MINIMIZING IMMERSIONS OF A HYPERBOLIC SURFACE IN A HYPERBOLIC 3-MANIFOLD

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Abstract. Let (S, h) be a closed hyperbolic surface and M be a quasi-Fuchsian 3-manifold. We consider incompressible maps from S to M that are critical points of an energy functional F which is homogeneous of degree 1. These "minimizing" maps are solutions of a non-linear elliptic equation, and reminiscent of harmonic maps – but when the target is Fuchsian, minimizing maps are minimal Lagrangian diffeomorphisms to the totally geodesic surface in M. We prove the uniqueness of smooth minimizing maps from (S, h) to M in a given homotopy class. When (S, h) is fixed, smooth minimizing maps from (S, h) are described by a simple holomorphic data on S: a complex self-adjoint Codazzi tensor of determinant 1. The space of admissible data is smooth and naturally equipped with a complex structure, for which the monodromy map taking a data to the holonomy representation of the image is holomorphic. Minimizing maps are in this way reminiscent of shear-bend coordinates, with the complexification of F analoguous to the complex length.

Contents

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1. Introduction and results

The aim of this paper is to study "minimizing immersions" of a compact hyperbolic surface inside germs of hyperbolic 3-manifolds, which are defined as critical points of a suitable 1-homogeneous functional F.

On one hand, such immersions generalize the notion of minimal Lagrangian maps between hyperbolic surfaces (which correspond to the target being a Fuchsian hyperbolic 3-manifold). On the other hand, as the target varies in the quasi-Fuchsian space, a very natural complexification of F can be used to define a holomorphic function that looks like a "smooth version" of the complex length of a measured lamination.

Throughout the paper we will always use the word *smooth* to mean C^∞.

1.1. Minimal Lagrangian diffeomorphisms between hyperbolic surfaces.

Consider a compact, connected, oriented surface S of genus at least two.

Given two hyperbolic metrics on S, a central problem in Teichmüller theory is to find the "best" diffeomorphism of S isotopic to the identity. Usually such a diffeomorphism is the unique map that minimizes a suitable functional defined in terms of the two hyperbolic metrics.

One remarkable example is that of harmonic maps, which play an important role in Teichmüller theory (see [7] and [24]).

Another example is given by minimal Lagrangian maps, which are quite relevant both for Teichmüller theory and for 3-dimensional manifolds of constant curvature, see e.g. [15, 14, 3, 4].

Definition 1.1 (Minimal Lagrangian maps)**.** A *minimal Lagrangian* map $m : (S,h) \to (S,h^\star)$ between hyperbolic surfaces is an area-preserving diffeomorphism such that its graph in $(S \times S, h \oplus h^\star)$ is minimal.

Here we present another variational characterization of minimal Lagrangian maps between hyperbolic surfaces.
Let $f : (S,h) \to (S,h^\star)$ be a smooth map between hyperbolic surfaces. There is a unique non-negative h-self-adjoint operator $b : TS \to TS$ such that $f^*h^\star(\bullet,\bullet) = h(b\bullet, b\bullet)$. We define the functional $F : C^\infty(S,S) \to \mathbb{R}$ on the space of C^∞ maps from S to S as

$$F(f) := \int_S \mathrm{tr}(b)\, \omega_h$$

where ω_h is the area form on S associated to h.
The following statement is almost implicit in some variational formulas in [3, 4], and also similar to results of Trapani and Valli [23].

Lemma 1.2 (Variational characterization of minimal Lagrangian maps)**.** *Let $f : (S,h) \to (S,h^*)$ be a smooth diffeomorphism between hyperbolic surfaces. Then f is minimal Lagrangian if and only if it is a critical point of F.*

One of the key motivations here is to extend the notion of minimal Lagrangian diffeomorphism to smooth maps from a hyperbolic surface to a hyperbolic 3-manifold.

1.2. **Minimizing immersions of surfaces in 3-manifolds.** Suppose now that the target surface is replaced by a hyperbolic 3-manifold M, which we assume to be complete and with injectivity radius positively bounded from below.
In a given homotopy class $[f]$ of embeddings of S into M that induce an injective homomorphism $f_* : \pi_1(S) \to \pi_1(M)$, there is still a unique harmonic map from (S,h) to M, but its relation to the moduli space of hyperbolic structures on M is not direct — for instance, the complex structure on this moduli is more readily visible if one fixes on S a *measured lamination* rather than a metric, and considers shear-bend coordinates associated to it, see [2].
The analog of minimal Lagrangian maps for embeddings of a hyperbolic surface in a hyperbolic 3-manifold is not clear, if one follows Definition 1.1. On the other hand, it is possible to adapt the variational approach suggested by Lemma 1.2.

Let (S,h) be a hyperbolic surface and (M,h_M) be a hyperbolic 3-manifold and let $f : S \to M$ be a smooth map. Again, there exists a unique non-negative h-self-adjoint operator $b : TS \to TS$ such that $f^*h_M(\bullet,\bullet) = h(b\bullet, b\bullet)$. We define the functional $F : C^\infty(S,M) \to \mathbb{R}$ as

$$F(f) := \int_S \mathrm{tr}(b)\, \omega_h\ .$$

Lemma 1.2 then suggests the following definition.

Definition 1.3 (Minimizing maps)**.** A smooth map $f : S \to M$ from a hyperbolic surface to a hyperbolic 3-manifold is *minimizing* if F achieves a local minimum at f.

We will see that minimizing immersions have a number of pleasant properties. Given (S,h), M and the homotopy class $[f]$ of a map $f:S\to M$, there is at most one minimizing immersion from (S,h) to M in $[f]$. Moreover, the moduli space of minimizing immersions of (S,h) in hyperbolic 3-manifolds has a complex structure, for which the map sending a minimizing immersion to the holonomy representation of the target manifold is holomorphic.

1.3. **Definition and notations.** We now fix the background hyperbolic metric h on S and let $\tilde{h}$ be the pull-back of h to the universal cover $\widetilde{S}\to S$.
Rather than considering immersions of S into a hyperbolic 3-dimensional manifold M, it is often more convenient to consider equivariant immersions of $\widetilde{S}$ into $\mathbb{H}^3$, so that deformations of M correspond to deformations of the representation.

Definition 1.4 (Equivariant immersions)**.** An *immersion of S in a germ of a hyperbolic 3-manifold* is a couple $(\tilde{f},\rho)$, where $\rho:\pi_1(S)\to \mathrm{PSL}_2(\mathbb{C})$ is a representation and $\tilde{f}:\widetilde{S}\to\mathbb{H}^3$ is a ρ-equivariant smooth immersion. The representation ρ is called the *monodromy* of the map $\tilde{f}$.

The set $\widetilde{\mathcal{I}}$ of smooth immersions of S in a germ of hyperbolic 3-manifold is a subset of $\mathrm{Hom}(\pi_1(S),\mathrm{PSL}_2(\mathbb{C}))\times C^\infty(\widetilde{S},\mathbb{H}^3)$ and so it inherits a subspace topology.
Note that $\mathrm{PSL}_2(\mathbb{C})$ acts on $\widetilde{\mathcal{I}}$ as $g\cdot(\tilde{f},\rho):=(g\circ\tilde{f},g\rho g^{-1})$. We denote by $[\tilde{f},\rho]$ the orbit of $(\tilde{f},\rho)$ under this action and by $\mathcal{I}$ the quotient $\mathrm{PSL}_2(\mathbb{C})\backslash\widetilde{\mathcal{I}}$, which is thus endowed with the quotient topology.
We also say that the family of equivalence classes $[\tilde{f}_t,\rho_t]_{t\in I}$ is smooth if it can be represented by a smooth family of immersions.

Remark 1.5 (Equivariant immersions and classes of immersions). In order to explain the above definition, consider two immersions $f_1:S\to M_1$ and $f_2:S\to M_2$ inside two (not necessarily compact) hyperbolic 3-manifolds M_1 and M_2. We declare the two immersions equivalent if there exists a third immersion $f_3:S\to M_3$ into a hyperbolic 3-manifold M_3 and local isometries $i_1:M_3\to M_1$ and $i_2:M_3\to M_2$ such that $f_1=f_3\circ i_1$ and $f_2=f_3\circ i_2$. Lifting such an immersion $f:S\to M$ to the universal covering $\widetilde{M}$ and composing with a developing map $\mathrm{dev}:\widetilde{M}\to\mathbb{H}^3$, one gets an immersion $\tilde{f}:\widetilde{S}\to\mathbb{H}^3$, which is equivariant under $\pi_1(S)$ that acts on $\widetilde{S}$ by deck transformations and on $\mathbb{H}^3$ via the representation $\rho:\pi_1(S)\to\mathrm{PSL}_2(\mathbb{C})$ obtained by composing the holonomy of M with the homomorphism $f_*:\pi_1(S)\to\pi_1(M)$. It is easy to see that equivalent immersions give rise to couples $(\tilde{f},\rho)$ in the same $\mathrm{PSL}_2(\mathbb{C})$-orbit.
Vice versa, given a couple $(\tilde{f},\rho)$, there exists $\epsilon>0$ such that the ρ-equivariant immersion $\tilde{f}:\widetilde{S}\times\{0\}\cong\widetilde{S}\to\mathbb{H}^3$ can be extended to an immersion $\hat{f}:\widetilde{U}=\widetilde{S}\times(-\epsilon,\epsilon)\to\mathbb{H}^3$ in such a way that the restriction to each segment $\{\tilde{p}\}\times(-\epsilon,\epsilon)$ is a geodesic of unit speed normal to $d\tilde{f}_{\tilde{p}}(T_{\tilde{p}}\widetilde{S})$ at $\tilde{f}(\tilde{p})$. Pulling back the metric of $\mathbb{H}^3$ via $\hat{f}$, we obtain a hyperbolic structure on $\widetilde{U}$. By ρ-equivariance, such $\hat{f}$ descends to an immersion of S inside a hyperbolic 3-manifold $U=S\times(-\epsilon,\epsilon)$.
Clearly, if $(\tilde{f}_1,\rho_1)$ and $(\tilde{f}_2,\rho_2)$ are in the same $\mathrm{PSL}_2(\mathbb{C})$-orbit, then they give rise to the same equivalence class of immersions.

Given the class $[\tilde{f}, \rho] \in \widetilde{\mathcal{I}}$ of an immersion of S in a germ of hyperbolic manifolds. We denote by $\tilde{a} : T\widetilde{S} \to T\widetilde{S}$ the shape operator of $\tilde{f}$, which is then self-adjoint with respect to the first fundamental form of the immersion. It is immediate to see by ρ-equivariance of $\tilde{f}$ that $\tilde{a}$ descends to an operator $a : TS \to TS$. Thus, to every $[\tilde{f}, \rho]$ we can associate a pair (b, a) of bundle morphisms $b, a : TS \to TS$, where b is the operator defined in Section 1.2.

Consider now the locus $\widetilde{\mathcal{MI}}$ the locus of minimizing immersions inside $\widetilde{\mathcal{I}}$ and let $\mathcal{MI}$ be its quotient by $\mathrm{PSL}_2(\mathbb{C})$.

Definition 1.6 (Immersion datum associated to a minimizing immersion)**.** For every $[\tilde{f}, \rho]$ in $\mathcal{MI}$ we define the 1-form on S with values on the bundle $T_{\mathbb{C}}S := \mathbb{C} \otimes_{\mathbb{R}} TS$

$$\Phi(\tilde{f}, \rho) := b - iJba \ ,$$

where J is the almost-complex structure on S associated to h.
Such $\Phi(\tilde{f}, \rho)$ is independent of the chosen representative in $[\tilde{f}, \rho]$.

Notice that $\Phi(\tilde{f}, \rho)$ can be uniquely extended to a complex-linear endomorphism of the bundle $T_{\mathbb{C}}S$. We will often consider such an extension and we denote it by the same symbol.
We will show in Section 3 that minimizing immersions of (S, h) into a (germ of a) hyperbolic manifold can be described in terms of their immersion data. The equations satisfied by such minimizing immersion data describe a complex space $\mathcal{D}$ defined below.

Definition 1.7 (Space of minimizing immersion data)**.** Fix a hyperbolic surface (S, h). The *space of minimizing immersion data* $\mathcal{D}$ is the space of smooth h-self-adjoint complex-linear operators $\phi : T_{\mathbb{C}}S \to T_{\mathbb{C}}S$ whose real part $\mathfrak{R}(\phi)$ is positive and that satisfy $d^{\nabla}\phi = 0$ and $\det \phi = 1$.

Here ∇ is the Levi-Civita connection of h, which can be extended as a connection on the complex bundle $T_{\mathbb{C}}S$ by $\mathbb{C}$-linearity. The operator d^{∇} is the exterior derivative on TS-valued 1-forms defined through ∇, so that $d^{\nabla}\phi$ is a 2-form on S with values on $T_{\mathbb{C}}S$: more explicitly, if v, w are two vector fields on S we have

$$(d^{\nabla}\phi)(v, w) = \nabla_v(\phi(w)) - \nabla_w(\phi(v)) - \phi([v, w]) \ .$$

Finally, h can be extended to a complex bilinear form on $T_{\mathbb{C}}S$, still denoted by h, so that ϕ is h-self-adjoint, i.e. $h(\phi(v), w) = h(v, \phi(w))$ for all vector fields v, w on S.

1.4. **Main results.** The first main result of this paper consists of a characterization of the immersion data corresponding to minimizing immersions.

Theorem A (Immersion data of minimizing immersions)**.** *Let (S, h) be a hyperbolic surface.*

(i) *For every class $[\tilde{f}, \rho] \in \mathcal{MI}$ of minimizing immersions, the immersion datum $\Phi([\tilde{f}, \rho])$ belongs to $\mathcal{D}$. Moreover, each $\phi \in \mathcal{D}$ is obtained from a unique minimizing class $[\tilde{f}, \rho] \in \mathcal{MI}$.*

(ii) *The map* $\Phi : \mathcal{MI} = \{[\tilde f, \rho]\} \to \mathcal{D}$ *that associates to a minimizing immersion* $[\tilde f, \rho]$ *its immersion datum is a homeomorphism.*

The correspondence established in Theorem A is in fact smooth in the following sense:

- if a family $(\tilde f_t, \rho_t)_{t\in I}$ in $\widetilde{\mathcal{MI}}$ depends C^∞ on t, then the corresponding embedding data $(\phi_t)_{t\in I}$ depends C^∞ on t too;
- if $(\phi_t)_{t\in I}$ is a C^∞ family of embedding data in $\mathcal{D}$ and $(\tilde f_0, \rho_0)$ corresponds to ϕ_0, then $(\tilde f_0, \rho_0)$ can be deformed to a C^∞ family $(\tilde f_t, \rho_t)_{t\in I}$ in $\widetilde{\mathcal{MI}}$ with embedding data $(\phi_t)_{t\in I}$.

