Entropy production in nonequilibrium steady states
A different approach and an exactly solvable canonical model

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D. ben-Avraham, S. Dorosz, and M. Pleimling, Phys. Rev. E 84, 011115 (2011).

Entropy production in nonequilibrium steady states is discussed by focusing on paths obtained by sampling at regular (small) intervals τ, instead of sampling on each change of the system's state.

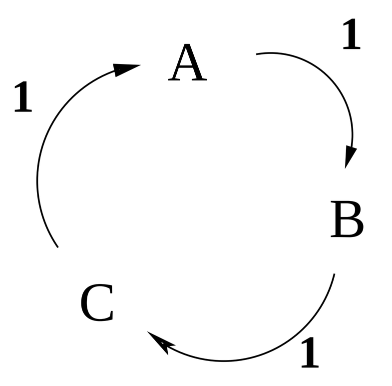
$$P(\mathbf{X}) = \rho(X_0)\cdot\omega(X_0|X_1;\tau)\cdot\omega(X_1|X_2;\tau)...\cdot\omega(X_{M-1}|X_M;\tau)$$

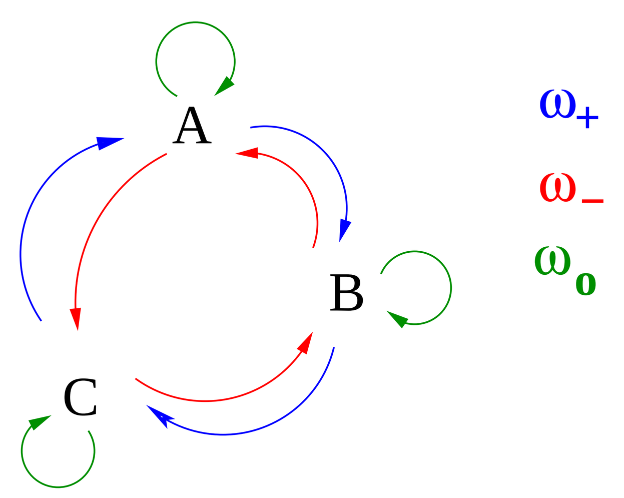
The total measurement time is $T = M\cdot\tau$
The fluctuation relation holds also for the novel paths [1].
Interval sampling allows direct analysis of systems with microscopic irreversibility,
and is more easily implemented in experiments.
We were able to identify universal features of entropy production.
The kink at zero entropy production arises from the irreversibility alone.

3 State Model

stationary probabilities $\rho(A) = \rho(B) = \rho(C)$

elementary transitions **rates as a function of time**





$$\omega_o \equiv \omega(A,A;\tau) = \omega(B,B;\tau) = \omega(C,C;\tau),$$
$$\omega_+ \equiv \omega(A,B;\tau) = \omega(B,C;\tau) = \omega(C,A;\tau),$$
$$\omega_- \equiv \omega(A,C;\tau) = \omega(B,A;\tau) = \omega(C,B;\tau).$$

$$\omega_o = \frac{1}{3} + \frac{2}{3}e^{-3\tau/2}\cos\left(\frac{\sqrt{3}}{2}\tau\right) \qquad \omega_o = 1 - \tau + \tau^2/2 + o(\tau^3) \tag{1}$$

$$\omega_+ = \frac{1}{3} + \frac{1}{3}e^{-3\tau/2}\left[-\cos\left(\frac{\sqrt{3}}{2}\tau\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\tau\right)\right] \qquad \omega_+ = \tau - \tau^2 + o(\tau^3) \tag{2}$$

$$\omega_- = \frac{1}{3} + \frac{1}{3}e^{-3\tau/2}\left[-\cos\left(\frac{\sqrt{3}}{2}\tau\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\tau\right)\right] \qquad \omega_- = \tau^2/2 - \tau^3/2 + +o(\tau^4) \tag{3}$$

The ratio $\omega_-/\omega_+ \approx \tau/2$, for the "forbidden" reverse direction, vanishes as $\tau \to 0$.

