and only if, for any $B \in \mathcal{B}(E^n)$, there exist an index $i_B \in [n]$ and a strictly monotone or constant mapping $g_B : P_{i_B}(B) \to \mathbb{R}$ such that*

$$f(x) = g_B(x_{i_B}) \qquad (x \in B), \tag{2}$$

where, for any $B, C \in \mathcal{B}(E^n)$, either $g_B = g_C$, or $\mathrm{Ran}(g_B) = \mathrm{Ran}(g_C)$ is singleton, or $\mathrm{Ran}(g_B) < \mathrm{Ran}(g_C)$, or $\mathrm{Ran}(g_B) > \mathrm{Ran}(g_C)$. (Note that $\mathrm{Ran}(g_B) < \mathrm{Ran}(g_C)$ means that for all $r \in \mathrm{Ran}(g_B)$ and all $s \in \mathrm{Ran}(g_C)$, we have $r < s$.)

Example 3.1. Put $E = [0, 1]$ and $n = 2$. Then there are eleven minimal invariant subsets in $\mathcal{B}([0, 1]^2)$, namely $B_1 = \{(0, 0)\}$, $B_2 = \{(1, 0)\}$, $B_3 = \{(1, 1)\}$, $B_4 = \{(0, 1)\}$, $B_5 =]0, 1[\times\{0\}$, $B_6 = \{1\}\times]0, 1[$, $B_7 =]0, 1[\times\{1\}$, $B_8 = \{0\}\times]0, 1[$, $B_9 = \{(x_1, x_2) \mid 0 < x_1 = x_2 < 1\}$, $B_{10} = \{(x_1, x_2) \mid 0 < x_1 < x_2 < 1\}$, $B_{11} = \{(x_1, x_2) \mid 0 < x_2 < x_1 < 1\}$. Let $i_{B_j} = 1$ and $g_{B_j}(x) = 1 - x$ for $j \in \{1, 2, 3, 5, 6, 9, 11\}$, and $i_{B_j} = 2$ and $g_{B_j}(x) = 2x - 3$ for $j \in \{4, 7, 8, 10\}$, where always $x \in P_{i_{B_j}}(B_j)$. Then the relevant comparison meaningful function $f : [0, 1]^2 \to [0, 1]$ is given by

$$f(x_1, x_2) = \begin{cases} 1 - x_1, & \text{if } x_1 \geq x_2, \\ 2x_2 - 3, & \text{if } x_1 < x_2. \end{cases}$$

Theorem 3.1 enables us to characterize strong comparison meaningful functions, too. Observe that while in the case of comparison meaningful functions, for any point $x \in E^n$ the set of all $\phi(x) = (\phi(x_1), \ldots, \phi(x_n))$, with $\phi \in \Phi(E)$, gives some minimal invariant set B, in the case of strong comparison meaningful functions we are faced to the set of all points $(\phi_1(x_1), \ldots, \phi_n(x_n))$, with $\phi_1, \ldots, \phi_n \in \Phi(E)$, which is exactly the invariant set B^* linked to the previous B, which together with Theorem 3.1 results in the next corollary.

Corollary 3.1. *The function* $f : E^n \to \mathbb{R}$ *is strongly comparison meaningful if and only if, for any* $B^* \in \mathcal{B}^*(E^n)$*, there exist an index* $i_{B^*} \in [n]$ *and a strictly monotone or constant mapping* $g_{B^*} : P_{i_{B^*}}(B^*) \to \mathbb{R}$ *such that*

$$f(x) = g_{B^*}(x_{i_{B^*}}) \qquad (x \in B^*),$$

where, for any $B^*, C^* \in \mathcal{B}^*(E^n)$*, either* $g_{B^*} = g_{C^*}$*, or* $\mathrm{Ran}(g_{B^*}) = \mathrm{Ran}(g_{C^*})$ *is singleton, or* $\mathrm{Ran}(g_{B^*}) < \mathrm{Ran}(g_{C^*})$*, or* $\mathrm{Ran}(g_{B^*}) > \mathrm{Ran}(g_{C^*})$*.*

4 Monotone comparison meaningful functions

In this section we will examine monotone comparison meaningful functions. Note that the monotonicity of a fusion function is a rather natural property.

Now, for any strictly monotone or constant real function $h : \mathbb{R} \to \mathbb{R}$, and any comparison meaningful function $f : E^n \to \mathbb{R}$, also the composite $h \circ f : E^n \to \mathbb{R}$ is comparison meaningful. Consequently, to get a complete description of monotone comparison meaningful functions it is enough to examine nondecreasing comparison meaningful functions only.

Theorem 4.1. *Let* $f : E^n \to \mathbb{R}$ *be a nondecreasing function. Then* f *is comparison meaningful if and only if it has the representation*

$$\{(i_B, g_B) \mid B \in \mathcal{B}(E^n)\},$$

as stated in Theorem 3.1, such that any g_B *is either constant or strictly increasing,* $\mathrm{Ran}(g_B) = \mathrm{Ran}(g_C)$ *if* $B \sim C$*, and* $\mathrm{Ran}(g_B) \not> \mathrm{Ran}(g_C)$ *if* $B \nsim C$ *and* $B \preceq C$*.*

Now, several results mentioned in Section 2 are immediate corollaries of Theorems 3.1 and 4.1. Interesting seems to be also the next result, in which $\mathcal{G}(E)$ means the system of all strictly increasing or constant real functions g defined either on E° or on singleton $\{e_0\} \cap E$ or on $\{e_1\} \cap E$ (if these singletons exist) and for $g_1, g_2 \in \mathcal{G}(E)$ we put $g_1 \preceq g_2$ if either $g_1 = g_2$, or $\mathrm{Ran}(g_1) = \mathrm{Ran}(g_2)$ is a singleton, or $\mathrm{Ran}(g_1) < \mathrm{Ran}(g_2)$.

Corollary 4.1. *A nondecreasing function* $f : E^n \to \mathbb{R}$ *is comparison meaningful if and only if there are nondecreasing mappings* $\xi : \mathcal{B}^*(E^n) \to \mathcal{M}_n(E)$ *and* $\gamma : \mathcal{B}^*(E^n) \to \mathcal{G}(E)$ *such that*

$$f(x) = \gamma(B^*)(L_{\xi(B^*)}(x)) \qquad (x \in B^* \in \mathcal{B}^*(E^n)). \tag{3}$$

Observe also that whenever B^* is not singleton then the relevant function $\gamma(B^*)$ from the representation (3) can be obtained (for all $z \in E^\circ$) by

$$\gamma(B^*)(z) = f(z_1, \dots, z_n),$$

where

$$z_i = \begin{cases} e_0, & \text{if } P_i(B^*) = \{e_0\}, \\ e_1, & \text{if } P_i(B^*) = \{e_1\}, \\ z, & \text{otherwise}. \end{cases}$$

For example, if E is open, then $\mathcal{B}^*(E^n) = \{E^n\}$ and then necessarily each monotone comparison meaningful $f : E^n \to \mathbb{R}$ is given by $f = g \circ L_\mu$, where $\mu \in \mathcal{M}_n(E)$ and $g(z) = f(z, \ldots, z)$ is strictly monotone or constant (see also [7]).

Based on Corollaries 3.1 and 4.1, we can characterize nondecreasing strong comparison meaningful functions as follows:

Corollary 4.2. *A nondecreasing function $f : E^n \to \mathbb{R}$ is strongly comparison meaningful if and only if there is a mapping $\delta : \mathcal{B}^*(E^n) \to [n]$ and a nondecreasing mapping $\gamma : \mathcal{B}^*(E^n) \to \mathcal{G}(E)$ such that*

$$f(x) = \gamma(B^*)(x_{\delta(B^*)}) \qquad (x \in B^* \in \mathcal{B}^*(E^n)),$$

where, if $\gamma(B^*) = \gamma(C^*)$, then also $\delta(B^*) = \delta(C^*)$ (unless $\gamma(B^*) = \gamma(C^*)$ is constant).

Continuity of a comparison meaningful function is even more restrictive and it forces the monotonicity. From Theorem 3.1 we have the next result (see also [7]).

Corollary 4.3. *A continuous function $f : E^n \to \mathbb{R}$ is comparison meaningful if and only if there is a continuous, strictly monotone or constant mapping $g : E \to \mathbb{R}$ and a function $\mu \in \mathcal{M}_n(E)$ such that*

$$f = g \circ L_\mu. \tag{4}$$

Note that in trivial cases when f is constant, f admits also representations different from (4), however, always in the form $f = g \circ f^*$, where g is a constant function on E and $f^* : E^n \to E$ is an arbitrary function. In all other cases the representation (4) is unique.

Corollary 4.4. *A continuous function $f : E^n \to \mathbb{R}$ is strongly comparison meaningful if and only if there is a continuous, strictly monotone or constant mapping $g : E \to \mathbb{R}$ and an index $i \in [n]$ so that*

$$f = g \circ P_i.$$

5 Conclusions

We have described the structure of a general comparison meaningful function. As corollaries, some results concerning special cases (monotone and/or continuous operators) were characterized. Moreover, our characterization can be understood also as a hint how to construct comparison meaningful operators.

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May's Theorem for Trees

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Abstract

Kenneth May in 1952 proved a classical theorem characterizing simple majority rule for two alternatives. The present paper generalizes May's theorem to the case of three alternatives, but where the voters' preference relations are required to be trees with the alternatives at the leaves.

1 Introduction

In 1952, Kenneth May gave an elegant characterization of simple majority decision based on a set with exactly two alternatives [9]. This work is a model of the classic voting situation where there is two candidates and the candidate with the most votes is declared the winner. May's theorem is a fundamental result in the area of social choice and it has inspired many extensions. See [2], [3], [4], [5], [8], and [10] for a sample of these results.