In our second main result we show that the space of immersion data of minimizing maps has a natural structure of complex manifold.

Denote by $\mathcal{Q}$ the space of J-holomorphic quadratic differentials on S, viewed as a real vector space, and by $\mathcal{Q}_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} \mathcal{Q}$ the complexification of $\mathcal{Q}$.
We now consider more closely the space of immersion data on (S, h). To do this, we use the decomposition given in Proposition 3.9: for every smooth $\phi : T_{\mathbb{C}}S \to T_{\mathbb{C}}S$ which is self-adjoint and Codazzi, there exists a unique triple (u, q, q'), where $u : S \to \mathbb{C}$ is a smooth function and $q, q' \in \mathcal{Q}$ such that

$$\phi = (u\mathbb{1} - \mathrm{Hess}(u)) + (b_q + ib_{q'}) ,$$

where

- $\mathbb{1}$ is the identity operator;
- $\mathrm{Hess}(u) = \nabla(\mathrm{grad}\, u) : TS \to TS$ is the bundle morphism associated (through the Riemannian metric h) to the covariant Hessian of u;
- $b_q : TS \to TS$ is the bundle morphism associated (through h) to the bilinear form $\mathfrak{R}(q)$ on TS.

In other words, the complex vector space $\mathcal{Cod}$ of smooth d^∇-closed h-self-adjoint 1-forms with values in $T_{\mathbb{C}}S$ (endowed with the smooth topology) splits as

$$\mathcal{Cod} = C^\infty(S, \mathbb{C}) \oplus \mathcal{Q}_{\mathbb{C}}$$

We will denote by $Q : \mathcal{Cod} \to \mathcal{Q}_{\mathbb{C}}$ the projection to $\mathcal{Q}_{\mathbb{C}}$ induced by this splitting.

Theorem B (Manifold structure on the space of minimizing maps)**.** *Let* (S, h) *be a hyperbolic surface. The space* $\mathcal{D}$ *of immersion data is a complex submanifold of* $\mathcal{Cod}$ *of complex dimension* $6g - 6$*. Moreover, the restriction of* Q *over* $\mathcal{D}$ *is a local biholomorphism.*

In our third result we show that the monodromy map that sends a minimizing immersion datum ϕ to the conjugacy class $[\rho_\phi]$ of the monodromy of the corresponding germ of hyperbolic 3-manifold is a biholomorphism onto its open image.

Definition 1.8 (Space of non-elementary $\mathrm{SL}_2(\mathbb{C})$-representations)**.** The space $\mathcal{X}$ of non-elementary representation is the locus in $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ of conjugacy classes of representations without fixed points in $\overline{\mathbb{H}}^3$.

Theorem C (Monodromy map is holomorphic)**.** *Let (S,h) be a hyperbolic surface. For every $\phi \in \mathcal{D}$, the conjugacy class $[\rho_\phi]$ is non-elementary. Moreover, the map $\mathsf{Mon} : \mathcal{D} \to \mathcal{X}$ that sends ϕ to $[\rho_\phi]$ is a biholomorphism onto an open subset of $\mathcal{X}$ that contains the Fuchsian locus.*

In view of Theorem C, we define a functional $\mathsf{F} : \mathcal{X} \to \mathbb{R}_{\geq 0}$ on the representation space $\mathcal{X}$ as

$$\mathsf{F}([\rho]) := \inf\{F(\tilde{f}) \mid [\tilde{f}, \rho] \in \mathcal{I}\}.$$

In our fourth result we show that, in view of Theorem C, for every hyperbolic metric h on S there exists a suitable open subset of $\mathcal{X}$ that contains the Fuchsian locus and on which the functional F is the real part of a holomorphic function.

Theorem D (Complexification of the functional)**.** *Let (S,h) be a hyperbolic surface. For every $\phi \in \mathcal{D}$, we have $\mathsf{F}([\rho_\phi]) = \Re \int_S \operatorname{tr}(\phi)\omega_h$. As a consequence, the restriction of F to the open subset $\mathsf{Mon}(\mathcal{D})$ is the real part of the function $\mathsf{F}_{\mathbb{C}} : \mathsf{Mon}(\mathcal{D}) \to \mathbb{C}$ defined as*

$$\mathsf{F}_{\mathbb{C}}([\rho]) := \int_S \operatorname{tr}(\mathsf{Mon}^{-1}(\rho))\omega_h\,,$$

which is holomorphic.

The conjectural link between the holomorphic function $\mathsf{F}_{\mathbb{C}}$ associated to a hyperbolic metric h and the complex length associated to a measured lamination λ on S is discussed in Section 5.3.

1.5. **Structure of the paper.** In Section 2 we define the main objects of investigation, such as equivariant maps, immersion data and the 1-energy functional F. Then we discuss first-order deformations of equivariant maps, we prove convexity of F along geodesic displacements and from that we deduce uniqueness of smooth minimizing immersions.

In Section 3 we compute the first-order variation of F and we deduce Euler-Lagrange equations for minimizing immersions, thus proving Theorem A. Then we obtain Theorem B through an implicit function theorem argument.

In Section 4 we resume the deformation theory of equivariant immersions developed in Section 2.7, and we rephrase it in terms of the bundle of local Killing vector fields. Using such rephrasing, we prove Theorem C and its immediate consequence, namely Theorem D.

In Section 5 we list some open problems and perspectives that came up naturally when working at the present article.

Finally, we collect in Appendix A some facts on 1-Schatten norms of matrices and of families of matrices, that are used in Section 2.

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2. A 1-energy functional

In this section we introduce certain spaces of equivariant maps and immersions from the universal cover $\widetilde{S}$ of the surface S to $\mathbb{H}^3$ and we define the 1-energy of such maps.
A key space of our constructions is the space $C^\infty(\widetilde{S}, \mathbb{H}^3)$ of smooth maps $\widetilde{S} \to \mathbb{H}^3$. While introducing a structure of Fréchet manifold on $C^\infty(\widetilde{S}, \mathbb{H}^3)$ is possible, for the sake of simplicity we limit ourself to introduce the minimum technical background to develop first-order computations at a single point of $C^\infty(\widetilde{S}, \mathbb{H}^3)$. For this reason, we study the deformation theory of (equivariant) maps $\widetilde{S} \to \mathbb{H}^3$ and we define appropriate tangent spaces to such (usually infinite-dimensional) spaces of maps. Our interest in such tangent spaces is twofold: on one hand, they will allow us to define the differential of a map from or to such infinite-dimensional spaces; on the other hand, they will be the natural setting to linearise differential operators defined in a neighbourhood of an equivariant map. One example of the former application will be the monodromy map from the space $\widetilde{\mathcal{C}}$ of smooth equivariant maps $\widetilde{S} \to \mathbb{H}^3$ with non-elementary monodromy to the manifold $\widetilde{\mathcal{X}}$ of non-elementary representations $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$.
Toward the end of the section, we show that the 1-energy has a convexity property with respect to geodesic displacements. We conclude by showing that equivariant minimal immersions (whenever they exist) are the unique critical point for the 1-energy functional among maps with the same monodromy, and in fact they are a point of absolute minimum with non-elementary reductive monodromy. We also show that in the Fuchsian case minimizing maps are exactly minimal Lagrangian maps between hyperbolic surfaces.

2.1. **Setting.** Let S be a compact, connected, oriented surface with $\chi(S) < 0$. Fix a universal cover $\pi : \widetilde{S} \to S$ and an identification between $\pi_1(S)$ and the group $\mathrm{Aut}(\pi)$ of the deck transformations of π.

Notation. We will use the symbol $\gamma \in \mathrm{Aut}(\pi) \cong \pi_1(S)$ to denote an automorphism $\gamma : \widetilde{S} \to \widetilde{S}$ over the covering space π, and by γ_* the push-forward operator on vector fields or other tensors on $\widetilde{S}$ induced by the diffeomorphism γ.

Fix also a hyperbolic metric h on S and let $\tilde{h}$ be its pull-back on $\widetilde{S}$, so that $(\widetilde{S}, \tilde{h})$ is isometric to the hyperbolic plane $\mathbb{H}^2$ and $\pi_1(S)$ acts on $(\widetilde{S}, \tilde{h})$ via hyperbolic isometries.

2.2. **Maps and immersions.** Given a complete hyperbolic 3-manifold M, we can identify its universal cover $\widetilde{M}$ to $\mathbb{H}^3$ and the group of orientation-preserving isometries of $\widetilde{M}$ to $\mathrm{PSL}_2(\mathbb{C})$. Note that a continuous map $f : S \to M$

can be lifted to a $\pi_1(S)$-equivariant continuous map $\tilde{f}: \widetilde{S} \to \widetilde{M} = \mathbb{H}^3$, where $\pi_1(S)$ acts on $\widetilde{S}$ by deck-transformations and on $\mathbb{H}^3$ through a representation $\rho: \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$.
The advantage of the following definition is in its flexibility, since looking at equivariant maps "allows M to vary".

Definition 2.1 (Equivariant maps)**.** An *equivariant (non-elementary) map* from $\widetilde{S}$ to $\mathbb{H}^3$ is a couple $(\tilde{f}, \rho)$, where $\rho: \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ is a non-elementary representation and $\tilde{f}: \widetilde{S} \to \mathbb{H}^3$ is a $\pi_1(S)$-equivariant smooth map.

If $(\tilde{f}, \rho)$ is an equivariant map, the representation ρ is called the *monodromy* of $\tilde{f}$. Notice that ρ is determined by $\tilde{f}$ provided that the rank of $d\tilde{f}$ is 2 at least at one point $\tilde{p} \in \widetilde{S}$.
Since $\pi_1(S)$ is generated by $2g$ loops, the evaluation at those loops identifies $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C}))$ to a closed algebraic variety inside $\mathrm{PSL}_2(\mathbb{C})^{2g}$. We denote by $\widetilde{\mathcal{X}}$ the locus in $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C}))$ consisting of non-elementary representations.
As a consequence of the above discussion, the space $\widetilde{\mathcal{C}}$ of smooth non-elementary equivariant maps inherits a topology as a subset of $C^\infty(\widetilde{S}, \mathbb{H}^3) \times \widetilde{\mathcal{X}}$.
The group $\mathrm{PSL}_2(\mathbb{C})$ of orientation-preserving isometries of $\mathbb{H}^3$ acts on $\widetilde{\mathcal{C}}$ as $g \cdot (\tilde{f}, \rho) \coloneqq (g \circ \tilde{f}, g\rho g^{-1})$. Let $\mathcal{C} \coloneqq \mathrm{PSL}_2(\mathbb{C}) \backslash \widetilde{\mathcal{C}}$ and denote by $[\tilde{f}, \rho]$ the class of smooth non-elementary equivariant maps up to this $\mathrm{PSL}_2(\mathbb{C})$-action.

We recall from the introduction that $\mathcal{X}$ is the space of $\mathrm{PSL}_2(\mathbb{C})$-conjugacy classes of representations in $\widetilde{\mathcal{X}}$. The following fact is well-known, see [11].

Lemma 2.2 (Smoothness of the representation space)**.** *The space $\widetilde{\mathcal{X}}$ is a complex manifold of dimension $3(1-\chi(S))$, the quotient map $\widetilde{\mathcal{X}} \to \mathcal{X}$ is a principal $\mathrm{PSL}_2(\mathbb{C})$-fibration and $\mathcal{X}$ is a complex manifold of dimension $-3\chi(S)$.*

Lemma 2.2 allows us to define smooth families of equivariant maps.

Definition 2.3 (Paths of (equivariant) maps)**.** A *(germ of a) path of maps* is a smooth map $\tilde{\boldsymbol{f}}: \widetilde{S} \times (-\varepsilon, \varepsilon) \to \mathbb{H}^3$ for some $\varepsilon > 0$. A *(germ of a) path of equivariant maps* is a couple $(\tilde{\boldsymbol{f}}, \boldsymbol{\rho})$, where $\tilde{\boldsymbol{f}}: \widetilde{S} \times (-\varepsilon, \varepsilon) \to \mathbb{H}^3$ and $\boldsymbol{\rho}: (-\epsilon, \epsilon) \to \mathcal{X}$ are smooth maps for some $\epsilon > 0$ such that for any $t \in (-\epsilon, \epsilon)$ the restriction $\tilde{f}_t \coloneqq \tilde{\boldsymbol{f}}(\bullet, t)$ and the representation $\rho_t \coloneqq \boldsymbol{\rho}(t)$ form a smooth equivariant map $(\tilde{f}_t, \rho_t)$. Such a path $(\tilde{\boldsymbol{f}}, \boldsymbol{\rho})$ is a *deformation* of the equivariant map $(\tilde{f}, \rho)$ if $\tilde{f} = \tilde{f}_0$ and $\rho = \rho_0$; moreover, it is an *isomonodromic deformation* if $\rho_t = \rho$ for all $t \in (-\epsilon, \epsilon)$.

In this paper we will be mainly interested in equivariant immersions, that is, equivariant maps $(\tilde{f}, \rho)$ such that $d\tilde{f}$ has rank 2 at every point. As in the introduction, we will denote by $\widetilde{\mathcal{I}}$ the space of equivariant smooth immersions, which is an open subset of $\widetilde{\mathcal{C}}$ for the smooth topology. This in particular implies that if $(\tilde{\boldsymbol{f}}, \boldsymbol{\rho})$ is a path of equivariant maps and f_0 is an immersion, then f_t is an immersion for t sufficiently small.
Clearly $\widetilde{\mathcal{I}}$ is preserved by the action of $\mathrm{PSL}_2(\mathbb{C})$. Equivariant immersions that differ by post-composition with an isometry of $\mathbb{H}^3$ are geometrically equivalent. For this reason, we introduce the quotient space $\mathcal{I} = \mathrm{PSL}_2(\mathbb{C}) \backslash \widetilde{\mathcal{I}}$.