Probability Distribution of Entropy Production

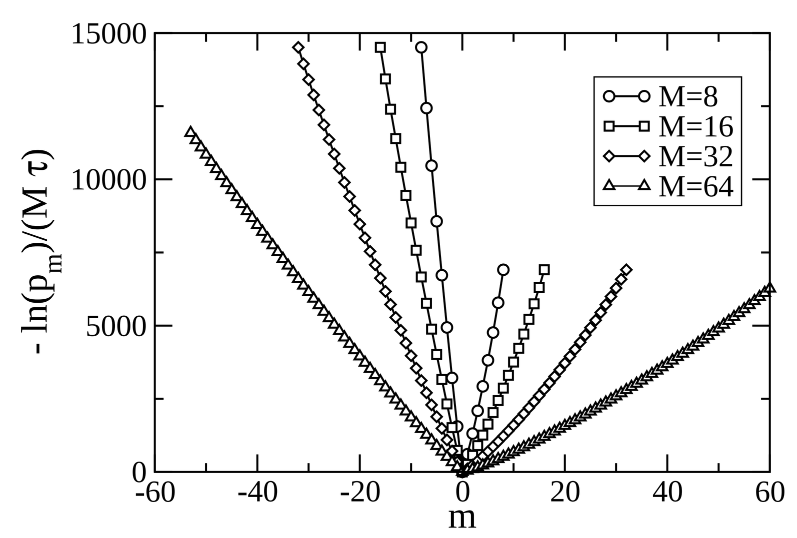
The total entropy production, in the steady state, is given by [2]

$$s_{tot} = \ln\frac{\rho(X_0)}{\rho(X_M)} + \ln\prod_i \frac{\omega(X_{i-1},X_i;\tau)}{\omega(X_i,X_{i-1};\tau)} := \Delta s + s = s\,, \tag{4}$$

$\omega(X,Y;\tau)$ denotes the conditional probability for finding the system in state Y after time τ, having started at state X.
The generating function of the stochastic process is

$$\langle\exp(-\mu s)\rangle = \left(\omega_0 + \omega_+^{1-\mu}\omega_-^{\mu} + \omega_+^{\mu}\omega_-^{1-\mu}\right)^M. \tag{5}$$

M is the number of measurements at constant time intervals τ.



m is the net number of forward transitions sampled over M intervals of length $\tau = 0.001$.

Long time behaviour

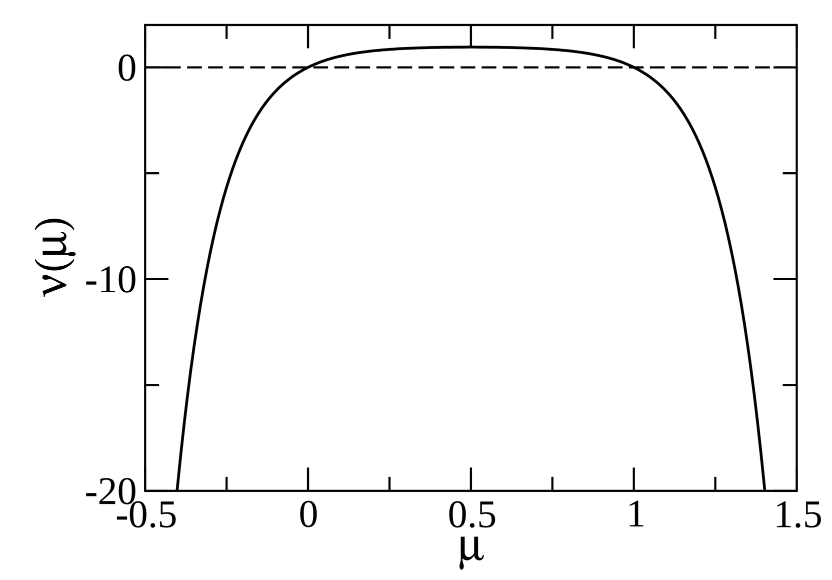
The lowest EV of the time evolution operator:

$$\nu(\mu) = -\tau^{-1}\ln\left(\omega_0 + \omega_+^{1-\mu}\omega_-^{\mu} + \omega_+^{\mu}\omega_-^{1-\mu}\right), \tag{6}$$

$$\tau << 1 \qquad \nu(\mu) \approx 1 - (\tau/2)^{\mu} - (\tau/2)^{1-\mu}. \tag{7}$$

Both expressions satisfy the fluctuation relation [3],

$$\nu(\mu) = \nu(1-\mu). \tag{8}$$



The eigenvalue $\nu(\mu)$ is plotted for $\tau = 0.001$.