The goal of the current paper is to state and prove a version of May's theorem in the context of trees. In what follows, **tree** will mean a rooted tree with labelled leaves and unlabelled interior vertices, and no vertex except possibly the root can have degree 2. In the biological literature, such a tree T might represent the evolutionary history of the set S of species, with interior vertices of T representing ancestors of the species in S. Clearly the simplest nontrivial case is when $|S| = 3$. In this case, there are exactly 4 distinct trees with leaves labelled by the set S. It is within this context that we define a version of simple majority decision for trees and characterize it in terms of three conditions. There is a clear connection between our conditions and those given by May.

This paper is divided into four sections with this introduction being the first section. Section 2 is background material on May's work and includes the statement of May's Theorem. Section 3 contains the definition of majority decision for trees, and the main result of this paper is stated and proved in Section 4.

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2 Background on May's Work

Let $S = \{x, y\}$ be a set with two alternatives. The binary relations $R_{-1} = \{(x,x),(y,y),(y,x)\}$, $R_0 = S \times S$, and $R_1 = \{(x,x),(y,y),(x,y)\}$ are the three weak orders on S. The relation R_{-1} represents the situation where y is strictly preferred to x, R_1 represents the situation where x is strictly preferred to y, and R_0 represents indifference between x and y.

Let $K = \{1, \ldots, k\}$ be a set with $k \geq 2$ individuals and let $\mathcal{W}(S)$ be the set $\{R_{-1}, R_0, R_1\}$. A function of the form

$$f : \mathcal{W}(S)^k \to \mathcal{W}(S)$$

is called a **group decision function** by May.

For any $p = (D_1, \ldots, D_k)$ in $\mathcal{W}(S)^k$ and for any $i \in \{-1, 0, 1\}$ let

$$N_p(i) = |\{D_j : D_j = R_i\}|.$$

That is, $N_p(i)$ is the number of times the relation R_i appears in the k-tuple p. It follows that $N_p(-1) + N_p(0) + N_p(1) = k$ and $N_p(i) \geq 0$ for each $i \in \{-1, 0, 1\}$. The group decision function

$$M : \mathcal{W}(S)^k \to \mathcal{W}(S)$$

defined by

$$M(p) = \begin{cases} R_{-1} & \text{if } N_p(1) - N_p(-1) < 0 \\ R_1 & \text{if } N_p(1) - N_p(-1) > 0 \\ R_0 & \text{if } N_p(1) - N_p(-1) = 0 \end{cases}$$

for any k-tuple p is called, for obvious reasons, **simple majority decision**. The consensus weak order $M(p)$ has y strictly preferred to x if more individuals rank y strictly over x than x strictly over y. There is indifference between x and y if the number of individuals that strictly prefer y over x is the same as the number of individuals that strictly prefer x over y. Finally, $M(p)$ has x strictly preferred to y if the number of individuals that rank x strictly over y is more than the number of individuals that rank y strictly over x.

May simplified the notation used above as follows. The relation R_{-1} is identified with the number -1, the relation R_0 is identified with the number 0, and the relation R_1 is identified with 1. Using this identification we can think of a group decision function as a function with domain $\{-1, 0, 1\}^k$ and range $\{-1, 0, 1\}$.

Let $f : \{-1, 0, 1\}^k \to \{-1, 0, 1\}$ be a group decision function. Then reasonable properties that f may or may not satisfy are the following.

(A) For any k-tuple $p = (D_1, \ldots, D_k)$ and for any permutation α of K,

$$f(D_{\alpha(1)}, \ldots, D_{\alpha(k)}) = f(D_1, \ldots, D_k).$$

(N) For any k-tuple $p = (D_1, \ldots, D_k)$,

$$f(-D_1, \ldots, -D_k) = -f(D_1, \ldots, D_k).$$

(PR) For any k-tuples $p = (D_1, \ldots, D_k)$ and $p' = (D'_1, \ldots, D'_k)$,

$$\text{if } f(D_1, \ldots, D_k) \in \{0, 1\}, D'_i = D_i \text{ for all } i \neq i_0, \text{ and } D'_{i_0} > D_{i_0},$$

then

$$f(D'_1, \ldots, D'_k) = 1.$$

The conditions (A), (N), and (PR) correspond to conditions II, III, and IV given on pages 681 and 682 in [9]. Condition (A) states that f is a symmetric function of its arguments and thus individual voters are anonymous. Condition (N) is called **neutrality**. This axiom is motivated by the idea that the consensus outcome should not depend upon any labelling of the alternatives. Condition (PR) is called **positive responsiveness** since it reflects the notion that a group decision function should respond in a positive way to changes in individual preferences. If the consensus outcome $f(p)$ does not rank y strictly preferred to x and one individual i_0 changes their vote in a favorable way toward x, then the consensus outcome $f(p')$ should strictly prefer x to y.

We now can state May's result.

Theorem 1 *A group decision function is the method of simple majority decision if and only if it satisfies (A), (N), and (PR).*

3 Trees with 3 Leaves

As we have noted, May studied majority decision for two alternatives, which is the simplest non-trivial case for weak orders. Since our goal is to prove a version of May's result for trees, we too restrict our attention to the simplest non-trivial case for trees; namely when $|S| = 3$. For $S = \{x, y, z\}$, and $\{u, v\} \subset S$, let $T_{\{u,v\}}$ denote the tree with one non-root vertex of degree three adjacent to the root, u, and v. Let $T_\emptyset$ be the tree whose only internal vertex is the root.

Let $\mathcal{T}(S)$ be the set $\{T_{\{x,y\}}, T_{\{x,z\}}, T_{\{y,z\}}, T_\emptyset\}$ of all trees with the leaves labelled by the elements of S. We will call a function of the form

$$C : \mathcal{T}(S)^k \to \mathcal{T}(S)$$

a **consensus function** to conform with current useage [6] . An element $P = (T_1, \ldots, T_k)$ in $\mathcal{T}(S)^k$ is called a **profile** and the output $C(P)$ is called a **consensus tree**. For any profile $P = (T_1, \ldots, T_k)$ and for any two element subset $\{u, v\}$ of S, let

$$N_P(uv) = |\{T_i : T_i = T_{\{u,v\}}\}|.$$

Also, let

$$N_P(\emptyset) = |\{T_i : T_i = T_\emptyset\}|.$$

So $N_P(xy) + N_P(xz) + N_P(yz) + N_P(\emptyset) = k$. The consensus function

$$Maj : \mathcal{T}(S)^k \to \mathcal{T}(S)$$

defined by

$$Maj(P) = \begin{cases} T_{\{u,v\}} & \text{if } N_p(uv) > \frac{k}{2} \\ T_\emptyset & \text{otherwise} \end{cases}$$

is called **majority rule** [7]. This consensus function is well known but it is not the best analog of simple majority decision *sensu* May. We feel that a better candidate is the consensus function

$$M : \mathcal{T}(S)^k \to \mathcal{T}(S)$$

defined by

$$M(P) = \begin{cases} T_{\{u,v\}} & \text{if } N_p(uv) > \max\{N_P(uw), N_P(vw)\} \\ T_\emptyset & \text{otherwise} \end{cases}$$

where $\{u, v, w\} = \{x, y, z\}$. It is easy to see that if $Maj(P) = T_{\{u,v\}}$ for some two element subset $\{u, v\}$ of S, then $M(P) = Maj(P)$. The converse is not true. For example, if $P = (T_1, \ldots, T_k)$ such that $T_1 = T_{\{x,y\}}$ and $T_i = T_\emptyset$ for all $i \neq 1$ in K, then $M(P) = T_{\{x,y\}}$ and $Maj(P) = T_\emptyset$. For the remainder of this paper the function M will be called **majority decision**.

4 Main Result

Following are translations of the conditions (A), (N), and (PR) to the context of trees. Let $C : \mathcal{T}(S)^k \to \mathcal{T}(S)$ be a consensus function, and consider the following conditions.

(A)$^+$ For any profile $P = (T_1, ..., T_k)$ and any permutation α of K,

$$C(P_\alpha) = C(P).$$

where $P_\alpha = (T_{\alpha(1)}, ..., T_{\alpha(k)})$.

Let $\beta : S \to S$ be a permutation. Then β induces a map on $\mathcal{T}(S)$ as follows: $\beta T_\emptyset = T_\emptyset$ and $\beta T_{\{u,v\}} = T_{\{\beta(u),\beta(v)\}}$ for any two element subset $\{u, v\}$ of S. If $P = (T_1, ..., T_k)$ is a profile, then set $\beta P = (\beta T_1, ..., \beta T_k)$.

(N)$^+$ For any profile $P = (T_1, ..., T_k)$ and any permutation β of S,

$$C(\beta P) = \beta C(P).$$

(PR)$^+$ This condition has three parts.
(1) For any profiles $P = (T_1, ..., T_k)$ and $P' = (T'_1, ..., T'_k)$, if $T'_i = T_i$ for all $i \neq i_0$ and $T'_{i_0} = T_{\{x,y\}}$, then $C(P) = T_{\{x,y\}}$ implies $C(P') = T_{\{x,y\}}$.

(2) For any profiles $P = (T_1, ..., T_k)$ and $P' = (T'_1, ..., T'_k)$, if $T'_i = T_i$ for all $i \neq i_0$, $T_{i_0} \notin \{T_\emptyset, T_{\{x,y\}}\}$, and $T'_{i_0} = T_\emptyset$, then $C(P) = T_{\{x,y\}}$ implies $C(P') = T_{\{x,y\}}$.