2.3. **Geometry of equivariant immersions.** Given an equivariant immersion $(\tilde{f}, \rho)$, the pullback $\tilde{I} = \tilde{f}^* h_{\mathbb{H}^3}$ on $\widetilde{S}$ of the hyperbolic metric $h_{\mathbb{H}^3}$ of $\mathbb{H}^3$ is a Riemannian metric on $\widetilde{S}$, which is invariant under the action of $\pi_1(S)$. So $\tilde{I}$ is the lift of a Riemannian metric I on S. This metric I is called the *first fundamental form* of the immersion.
On the other hand, associated to $\tilde{f}$ there is a normal field $\tilde{N}$ on $\widetilde{S}$, which is defined by the conditions that at every $\tilde{p} \in \widetilde{S}$

- $\tilde{N}(\tilde{p}) \in T_{\tilde{f}(\tilde{p})}\mathbb{H}^3$ is a unitary vector orthogonal to the image of $d\tilde{f}_{\tilde{p}}$.
- if $(e_1(\tilde{p}), e_2(\tilde{p}))$ is a positively oriented basis of $T_{\tilde{p}}\widetilde{S}$, then $\big(\tilde{N}(\tilde{p}), d\tilde{f}_{\tilde{p}}(e_1(\tilde{p})), d\tilde{f}_{\tilde{p}}(e_2(\tilde{p}))\big)$ is a positively oriented basis of $T_{\tilde{f}(\tilde{p})}\mathbb{H}^3$

Formally, $\tilde{N}$ is a section of the bundle $\Theta_{\tilde{f}} := \tilde{f}^* T\mathbb{H}^3$, which comes endowed with the pull-back $\nabla^{\mathbb{H}^3}$ of the Levi-Civita connection of $\mathbb{H}^3$. Thus $\nabla^{\mathbb{H}^3}\tilde{N}$ is a 1-form on $\widetilde{S}$ with values in $\Theta_{\tilde{f}}$. Standard arguments show that $\nabla^{\mathbb{H}^3}_{\tilde{w}}\tilde{N}$ is tangent to the immersion for every $\tilde{w} \in T\widetilde{S}$. Thus an endomorphism $\tilde{a} : T\widetilde{S} \to T\widetilde{S}$ is defined by requiring that

$$\nabla^{\mathbb{H}^3}_{\tilde{w}}\tilde{N} = d\tilde{f}(\tilde{a}(\tilde{w}))$$

for every $\tilde{w} \in T\widetilde{S}$. It is a classical fact that $\tilde{a}$ is $\tilde{I}$-self-adjoint. Moreover the equivariance of $\tilde{N}$ with respect to the action of $\pi_1(S)$ implies that $\tilde{a}$ is $\pi_1(S)$-invariant, so that it is the lift of an endomorphism $a : TS \to TS$ called the *shape operator* of the immersion.
The pair (I, a) are called the *immersion datum* of $\tilde{f}$. Since $\mathbb{H}^3$ has curvature -1, the couple (I, a) obeys a system of integrability conditions called the Gauss-Codazzi equations:

$$\begin{cases} K_I = \det(a) - 1\,, \\ d^{\nabla^I} a = 0\,, \end{cases} \tag{1}$$

where K_I is the intrinsic curvature of the metric I and d^{∇^I} is the exterior differential associated to the Levi-Civita connection of I. Namely, $d^{\nabla^I}a$ is the 2-form with values in TS defined by $(d^{\nabla^I}a)(v, w) = (\nabla^I_v a)(w) - (\nabla^I_w a)(v)$.

Remark 2.4. Once we have fixed a reference metric h over S, any other Riemannian metric h' over S can be described by an h-self-adjoint positive operator b by requiring that $h'(v, w) = h(bv, bw)$ for every $v, w \in TS$. Notice that b is the square root of the operator obtained by "rising" an index of h' with respect to the background metric h. More precisely, the Riemannian metric h induces a bundle isomorphism between the bundle of positive bilinear forms h' and the bundle of positive h-self-adjoint endomorphisms b such that $h' = h(b, b)$, so that a family of metrics (h'_t) is smooth if and only if the corresponding family of endomorphisms (b_t) is.

The following classical result states that the space $\mathcal{I}$ of equivariant immersions up to the action of $\mathrm{PSL}_2(\mathbb{C})$ is naturally identified to the space of solutions of (1) through the correspondence that sends $(\tilde{f}, \rho)$ to its immersion datum. It is a direct consequence of the Fundamental Theorem of Surface theory, see e.g. [8, Section 64] or [20, Chapter 1], but also [21, 13] for other applications to equivariant surfaces in $\mathbb{H}^3$.

Proposition 2.5 (Immersions and immersion data). *Two equivariant immersions correspond to the same immersion datum (I, a) if and only if they differ by post-composition by an element of $\mathrm{PSL}_2(\mathbb{C})$.*
Moreover, if (I, a) is a solution of the Gauss-Codazzi equations (1), where I is a Riemannian metric and a is an I-self-adjoint endomorphism of TS, then (I, a) is the immersion datum of some equivariant immersion.

The above correspondence between immersions and immersion data can be promoted to a correpondence between paths of immersions.

Proposition 2.6 (Paths of immersions and of immersion data). *If $(\tilde{\boldsymbol{f}}, \boldsymbol{\rho})$ is a smooth path of equivariant immersions, then the corresponding family of immersion data $(I_t, a_t)_{t\in(-\epsilon,\epsilon)}$ is smooth, i.e. for any couple of vector fields X, Y over S the functions $(t, p) \mapsto I_t(X(p), Y(p))$ and $(t, p) \to a_t(X(p), Y(p))$ defined on $(-\epsilon, \epsilon) \times S$ are smooth.*
Conversely, if $(I_t, a_t)_{t\in(-\epsilon,\epsilon)}$ is a smooth family of immersion data, there is a smooth path $(\tilde{\boldsymbol{f}}, \boldsymbol{\rho})$ of equivariant immersions such that (I_t, a_t) are the embedding data of f_t for every $t \in (-\epsilon, \epsilon)$.

Before getting into the proof of the above proposition, let us make some observations about *any* immersion $\tilde{f} : \widetilde{S} \to \mathbb{H}^3$.
Fix global coordinates $u = (u_1, u_2)$ on $\widetilde{S}$ and $x = (x_1, x_2, x_3)$ on $\mathbb{H}^3$, and let $\partial_i := \frac{\partial}{\partial u_i}$. Consider any immersion $\tilde{f} = (\tilde{f}^1, \tilde{f}^2, \tilde{f}^3)$ and choose a unit vector field $\tilde{N}_{\tilde{f}}$ orthogonal to $\tilde{f}(\widetilde{S})$, and let $\tilde{I}$ be its first fundamental form and $\tilde{a}$ its shape operator. Denote by $\nabla^{\tilde{I}}$ be the Levi-Civita connection on $\widetilde{S}$ associated to $\tilde{I}$ and by $\nabla^{\tilde{f}}$ the connection on $\tilde{f}^*T\mathbb{H}^3$ obtained by pulling back the Levi-Civita connection of $\mathbb{H}^3$ via $\tilde{f}$. The partial derivatives $\partial_i\tilde{f}$ are sections of $\tilde{f}^*T\mathbb{H}^3$ and the covariant derivatives $\nabla^{\tilde{f}}(\partial_i\tilde{f})$ are determined by $\tilde{I}$ and $\tilde{a}$ as follows

$$\nabla^{\tilde{f}}_{\partial_j}(\partial_i\tilde{f}) = d\tilde{f}(\nabla^{\tilde{I}}_{\partial_j}\partial_i) + \tilde{I}(\partial_j, \tilde{a}(\partial_i))\tilde{N}_{\tilde{f}}$$

Writing such identity in coordinates, we deduce that the map $\tilde{f} = (\tilde{f}^1, \tilde{f}^2, \tilde{f}^3)$ satisfies a system of equations of the form

$$\frac{\partial^2}{\partial u_i\,\partial u_j}\tilde{f}^\alpha = C^\alpha_{ij}\left(u_k, \tilde{f}^\beta, \frac{\partial \tilde{f}^\gamma}{\partial u_k}\right) \tag{2}$$

and C^α_{ij} are functions that smoothly depend on the coefficients of $\tilde{I}$ and $\tilde{a}$. Here we adopt the convention that Latin indices range in $\{1, 2\}$, whereas Greek indices take values in $\{1, 2, 3\}$. The fundamental theorem of Riemannian geometry asserts that the Gauss-Codazzi equations are equivalent to the integrability of the system.
Now, denote by $\mathcal{M}_{3,2}$ the vector space of 3×2 real matrices and consider the locus of $\widetilde{S} \times \mathbb{H}^3 \times \mathcal{M}_{3,2}$ defined by

$$\mathcal{J}_{\tilde{I}} := \left\{(u, x, \xi) \in \widetilde{S} \times \mathbb{H}^3 \times \mathcal{M}_{3,2} \;\middle|\; g_{\alpha,\beta}(x)\xi^\alpha_i\xi^\beta_j = \tilde{I}_{ij}(u)\right\},$$

where $g_{\alpha\beta}$ is the hyperbolic metric on $\mathbb{H}^3$.

First note that $\mathcal{J}_{\tilde{I}}$ is a submanifold: this is a consequence of the implicit function theorem, since $\mathcal{J}_{\tilde{I}}$ is the fiber over 0 of the map $\widetilde{S}\times\mathbb{H}^3\times\mathcal{M}_{3,2}\to\mathcal{M}_{2,2}^{sym}$ defined as $(u,x,\xi)\mapsto\xi^T g(x)\xi-\tilde{I}(u)$, which is easily seen to be a submersion as $\tilde{I}(u)$ is non-degenerate.

Note also that the graph of the 1-jet map $j^1\tilde{f}=(\tilde{f}^\alpha,\partial_i\tilde{f}^\alpha):\widetilde{S}\to\mathbb{H}^3\times\mathcal{M}_{3,2}$ associated to $\tilde{f}$ is contained in $\mathcal{J}_{\tilde{I}}$. Actually, once $\tilde{I},\tilde{a}$ are fixed, the manifold $\mathcal{J}_{\tilde{I}}$ is foliated by the graphs of the 1-jets of all the immersions with embedding data $(\tilde{I},\tilde{a})$: by Proposition 2.5 for every element (u,x,ξ) of $\mathcal{J}_{\tilde{I}}$ there is a unique immersion $\tilde{f}_{(u,x,\xi)}:\widetilde{S}\to\mathbb{H}^3$ such that $\tilde{f}_{(u,x,\xi)}(u)=x$ and the differential $df_{(u,x,\xi)}$ at u is ξ.

In fact, there is a distribution of planes $\mathcal{P}_{\tilde{I},\tilde{a}}$ on $\mathcal{J}_{\tilde{I}}$ so that the graphs of the 1-jets of the immersions with embedding data $(\tilde{I},\tilde{a})$ are the corresponding integral surfaces. Such a distribution $\mathcal{P}_{\tilde{I},\tilde{a}}$ is clearly generated by the two vector fields

$$G_i(u,x,\xi)=\partial_i+d(j^1\tilde{f}_{(u,x,\xi)})(\partial_i)\qquad\text{for } i=1,2,$$

and, using (2), we see that at every (u,x,ξ) we have

$$G_i(u,x,\xi)=\frac{\partial}{\partial u_i}+\sum_{\alpha=1}^{3}\xi_i^\alpha\frac{\partial}{\partial x_\alpha}+\sum_{j=1}^{2}\sum_{\alpha=1}^{3}C_{ij}^\alpha(u,x,\xi)\frac{\partial}{\partial\xi_j^\alpha}\qquad\text{for } i=1,2.$$

Proof of Proposition 2.6. The first part is straightforward. As for the second part, consider a smooth family of immersion data $(\tilde{I}_t,\tilde{a}_t)$ obtained by pulling (I_t,a_t) back via $\widetilde{S}\to S$. Let us set $\mathcal{J}_t:=\mathcal{J}_{\tilde{I}_t}$ and $\mathcal{P}_t:=\mathcal{P}_{\tilde{I}_t,\tilde{a}_t}$ the corresponding distribution of planes.

Notice that on $\widetilde{S}$ there is a field of positive-definite 2×2 symmetric matrices b_t, smoothly depending on t, such that

$$(\tilde{I}_t)_{ij}=(\tilde{I}_0)_{kh}(b_t)_i^k(b_t)_j^h.$$

In this way we see that the map $\Psi_t:\widetilde{S}\times\mathbb{H}^3\times\mathcal{M}_{3,2}\to\widetilde{S}\times\mathbb{H}^3\times\mathcal{M}_{3,2}$ defined by $\Psi_t(u,x,\xi):=(u,x,\xi\cdot b_t(u))$ restricts to a diffeomorphism of $\mathcal{J}_0\xrightarrow{\sim}\mathcal{J}_t$. From the discussion above it turns out that the family $(\Psi_t^*\mathcal{P}_t)$ of distributions on $\mathcal{J}_0$ is smooth in t.

Now, from the very definition of Ψ_t it follows that, if $\tilde{f}:\widetilde{S}\to\mathbb{H}^3$ is an immersion with embedding data $(\tilde{I}_t,\tilde{a}_t)$, then

$$\Psi_t^{-1}(\mathrm{graph}(j^1\tilde{f}))=\left\{\left(u,\tilde{f}(u),\tfrac{\partial\tilde{f}}{\partial u}(u)\cdot b(u)^{-1}\right)\,\Big|\,u\in\widetilde{S}\right\},$$

and so its projection on $\widetilde{S}\times\mathbb{H}^3$ is still the graph of $\tilde{f}$. Moreover, $\Psi_t^{-1}(\mathrm{graph}(j^1\tilde{f}))$ is an integral surface for $\Psi_t^*\mathcal{P}_t$ if and only if $\tilde{f}$ is an isometric immersion with embedding data $(\tilde{I}_t,\tilde{a}_t)$.

As a consequence of the above observations, we can construct a smooth family of immersions $(\tilde{f}_t)$, as required in the statement, by fixing any (u,x,ξ) in the graph of $j^1\tilde{f}_0$ and considering the family of integral surfaces for $\Psi_t^*\mathcal{P}_t$ passing through such (u,x,ξ). Smoothness of $\tilde{f}_t(u)$ in u and t follows from the Frobenius theorem.

Finally, since $(\tilde{I}_t, \tilde{a}_t)$ are $\pi_1(S)$-invariant, by Proposition 2.5 there exists a unique $\rho_t : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ such that the maps $\tilde{f}_t$ are ρ_t-equivariant. It is easy to see that, since $\tilde{f}_t(u)$ depends smoothly on u and t, the family (ρ_t) is smooth too. $\square$

Finally, we will associate to any section of $\Theta_{\tilde{f}}$ a self-adjoint and a skew-self-adjoint endomorphism of $\Theta_{\tilde{f}}$, using the following simple linear algebra lemma (see [3, Section 5.2]).