Rate Function for Entropy

The mean rate of entropy change is

$$\langle\dot{s}\rangle = d\nu/d\mu|_{\mu=0} = \tau^{-1}(\omega_+ - \omega_-)\ln(\omega_+/\omega_-) \approx \ln(2/\tau). \tag{9}$$

The rate function, $\chi(\sigma)$: $\qquad \sigma = s/\langle\dot{s}_{tot}\rangle T$ is the *scaled* entropy

$$\chi(\sigma) = \max_{\mu}\{\nu(\mu) - \langle\dot{s}\rangle\sigma\mu\}. \tag{10}$$

$$\Rightarrow \qquad \chi(\sigma) = 1 - \sqrt{\sigma^2 + 2\tau} + \sigma\ln\left(\frac{\sigma + \sqrt{\sigma^2 + 2\tau}}{2}\right). \tag{11}$$

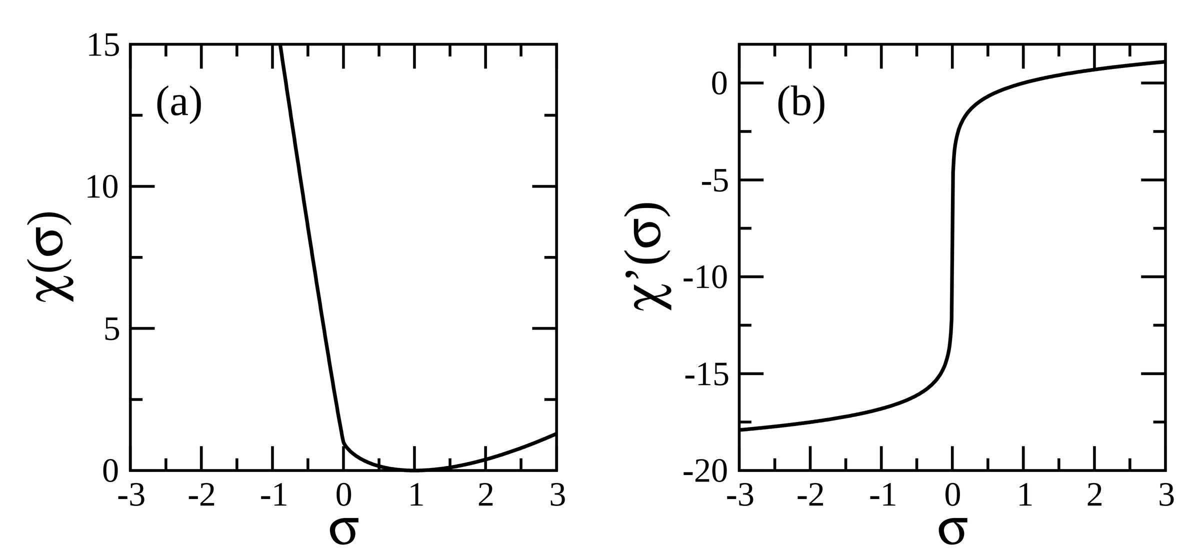
The limiting form of $\chi(\sigma)$ is universal:

$$\tau \to 0 \qquad \sigma > 0 \quad \chi(\sigma) \longrightarrow 1 - \sigma + \sigma\ln\sigma, \tag{12}$$
$$\tau \to 0 \qquad \sigma < 0 \quad \chi(\sigma) \to \infty. \tag{13}$$

Origin of the kink [4]

$$\chi'(\sigma) = \ln\left(\frac{\sigma + \sqrt{\sigma^2 + 2\tau}}{2}\right),$$
$$\chi''(\sigma) = \frac{1}{\sqrt{\sigma^2 + 2\tau}} \longrightarrow^{\tau\to 0} |\sigma|^{-1}, \tag{14}$$

. For finite τ : $\chi'(1) - \chi'(-1) \to \ln(2/\tau)$.



$\tau = 10^{-7}$. The kink is more pronounced the smaller the value of τ.

References

[1] D. ben-Avraham, S. Dorosz, and M. Pleimling, Phys. Rev. E. **83**, 041129 (2011).

[2] U. Seifert, Eur. Phys. J. B **64**, 423 (2008).

[3] J. L. Lebowitz and H. Spohn, J. Stat. Phys. **95**, 333 (1999).

[4] J. Mehl, T. Speck, and U. Seifert, Phys. Rev. E **78**, 011123 (2008).