(3) Let $P = (T_1, ..., T_k)$ be a profile such that $C(P) = T_\emptyset$ and $T_{i_0} \in \{T_\emptyset, T_{\{x,y\}}\}$. Then there exists a profile $P' = (T'_1, ..., T'_k)$ such that $T'_i = T_i$ for all $i \neq i_0$, $T'_{i_0} \neq T_{i_0}$, and $C(P') \notin \{T_\emptyset, T_{\{x,y\}}\}$.

It is easy to make direct comparisons between conditions (A) and (N) for group decision functions and conditions (A)$^+$ and (N)$^+$ for consensus functions. A comparison between conditions (PR) and (PR)$^+$ requires a bit more thought. The hypotheses of condition (PR) allow for different possibilities. One possibility, for example, is when $f(p) = 1$, $D'_{i_0} = 1$, and $D_{i_0} \in \{-1, 0\}$. Another possibility is $f(p) = 1$, $D'_{i_0} = 0$, and $D_{i_0} = -1$. These two possibilities translate into items (1) and (2) in (PR)$^+$. The tree $T_{\{x,y\}}$ is identified with 1 and the tree $T_\emptyset$ is identified with 0.

The final item (3) in (PR)$^+$ corresponds to the case when $f(p) = 0$ in (PR). Now $D'_{i_0} > D_{i_0}$ implies that $D_{i_0} \in \{0, -1\}$. This in turn is motivation for the hypothesis $T_{i_0} \in \{T_\emptyset, T_{\{x,y\}}\}$. Notice the change in identification with the tree $T_{\{x,y\}}$ now corresponding to -1. The conclusion in (PR) can be written as $f(p') \notin \{-1, 0\}$ which corresponds to the conclusion $C(P') \notin \{T_\emptyset, T_{\{x,y\}}\}$ in (PR)$^+$.

We need a lemma before we can state and prove our main result.

Lemma 2 *Suppose* $C : \mathcal{T}(S)^k \to \mathcal{T}(S)$ *satisfies (A)*$^+$*, (N)*$^+$*, and (PR)*$^+$*. If* $P = (T_1, \ldots, T_k)$ *is a profile where* $C(P) = H_{\{x,y\}}$*, then* $N_P(xy) > \max\{N_P(xz), N_P(yz)\}$.

Proof. Assume that $N_P(xy) = N_P(xz)$. Then $|K_1| = |K_2|$ where $K_1 = \{i \in K : T_i = T_{\{x,y\}}\}$ and $K_2 = \{i \in K : T_i = T_{\{x,z\}}\}$. Choose a permutation α of K such that α maps K_1 onto K_2, K_2 onto K_1, and $\alpha(i) = i$ for all $i \in K \setminus (K_1 \cup K_2)$. Define $\beta : S \to S$

by $\beta(x) = x, \beta(y) = z$, and $\beta(z) = y$ and note that $P = \beta P_\alpha$. It follows from (A)$^+$ and (N)$^+$ that

$$C(P) = C(\beta P_\alpha) = \beta C(P_\alpha) = \beta C(P).$$

But $C(P) = T_{\{x,y\}}$ and $T_{\{x,y\}} \neq \beta T_{\{x,y\}}$. This contradiction implies that $N_P(xy) \neq N_P(xz)$.

A similar argument shows that $N_P(xy) \neq N_P(yz)$.

Let $r = N_P(xz) - N_P(xy)$ and assume $r > 0$. Choose $i_0 \in \{i \in K : T_i = T_{\{x,z\}}\}$ and define $P' = (T'_1, \dots, T'_k)$ by $T'_i = T_i$ for all $i \neq i_0$ and

$$T'_{i_0} = \begin{cases} T_\emptyset & \text{if } r = 1 \\ T_{\{x,y\}} & \text{if } r \geq 2 \end{cases}$$

It follows from (PR)$^+$ that $C(P') = T_{\{x,y\}}$. Note that

$$N_P(xy) \leq N_{P'}(xy) \leq N_{P'}(xz) < N_P(xz).$$

Since K is finite this process can be continued (if necessary) until we find a profile P^* such that $C(P^*) = T_{\{x,y\}}$ and $N_{P^*}(xy) = N_{P^*}(xz)$. This contradicts the first part of the proof. Therefore, $N_P(xy) > N_P(xz)$.

A similar argument shows that $N_P(xy) > N_P(yz)$ and the proof is complete. □

Theorem 3 *The consensus function $C : \mathcal{T}(S)^k \to \mathcal{T}(S)$ is the majority decision function if and only if C satisfies (A)$^+$, (N)$^+$, and (PR)$^+$.*

Proof. First, it is straightforward to verify that the consensus function M satisfies (A)$^+$, (N)$^+$, and (PR)$^+$.

Suppose $C : \mathcal{T}(S)^K \to \mathcal{T}(S)$ satisfies (A)$^+$, (N)$^+$, and (PR)$^+$. Let $P = (T_1, \dots, T_k)$ be an arbitrary profile. The goal is to show that $C(P) = M(P)$.

If $C(P) = T_{\{x,y\}}$, then, by Lemma 2, $N_P(xy) > \max\{N_P(xz), N_P(yz)\}$. By the definition of M, $M(P) = T_{\{x,y\}}$ and so $C(P) = M(P)$.

If $C(P) = T_{\{x,z\}}$, then define $\beta : S \to S$ by $\beta(x) = x, \beta(y) = z$, and $\beta(z) = y$. It follows from (N)$^+$ that

$$C(\beta P) = \beta C(P) = \beta T_{\{x,z\}} = T_{\{x,y\}}.$$

Since $C(\beta P) = T_{\{x,y\}}$ it follows from above that $M(\beta P) = T_{\{x,y\}}$. A second application of (N)$^+$ yields

$$\beta C(P) = C(\beta P) = T_{\{x,y\}} = M(\beta P) = \beta M(P).$$

Since β induces a bijection on $\mathcal{T}(S)$ it follows that $C(P) = M(P)$.

If $C(P) = T_{\{y,z\}}$, then use a variation of the previous argument to establish that $C(P) = M(P)$.

The final case is when $C(P) = T_\emptyset$. Assume that $M(P) \neq T_\emptyset$. By using condition (N)$^+$ if necessary we may assume $M(P) = T_{\{x,y\}}$. By the definition of M, $N_P(xy) > \max\{N_P(xz), N_P(yz)\}$. Let $i_0 \in \{i \in K : T_i = T_{\{x,y\}}\}$. It follows from (PR)$^+$ that there exists a profile $P' = (T'_1, \ldots, T'_k)$ such that $T'_i = T_i$ for all $i \neq i_0$, $T'_{i_0} \neq T_{i_0}$, and $C(P') \notin \{T_\emptyset, T_{\{x,y\}}\}$. Then $C(P') = T_{\{x,z\}}$ or $T_{\{y,z\}}$. Assume without loss of generality that $C(P') = T_{\{x,z\}}$. By Lemma 2 and (N)$^+$, $N_{P'}(xz) > \max\{N_{P'}(xy), N_{P'}(yz)\}$. Thus $T'_{i_0} = T_{\{x,z\}}$ and $T_{i_0} = T_{\{x,y\}}$. In fact, $N_{P'}(xz) = N_{P'}(xy) + 1$. If $K_1 = \{i \in K : T'_i = T_{\{x,z\}}\}$ and $K_2 = \{i \in K : T'_i = T_{\{x,y\}}\}$, then $|K_1| = |K_2| + 1$. Choose a permutation α of K such that α maps K_2 onto $K_1 \setminus \{i_0\}$, $K_1 \setminus \{i_0\}$ onto K_2, and $\alpha(i) = i$ for all $i \in K \setminus (K_2 \cup K_1 \setminus \{i_0\})$. In particular, $\alpha(i_0) = i_0$. Note that $P'_\alpha = \beta P$. It follows from (A)$^+$ and (N)$^+$ that

$$C(P') = C(P'_\alpha) = C(\beta P) = \beta C(P) = \beta T_\emptyset = T_\emptyset,$$

contrary to $C(P') = T_{\{x,z\}}$. This last contradiction completes the proof of our main result. □

It is not possible to drop any one of (A)$^+$, (N)$^+$, (PR)$^+$ and still uniquely determine the consensus function M. The projection function $C_1 : \mathcal{T}(S)^k \to \mathcal{T}(S)$ defined by $C_1(P) = T_1$ for any profile $P = (T_1, \ldots, T_k)$ satisfies (N)$^+$ and (PR)$^+$ but it does not satisfy (A)$^+$. The constant function $C_2 : \mathcal{T}(S)^k \to \mathcal{T}(S)$ defined by $C_2(P) = T_{\{x,y\}}$ for any profile P satisfies (A)$^+$ and (PR)$^+$ but not (N)$^+$. The majority consensus rule Maj satisfies (A)$^+$, (N)$^+$, and items (1) and (2) in (PR)$^+$ but does not satisfy item (3) in (PR)$^+$.

It would be interesting to extend the consensus function M to trees with more than 3 leaves. However, it turns out that there is not a unique extension and in some cases the consensus outcome is not even a tree. The details of this work will be given in a future paper.

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An algorithmic solution for an optimal decision making process within emission trading markets

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Abstract

We present an algorithmic solution for optimal decision making in emission trading markets. The economy is modelled as a time discrete system. The trajectories correspond to possible strategies of the players. In this paper, we treat the strategies as control parameters and require that they lie in the core of a given cost-game. The suggested algorithmic solution principle is based on dynamic programming techniques. The uniqueness of the solution which is represented by the core is proved.