Lemma 2.7 (A-S decomposition of a linear immersion)**.** *Let V be an Euclidean vector space of dimension 3 and let W be a 2-dimensional linear subspace. Then any linear map $L : W \to V$ can be uniquely decomposed as $L = \mathsf{A}^L + \mathsf{S}^L$, where $\mathsf{A}^L : W \to W$ is self-adjoint and S^L can be written as $\mathsf{S}^L(\bullet) := v \times \bullet$ for some fixed $v \in V$.*

Given a section $\tilde{X}$ of $\Theta_{\tilde{f}}$, at every point $\tilde{p}$ of $\widetilde{S}$ we can view $(\nabla^{\mathbb{H}^3} \tilde{X})_{\tilde{p}}$ as a linear map from $T_{\tilde{f}(\tilde{p})}\tilde{f}(\widetilde{S})$ to $T_{\tilde{f}(\tilde{p})}\mathbb{H}^3$, where we have identified $T_{\tilde{p}}\widetilde{S}$ and $T_{\tilde{f}(\tilde{p})}\tilde{f}(\widetilde{S})$ via $d\tilde{f}_{\tilde{p}}$.

Definition 2.8 (Self-adjoint and skew-self-adjoint derivative of a section of $\Theta_{\tilde{f}}$)**.** Let $\tilde{f}$ be an immersion and $\tilde{X}$ be a section of $\Theta_{\tilde{f}}$. The *self-adjoint derivative* of $\tilde{X}$ is the endomorphism $\mathsf{A}^{\tilde{X}}_{\tilde{f}} : T\widetilde{S} \to T\widetilde{S}$ defined as $\left(\mathsf{A}^{\tilde{X}}_{\tilde{f}}\right)_{\tilde{p}} := \mathsf{A}^{(\nabla^{\mathbb{H}^3}\tilde{X})_{\tilde{p}}}$ for all $\tilde{p} \in \widetilde{S}$. The *skew-self-adjoint derivative* of $\tilde{X}$ is the linear morphism $\mathsf{S}^{\tilde{X}}_{\tilde{f}} : T\widetilde{S} \to \Theta_{\tilde{f}}$ defined as $\left(\mathsf{S}^{\tilde{X}}_{\tilde{f}}\right)_{\tilde{p}} := \mathsf{S}^{(\nabla^{\mathbb{H}^3}\tilde{X})_{\tilde{p}}}$ for all $\tilde{p} \in \widetilde{S}$.

We will usually denote by $\tilde{X}'$ the section of $\Theta_{\tilde{f}}$ such that $\mathsf{S}^{\tilde{X}}_{\tilde{f}}(\bullet) = \tilde{X}' \times \bullet$.

2.4. **The 1-Schatten energy.** In this subsection we introduce a functional on F on the space $\mathcal{C}$ of smooth equivariant maps from $\widetilde{S}$ to $\mathbb{H}^3$ that will be a central object of our investigation in this paper. We incidentally mention that such functional can be defined on a space of maps of lower regularity (for example, Lipschitz maps).

Given a smooth equivariant map $(\tilde{f}, \rho)$, the 1-Schatten norm of f is defined as the function on $\widetilde{S}$ given by $\tilde{p} \mapsto \|d\tilde{f}_{\tilde{p}}\|_1$, where $\|d\tilde{f}_{\tilde{p}}\|_1$ denotes the 1-Schatten norm of the linear map $d\tilde{f}_{\tilde{p}}$ as defined in Section A.1.
Clearly, this norm is unchanged if we replace $\tilde{f}$ by $g \circ \tilde{f}$ with $g \in \mathrm{PSL}_2(\mathbb{C})$. Hence, the function $\|d\tilde{f}\|_1$ on $\widetilde{S}$ descends to a function on S, denoted as $\|df\|_1$.

Definition 2.9 (1-Schatten norm of an equivariant map)**.** The 1-*Schatten norm of the equivariant map* $(\tilde{f}, \rho)$ is the function $\|df\|_1 : S \to \mathbb{R}_{\geq 0}$ defined in such a way that for every $p \in S$ the value $\|df_p\|_1$ agrees with the 1-Schatten norm $\|d\tilde{f}_{\tilde{p}}\|_1$ of the linear map $d\tilde{f}_{\tilde{p}} : T_{\tilde{p}}\widetilde{S} \to T_{\tilde{f}(\tilde{p})}\mathbb{H}^3$ where $\tilde{p} \in \widetilde{S}$ is any lift of p.

Notation. The symbol $\|df\|_1$ associated to an equivariant map $(\tilde{f}, \rho)$ aims at helping the reader in remembering that $\|df\|_1$ is a well-defined function on S, and not just on $\widetilde{S}$. In general, though, no map f is involved in its definition.

However, if $\tilde{f}$ is a lift of a map $f : S \to M$ to a complete hyperbolic 3-manifold M, then $\|df_p\|_1$ is exactly the 1-Schatten norm of $df_p : T_pS \to T_{f(p)}M$.

The following is a direct consequence of Lemma A.3.

Lemma 2.10 (Regularity of 1-energy density). *Let $(\tilde{\boldsymbol{f}}, \boldsymbol{\rho})$ be a path of equivariant maps. The function $S \times (-\epsilon, \epsilon) \ni (p,t) \mapsto \|(df_t)_p\|_1 \in \mathbb{R}_{\geq 0}$ is Lipschitz; moreover, it is smooth at all points $p \in S$ such that $(d\tilde{f}_t)_{\tilde{p}}$ has rank 2.*

Remark 2.11 (1-energy density and b-operator). If $(\tilde{f}, \rho)$ is an equivariant immersion with first fundamental form I and let b be the h-self-adjoint operator on TS such that $I(v,w) = h(bv, bw)$. Then its pull-back $\tilde{b}$ to $T\widetilde{S}$ is the h-self-adjoint component in the polar decomposition of $d\tilde{f}$. Thus $\|df\|_1 = \operatorname{tr}(b)$.

Definition 2.12 (1-Schatten energy). The 1-*Schatten energy* of a smooth equivariant map $(\tilde{f}, \rho)$ in $\mathbb{H}^3$ is defined as

$$F(\tilde{f}) \coloneqq \int_S \|df\|_1 \, \omega_h .$$

Remark 2.13. The 1-Schatten energy $F(\tilde{f})$ can be defined for equivariant maps $\tilde{f}$ of lower regularity, such as Lipschitz maps (in this case $\|df\|_1$ is bounded measurable).

Clearly, $F(g \circ \tilde{f}) = F(\tilde{f})$ for every $g \in \mathrm{PSL}_2(\mathbb{C})$. We also note that, for a path of equivariant maps $(\tilde{\boldsymbol{f}}, \boldsymbol{\rho})$, the function $t \mapsto F(\tilde{f}_t)$ is smooth at t_0 provided that the map $\tilde{f}_{t_0}$ is an immersion.
The following simple and important property is a consequence of Lemma A.4.

Lemma 2.14 (1-Schatten energy and Lipschitz maps). *If $\tilde{f} : \widetilde{S} \to \mathbb{H}^3$ is a smooth equivariant immersion and $g : \mathbb{H}^3 \to \mathbb{H}^3$ is C-Lipschitz, then the 1-Schatten energy (in the sense of Remark 2.13) of the Lipschitz map $g \circ \tilde{f}$ satisfies $\|d(g \circ \tilde{f})\|_1 \leq C \cdot \|d\tilde{f}\|_1$ at almost every point of $\widetilde{S}$. Hence, $F(g \circ \tilde{f}) \leq C \cdot F(\tilde{f})$.*

2.5. **Minimizing maps and critical points of F.** Fix a non-elementary representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ throughout the whole section.
Let us denote by $\widetilde{\mathcal{C}}_\rho$ the space of smooth equivariant maps of $\widetilde{S}$ into $\mathbb{H}^3$ with monodromy ρ, equipped with the C^∞ topology, and by $\widetilde{\mathcal{I}}_\rho$ the subspace of smooth equivariant immersions with monodromy ρ.

Definition 2.15 (Minimizing ρ-equivariant maps). A ρ-equivariant map $\tilde{f} \in \widetilde{\mathcal{C}}_\rho$ is *minimizing* if it realizes the minimum of the functional F on $\widetilde{\mathcal{C}}_\rho$.

By Lemma A.3, the functional F on $\widetilde{\mathcal{I}}_\rho$ is smooth in the following sense: if $\tilde{\boldsymbol{f}} : (-\epsilon, \epsilon) \times \widetilde{S} \to \mathbb{H}^3$ is a smooth path of equivariant immersions with constant monodromy ρ, then the function $t \mapsto F(f_t)$ is smooth. The following is then immediate.

Lemma 2.16 (Minimizing immersions are critical points of F). *If $(\tilde{f}, \rho) \in \widetilde{\mathcal{I}}_\rho$ is a ρ-equivariant minimizing immersion, then it is a critical point of F, i.e. for any isomonodromic deformation $\tilde{\boldsymbol{f}} : (-\epsilon, \epsilon) \times \widetilde{S} \to \mathbb{H}^3$ of $\tilde{f}$ we have*

$$\left. \frac{d}{dt} F(\tilde{f}_t) \right|_{t=0} = 0 \ .$$

In Section 2.8 we will discuss the convexity of F showing that any critical immersion is in fact minimizing (Corollary 2.38).

2.6. **Minimizing maps with Fuchsian monodromy.** In this section we consider the case where the representation ρ is *Fuchsian*, that is, ρ is a discrete and faithful representation of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C})$. We identify $\mathbb{H}^2$ to the totally geodesic plane of $\mathbb{H}^3$ stabilised by $\mathrm{PSL}_2(\mathbb{R})$.
We begin with a more general remark (see [6, Section II.1.3] for more details on the nearest point retraction).

Lemma 2.17 (Nearest point retraction)**.** *Let K be a non-empty closed convex subset of $\mathbb{H}^3$, invariant under the action of $\pi_1(S)$ induced by a representation $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$. Then the nearest point retraction $r : \mathbb{H}^3 \to K$ is $\pi_1(S)$-equivariant and 1-Lipschitz. Moreover, if the representation takes values in $\mathrm{PSL}_2(\mathbb{R})$ and $K = \mathbb{H}^2$, then r is smooth.*

Thus for any ρ-equivariant map $\tilde{f}$, the composition $r \circ \tilde{f}$ is ρ-equivariant and $F(r \circ \tilde{f}) \leq F(\tilde{f})$ by Lemma 2.14. This implies that, if $(\tilde{f}_n)$ is any minimizing sequence in $\widetilde{\mathcal{C}}_\rho$, then $(r \circ \tilde{f}_n)$ is still a minimizing sequence in the space of ρ-equivariant Lipschitz maps from $\widetilde{S}$ to K.
Now let ρ be a Fuchsian representation, so that $K = \mathbb{H}^2$ is a ρ-invariant closed convex set. If $(\tilde{f}_n)$ is any minimizing sequence in $\widetilde{\mathcal{C}}_\rho$, then $(r \circ \tilde{f}_n)$ is still a minimizing sequence in $\widetilde{\mathcal{C}}_\rho$ consisting of maps that take values in $\mathbb{H}^2$. Hence, minimizers of F among ρ-equivariant maps with values into $\mathbb{H}^2$ are indeed minimizers of F.
We will prove in Section 3.1 the following characterization of ρ-equivariant local diffeomorphisms $\widetilde{S} \to \mathbb{H}^2$ that are F-minimizers.

Lemma 2.18 (Fuchsian minimizers are minimal Lagrangian)**.** *Let ρ be a Fuchsian representation. A ρ-equivariant local diffeomorphism $\tilde{f} : \widetilde{S} \to \mathbb{H}^2$ is minimizing if and only it is minimal Lagrangian.*

In this setting we can interpret the result proved by Schoen in [19] in terms of the existence result of good minimizers for F when ρ is Fuchsian.

Theorem 2.19 (Fuchsian minimizers)**.** *Let $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian representation. There exists a unique smooth ρ-equivariant map $\tilde{f}$ which is a critical point of F. Such $\tilde{f}$ takes values in $\mathbb{H}^2$ and it is the lift of the unique minimal Lagrangian map $f : S \to S^\star := \mathbb{H}^2/\rho$ between hyperbolic surfaces. As a consequence, $\tilde{f}$ is a real-analytic diffeomorphism.*

Uniqueness is in fact a consequence the convexity of the functional proved in Section 2.8.

2.7. **Infinitesimal deformations.** In this section we study first-order deformations of a (not necessarily equivariant) smooth map $\tilde{f} : \widetilde{S} \to \mathbb{H}^3$, namely smooth paths of maps $\boldsymbol{\tilde{f}}$ such that $\tilde{f}_0 = \tilde{f}$.

2.7.1. *Deformations of maps.* We recall that $\tilde{f}^* T\mathbb{H}^3$ is the vector bundle on $\widetilde{S}$ consisting of pairs $(\tilde{p}, v)$, where $\tilde{p} \in \widetilde{S}$ and $v \in T_{\tilde{f}(\tilde{p})}\mathbb{H}^3$. As in Section 2.3, we also denote such vector bundle by $\Theta_{\tilde{f}}$.

Definition 2.20 (Velocity vectors of smooth paths of maps)**.** Let $\tilde{\boldsymbol{f}} : \widetilde{S} \times (-\varepsilon, \varepsilon) \to \mathbb{H}^3$ be a path of maps. The *velocity vector of* $\tilde{\boldsymbol{f}}$ is the smooth section of $\Theta_{\tilde{f}}$ obtained by differentiating $\tilde{f}_t$ at $t = 0$.

The velocity vector of $\tilde{\boldsymbol{f}}$ is also classically called *variational field* of the smooth deformation $\tilde{\boldsymbol{f}}$ of $\tilde{f}_0$. We remark that, for any map $\tilde{f}$, any smooth section $\tilde{X}$ of $\Theta_{\tilde{f}}$ is the variational field of a smooth deformation of $\tilde{f}$: for example, $\tilde{X}$ is the variational field of the geodesic displacement $\tilde{\boldsymbol{f}}_{\tilde{X}}$ introduced in Definition 2.22 below. Thus, we can identify first-order deformations of $\tilde{f}$ with their variational fields.
The *tangent cone* to the space of maps at $\tilde{f}$ is the set of all velocity vectors of deformations of $\tilde{f}$. We have seen above that, in the present case, such tangent cone at $\tilde{f}$ is given by the vector space of all sections of $\Theta_{\tilde{f}}$: because of such linear structure, it will then be natural to call it "tangent space".

Definition 2.21 (Tangent space to the space of maps)**.** The tangent space $T_{\tilde{f}}C^\infty(\widetilde{S}, \mathbb{H}^3)$ to the space $C^\infty(\widetilde{S}, \mathbb{H}^3)$ at the map $\tilde{f}$ is the vector space $\Gamma(\Theta_{\tilde{f}})$.

A geodesic displacement is a very natural deformation of a map to a Riemannian target. Its relevance also relies on the fact that many energy functionals turn out to be convex when such target is non-positively curved. Here we briefly recall such notion.

Definition 2.22 (Geodesic displacement)**.** Let $\tilde{X}$ be a smooth section of $\Theta_{\tilde{f}}$. The *geodesic displacement* of $\tilde{f}$ along $\tilde{X}$ is the path of smooth maps $\tilde{\boldsymbol{f}}_{\tilde{X}} : \mathbb{R} \times \widetilde{S} \to \mathbb{H}^3$ defined as $\tilde{\boldsymbol{f}}_{\tilde{X}}(t, \tilde{p}) := \exp_{\tilde{f}(\tilde{p})}(t \cdot \tilde{X}(\tilde{p}))$.

Since the exponential map of the hyperbolic space induces a diffeomorphism between $\mathbb{H}^3$ and $T_x\mathbb{H}^3$ for any $x \in \mathbb{H}^3$, we get the following property.

Lemma 2.23 (Uniqueness of geodesic displacements between given maps)**.** *If $\tilde{f}_0, \tilde{f}_1$ are ρ-equivariant smooth maps, then there is a unique first-order smooth deformation $\tilde{X}$ of $\tilde{f}_0$ such that $\tilde{\boldsymbol{f}}_{\tilde{X}}(1, \tilde{p}) = \tilde{f}_1(p)$.*

Proof. It is immediate to check that the only infinitesimal deformation of $\tilde{f}_0$ with the stated properties is defined by the formula $\tilde{X}(\tilde{p}) := (\exp_{\tilde{f}_0(\tilde{p})})^{-1}(\tilde{f}_1(\tilde{p}))$. □

2.7.2. *Deformation of representations.* Fix $\rho \in \widetilde{\mathcal{X}}$ and let $\boldsymbol{\rho}$ be a smooth deformation of ρ, namely a smooth path $\boldsymbol{\rho} : (-\epsilon, \epsilon) \to \mathcal{X}$ such that $\rho_0 = \rho$.
For every $\gamma \in \pi_1(S)$, let $\varsigma_\gamma \in \mathfrak{sl}_2(\mathbb{C})$ be defined as

$$\varsigma_\gamma := \frac{d}{dt}\rho_t(\gamma)\rho(\gamma)^{-1}\Big|_{t=0}.$$

We recall that $\mathfrak{sl}(2, \mathbb{C})$ can be identified to the Lie algebra of Killing vector fields on $\mathbb{H}^3$. Under this identification, ς_γ can be regarded as the Killing vector field over $\mathbb{H}^3$ whose value at $x \in \mathbb{H}^3$ is the velocity of the curve $t \mapsto \rho_t(\gamma)\rho(\gamma)^{-1}(x)$ at time $t = 0$. Thus, to the given deformation $\boldsymbol{\rho}$ of ρ we can associate the $\mathfrak{sl}_2(\mathbb{C})$-valued function

$$\begin{aligned}\varsigma:\ \pi_1(S) &\longrightarrow \mathfrak{sl}_2(\mathbb{C})\\ \gamma &\longmapsto \varsigma_\gamma\end{aligned}$$

It is easy to check that, since ρ_t is a representation for all t, the function ς satisfies the condition

$$\varsigma_{\gamma_1\gamma_2} = \varsigma_{\gamma_1} + \mathrm{Ad}_{\rho(\gamma_1)}\varsigma_{\gamma_2}. \tag{3}$$

Remark 2.24. Elements of $\mathfrak{sl}_2(\mathbb{C})$ can be considered as Killing vector fields on $\mathbb{H}^3$. For any element $\zeta \in \mathfrak{sl}_2(\mathbb{C})$, and any $x \in \mathbb{H}^3$ we can put

$$\zeta(x) = \frac{d\exp(t\zeta)(x)}{dt}|_0\,.$$

With this definition notice that $\mathrm{Ad}_g(\zeta)(x)$ is the push-forward of ζ through the map g:

$$\mathrm{Ad}_g(\zeta)(x) = \frac{d(g\exp(t\zeta)(g^{-1}(x)))}{dt}|_0 = dg_{g^{-1}x}(\zeta(g^{-1}(x)))\,.$$

Definition 2.25 (ρ-twisted 1-cocycles)**.** A $\mathfrak{sl}_2(\mathbb{C})$*-valued* ρ*-twisted* 1*-cocycle* is a function $\varsigma : \pi_1(S) \to \mathfrak{sl}_2(\mathbb{C})$ that satisfies (3). The vector space of such functions is denoted by $\mathcal{Z}^1_\rho$.

The following is rather classical, see [10].

Lemma 2.26 (Deformations of representations as cocycles)**.** *For every $\rho \in \widetilde{\mathcal{X}}$, the map*

$$T_\rho\widetilde{\mathcal{X}} \longrightarrow \mathcal{Z}^1_\rho$$

that sends a first-order deformation of ρ to its associated 1*-cocycle is a bijection.*

Because of the above lemma, we will identify the first-order deformation $\boldsymbol{\rho}$ of ρ with its associated cocycle function ς.
The following fact will be useful later.

Lemma 2.27 (Existence of an equivariant deformation)**.** *Let $(\tilde f,\rho)$ be an equivariant map and let $\boldsymbol{\rho}$ be a deformation of ρ. Then there exists a deformation $\tilde{\boldsymbol{f}} : (-\epsilon,\epsilon)\times\widetilde{S} \to \mathbb{H}^3$ of $\tilde f$ such that $\tilde f_t$ is ρ_t-equivariant for all $t \in (-\epsilon,\epsilon)$.*

Proof. Let $\zeta_t \in \mathcal{Z}^1_{\rho_t}$ the cocycle representing the derivative of $\boldsymbol{\rho}$ at t, and denote by $\pi : \widetilde{S}\times\mathbb{H}^3 \to \mathbb{H}^3$ the natural projection. We first construct a continuous family of sections σ_t of the bundle $\pi^*(T\mathbb{H}^3)$ which satisfy the following coboundary condition

$$\sigma_t(\gamma(p),x) = d(\rho_t(\gamma))(\sigma_t(p,\rho_t^{-1}(\gamma)(x))) + \zeta_t(\gamma)(x) \tag{4}$$

for all $\gamma \in \pi_1(S)$, $p \in \widetilde{S}$ and $x \in \mathbb{H}^3$. The construction is pretty standard, based on the partition of the unity. Fix a covering $\{U_\alpha\}$ of S, made by simply connected open subsets. For each α let $\tilde U_\alpha$ a lifting of U_α in $\widetilde{S}$, through the covering map $\mathscr{P} : \widetilde{S} \to S$ so that

$$\mathscr{P}^{-1}(U_\alpha) = \sqcup_{\gamma\in\pi_1(S)}\gamma(\tilde U_\alpha)$$

Define the family of section σ_t^α of $\pi^*(T\mathbb{H}^3)$ only on $\pi^{-1}(U_\alpha)\times\mathbb{H}^3$ by putting on $\gamma(\tilde{U}_\alpha)\times\mathbb{H}^3$

$$\sigma_t^\alpha(p,x) \coloneqq \zeta_t(\gamma)(x)\,.$$

Notice that (4) is verified for all $p\in\mathscr{P}^{-1}(U_\alpha)$ and $x\in\mathbb{H}^3$. Now if (μ_α) is a partition of unity subordinate to $\{U_\alpha\}$, then the global family of sections

$$\sigma_t(p,x) = \sum \mu_\alpha(\mathscr{P}(p))\sigma_t^\alpha(p,x)$$

satisfies (4).

Now for each $p\in\widetilde{S}$ consider the path $c_p:(-\epsilon(p),\epsilon(p))\to\mathbb{H}^3$ solution of the following Cauchy problem:

$$\begin{cases} \dot{c}_p(t) = \sigma_t(p,c_p(t)) \\ c_p(0) = f_0(p) \end{cases}$$

For a fixed $\gamma\in\pi_1(S)$ we observe that the velocity of the path $\beta(t)\coloneqq\rho_t(\gamma)c_p(t)$ is

$$\dot{\beta}(t) = d(\rho_t(\gamma))(\dot{c}(t))+\zeta_\gamma(\beta(t)) = d(\rho_t(\gamma))(\sigma_t(p,c_p(t)))+\zeta_\gamma(\beta(t)) = \sigma_t(\gamma(p),\beta(t))$$

where the last equality holds by (4). So we deduce that $c_{\gamma(p)}(t)=\beta(t)$ since they are solutions of the same Cauchy problem. In particular $\epsilon(\gamma(p))=\epsilon(p)$, so there exists $\epsilon>0$ such that $\epsilon(p)>\epsilon$ for all $p\in\widetilde{S}$. Finally let us put

$$\boldsymbol{f}(t,p)\coloneqq c_p(t)$$

By the standard smooth dependence results, this function is smooth, and from what we have proved $\boldsymbol{f}(t,\gamma(p)) = \boldsymbol{f}(t,\rho_t(\gamma)(p))$ for all $t\in(-\epsilon,\epsilon)$ and $p\in\widetilde{S}$. □

2.7.3. *Deformation of equivariant maps.* If $\tilde{f}$ is an equivariant map with non-elementary monodromy ρ, there is a natural action of $\pi_1(S)$ on $\Theta_{\tilde{f}}$, plays an important role to detect the variational fields of deformations through equivariant maps.

Namely, if $\tilde{X}\in\Gamma(\Theta_{\tilde{f}})$ and $\gamma\in\pi_1(S)$, we set

$$\gamma_*\tilde{X}(\tilde{p})\coloneqq d(\rho(\gamma))_{\tilde{f}(\gamma^{-1}(\tilde{p}))}\tilde{X}(\gamma^{-1}(\tilde{p}))$$

for every $\tilde{p}\in\widetilde{S}$. We denote by Θ_f the vector bundle on S obtained as the quotient of $\Theta_{\tilde{f}}$ by the action of $\pi_1(S)$. Quite similarly, given a deformation $(\tilde{\boldsymbol{f}},\boldsymbol{\rho})_{t\in(-\epsilon,\epsilon)}$ of $(\tilde{f},\rho)$, we denote by $\boldsymbol{\Theta_f}$ the vector bundle on $(-\epsilon,\epsilon)\times S$ defined as the quotient of $\boldsymbol{\Theta_{\tilde{f}}}$ by the action of $\pi_1(S)$.

It follows that the vector space $\Gamma(\Theta_f)$ can be identified to the space $\Gamma(\Theta_{\tilde{f}})^\rho$ of ρ-invariant elements in $\Gamma(\Theta_{\tilde{f}})$. Thus sections X of $\Gamma(\Theta_f)$ correspond to ρ-invariant sections $\tilde{X}$ of $\Theta_{\tilde{f}}$.

Notation. In general, the symbol Θ_f is defined without using a map f from S to a hyperbolic 3-manifold. However, if $\tilde{f}$ is the lift of a map $f:S\to M$ to a hyperbolic 3-manifold M, then Θ_f identifies to f^*TM. The same considerations hold for a vector bundle of type $\boldsymbol{\Theta_f}$.

Note that $\widetilde{\mathcal{C}}_\rho$ can be naturally viewed as a subset of $C^\infty(\widetilde{S},\mathbb{H}^3)$. Thus we can define the *tangent cone* to $\widetilde{\mathcal{C}}_\rho$ at a point $\tilde{f}$ as the subset of $T_{\tilde{f}}C^\infty(\widetilde{S},\mathbb{H}^3) = \Gamma(\Theta_{\tilde{f}})$ consisting of all velocity vectors of deformations $\tilde{\boldsymbol{f}}$ of $\tilde{f}$ entirely contained inside $\widetilde{\mathcal{C}}_\rho$.

Remark 2.28 (Connectedness of $\widetilde{\mathcal{C}}_\rho$). We also note that $\widetilde{\mathcal{C}}_\rho$ is path-connected: indeed, if $\tilde{f},\tilde{f}'$ are points in $\widetilde{\mathcal{C}}_\rho$, then the geodesic displacement between $\tilde{f}$ and $\tilde{f}'$ gives a path of maps in $\widetilde{\mathcal{C}}_\rho$ between them. In fact, $\widetilde{\mathcal{C}}_\rho$ is also locally path-connected: it can be easily checked that, if $\tilde{f}'$ is close to $\tilde{f}$, then the geodesic displacement from $\tilde{f}$ to $\tilde{f}'$ remains close to $\tilde{f}$.

We will see in the following lemma that $T_{\tilde{f}}\widetilde{\mathcal{C}}_\rho$ is indeed a vector subspace.

Lemma 2.29 (Isomonodromic deformations)**.** *Let $(\tilde{f},\rho)\in\widetilde{\mathcal{C}}$ be a smooth equivariant map. Then the map*

$$T_{\tilde{f}}\widetilde{\mathcal{C}}_\rho \longrightarrow \Gamma(\Theta_{\tilde{f}})^\rho$$

that sends a first-order deformations of $(\tilde{f},\rho)$ inside $\widetilde{\mathcal{C}}_\rho$ to its corresponding (ρ-invariant) variational field $\tilde{X}$ is a bijection.

Proof. Consider a deformation $\tilde{\boldsymbol{f}}$ of $\tilde{f}$ inside $\widetilde{\mathcal{C}}_\rho$. Since all $\tilde{f}_t$ are ρ-equivariant, so is the corresponding variational field $\tilde{X}$, which thus belongs to $\Gamma(\Theta_{\tilde{f}})^\rho$. Vice versa, given $\tilde{X}\in\Gamma(\Theta_{\tilde{f}})^\rho$, we let $\tilde{\boldsymbol{f}}$ be the geodesic displacement associated to $\tilde{X}$. Since $\tilde{X}$ is ρ-invariant, so are the maps $\tilde{f}_t$ for all t. Hence, $\tilde{\boldsymbol{f}}$ is a smooth deformation of $\tilde{f}$ with fixed monodromy ρ and variational field $\tilde{X}$. □

Consider now a smooth deformation $(\tilde{\boldsymbol{f}},\boldsymbol{\rho})$ of $(\tilde{f},\rho)$ in which the monodromy ρ_t need not be the same at all t. Let $\tilde{X}$ be the variational field of $\tilde{\boldsymbol{f}}$ and ς the 1-cocycle attached to $\boldsymbol{\rho}$. For every $\gamma\in\pi_1(S)$, the relation $\rho_t(\gamma)\circ\tilde{f}_t = \tilde{f}_t\circ\gamma$ can be rewritten as

$$\left(\tilde{f}^{-1}\circ\left(\rho_t(\gamma)\rho^{-1}(\gamma)\right)\circ\tilde{f}\right)\circ\left(\tilde{f}^{-1}\rho(\gamma)\tilde{f}\right)\circ\left(\tilde{f}^{-1}\tilde{f}_t\right) = \left(\tilde{f}^{-1}\tilde{f}_t\right)\circ\gamma.$$

Differentiating it at $t=0$ and evaluating it at $\gamma^{-1}(\tilde{p})$, we obtain

$$(\tilde{f}^*\varsigma_\gamma)(\tilde{p}) + (\gamma_*\tilde{X})(\tilde{p}) = \tilde{X}(\tilde{p}),$$

namely

$$\gamma_*\tilde{X} = \tilde{X} - \varsigma_\gamma\big|_{\widetilde{S}} \tag{5}$$

where $\varsigma_\gamma\big|_{\widetilde{S}} = \tilde{f}^*\varsigma_\gamma$ is pull-back to $\widetilde{S}$ of the Killing vector field ς_γ in $\mathbb{H}^3$, viewed as a section of $\tilde{f}^*T\mathbb{H}^3$.