Key words : Decision making process, algorithmic solution, knowledge interaction

1 Introduction

The conference of Kyoto 1997 institutionalized a new and important economic instrument for environmental protection, the *Joint-Implementation Program* (JI), see *Kyoto (1997)*. The program intends to strengthen international cooperations between enterprises in order to reduce greenhouse gas emissions. Specifically, the concept of Joint Implementation involves a bilateral or multilateral deal in which countries facing high pollution abatement costs invest in abatement in countries with lower costs, and receive credit for the resulting reduction in greenhouse gas emissions. The reduction in emissions resulting from technical cooperations are recorded at the *Clearing House*. The realization of Joint-Implementation (JI) is subject to technical and financial constraints. The so-called TEM model was developed to capture these constraints in a empirically practicable way. For more details, see *Pickl (1999)*, where the TEM model is treated as a time-discrete control problem, and *Pickl (2001)*, who analyses the feasible control set.

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In the following sections, we present a new bargaining approach to international emissions trading markets within the so-called Kyoto game. The Kyoto game is part of the TEM model. Nash equilibria and Pareto optima are characterized and calculated applying dynamic programming techniques.

2 Economic Motivation: Multistep Investments

In order to get an intuition of the following allocation problem, let us begin with a very simple case where we have only two players. The two players have two alternative strategies to invest. The origin of the coordinate system is the starting point of the two players. Each player tries to reach the black square which stands for the level of reductions of emissions mentioned in Kyoto Protocol in a given number of time-steps. In the following, we assume only 3 time-steps:

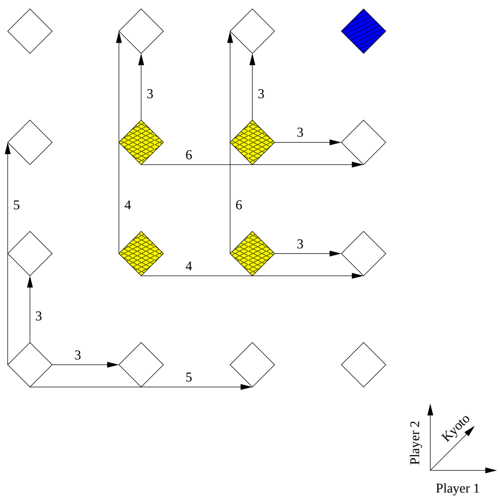

Figure 1: Multistep investments

The different paths are related with independent costs which are recorded at the clearing house. This institution is mentioned explicitly in Kyoto protocol.

2.1 A Non-Cooperative Approach

In the figure 3.1. the players make their choice independently and simultaneously. Each player can choose between two alternatives strategies (1, 2) or (2, 1), meaning moving first 1-step then 2-steps or first 2-steps then1-step. One of the grey squares will be attained after the first time-step. The directions of movement are shown on the figure in the small diagram. Player 1 goes to the right. To reduces his CO_2 emissions by 1 unit, he has to invest 3 monetary units. To attain a reduction of 2 units, then he has to invest 5 units. The strategy (2, 1) leads to a greater reduction at the beginning and a smaller investment at the end of the period. The costs are lower than in the (1, 2) case, reflecting the fact that early innovations are favorable. We can transfer this simple model with two players and two time-steps to a simple matrix game, which we call the *Kyoto-game*.

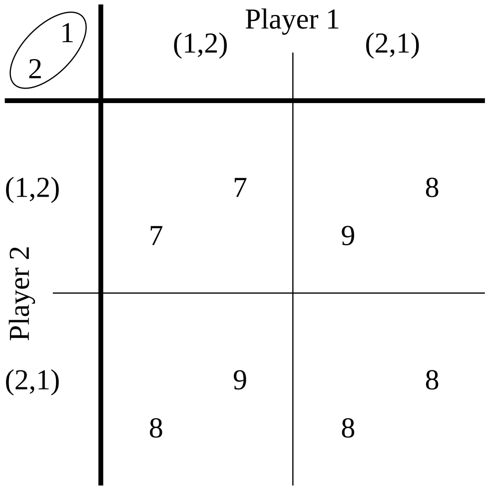

Figure 2: Nash Equilibria

The combinations of strategies $\{(1,2),(1,2)\}$ and $\{(2,1),(2,1)\}$ are both Nash equilibria. Unilateral deviations do not profit any player. Nevertheless, both players prefer the combination of strategies $\{(1,2),(1,2)\}$.

If the clearing house wants to support a given combination of strategies, say $\{(1,2),$ $(1,2)\}$, then it can induce it by adding taxes to specific payoff combinations. This very simple example gives an intuition about the situation if many players are involved. Symmetry aspects are observable in that example. If the relationships are not symmetric, we construct a worst-case scenario and consider the minimum of the two parameters. Furthermore, as the necessary data is given to the clearing house, real time monitoring of the actual developments is possible. We now extend the analysis to a general n-player situation with an arbitrary number of time-steps.

2.2 A Cooperative Approach

In such a general n-player multistep situation the distribution of the costs along the paths (strategies) can be interpreted as imputations of an underlying time-discrete dynamical game. For such a game we consider now the core as suitable solution concept:

Let us introduce the well known allocation concept of the core:

Definition 2.1 *Let v_t a cooperative n-person game with $y(K) = \sum_{i \in K} y_i, K \subseteq \mathcal{N}$, where y is an imputation of the game. Then we define the* ***Core*** *by:*

$$\mathit{Core}_{v_t} := \{y \in \mathbb{R}^n \mid y(\mathcal{N}) = v_t(\mathcal{N}) \quad \textit{and} \quad y(K) \geq v_t(K) \quad \textit{for all} \quad K \subseteq \mathcal{N}\}\,.$$

The expression for Core_{v_t} indicates that each time-step the core depends on the state $x(t)$ of an underlying time-discrete system. In the following we assume that the Core exists at each time-step. Then we get the following problem formulation if we regard the general multistep Kyoto game ($\tilde{x}_i(t)$ indicates the state of the i-th player at time-step= t; this state defines the core):

$$\tilde{x}_i(t) \in X_i \subseteq \mathbb{R}^{n_i} \quad (i = 1, \ldots, n\,, \quad t = 0, \ldots, N)$$

$$\tilde{u}_i(t) \in Core(\tilde{x}(t)) \subseteq \mathbb{R}^{m_i} \quad (i = 1, \ldots, n\,, \quad t = 0, \ldots, N-1) \tag{1}$$

$$\tilde{x}_i(t+1) = \tilde{x}_i(t) + f_i(\tilde{x}(t), \tilde{u}(t)) \quad .$$

In vector notation, we write (1) as follows:

$$\begin{aligned} x_{t+1} &= T_t(x_t, u_t),\ T_t : X_t \times U_t \\ & \quad (t = 0, \ldots, N-1), \quad \text{where} \\ x_t &:= (\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t)), \\ u_t &:= (\tilde{u}_1(t), \tilde{x}_2(t), \ldots, \tilde{u}_n(t)). \end{aligned}$$

We call $x_{t+1} = T_t(x_t, u_t)$ a *general multistep process* with x_0 as start vector and where T_t is a suitable vector transformation. It is a generalization of figure 3.1.

The states of the Kyoto game $x_i = (x_i^{(1)}, x_i^{(2)}, \ldots x_i^{(\alpha)})^T$ for $i = 1, \ldots, N$ are elements of the nonempty set $X_i,\ x_i \in \mathbb{R}^{\alpha}$. The parameter α describes the dimension of the state vector. We call the states which can be realized *feasible*.

The process is restricted to a finite time-period $[t_0, T]$. Starting point is $t_0 = 0$.

2.3 Time-Discrete Kyoto Game

We introduce intervals $I_p = [t_{p-1}, t_p]$ of length Δ_p with $t_T = T$. Each interval I_i is the $i-$ step of the generalized Kyoto Game with N steps. The intervals indicate the stages of the intuitive example. The advantage of taking variable intervals lies in the fact that variable bargaining situations can be described. Now, we introduce the following objective function

$$Z(x,u) = \sum_{p=0}^{P-1} V_p(x_p, u_p) + V_P(x_P), \tag{2}$$

V_p objective function of the p-th step:it depends on the state x_p the decision parameter u_p,

V_P objective function of the P-th step: it depends upon the input value x_P assuming that the decision value on the last step is zero

We consider $Z(x,u)$ depending on the feasible multistep decision process

$$PR = (x,u) := (x_0, x_1, \ldots, x_P, u_0, u_1, \ldots, u_{P-1})$$
$$\text{where}, x_t \in X_t \text{ and } \quad u_t \in U_t(x_t) := \text{Core}(x_t),$$
$$\text{and} x_{t+1} = T_t(x_t, u_t) \in X_{t+1}(t = 0, \ldots, P-1).$$

We get an algorithmic solution of the problem if we introduce subprocesses PR_j just at the $(j+1)$-th step of the whole process $0 \leq j \leq P-1 \;\; P-j$:

$$PR^j := (x,u)^j := (x_j, x_{j+1}, \ldots, x_P, u_j, u_{j+1}, \ldots, u_P),$$
$$\text{where } x_t \in X_t, \quad u_t \in U_t(X_t) := \text{Core}(x_t), \text{ and}$$
$$x_{t+1} = T_t(x_t, u_t) \in X_{t+1} \quad (t = j, j+1, \ldots, P-1)$$

Based on the sequence PR_j we have a sequence of objective functions

$$\begin{aligned} Z_j(PR^j) &= Z_j((x,u)^j) \\ &= \sum_{i=j}^{P-1} V_i(x_i, u_i) + V_P(x_P)\,. \end{aligned}$$

It is obvious that for $j = 0$ we have again the whole process PR:

$$Z(P) = Z(x, u) = \sum_{i=0}^{N-1} V_i(x_i, u_i) + V_P(x_P). \tag{3}$$

In the next part we present a dynamic programming approach which leads to a solution of that game.