Definition 2.30 (1-cocycle attached to a deformation of an equivariant map)**.** A 1-*cocycle* associated to $(\tilde{f},\rho)$ is a couple $(\tilde{X},\varsigma)$ such that $\varsigma\in\mathcal{Z}^1_\rho$ and Equation (5) is satisfied. Such vector space of 1-cocycles is denoted by $\mathcal{Z}^1_{(\tilde{f},\rho)}$.

Since $\widetilde{\mathcal{C}}$ is a subset of $C^\infty(\widetilde{S},\mathbb{H}^3)\times\widetilde{\mathcal{X}}$, the tangent cone at a point $(\tilde{f},\rho)$ to $\widetilde{\mathcal{C}}$, namely the set of all velocity vectors of deformations of $(\tilde{f},\rho)$ entirely contained in $\widetilde{\mathcal{C}}$, can be viewed as a subset of $T_{\tilde{f}}C^\infty(\widetilde{S},\mathbb{H}^3)\oplus T_\rho\widetilde{\mathcal{X}}$.

Remark 2.31 (Connectedness of $\widetilde{\mathcal{C}}$). Since $\mathcal{X}$ is a connected manifold, Lemma 2.27 implies that (local) path-connectedness of $\widetilde{\mathcal{C}}$ is a consequence of (local) path-connectedness of each $\widetilde{\mathcal{C}}_\rho$, which was observed in Remark 2.28.

Again, we will see in the following lemma that tangent cones to $\widetilde{\mathcal{C}}$ are in fact vector spaces.

Lemma 2.32 (First-order deformations of equivariant maps)**.** *Let* $(\tilde{f},\rho) \in \widetilde{\mathcal{C}}$ *be a smooth equivariant map. The application*

$$T_{(\tilde{f},\rho)}\widetilde{\mathcal{C}} \longrightarrow \mathcal{Z}^1_{(\tilde{f},\rho)}$$

that sends a first-order deformation to its associated 1*-cocycle is a bijection.*

We have already seen how to associate a cocycle to a deformation: such application is injective essentially by definition. Surjectivity of such map is more subtle. The point is that it is not true in general that, if $\tilde{X}$ is a variational field on $\tilde{f}$ which satisfies (5) for some ς, then the geodesic displacement $\tilde{\boldsymbol{f}}_{\tilde{X}}$ is a family of maps that are equivariant with respect to some deformation $\boldsymbol{\rho}$ of ρ.

Proof of Lemma 2.32. Let $(\tilde{X},\varsigma)$ be an element of $\mathcal{Z}^1_{(\tilde{f},\rho)}$ and let $\boldsymbol{\rho}$ be a deformation of ρ with associated 1-cocycle ς. By Lemma 2.27, there exists $\tilde{\boldsymbol{\phi}} : (-\epsilon,\epsilon)\times\widetilde{S} \to \mathbb{H}^3$ that deforms $\tilde{\phi}$ and such that $\tilde{\phi}_t$ is ρ_t-equivariant. Let $\tilde{X}^\star$ be the variational field of $\tilde{\boldsymbol{\phi}}$. Since both $\tilde{X}$ and $\tilde{X}^\star$ solve Equation (5) with respect to ρ and ς, their difference $\tilde{Y}_0 := \tilde{X} - \tilde{X}^\star$ is $\pi_1(S)$-invariant, and so descends to a section Y_0 of Θ_ϕ. Since $\boldsymbol{\Theta}_{\boldsymbol{\phi}}$ is smoothly isomorphic to $(-\epsilon,\epsilon)\times\Theta_\phi$, the section Y_0 of $\{0\}\times\Theta_\phi$ can be extended to a smooth section $\boldsymbol{Y}$ of $\boldsymbol{\Theta}_{\boldsymbol{\phi}}$. Consider now the path of maps $\tilde{\boldsymbol{f}} : (-\epsilon,\epsilon)\times\widetilde{S}\to\mathbb{H}^3$ defined as $\tilde{f}_t(\tilde{p}) := \exp_{\tilde{\phi}_t(\tilde{p})}(t\cdot\tilde{Y}_t(\tilde{p}))$ Such family $(\tilde{\boldsymbol{f}},\boldsymbol{\rho})$ is a smooth deformation of $(\tilde{f},\rho)$ with variational field $\tilde{X}^\star + \tilde{Y}_0 = \tilde{X}$, as desired. □

As we wrote above, we consider two equivariant maps in the same $\mathrm{PSL}_2(\mathbb{C})$-orbit as "geometrically equivalent". A typical geometrically trivial deformation $(\tilde{\boldsymbol{f}},\boldsymbol{\rho})$ of $(\tilde{f},\rho)$ can be obtained by setting $\tilde{f}_t := g_t\circ\tilde{f}$ and $\rho_t := \mathrm{Ad}_{g_t}\rho$, where $\boldsymbol{g} : (-\epsilon,\epsilon)\to\mathrm{PSL}_2(\mathbb{C})$ satisfies $g_0 = \mathbb{1}$ and $\dot{g}_0 \in \mathfrak{sl}_2(\mathbb{C})$. In this case, a straightforward computation shows that its associated $(\tilde{X},\varsigma)$ satisfy

$$\tilde{X} = \dot{g}_0\Big|_{\widetilde{S}}, \quad \varsigma_\gamma = \dot{g}_0 - \mathrm{Ad}_{\rho(\gamma)}\dot{g}_0.$$

Definition 2.33 (1-coboundary associated to an equivariant map)**.** A 1-*coboundary* associated to $(\tilde{f},\rho)$ is a couple $(\tilde{X},\varsigma)$ such that $\tilde{X} = \dot{g}\Big|_{\widetilde{S}}$ and $\varsigma = \dot{g} - \mathrm{Ad}_\rho\dot{g}$ for some $\dot{g}\in\mathfrak{sl}_2(\mathbb{C})$. The vector space of 1-coboundaries is denoted by $\mathcal{B}^1_{(\tilde{f},\rho)}$.

As a subset of $\widetilde{\mathcal{C}}$, the tangent cone at a point $(\tilde{f},\rho)$ to its $\mathrm{PSL}_2(\mathbb{C})$-orbit is a subset of $T_{(\tilde{f},\rho)}\widetilde{\mathcal{C}}$. The above discussion can be condensed into the following.

Lemma 2.34 (Geometrically trivial first-order deformations)**.** *Let* $(\tilde f,\rho)\in\widetilde{\mathcal C}$ *be an equivariant map. The map*

$$T_{(\tilde f,\rho)}(\mathrm{PSL}_2(\mathbb C)\cdot(\tilde f,\rho))\longrightarrow\mathcal B^1_{(\tilde f,\rho)}$$

is a bijection.

We say that a path in $\mathcal C=\mathrm{PSL}_2(\mathbb C)\backslash\widetilde{\mathcal C}$ is smooth if it is induced by a smooth path in $\widetilde{\mathcal C}$, and that two smooth paths $(\tilde{\boldsymbol f},\boldsymbol\rho)$ and $(\tilde{\boldsymbol f}',\boldsymbol\rho')$ determine the same tangent vector to $\mathcal C$ if, for every t, there exists $g_t\in\mathrm{PSL}_2(\mathbb C)$ such that the path $g_t\cdot(\tilde f_t,\rho_t)$ is smooth and determines the same tangent vector in $\widetilde{\mathcal C}$ as $(\tilde{\boldsymbol f}',\boldsymbol\rho')$. It will follow from Lemma 2.35 below that the path $\boldsymbol g=(g_t)$ in $\mathrm{PSL}_2(\mathbb C)$ is necessarily smooth and so $T_{[\tilde f,\rho]}\mathcal C$ can be identified to the quotient of $T_{(\tilde f,\rho)}\widetilde{\mathcal C}$ by the vector subspace $T_{(\tilde f,\rho)}(\mathrm{PSL}_2(\mathbb C)\cdot(\tilde f,\rho))$. As a consequence, $T_{[\tilde f,\rho]}\mathcal C$ is a vector space.

Lemma 2.35 (Smoothness of the path $\boldsymbol g$)**.** *Let* $(\tilde{\boldsymbol f},\boldsymbol\rho)$ *be a smooth path in* $\widetilde{\mathcal C}$ *and let* $\boldsymbol g=(g_t)$ *be any path in* $\mathrm{PSL}_2(\mathbb C)$*. Then* $t\mapsto g_t\cdot(\tilde f_t,\rho_t)$ *is a smooth path in* $\widetilde{\mathcal C}$ *if and only if* $\boldsymbol g$ *is a smooth path in* $\mathrm{PSL}_2(\mathbb C)$*.*

Proof. If $\boldsymbol g$ is smooth, then clearly $(g_t\cdot(\tilde f_t,\rho_t))$ is smooth. So suppose now that $(g_t\cdot(\tilde f_t,\rho_t))$ is smooth. Since ρ_0 is non-elementary, the image of $\tilde f_0$ cannot be contained inside a geodesic. Hence, there exist three points $\tilde p_1,\tilde p_2,\tilde p_3$ in $\widetilde S$ such that $\tilde f_0(\tilde p_1),\tilde f_0(\tilde p_2),\tilde f_0(\tilde p_3)$ are not contained inside the same geodesic of $\mathbb H^3$. Call $x_{i,t}\coloneqq\tilde f_t(\tilde p_i)$ and $y_{i,t}\coloneqq g_t(x_{i,t})$ for $i=1,2,3$. Since the path $\tilde{\boldsymbol f}$ is continuous, the triple $(x_{1,t},x_{2,t},x_{3,t})$ does not sit on the same geodesic of $\mathbb H^3$ for $|t|$ small enough. It follows that g_t is the only element of $\mathrm{PSL}_2(\mathbb C)$ that takes $(x_{1,t},x_{2,t},x_{3,t})$ to $(y_{1,t},y_{2,t},y_{3,t})$. Since the three paths $(x_{i,t})$ in $\mathbb H^3$ are smooth (because $\tilde{\boldsymbol f}$ is smooth), and the three paths $(y_{i,t})$ in $\mathbb H^3$ are smooth too (because $(g_t\cdot\tilde f_t)$ is smooth), the path $\boldsymbol g$ is smooth. □

We can finally summarize the statement of Lemma 2.32 and Lemma 2.34 in this way.

Corollary 2.36 (Encoding first-order deformations)**.** *Let* $(\tilde f,\rho)\in\widetilde{\mathcal C}$ *be a smooth equivariant map.*

- *The tangent space* $T_{\tilde f}\widetilde{\mathcal C}_\rho$ *can be identified to* $\Gamma(\Theta_f)$*.*
- *The tangent space* $T_{(\tilde f,\rho)}\widetilde{\mathcal C}$ *can be identified to* $\mathcal Z^1_{(\tilde f,\rho)}$*.*
- *The tangent space at* $(\tilde f,\rho)$ *to the* $\mathrm{PSL}_2(\mathbb C)$*-orbit inside* $\widetilde{\mathcal C}$ *can be identified to* $\mathcal B^1_{(\tilde f,\rho)}$*.*

Hence, $T_{[\tilde f,\rho]}\mathcal C$ *can be identified to* $\mathcal H^1_{(\tilde f,\rho)}\coloneqq\mathcal Z^1_{(\tilde f,\rho)}/\mathcal B^1_{(\tilde f,\rho)}$*.*

2.8. **Convexity.** Given a (not necessarily non-elementary) representation ρ, we denote by $C^\infty(\widetilde S,\mathbb H^3)_\rho$ the subset of $C^\infty(\widetilde S,\mathbb H^3)$ consisting of maps that are ρ-equivariant.
One of the main properties of the 1-Schatten energy is that it is convex along geodesic deformations. The rest of this section will be devoted to proving the following statement.

Proposition 2.37 (Convexity of the 1-Schatten energy)**.** *Let ρ be a representation and $\tilde{\boldsymbol{f}}$ a smooth path inside $C^\infty(\widetilde{S},\mathbb{H}^3)_\rho$ and let $\tilde{X} \in \Gamma(\Theta_{\tilde{f}})^\rho$ be its variational field. If $\tilde{\boldsymbol{f}}_{\tilde{\boldsymbol{X}}}$ is the geodesic displacement of $\tilde{f} = \tilde{f}_0$ along $\tilde{X}$, then the 1-Schatten energy of the ρ-equivariant map $\tilde{f}_{\tilde{X},t}$ satisfies*

(i) *the function $t \to F(\tilde{f}_{\tilde{X},t})$ is convex;*
(ii) *if $\tilde{f}_0$ is an immersion, then $\frac{d^2}{dt^2}F(\tilde{f}_{\tilde{X},t})\Big|_{t=0} > 0$.*

The proof of Proposition 2.37 is based on some technical lemmas on a smooth perturbation of the Schatten norm, contained in Section A.2. Essentially the same proof shows that the result still holds for equivariant maps inside a negatively curved complete manifold.

Proof of Proposition 2.37. Let us fix $\tilde{p} \in \widetilde{S}$. We will prove that the function $t \mapsto \|d(\tilde{f}_{\tilde{X},t})_{\tilde{p}}\|_1$ is convex; and moreover that it is strictly convex at $t = 0$, provided that $d\tilde{f}_{\tilde{X},0}$ has rank 2 at $\tilde{p}$ and $\tilde{X}(\tilde{p}) \neq 0$.
Consider the parallel transport $\tilde{\tau}_t : \Theta_{\tilde{f}_{\tilde{X},t}} \to \Theta_{\tilde{f}_{\tilde{X},0}}$ along geodesics in $\mathbb{H}^3$. We can define a smooth family $\tilde{\boldsymbol{s}}$ of sections of $\mathrm{Hom}(T\widetilde{S},\Theta_{\tilde{f}})$ by setting $\tilde{s}_t := \tilde{\tau}_t \circ d\tilde{f}_{X,t} : T\widetilde{S} \to \Theta_{\tilde{f}}$. Clearly we have that $\|d\tilde{f}_{X,t}\|_1 = \|\tilde{s}_t\|_1$. So we need to prove that the function $t \mapsto \|\tilde{s}_t\|_1$ is convex.

Claim. *The family $\tilde{\boldsymbol{s}} : \mathbb{R} \to \mathrm{Hom}(T\widetilde{S},\Theta_{\tilde{f}})$ is a solution of the following Cauchy problem*

$$\begin{cases} \ddot{\tilde{s}} = \Xi \circ \tilde{s} \\ \tilde{s}_0 = d\tilde{f}_{\tilde{X},0} \\ \dot{\tilde{s}}_0 = \nabla^{\mathbb{H}^3}\tilde{X} \end{cases}$$

where $\nabla^{\mathbb{H}^3}$ is the connection on $\mathbb{H}^3$ and $\Xi : \Theta_{\tilde{f}_{X,0}} \to \Theta_{\tilde{f}_{X,0}}$ is the self-adjoint operator defined by

$$\Xi(\bullet) := -R(\bullet,\tilde{X})\tilde{X}$$

where R is the Riemann curvature tensor of $\mathbb{H}^3$.

Assuming the above claim, the operator Ξ is nonnegative because the curvature of $\mathbb{H}^3$ is, and Proposition A.5 shows that the function $t \mapsto \|\tilde{s}_t\|_1$ is convex, thus proving (i).

In order to verify the above claim, fix a point $\tilde{p} \in \widetilde{S}$ and let $\tilde{\alpha}$ be the geodesic in $\mathbb{H}^3$ defined by $\tilde{\alpha}(t) := \tilde{f}_t(\tilde{p}) = \exp_{\tilde{f}_{\tilde{X},0}(\tilde{p})} t\tilde{X}(\tilde{p})$.
For every fixed $\tilde{v} \in T_{\tilde{p}}\widetilde{S}$. define a vector field $\tilde{J}$ along α as $\tilde{J}(t) := d\tilde{f}_{\tilde{X},t}(\tilde{v})$. Then $\tilde{J}$ is a Jacobi field with initial conditions $\tilde{J}(0) = d\tilde{f}_{\tilde{X},0}(\tilde{v})$ and $\dot{\tilde{J}}(0) = (\nabla^{\mathbb{H}^3}_{\tilde{v}}\tilde{X})_{\tilde{p}}$.
Now, let $\{e_i\}$ be a parallel orthonormal frame along $\tilde{\alpha}$. Putting $\tilde{J}(t) = \sum_i c^i(t)e_i(t)$ we have that

$$\ddot{c}^i(t) = -\langle R(\tilde{J}(t),\dot{\tilde{\alpha}}(t))\dot{\tilde{\alpha}}(t), e_i(t)\rangle\ .$$

Now considering that $\tilde{s}_t(\tilde{v}) = \sum_i c^i e_i(0)$, we deduce that

$$\ddot{\tilde{s}}_t(\tilde{v}) = -\tilde{\tau}_t\left(R(\tilde{\tau}_t^{-1}(\tilde{v}), \dot{\tilde{\alpha}}(t))\dot{\tilde{\alpha}}(t)\right) = -R(\tilde{v}, \tilde{X})\tilde{X} = \Xi(\tilde{s}_t(\tilde{v}))\ ,$$

where the second equality holds because R and $\dot{\tilde{\alpha}}$ are parallel along the geodesic displacement and $\dot{\tilde{\alpha}}(0) = \tilde{X}$. Since $\tilde{p}$ and $\tilde{v}$ were arbitrary, the claim is proven.

Finally, in order to prove (ii), notice that negativity of the curvature of $\mathbb{H}^3$ has the following consequence: at every point $\tilde{p} \in \widetilde{S}$ where $\tilde{X}(\tilde{p}) \neq 0$, the quadratic form $T_{\tilde{p}}\widetilde{S} \ni \tilde{v} \mapsto \langle \Xi(\tilde{v}), \tilde{v}\rangle = -\langle R(\tilde{v}, \tilde{X}(\tilde{p}))\tilde{X}(\tilde{p}), \tilde{v}\rangle$ is semi-positive definite. Moreover, if $d\tilde{f}_{\tilde{X},0}$ has rank 2 at $\tilde{p}$ and $\tilde{X}(\tilde{p}) \neq 0$, then $\Xi \circ \tilde{s}_0 \neq 0$. Hence, by Proposition A.5 we deduce that

$$\left.\frac{d^2\|d\tilde{f}_t(\tilde{p})\|_1}{dt^2}\right|_{t=0} > 0$$

and the proof is complete. □

Let us draw the first consequences of the convexity property proven above.

Corollary 2.38 (ρ-equivariant critical immersions as minima)**.** *Let ρ be a representation. If $\tilde{f}$ is an immersion and critical point for the restriction of F to $C^\infty(\widetilde{S}, \mathbb{H}^3)_\rho$, then $\tilde{f}$ is the unique critical point and the absolute minimum of the restriction of F to $C^\infty(\widetilde{S}, \mathbb{H}^3)_\rho$.*

Proof. Let $\tilde{f}_1 \in C^\infty(\widetilde{S}, \mathbb{H}^3)_\rho$ be map different from $\tilde{f}_0 = \tilde{f}$. There is an invariant vector field $\tilde{X} \neq 0$ along $\tilde{f}_0$ such that the geodesic displacement $\boldsymbol{\tilde{f}_{\tilde{X}}}$ connects $\tilde{f}_0$ to $\tilde{f}_1$. By Proposition 2.37, the function $t \mapsto F(\tilde{f}_{\tilde{X},t})$ is strictly convex. Such a function though has vanishing first derivative at $t = 0$, because $\tilde{f}_0$ is a critical point for F. Hence, its derivative at $t = 1$ is strictly positive. Hence, we deduce that $\tilde{f}_1$ is not a critical point for F and that $F(\tilde{f}_0) < F(\tilde{f}_1)$. □

Another consequence of the convexity of the 1-Schatten energy is the following observation, which also motivates why we usually limit ourselves to investigate equivariant maps with non-elementary monodromy.

Lemma 2.39 (Monodromy of critical immersions)**.** *Let $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ be a representation and suppose that $\tilde{f}$ is a critical point for the restriction of F to $C^\infty(\widetilde{S}, \mathbb{H}^3)_\rho$.*

(i) *The representation ρ is reductive, i.e. the closure of the image of ρ is a reductive subgroup of* $\mathrm{PSL}_2(\mathbb{C})$.
(ii) *If $d\tilde{f}$ has rank 2 at some point, then ρ is non-elementary.*

Proof. We analyze case by case the possible types of monodromy representations.
If ρ fixes a point x in $\mathbb{H}^3$, then it is elementary and reductive and the unique critical point is the constant map with value x. Hence, both (i) and (ii) hold.
If ρ fixes exactly one point $x_\infty \in \partial\mathbb{H}^3$, then it is elementary but not reductive and a ρ-equivariant map $\tilde{f}$ cannot have image contained in a geodesic that limits to x_∞. As a consequence, the convexity properties of F imply that F is

strictly descreasing along the geodesic displacement from $\tilde{f}$ towards x_∞. Thus, there is no ρ-equivariant $\tilde{f}$ which is critical for F, and so (i) and (ii) hold.

If ρ fixes exactly two points in $\partial\mathbb{H}^3$, then it fixes a geodesic L in $\mathbb{H}^3$. Hence, ρ is elementary and reductive and the closest-point retraction Π from $\mathbb{H}^3$ to L is 1-Lipschitz, smooth and ρ-equivariant. As a consequence, minimizers must take values in L and so both (i) and (ii) hold.

If ρ does not fix any point in $\mathbb{H}^3$ or in $\partial\mathbb{H}^3$, then ρ is not elementary. In this case, the image of ρ cannot have a finite-index subgroup that fixes a point. Hence, ρ is reductive. □

3. Minimizing immersions

The aim of this section is to study the space of smooth minimizing immersions of a given hyperbolic surface (S,h) into germs of hyperbolic manifolds. This is equivalent to studying the space $\mathcal{MI}$ of equivalence classes $[\tilde{f},\rho]$ of minimizing smooth equivariant immersions. The first step is to describe an element of $\mathcal{MI}$ as a pair (b,a) of endomorphisms of TS that satisfy some relatively elementary properties. The second step is to notice that the package consisting of all such properties can be expressed in a particularly simple way in terms of the $\mathbb{C}$-linear operator $\phi = b - iJba$ on the complexified tangent bundle $T_{\mathbb{C}}S$. The last step is to describe the local structure of the space $\mathcal{D}$ of all such operators ϕ, and show that $\mathcal{D}$ is a complex manifold of dimension $-3\chi(S)$.

3.1. **Euler-Lagrange equations.** In this section, we fix an equivariant immersion $(\tilde{f},\rho) \in \widetilde{\mathcal{I}}$, consisting of a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ together with a smooth ρ-equivariant immersion $\tilde{f}$ of $\widetilde{S}$ into $\mathbb{H}^3$. We further consider a smooth isomonodromic deformation $(\tilde{\boldsymbol{f}})$ of $\tilde{f}$ in $\widetilde{\mathcal{I}}_\rho$ and we determine here the first-order variation of $F(f_t)$ at $t=0$.

As in the previous section, denote by $\tilde{X} = \frac{d}{dt}\tilde{f}_t\Big|_{t=0}$ the variational field of $\tilde{f}$, which descends to a section $X \in \Gamma(\Theta_f)$. Since $\tilde{f}$ is an immersion, the following is immediate.

Lemma 3.1 (Decomposition of the variational field)**.** *The variational field X can be decomposed as $X = X^T + \nu N$, where $\nu : S \to \mathbb{R}$ is a function, X^T is tangent to S and $N \in \Gamma(\Theta_f)$ is the positively-oriented unit normal vector. Moreover, the almost-complex structure J^I on S induced by I coincides with the operator $N \times \bullet$ on S.*

After the notation is set as above, we can state a formula for the first variation of F.

Proposition 3.2 (First-order variation of F at an equivariant immersion)**.** *The first-order variation of F along the isomonodromic family $(\tilde{\boldsymbol{f}})$ with variational field X satisfies*

$$\frac{d}{dt}F(\tilde{f}_t)\Big|_{t=0} = \int_S (\nu \cdot \mathrm{tr}(ba) + I(bW, J^I X^T))\omega_h \ ,$$

where $W := \det(b)^{-1}b^{-1} * d^\nabla b$ and $* : \Lambda^2(T^*S) \otimes TS \to TS$ is the Hodge $*$-operator associated with h.

Proof. Clearly,

$$\frac{dF(\tilde{f}_t)}{dt}\Big|_{t=0} = \int_S \mathrm{tr}(\dot{b})\omega_h$$

so we need to compute $\dot{b}$. In order to do so, we compute the first-order variation of the metric I_t induced on S by $\tilde{f}_t$ at $t = 0$ in two different way. On one hand, differentiating the identity $I_t = h(b_t, b_t)$ at $t = 0$, we get

$$\dot{I} = I(b^{-1}\dot{b}\bullet, \bullet) + I(\bullet, b^{-1}\dot{b}\bullet) \ .$$

On the other hand, the self-adjoint derivative $\mathsf{A}_{\tilde{f}}^{\tilde{X}}$ of $\tilde{X}$ (see Defintion 2.8) satisfies $\mathsf{A}_{\tilde{f}}^{\tilde{X}} = \nabla^{\tilde{I}}\tilde{X}^T + \tilde{\nu}\tilde{a}$, and it thus descends to an operator on TS, which we denote by A^X. It follows that $\dot{I}$ can be also written as

$$\dot{I} = I(A^X\bullet, \bullet) + I(\bullet, A^X\bullet) \ .$$

Comparing those two equations, we find that $b^{-1}\dot{b}$ and A^X have the same self-adjoint component, and therefore

$$b^{-1}\dot{b} = A^X + gJ^I = \nabla^I X^T + \nu a + \eta J^I \ ,$$

where $\eta : S \to \mathbb{R}$ is a function. Since $bJ^I = Jb$ is traceless, it follows that

$$\mathrm{tr}(\dot{b}) = \mathrm{tr}(bA^X) = \mathrm{tr}(b\nabla^I X^T) + \nu\mathrm{tr}(ba) \ .$$

By Lemma 3.7 of [4], given any vector field w on S, we have $\nabla^I_\bullet w = b^{-1}\nabla_\bullet(bw) + I(\bullet, W)J^I w$. Taking $w = X^T$, we obtain

$$\mathrm{tr}(b\nabla^I X^T) = \mathrm{div}_h(bX^T) + I(bW, J^I X^T)$$

and so

$$\mathrm{tr}(\dot{b}) = \mathrm{div}_h(bX^T) + I(bW, J^I X^T) + \nu\,\mathrm{tr}(ba)$$

Since the integral of $\mathrm{div}_h(bX^T)\omega_h$ over S vanishes, the conclusion follows. □

The following is then immediate.

Corollary 3.3 (Characterization of critical equivariant immersions)**.** *An equivariant immersion $(\tilde{f}, \rho)$ is a critical point of F if and only if the corresponding operator b satisfies the Codazzi equation $d^\nabla b = 0$, and the second fundamental form a satisfies* $\mathrm{tr}(ba) = 0$.

Proof. Consider the first-order variation formula as in Proposition 3.2. If b satisfies the Codazzi equation, then $W = 0$. If furthermore $\mathrm{tr}(ba) = 0$, then clearly $(\tilde{f}, \rho)$ is a critical point.
Vice versa, suppose that $(\tilde{f}, \rho)$ is a critical point. Considering first-order deformations of $(\tilde{f}, \rho)$ that are tangent to the image of $\tilde{f}$, namely with $\nu = 0$ and arbitrary X^T, we obtain that $W = 0$ and so $d^\nabla b = 0$. Considering then first-order deformations with arbitrary ν, we obtain $\mathrm{tr}(ba) = 0$. □

Recall that b is h-self-adjoint by definition, and that it is positive-definite because $\tilde{f}$ is an immersion. Taking in account also the Gauss-Codazzi equation of the immersions, we obtain the following statement, that implements the first step announced at the beginning of the section.