2.4 Sequencing and Dynamic Programming

We introduce the following decision function which indicates that the players can decide between several strategies. We assume that this decision function depends on the state of the system. This leads to the following representation:

$$s^j := [s_j(x_j), s_{j+1}(x_{j+1}), \ldots, s_{P-1}(x_{P-1})] \quad \text{decision function} \quad 0 \leq j \leq P-1.$$

In the next part we present a solution principle which can be obtained by the well-known *Bellmann Principle of Dynamic Programming*.

For the objective function we get the following representation:

$$\begin{aligned} Z_j &= \sum_{t=j}^{P-1} V_t(x_t, u_t) + V_P(x_P) \\ &= Z_j(x_j, x_{j+1}, \ldots, x_P, s_j(x_j), s_{j+1}(x_{j+1}), \ldots) \\ &\quad s_{P-1}(x_{P-1})) \\ &= \sum_{t=j}^{P-1} V_t(x_t, s_t(x_t)) + V_P(x_P) \end{aligned}$$

There is only a dependence on the state vector and the decision strategy. We are independent on the control parameter which was not part of the Kyoto game. Additionally, in the next section we introduce the concept of optimality related to that strategy s^j. First, we call a decision function *feasible*, if $s_t(x_t) \in U_t(x_t)$ for all $x_t \in X_t$. Instead of $x_t = T_{t-1}(x_{t-1}, u_{t-1})$ we get $x_t = T_{t-1}(x_{t-1}, s_{t-1}(x_{t-1}))$.

Introducing the following abbreviations

$$\begin{aligned} Z^*_{P,s^P}(x_P) &= V_P(x_P)\,, \\ Z^*_{P-1,s^{P-1}}(x_{P-1}) &= V_{P-1}(x_{P-1}, s_{P-1}(x_{P-1} =)) + \\ &\qquad \underbrace{V_P(x_P)}_{Z^*_{P,s^P}(T_{P-1}(x_{P-1},s_{P-1}(x_{P-1})))} \\ Z^*_{t,s^t}(x_t) &= V_t(x_t, s_t(x_t)) \\ &\quad + Z^*_{t+1,s^{t+1}}(T_t(x_t, s_t(x_t))), \end{aligned}$$

we obtain the following definition for optimality:

Definition 2.2 *A feasible decision strategy*

$$\tilde{s}^j = [\tilde{s}_j(x_j), = \tilde{s}_{j+1}(x_{j+1}), \ldots, \tilde{s}_{P-1}(x_{P-1})]$$

*of the process P is called **optimal** if the following inequality is valid:*

$$Z^*_{j,\tilde{s}^j}(x_j) \geq Z^*_{j,s^j}(x_j) \quad \textit{for all} \quad x_j \in X_j, \quad \textit{for each feasible decision strategy } s^j.$$

We call the functions which realize an optimal decision strategy, *optimal decision functions*. If we take $t = 0$, we apply the same terminology for the whole process of the Kyoto game. The paths of the game are now interpreted as several strategies.

This functions depends only on the start vector $x_j \in X_j$. Furthermore, they describe the maximum of the characteristic function Z_j for the process $j,\ j+1, \ldots, N$. We assume that the decision strategy s^j is feasible. It is obvious, that each optimal strategy $\tilde{s}^t$ represents an optimal process $\tilde{P}R_t$.

We call the functions $\tilde{s}_t, \quad t = j, j+1, \ldots, N-1$, which represent an optimal decision policy, *optimal decision functions*. In the same way we call the states of only a part of a process P_t *optimal*, too. If $t = 0$, then we have the same situation for the whole process. Instead of varying over all possible strategies we may vary over all feasible processes:

Let us furthermore introduce:

$$\begin{aligned} f_{P-j}(x_j) &= \max_{u_t \in U_t(x_t) t=j,\ldots,P-1} Z_j((x,u)^j) \\ &= \max_{u_t \in U_t(x_t) t=j,\ldots,P-1} Z^*_j(x_j, u_j) \\ &\quad (j = 0, 1, \ldots, P-1), \end{aligned}$$

where $x_{t+1} = T_t(x_t, u_t)$ and $f_0(x_P) = V_P(x_P)$. If we use :

$$Z_j((x,u)^j) = \sum_{t=j}^{P-1} V_t(x_t, u_t) + V_P(x_P)$$

then we get:

$$f_{P-j}(x_j) = \max_{u_t \in U_t(x_t) t=j,\ldots,P-1} \left[\sum_{t=j}^{P-1} V_t(x_t, u_t) + V_P(x_P) \right]$$
$$(j = 0, 1, \ldots, P-1).$$

Applying the following auxiliary lemma (*P*ickl 1999) we obtain a method for a successive and algorithmic solution principle.

Lemma 2.1 *Let* $Y_1 \subset \mathbb{R}$ *and* $Y_2 \subset \mathbb{R}$. *The functions* $g_1 : Y_1 \to \mathbb{R}$ *and* $g_2 : Y_1 \times Y_2 \to \mathbb{R}$ *are assumed to be continuous, the sets* Y_1 *and* Y_2 *are compact. Then, this yields*

$$\max_{y_1 \in Y_1 y_2 \in Y_2} [g_1(y_1) + g_2(y_1, y_2)] = \max_{y_1 \in Y_1} [g_1(y_1) + \max_{y_2 \in Y_2} g_2(y_1, y_2)] .$$

This means

$$\begin{aligned} f_{P-j}(x_j) &= \max_{u_j \in U_j(x_j)} \left\{ V_j(x_j, u_j) + \max_{\substack{u_t \in U_t(x_t) \\ t=j+1,\ldots,P-1}} \left[\sum_{t=j+1}^{P-1} V_t(x_t, u_t) + V_P(x_P) \right] \right\} \\ &= \max_{u_j \in U_j(x_j)} \{ V_j(x_j, u_j) + \max_{\substack{u_t \in U_t(x_t) \\ t=j+1,\ldots,P-1}} Z^*_{j+1}(x_{j+1}, u^{j+1}) \} \\ &= \max_{u_j \in U_j(x_j)} \left\{ V_j(x_j, u_j) + f_{P-(j+1)}(x_{j+1}) \right\} \quad (j = 0, 1, \ldots, P-1). \end{aligned}$$

Replacing $x_{t+1} = T_t(x_t, u_t)$, we get the *Bellmann functional equation*:

$$f_0(x_P) = V_P(x_P)$$
$$f_{P-t}(x_t) = \max_{u_t \in U_t(x_t)} \left\{ V_t(x_t, u_t) + f_{P-(t+1)}(T_t(x_t, u_t)) \right\} .$$

It is obvious that the functional equations are necessary and sufficient conditions for an optimal decision parameter $\tilde{u}^t$. Each solution of the Bellmann equation is an optimal solution of the process and each optimal solution of the process is a solution of (4). Furthermore this representation contains an algorithmic solution principle which results from the following two theorems:

Theorem 2.1 *Let (4) be the Bellmann functional equation. Then, there exists an optimal strategy* $[\tilde{s}_j(x_j), \ldots, \tilde{s}_{N-1}(x_{N-1})]$ *of the process* PR_j *, which starts at step* $j+1$ *. The strategy depends only on the input parameter* $x_j \in X_j$ *.*

The proof is obvious. More important is the following theorem:

Theorem 2.2 *Let the Bellmann functional equation of (4) be given. Let*

$$\tilde{s}^j(x_j) = [\tilde{s}_j(x_j), \ldots, \tilde{s}_{N-1}(x_j)]\,(j = 0, 1, \ldots, N-1) \tag{4}$$

be an optimal decision strategy for the process PR_j *with* $x_j \in X_j$ *.*
The process

$$\tilde{s}^{j+1} = [\tilde{s}_{j+1}(x_j), \tilde{s}_j(x_j), \ldots, \tilde{s}_{N-1}(x_j)] \tag{5}$$

which results from (3) is an optimal decision policy for the process PR_{j+1}*.*

Proof 2.1 *The proof is done by induction. Let us assume that* $\tilde{u}_{N-1} = \tilde{s}_{N-1}(x_{N-1})$ *is an optimal strategy and for* $u_{N-1} = s_{N-1}(x_{N-1})$ *it is valid*

$$\begin{aligned} f_1(x_{N-1}) &= \max_{u_{N-1} \in U_{N-1}(x_{N-1})} [V_{N-1}(x_{N-1}, u_{N-1}) + V_N(T_{N-1}(x_{N-1}, u_{N-1}))] \\ &= V_{N-1}(x_{N-1}, \tilde{s}_{N-1}(x_{N-1})) + V_N(T_{N-1}(x_{N-1}, \tilde{s}_{N-1}(x_{N-1}))) \\ &\geq V_{N-1}(x_{N-1}, s_{N-1}(x_{N-1})) + \underbrace{V_N(T_{N-1}(x_{N-1}, s_{N-1}(x_{N-1})))}_{Z^*_{N,s^N}(x_N)} \\ &= Z^*_{N-1,s^{N-1}}(x_{N-1}) \end{aligned} \tag{6}$$

$f_1(x_{N-1})$ *is per definitionem the maximum over*

$$\max_{s_{N-1}(x_{N-1}) \in U_{N-1}(x_{N-1})} Z^*_{N-1,s^{N-1}}(x_{N-1}) = Z^*_{N-1,\tilde{s}^{N-1}}(x_{N-1}) \quad .$$

Then we get

$$Z^*_{N-1,\tilde{s}^{N-1}}(x_{N-1}) \geq Z^*_{N-1,s^{N-1}}(x_{N-1}) \tag{7}$$

Thereby $\tilde{s}_{N-1}(x_{N-1})$ *is an optimal strategy. This terminates the first induction step. Let us assume that for one* i *we have,* $N-1 > i > j$ *. These assumptions are valid for a certain* i*. In a next step we have to prove that the induction assumption is valid for* $i-1$ *:*

$$f_{N-i}(x_i) = Z^*_{i,\tilde{s}^i}(x_i) \geq Z^*_{i,s^i}(x_i) \quad \text{for} \quad x_i \in X_i$$

and every suitable strategy s^i *. Applying (4) we get*

$$Z^*_{i-1,s^{i-1}}(x_{i-1}) = V_{i-1}(x_{i-1}, s_{i-1}(x_{i-1})) + Z^*_{i,s^i}[T_{i-1}(x_{i-1}, s_{i-1}(x_{i-1}))] \tag{8}$$

In the following we consider $u_{i-1} = s_{i-1}(x_{i-1})$. *The variable* $x_{i-1} \in X_{i-1}$ *is fixed but variably chosen. Applying the induction hypothesis we get:*

$$V_{i-1}(x_{i-1}, u_{i-1}) + Z^*_{i,\tilde{s}^i}(T_{i-1}(x_{i-1}, u_{i-1})) \geq V_{i-1}(x_{i-1}, u_{i-1}) + Z^*_{i,s^i}(T_{i-1}(x_{i-1}, u_{i-1}))$$

This expression is valid for all $u_{i-1} \in U_{i-1}(x_{i-1})$ *and all feasible* s^i . *We search for a maximum on the left side,* $u_{i-1} \in U_{i-1}(x_{i-1})$. *An optimal* u_i *is indicated by* $\tilde{u}_i$. *Hereby, we get*

$$\underbrace{V_{i-1}(x_{i-1}, \tilde{u}_{i-1}) + Z^*_{i,\tilde{s}^i}(T_{i-1}(x_{i-1}, \tilde{u}_{i-1}))}_{Z^*_{i-1,\tilde{s}^{i-1}}(x_{i-1})} \geq \underbrace{V_{i-1}(x_{i-1}, u_{i-1}) + Z^*_{i,s^i}(T_{i-1}(x_{i-1}, u_{i-1}))}_{Z^*_{i-1,s^{i-1}}(x_{i-1})}$$

$$\Rightarrow \tag{9}$$

$$\begin{aligned} Z^*_{i-1,\tilde{s}^{i-1}}(x_{i-1}) = V_{i-1}(x_{i-1}, \tilde{u}_{i-1}) + f_{N-i}(T_{i-1}(x_{i-1}, \tilde{u}_{i-1})) &= \\ \max_{u_{i-1} \in U_{i-1}(x_{i-1})} [V_{i-1}(x_{i-1}, u_{i-1}) + f_{N-i}(T_{i-1}(x_{i-1}, u_{i-1}))] &= \\ f_{N-(i-1)}(x_{i-1}) = Z^*_{i-1,\tilde{s}^{i-1}}(x_{i-1}) &\geq Z^*_{i-1,s^{i-1}}(x_{i-1}) \end{aligned}$$

If we assume that x_{i-1} *is variable we can determine* $\tilde{u}_{i-1} = \tilde{s}_{i-1}(x_{i-1})$ *and* $u_{i-1} = s_{i-1}(x_{i-1})$ *in such a way that the last condition is valid for all (!)* $x_{i-1} \in X_i$.

This results expresses the fact that an optimal strategy of a sub-process $[\tilde{s}_j(x_j), \ldots, \tilde{s}_{N-1}(x_{N-1})]$ is only dependent on the value x_j . This leads to the following fact which can be seen as a version of the Bellmann Optimization principle:

Theorem 2.3 *We consider the Bellmann functional equations (4). If we regard a subprocess* P_j *, which begins on the stage* $j+1$ *, there exists an optimal strategy* $[\tilde{s}_j(x_j), \ldots, \tilde{s}_{N-1}(x_{N-1})]$ *which depends only of the stage* $x_j \in X_j$.

The whole process before the actual state $(j+1)$ has no effect on the optimal strategy. Applying this results we can prove:

Theorem 2.4 *Let (4) be given as the Bellman functional equations. Let* $\tilde{s}^j(x_j) = [\tilde{s}_j(x_j), \ldots, \tilde{s}_{N-1}(x_j)]\,(j = 0, 1, \ldots, N-1)$ *be an optimal strategy of the subprocess* P_j *mit* $x_j \in X_j$.
If we consider the subprocess $\tilde{s}^{j+1} = (\tilde{s}_{j+1}, \tilde{s}_j(x_j), \ldots, \tilde{s}_{N-1}(x_j))$ *constructed by (3) we get an optimal strategy for the subprocess* P_{j+1}.

Proof 2.2 *We proof this result again by an indirect construction. Let us therefore assume that* s^{*j+1} *is an optimal policy, which is different from* $\tilde{s}^{j+1}$ *in at least one* $k, k = j+1, \ldots, N-1$. *Then we get:*

$$\begin{array}{rcl} Z^*_{j+1,s^{*j+1}}(x_{j+1}) & > & Z^*_{j+1,\tilde{s}^{j+1}}(x_{j+1}) \\ & \Longrightarrow & \\ \underbrace{V_j(x_j,\tilde{s}_j(x_j)) + Z^*_{j+1,s^{*j+1}}(x_{j+1})}_{Z^*_{j,s^{*j}}(x_j)} & > & \underbrace{V_j(x_j,\tilde{s}_j(x_j)) + Z^*_{j+1,\tilde{s}^{j+1}}(x_{j+1})}_{Z^*_{j,\tilde{s}^j}(x_j)} \end{array}$$

This is a contradiction to the fact that $\tilde{s}^j(x_j)$ *was assumed to be an optimal strategy.*

3 Existence Theorem

In the following, we assume that our problem has at least one feasible solution. The state regions $X_i \in \mathbb{R}^\alpha, i = 0, \ldots, P$, are bounded and closed. The regions are described by the core of the Kyoto interval game: The decision regions $U_i(x_i) \subseteq \mathbb{R}^\beta$ with $x_i \subseteq X_i, i = 0, \ldots, P-1$, will be represented by the *Core* of the game. Furthermore we assume the core to be bounded. Applying the well known theorem of *Krein-Milman*, each general ϵ-core (*Driessen (1986)*) is the convex hull of the extremal points. For each state x_i the region $U_i(x_i)$ is a polyhedron and a compact set. Note that in the Kyoto Game we have $\epsilon = 0$.

Additionally, we assume that the functions V_i and the state transformations T_i are continuous and restricted to the following regions:

$$\{(x_i, u_i) \,|\, x_i \in X_i,\, u_i \in U_i(x_i)\}, i = 0, 1, \ldots, P-1$$

$$\{x_P \,|\, x_P \in X_P\,.\}$$

Now, we formulate the following main theorem:

Theorem 3.1 *Let* $U_i(x_i),\ i = 0, 1, \ldots, P-1$, *be continuous set functions. The assumptions mentioned above are valid for our process which is described by (2) and (3). The decision set is presented by the core of the cost-game. Then we state that our problem has one solution.*

Proof 3.1 *Let us take* $Z(v) = Z(x,u) = \sum_{i=0}^{N-1} V_i(x_i,u_i) + V_N(x_N)$ *and* $x_{i+1} = T_i(x_i,u_i)$. *The variable* x_0 *is fixed. As* V_i, V_N *and* T_i *for* $i = 1,\ldots,N$ *are continuous functions, the process* $Z(x,u)$ *is a continuous mapping. The assumptions states that the control sets and the sets of the feasible states are compact. Applying the well-known theorem of Weierstrass we get an existence result.*

4 Conclusion

We present an algorithmic solution for a time-discrete investment model occurring in emission trading markets. The properties of the economic background lead to the described mathematical model of a time-discrete dynamical process. Applying dynamic programming techniques we can prove an existence theorem. It states that a suitable solution is represented by the core.

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Preferences On Intervals: a general framework

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Abstract

The paper presents a general framework for interval comparison for preference modelling purposes. Two dimensions are considered in order to establish such a framework: the type of preference structure to be considered and the number of values associated to each interval. It turns out that is possible to characterise well known preference structures as special cases of this general framework.

Key words : Intervals, Preferences, Orders

1 Introduction

Preferences are usually considered as binary relations applied on a set of objects, let's say A. Preference modelling is concerned by two basic problems (see [31]).

The first can be summarised as follows. Consider a decision maker replying to a set of preference queries concerning a the elements of the set A: "do you prefer a to b?", "do you prefer b to c?" etc.. Given such replies the problem is to check whether exists (and under which conditions) one or more real valued functions which, when applied to A, will return (faithfully) the preference statements of the decision maker. As an example consider a decision maker claiming that, given three candidates a, b and c, he is indifferent between a and b as well as between b and c, although he clearly prefers a to c. There are several different numerical representations which could account for such preferences. For instance we could associate to a the interval $[5, 10]$, to b the interval $[3, 6]$ and to c the interval $[1, 4]$. Under the rule that x is preferred to y iff the interval associated to x is

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completely to the right (in the sense of the reals) of the one associated to y and indifferent otherwise, the above numerical representation faithfully represents the decision makers preference statements.

The second problem goes the opposite way. We have a numerical representation for all elements of the set A and we would like to construct preference relations for a given decision maker. As an example consider three objects a, b and c whose cost is 10, 12 and 20 respectively. For a certain decision maker we could establish that a is better than b which is better than c. For another decision maker the model could be that both a and b are better than c, but they are indifferent among them since the difference is too small. In both cases the adoption of a preference model implies the acceptation of a number of properties the decision maker should be aware of.

In this paper we focus our attention on both cases, but with particular attention to the situations where the elements of the set A can or are actually represented by intervals (of the reals). In other terms we are interested on the one hand to the necessary and sufficient conditions for which the preference statements of a decision maker can be represented through the comparison of intervals and on the other hand on general models through which the comparison of intervals can lead to the establishment of preference relations.

The paper's subject is not that new. Since the seminal work of Luce ([16]) there have been several contributions in literature including the classics [10], [26] and [22], as well as some key papers: [1], [6], [9], [11], [12]. Our main contribution in this paper is to suggest a general framework enabling to clarify the different preference models that can be associated to the comparison of intervals including situations of crisp or continuous hesitation of the decision maker.

The paper is organised as follows. In Section 2 we introduce all basic notation and all hypotheses that hold in the paper. In section 3 we introduce the structure of the general framework we suggest, based on two dimensions: the type of preference structure to be used and the structure of the intervals. In section 4 we introduce some further conditions enabling to characterise well known preference structures in the literature. We conclude showing the future research directions of this work.

2 Notation and Hypotheses

In the following we consider a countable set of objects which we denote with A. Variables ranging within A will be denoted with $x, y, z, w \cdots$, while specific objects will be denoted $a, b, c \cdots$. Letters $P, Q, I, R, L \cdots$, possibly subscribed, will denote preference relations on A, that is binary predicates on the universe of discourse $A \times A$ (each binary relation being a subset of $A \times A$). Letters $f, g, h, r, l \cdots$, possibly subscribed, will denote real valued functions mapping A to the reals. Since we work with intervals we will reserve

the letters r and l for the functions representing, respectively, the right and left extreme of each interval. Letters $\alpha, \beta, \gamma \cdots$ will represent constants. The usual logical notation applies including its equivalent set notation. Therefore we will have:
- $P \cap R$ equivalent to $\forall x, y \;\; P(x, y) \wedge R(x, y)$;
- $P \subseteq R$ equivalent to $\forall x, y \;\; P(x, y) \rightarrow R(x, y)$.
We will add the following definitions:
- $P.R$ equivalent to $\forall x, y \exists z \;\; P(x, z) \wedge R(z, y)$;
- $I_o = \{(x, x) \in A \times A\}$, the set of all identities in $A \times A$.

As far as the properties of binary relations are concerned we will adopt the ones introduced in [23]. For specific types of preference structures such as total orders, weak orders etc. we will equally adopt the definitions within [23].

We introduce the following definition:

Definition 2.1 *A preference structure is a collection of binary relations $P_j \; j = 1, \cdots n$, partitioning the universe of discourse $A \times A$:*
- $\forall x, y, j \; P_j(x, y) \rightarrow \neg P_{i \neq j}(x, y)$;
- $\forall x, y \exists j \; P_j(x, y) \vee P_j(y, x)$

Further on we will often use the following proposition:

Proposition 2.1 *Any symmetric binary relation can be seen as the union of two asymmetric relations, the one being the inverse of the other, and I_o.*

Proof. Obvious.

We finally make the following hypotheses:

- H1 We consider only intervals of the reals. Therefore there will be no incomparability in the preference structures considered.
- H2 If necessary we associate to each interval a flat uncertainty distribution. Each point in an interval may equally be the “real value”.
- H3 Without loss of generality we can consider only asymmetric relations.
- H4 We consider only discrete sets. Therefore we can consider only strict inequalities.

Remark 2.1 *Hypothesis 3 is based on proposition 2.1. The reason for eliminating symmetric relations from our models will become clear later on in the paper. However, we can anticipate that the use of asymmetric relations allows to better understand the underlying structure of intervals comparison.*

Remark 2.2 *Hypothesis 4 makes sense only when the purpose is to establish a representation theorem for a certain type of preference statements. The basis idea is that, since numerical representations of preferences are not unique, A being countable, is always possible to choose a numerical representation for which it never occurs that any of the extreme values of the intervals associated to two elements of A are the same. However, in the case the numerical representation is given and the issue is to establish the preference structure holding, the possibility that two extreme values coincide cannot be excluded.*

3 General Framework

In order to analyse the different models used in the literature in order to compare intervals for preference modelling purposes we are going to consider two separate dimensions.

1. *The type of preference structure*. We basically consider the following cases.

   - Use of two asymmetric preference relations P_1 and P_2. Such a preference structure is equivalent to the classic preference structure (in absence of incomparability) considering only strict preference (P_2 in our notation) and indifference ($P_1 \cup P_1^{-1} \cup I_o$ in our notation). For more details see [23].
   - Use of three asymmetric preference relations P_1, P_2 and P_3. Such structures are known under the name of PQI preference structures (see [30]), allowing for a strict preference (P_3 in our notation), a "weak preference" (P_2 in our notation), representing an hesitation between strict preference and indifference and an indifference ($P_1 \cup P_1^{-1} \cup I_o$ in our notation).
   - Use of n asymmetric relations $P_1, \cdots P_n$. Usually P_n to P_2 represent $n - 1$ preference relations of decreasing "strength", while $P_1 \cup P_1^{-1} \cup I_o$ is sometimes considered as indifference. For more details the reader can see [9].
   - Use of a continuous valuation of hesitation between strict preference and indifference. In this case we consider valued preference structures, that is preference relations are considered fuzzy subsets of $A \times A$. The reader cas see more in [20].

2. *The structure of the numerical representation of the interval*. We consider the following cases:

   - Use of two values. Such two values can be equivalently seen as the left and the right extreme of each interval associated to each element of A or as a value associated to each element of A and a threshold allowing to discriminate any two values.

- Use of three values. Again several different interpretations can be considered. For instance the three values can be seen as the two extremes of each interval plus an intermediate value aiming to represent a particular feature of the interval. They can be seen as a value associated to each element of A and two thresholds aiming to describe two different states of discrimination. They can also be seen as representing an extreme value of the interval, while the other extreme is represented by an interval.
- Use of four or more values. The reader will realise that we are extending the previous structures. The four values can be seen as the two extremes and two "special" intermediate values or as two imprecise extremes such that each of them is represented by an interval. The use of n values can be seen as a value associated to each element of A and $n-1$ thresholds representing different intensities of preference. Possibly we can extend such a structure to the whole length of any interval associated to each element of A such that we may obtain a continuous valuation of the preference intensity.

In table 1 we summarise the possible combinations of preference structures and interval structures.

	2 values	3 values	>3 values
2 asymmetric relations	Interval Orders and Semi Orders	Split Interval Orders and Semi Orders	Tolerance and Bi-tolerance orders
3 asymmetric relations	PQI Interval Orders and Semi Orders	Pseudo orders and double threshold orders	-
n asymmetric relations	-	-	Multiple Interval Orders and Semi Orders
valued relations	Valued Preferences Fuzzy Interval Orders and Semi Orders Continuous PQI Interval Orders		

Table 1: A general framework for interval comparison

The reader can see more details in the following references:
- Interval Orders and Semi Orders: [10], [11], [16], [22];
- Split Interval Orders and Semi Orders: [2], [13];
- Tolerance and Bi-tolerance orders: [3], [4], [5], [14], [15];
- PQI Interval Orders and Semi Orders: [17], [18], [28];
- Pseudo Orders and Double Threshold Orders: [24], [25], [27], [30];

- Multiple Interval Orders and Semi Orders: [6], [8], [9];
- Valued Preference Structures: [7], [19], [20], [21], [29].

4 Further Conditions

The general framework discussed in the previous section suggests that there exist several different ways to compare intervals in order to model preferences. Each of such preference models could correspond to different interpretations associated to the values representing each interval. For instance consider the case where only the two extreme values of each interval are available and only two asymmetric relations are used. We can establish:
- $P_2(x, y) \Leftrightarrow l(x) > r(y)$
- $P_1(x, y) \Leftrightarrow r(y) > l(x) > l(y)$
and we obtain a classic Interval Order preference structure
or we can establish:
- $P_2(x, y) \Leftrightarrow l(x) > l(y) \wedge r(x) > r(y)$
- $P_1(x, y) \Leftrightarrow r(x) > r(y) > l(y) > l(x)$
and we obtain a partial order of dimension 2 (P_2).

A first general question is the following:
- given a set A, if it is possible to associate to each element x of A n functions $f_i(x),\ i = 1, \cdots n$, such that $f_n(x) > \cdots > f_1(x)$, how many preference relations can be established?

In order to reply to this question we consider different conditions which may apply to the values of each interval and their differences. For notation purposes, given an interval to which n values are associated, we denote the i-th sub-interval of any element $x \in A$ (from value $f_i(x)$ to value $f_{i+1}(x)$) as x_i. When there is no risk of confusion x_i will also represent the "length" of the same sub-interval (the quantity $f_{i+1}(x) - f_i(x)$). We are now ready to consider the following cases:

1. No conditions. We consider that the functions describing the intervals are free to take any value.

2. Coherence conditions. We impose that $\forall i\ f_1(x) > f_1(y) \rightarrow f_i(x) > f_i(y)$. This is equivalent to claim that $\forall i\ x_i > y_i$.

3. Weak monotonicity conditions. We now impose that $\forall i, j, i \geq j\ x_i \geq y_j$. In other terms we demand that there are no sub-intervals of x included to any sub-interval of y. Such a condition implies coherence (but not vice-versa).

	free	coherent	weak monotone	monotone
2 values:	3	2	2	2
3 values:	10	5	4	3
4 values:	35	14	8	4
n values:	$\frac{(2n)!}{2(n!)^2}$	$\frac{1}{n+1}\binom{2n}{n}$	?(2^{n-1})?	n

Table 2: Number of possible relations comparing intervals

4. Monotonicity conditions. We now impose that $\forall i \ x_i \geq y_i \geq x_{i-1} \geq y_{i-1}$ (sub-intervals of x or y are never included and they do not decrease as the index i increases). Such a condition implies weak monotonicity (but not viceversa). The reader can easily check that a representation which satisfies such a condition is the one where all sub-intervals have the same constant length.

In table 2 we summarise the situation for all the above cases.

A second question concerns the existence of a general structure among the possible relations that the comparison of intervals allow. Consider for instance the ten possible relations allowed by the use of three values associated to each interval. Is there any relation among them?

In order to reply to this question we consider any preference relation as a vector of $2n$ elements. Indeed, since $P_j(x,y)$ compares two vectors (x and y) of n elements each ($\langle f_1(x), \cdots f_n(x)\rangle$ and $\langle f_1(y), \cdots f_n(y)\rangle$), there is a unique sequence of such $2n$ values which exactly describes each relation P_j. Consider the case of three values and the ten possible relations. These can be described as follows:
$P_1(x,y) : \langle f_1(y), f_1(x), f_2(x), f_3(x), f_2(y), f_3(y)\rangle$
$P_2(x,y) : \langle f_1(y), f_1(x), f_2(x), f_2(y), f_3(x), f_3(y)\rangle$
$P_3(x,y) : \langle f_1(y), f_1(x), f_2(y), f_2(x), f_3(x), f_3(y)\rangle$
$P_4(x,y) : \langle f_1(y), f_1(x), f_2(x), f_2(y), f_3(y), f_3(x)\rangle$
$P_5(x,y) : \langle f_1(y), f_1(x), f_2(y), f_2(x), f_3(y), f_3(x)\rangle$
$P_6(x,y) : \langle f_1(y), f_2(y), f_1(x), f_2(x), f_3(x), f_3(y)\rangle$
$P_7(x,y) : \langle f_1(y), f_1(x), f_2(y), f_3(y), f_2(x), f_3(x)\rangle$
$P_8(x,y) : \langle f_1(y), f_2(y), f_1(x), f_2(x), f_3(y), f_3(x)\rangle$
$P_9(x,y) : \langle f_1(y), f_2(y), f_1(x), f_3(y), f_2(x), f_3(x)\rangle$
$P_{10}(x,y) : \langle f_1(y), f_2(y), f_3(y), f_1(x), f_2(x), f_3(x)\rangle$

We now introduce the following definition.

Definition 4.1 *For any two relations $P_l, P_k, l, k \in I$ we write $P_l \rhd P_k$ and we read "re-*

lation P_l is stronger than relation P_k" iff relation P_k can be obtained from P_l by a single shift of values of x and y or it exists a sequence of P_i such that $P_l \rhd \cdots P_i \rhd \cdots P_k$.

The reader will easy verify the following proposition.

Proposition 4.1 *Relation $\rhd$ is a partial order defining a complete lattice on the set of possible preference relations.*

In figure 1 we show the lattice for the cases where $n = 2$ (3 relations) and $n = 3$ (10 relations).

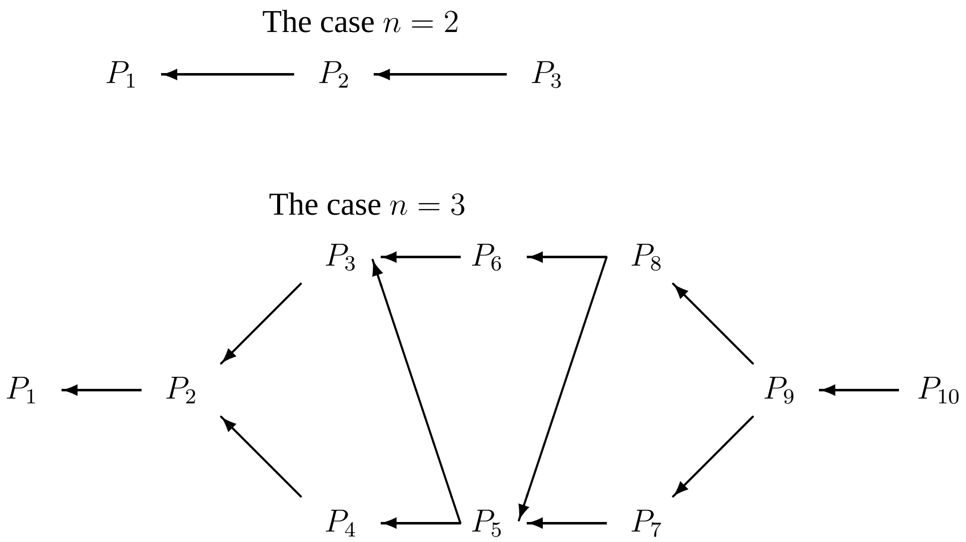

Figure 1: Partial Order among Preference Relations

How do well known in the literature preference structures fit the above presentation? The reader can easily check the following equivalences.

Interval orders:
$P = P_3$, $I = P_1 \cup P_2 \cup I_o \cup P_1^{-1} \cup P_2^{-1}$
Partial Orders of dimension. 2:
$P = P_3 \cup P_2$, $I = P_1 \cup I_o \cup P_1^{-1}$
Semi Orders:
$P = P_3$, $I = P_2 \cup I_o \cup P_2^{-1}$, P_1 empty
PQI Interval orders:
$P = P_3$, $Q = P_2$, $I = P_1 \cup I_o \cup P_1^{-1}$
PQI Semi orders:
$P = P_3$, $Q = P_2$, $I = I_o$, P_1 empty

Split Interval orders:
$P = P_{10} \cup P_9$, I the rest
Double Threshold orders:
$P = P_{10}\ Q = P_9 \cup P_8 \cup P_6$, I the rest
Pseudo Orders:
$P = P_{10}\ Q = P_9 \cup P_8$, $I = P_5 \cup P_7 \cup I_o \cup P_7^{-1} \cup P_5^{-1}$,
P_1, P_2, P_3, P_4, P_6 empty
Constant thresholds:
$P = P_{10}\ Q = P_9$, $I = P_5 \cup I_o \cup P_5^{-1}$,
$P_1, P_2, P_3, P_4, P_6, P_7, P_8$ empty

Remark 4.1 *The reader should note that in representing an Interval Order under the equivalence $P = P_3$ and $I = P_2 \cup P_1 \cup I_o \cup P_1^{-1} \cup P_2^{-1}$ we did an implicit hypothesis that I is separable in the relations P_2, P_1 and I_o. However, this is not always possible. The general representation of an Interval Order within our framework requires the existence of only two asymmetric relations P_2 and P_1 such that $P = P_2$ and $I = P_1 \cup I_o \cup P_1^{-1}$.*

How well known preference structures are characterised within our framework? We give here as an example the translation (within our frame) of two well known preference structures: interval orders and PQI interval orders.

Theorem 4.1 *An interval order is a $\langle P_2, P_1, I_o \rangle$ preference structure such that:*
- $P_2P_2 \subseteq P_2$
- $P_2P_1 \subseteq P_2$
- $P_1^{-1}P_2 \subseteq P_2$

Proof.
From $P_2P_2 \subseteq P_2$ we get $P_2I_oP_2 \subseteq P_2$
From $P_2P_1 \subseteq P_2$ we get $P_2P_1P_2 \subseteq P_2$
From $P_1^{-1}P_2 \subseteq P_2$ we get $P_2P_1{-}1P_2 \subseteq P_2$
Since $P_1 \cup I_o \cup P_1{-}1 = I$ and $P_2 = P$ we get $PIP \subseteq P$
this condition characterising interval orders (see [10]).

■

Theorem 4.2 *An interval order is a $\langle P_3, P_2, P_1, I_o \rangle$ preference structure such that:*
- $P_3P_3 \subseteq P_3$
- $P_2P_3 \subseteq P_3$
- $P_3P_2 \subseteq P_3$
- $P_3P_1 \subseteq P_3$
- $P_1^{-1}P_3 \subseteq P_3$

- $P_2P_2 \subseteq P_2 \cup P_3$
- $P_1P_2 \subseteq P_1 \cup P_2$
- $P_2P_1^{-1} \subseteq P^{-1} \cup P_2$

Proof.
From $P_3P_3 \subseteq P_3$, $P_3P_2 \subseteq P_3$, $P_3P_1 \subseteq P_3$ we get $P_3(P_3 \cup P_2 \cup P_1) \subseteq P_3$
From $P_3P_3 \subseteq P_3$, $P_2P_3 \subseteq P_3$, $P_1^{-1}P_3 \subseteq P_3$ we get $(P_3 \cup P_2 \cup P_1^{-1})P_3 \subseteq P_3$
From $P_2P_3 \subseteq P_3$, $P_2P_2 \subseteq P_2 \cup P_3$ $P_2P_1^{-1} \subseteq P^{-1} \cup P_2$ we get $P_2(P_3 \cup P_2 \cup P_1^{-1}) \subseteq P_3 \cup P_2 \cup P_1^{-1}$
From $P_3P_2 \subseteq P_3$, $P_2P_2 \subseteq P_2 \cup P_3$, $P_1P_2 \subseteq P_1 \cup P_2$ we get $(P_3 \cup P_2 \cup P_1)P_2 \subseteq P_3 \cup P_2 \cup P_1$ the above four conditions characterising a PQI interval order (see [28].

■

5 Conclusions

In this paper we introduce a general framework for the comparison of intervals under preference modelling purposes. Two possible extensions of such a framework can be envisaged. The first concerns the comparison of intervals for other purposes such as comparing time intervals. The second concerns the possibility to derive a general structure for representation theorems concerning any preference structure which can be conceived within the above framework.

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