Corollary 3.4 (Euler-Lagrange equations for a critical immersion)**.** *A pair of operators* (b, a) *on* (S, h) *corresponds to a critical equivariant immersion* $(\tilde{f}, \rho)$ *if and only if the following equations are satisfied:*

$$d^{\nabla} b = 0\,, \qquad d^{\nabla}(ba) = 0\,, \tag{6}$$

$$\mathrm{tr}(Jb) = 0\,, \qquad \mathrm{tr}(ba) = 0\,, \tag{7}$$

$$\mathrm{tr}(Jb^2 a) = 0\,, \qquad \det b - \det(ba) = 1\,, \tag{8}$$

$$b > 0. \tag{9}$$

Proof. Suppose that (b, a) corresponds to a critical equivariant immersion. The first equation of (7) is equivalent to the fact that b is h-self-adjoint. Condition (9) is equivalent to $\tilde{f}$ being an immersion. The first equation of (6) and the second equation of (7) follow from Corollary 3.3. Since $\tilde{a}$ is the shape operator of an immersion with first fundamental form $\tilde{I}$, we have that $d^{\nabla^I} a = 0$. On the other hand, since b satisfies the Codazzi equation, we have $d^{\nabla^I}(\bullet) = b^{-1} d^{\nabla}(b\bullet)$ and so $d^{\nabla^I} a = b^{-1} d^{\nabla}(ba)$: the second equation of (6) follows. Imposing that a is I-self-adjoint, we have that $\mathrm{tr}(J^I a) = 0$. But $J^I = b^{-1} J b$ and it follows that $\mathrm{tr}(b^{-1} J b a) = 0$. Since b is h-self-adjoint, $b^{-1} = -(\det b)^{-1} J b J$, and the first equation of (8) follows. Finally by the Gauss equation, $K_I = K_{\mathbb{H}^3} + \det(a) = -1 + \det(a)$. On the other hand, since b is a Codazzi operator and h is hyperbolic, $K_I = K_h / \det(b) = -1/\det b$. The second equation of (8) easily follows by comparing these identities.
Vice versa, suppose that (b, a) satisfies Equations (6)-(9). Reversing the above argument, we have that b is h-self-adjoint, positive-definite and it satisfies the Codazzi equation, that $I = h(b\bullet, b\bullet)$ is a Riemannian metric on S, that a is I-self-adjoint and satisfies the I-Codazzi equation $d^{\nabla^I} a = 0$, and that (I, a) satisfies the Gauss equation $K_I = -1 + \det(a)$. By Proposition 2.5 such (I, a) correspond to an immersion $\tilde{f}$ inside $\mathbb{H}^3$. Invariance of (b, a) implies that $\tilde{f}$ is equivariant with respect to some representation ρ. Since b satisfies the Codazzi equation and $\mathrm{tr}(ba) = 0$ by the second equation of (7), it follows that $(\tilde{f}, \rho)$ is critical by Corollary 3.3. □

The proof of Lemma 2.18 follows directly from this corollary.

Proof of Lemma 2.18. If $f : \tilde{S} \to \mathbb{H}^2$ is a ρ-equivariant local diffeomorphism, then $a = 0$ so Equation (6) reduces to $d^{\nabla} b = 0$, (7) is equivalent to b being self-adjoint for h, and (8) to $\det(b) = 1$. Those are precisely the conditions for f to be minimal Lagrangian, see e.g. [14]. □

3.2. **Complex interpretation of the Euler-Lagrange equations.** Given a pair of linear operators (b, a) on TS, it is convenient to introduce the operator $\phi : T_{\mathbb{C}} S \to T_{\mathbb{C}} S$ on the complexified tangent bundle $T_{\mathbb{C}} S = \mathbb{C} \otimes_{\mathbb{R}} TS$ defined as

$$\phi := b - iJba\ .$$

Denoting (with some abuse) by ∇ the $\mathbb{C}$-linear extension to $T_{\mathbb{C}} S$ of the Levi Civita connection of h, the Euler-Lagrange equations in Corollary 3.4 can be rephrased in a more compact way, a more precise version of Theorem A.

Corollary 3.5 (Complex Euler-Lagrange equations for a critical immersion)**.** *The operator $\phi = b - iJba$ corresponds to a critical equivariant immersion if and only if $\phi \in \mathcal{Cod}$, i.e.*

$$d^\nabla \phi = 0 \ , \tag{10}$$

$$\phi \textit{ is } h\textit{-self-adjoint} \tag{11}$$

$$\det(\phi) = 1 \ , \tag{12}$$

$$\Re(\phi) \textit{ is positive definite.} \tag{13}$$

Proof. The fact that (10) and (11) are equivalent to (6) and (7) respectively is straightforward.
Equation (12) is equivalent to Equation (8). In fact, if B, C are complex 2×2 matrices and J is a complex skew-symmetric 2×2 matrix such that $J^2 = -\mathbb{1}$, then $\det(B)\mathbb{1} = -B^T JBJ$, and so $\det(B+C) = \det(B) + \det(C) - \operatorname{tr}(JB^T JC)$. If moreover B is symmetric, we have

$$\det(B + C) = \det(B) + \det(C) - \operatorname{tr}(JBJC)\,.$$

As a consequence,

$$\det(\phi) = \det(b) - \det(ba) + i\operatorname{tr}(Jb^2 a)\,.$$

Finally (13) is clearly equivalent to (9). □

The following observation will be used in the proof of Theorem B.

Corollary 3.6. *In Corollary 3.5 Equation* (12) *can be replaced by*

$$\operatorname{tr}\left((J\phi)^2\right) = -2. \tag{12'}$$

Proof. Let p be a point of S. In h-orthonormal coordinates at p we can write

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{12} & \phi_{22} \end{pmatrix},$$

and a direct computation gives

$$J\phi J\phi = -\det(\phi)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that $\operatorname{tr}\left((J\phi)^2\right) = -2\det(\phi)$ and the conclusion follows. □

3.3. **Codazzi operators.** Before addressing Theorem B, it will be useful to consider a natural decomposition of Codazzi operators.

Definition 3.7 (Space of Codazzi operators)**.** The *space of complex Codazzi operators $\mathcal{Cod}$* is the space of smooth h-self-adjoint bundle morphism $\phi : T_{\mathbb{C}}S \to T_{\mathbb{C}}S$ satisfying the Codazzi equation $d^\nabla \phi = 0$.

Clearly $\mathcal{Cod}$ is infinite-dimensional. It contains a subspace that is in one-to-one correspondence with the space $C^\infty(S, \mathbb{C})$ of C^∞ complex-valued function on S, as shown in the following lemma.

Lemma 3.8 (The operator cod). *For every $u \in \mathbb{C}^\infty(S, \mathbb{C})$ define*

$$\mathrm{cod}(u) := u\mathbb{1} - \mathrm{Hess}(u) \in \mathrm{End}(T_{\mathbb{C}}S)$$

where $\mathbb{1} \in \mathrm{End}(T_{\mathbb{C}}S)$ is the identity map, $\mathrm{Hess}(u) \in \mathrm{End}(T_{\mathbb{C}}S)$ *is defined as* $\mathrm{Hess}(u) = \nabla(\mathrm{grad}\, u)$ *and the gradient is computed with respect to the metric h. Then*

(a) $\mathrm{cod}(u)$ *is h-self-adjoint and satisfies the Codazzi equation;*
(b) *the map* $\mathrm{cod} : C^\infty(S, \mathbb{C}) \to \mathcal{C}od$ *is $\mathbb{C}$-linear, continuous and injective onto a closed subspace $\mathcal{C}od_{tr}$.*

Proof. Concerning claim (a), note that $\mathrm{cod}(u)$ is h-self-adjoint by definition, because $\mathrm{Hess}(u)$ is. Consider now two vector fields v, w on S. Then

$$\begin{aligned} (d^\nabla(u\mathbb{1}))(v,w) &= \nabla_v((u\mathbb{1})(w)) - \nabla_w((u\mathbb{1})(v)) - u\mathbb{1}([v,w]) \\ &= \nabla_v(uw) - \nabla_w(uv) - u(\nabla_v w - \nabla_w v) \\ &= du(v)w - du(w)v \ . \end{aligned}$$

It follows that

$$(d^\nabla(u\mathbb{1}))(v,w) = \omega_h(v,w) J \,\mathrm{grad}\, u \ .$$

On the other hand, since h is a hyperbolic metric on S,

$$\begin{aligned} (d^\nabla \mathrm{Hess}(u))(v,w) &= \nabla_v(\nabla_w \mathrm{grad}\, u) - \nabla_w(\nabla_v \mathrm{grad}\, u) - \nabla_{[v,w]} \mathrm{grad}\, u \\ &= -R(v,w)\, \mathrm{grad}\, u \\ &= \omega_h(v,w) J \,\mathrm{grad}\, u \ . \end{aligned}$$

As a consequence, $d^\nabla(u\mathbb{1} - \mathrm{Hess}(u)) = 0$ and so $\mathrm{cod}(u) \in \mathcal{C}od$.
Concerning claim (b), continuity is obvious, since both spaces are endowed with the smooth topology. As for the injectivity, suppose that $\mathrm{cod}(u) = 0$ and so $\mathrm{tr}(\mathrm{cod}(u)) = 0$. Since $\mathrm{tr}(\mathrm{cod}(u)) = 2u - \Delta u$ (where the Laplacian is computed with respect to the metric h), an easy application of the maximum principle shows that $u = 0$.
The closure of the image $\mathcal{C}od_{tr}$ of the map cod is a consequence of Proposition 3.9 below. □

It is convenient to endow $\mathcal{C}od$ with the structure of tame Fréchet space. Let us recall that a *Fréchet space* is a completely metrizable, locally convex topological vector space, whose topology is induced by a countable family of seminorms. A *tame Fréchet space* is a Fréchet space endowed with choice of an increasing family of seminorms $\|\cdot\|_n$ as above, which also satisfies some additional technical property. We will refer to [12] for an introduction on the topic.
For the aim of the present paper we only remark the following points.

- The relevance for us of these spaces is due to the fact that the Nash-Moser inverse function theorem ([18], [17]) works in this category.
- In the category of tame Fréchet spaces, a linear map $L : E \to F$ is *tame* if there exists n_0, k such that $\|L(v)\|_n < C_n \|v\|_{n+k}$ for all $v \in E$, all $n \geq n_0$ and for some constant C_n that may depend on n. In a similar way one can define a (non-necessarily linear) tame map between Fréchet spaces (see Definition II.2.1.1 of [12]).

- The space of C^∞-smooth sections of any vector bundle on a compact manifold is naturally equipped with a structure of tame Fréchet space (see Corollary II.1.3.9 of [12]). Here the family of seminorms is given by the C^k-norm of the section computed with respect to some background metric and connection. While the tame Fréchet structure depends on those choices, different choices produce isomorphic structures in the category of tame Fréchet spaces.

In particular, $\mathcal{Cod}$ is a closed linear subspace of the space of smooth sections of the vector bundle $\mathrm{End}(T_{\mathbb{C}}S)$, and so it carries a structure of tame Fréchet space.

Notation. We will denote by $\mathcal{Q}_{\mathbb{C}} := \mathcal{Q} \otimes_{\mathbb{R}} \mathbb{C}$ the complex vector space obtained by complexifying the space of J-holomorphic quadratic differential on S (viewed as a real vector space). An element of $\mathcal{Q}_{\mathbb{C}}$ with real part q and imaginary part q' will be denote by (q, q'), so that $i \cdot (q, q') = (-q', q)$. For every $q \in \mathcal{Q}$ we will denote by b_q the endomorphism of TS associated to the quadratic form $\mathfrak{R}(q)$ via h. Note that $\mathcal{Q}$ and $\mathcal{Q}_{\mathbb{C}}$ can be seen as a closed linear subspaces (of finite dimension!) of the space of smooth sections of the vector bundle of quadratic differentials on S and of its complexification respectively.

We can now provide a canonical decomposition, in the category of tame Fréchet spaces, of self-adjoint complex Codazzi tensors on S which will be useful for the proof of Theorem B, see [5].

Proposition 3.9 (Canonical decomposition of Codazzi operators)**.** *The map*

$$\begin{aligned} \mathcal{H} :\ C^\infty(S, \mathbb{C}) \times \mathcal{Q}_{\mathbb{C}} &\longrightarrow \mathcal{Cod} \\ (u, q, q') &\longmapsto \mathrm{cod}(u) + b_q + i b_{q'} \end{aligned}$$

is a $\mathbb{C}$-linear tame isomorphism of tame Fréchet spaces.

Proof. We recall that endomorphisms of TS which are h-self-adjoint, traceless and satisfy Codazzi are exactly of type b_q for some holomorphic quadratic differential q. Hence, $\mathcal{H}$ is well-defined and it is manifestly linear. We need to show that $\mathcal{H}$ is tame, bijective, and that $\mathcal{H}^{-1}$ is tame. In order to do that, we endow $C^\infty(S, \mathbb{C}) \times \mathcal{Q}_{\mathbb{C}}$ with the family of seminorms $\|(u, q, q')\|_n = \|u\|_n + \|q\|_n + \|q'\|_n$.

Tameness of $\mathcal{H}$. It is enough to note that $\|\mathrm{cod}(u) + b_q + i b_{q'}\|_n \le C'_n \|u\|_{n+2} + C''_n (\|q\|_n + \|q'\|_n) \le C'''_n \|(u, q, q')\|_{n+2}$, and so the linear map $\mathcal{H}$ is tame.

Bijectivity of $\mathcal{H}$. For every $u \in C^\infty(S, \mathbb{C})$, denote by $\Delta u := \mathrm{tr}(\mathrm{Hess}(u))$ the (negative semi-definite) Laplacian (with respect to h). Consider now the linear operator

$$\mathrm{tr}(\mathrm{cod}) = (2 - \Delta) : C^\infty(S, \mathbb{C}) \to C^\infty(S, \mathbb{C}).$$

For every $\tau \in C^\infty(S, \mathbb{C})$ the equation $-\Delta u + 2u = \tau$ has a unique solution, which is in fact smooth. It follows that $\mathrm{tr}(\mathrm{cod})$ is invertible. We want to show that the inverse of $\mathcal{H}$ is given by

$$\begin{aligned} \mathcal{C}od &\longrightarrow C^\infty(S,\mathbb{C}) \times Q_{\mathbb{C}} \\ \phi &\longmapsto (u_\phi, \Re(\phi - \mathrm{cod}(u_\phi)), \Im(\phi - \mathrm{cod}(u_\phi))) \end{aligned}$$

with $u_\phi := (\mathrm{tr} \circ \mathrm{cod})^{-1}\mathrm{tr}(\phi)$. In order to check that the above map is well-defined, observe that $\phi - \mathrm{cod}(u_\phi)$ is self-adjoint and Codazzi, and is also traceless since $\mathrm{tr}\left(\phi - \mathrm{cod}(u_\phi)\right) = \mathrm{tr}(\phi) - (\mathrm{tr} \circ \mathrm{cod})(u_\phi) = \mathrm{tr}(\phi) - \mathrm{tr}(\phi) = 0$. The real and imaginary parts of $\phi - \mathrm{cod}(u_\phi)$ are therefore real, self-adjoint, Codazzi and traceless, and so each of them is equal to the endomorphism associated to the real part of a holomorphic quadratic differential. It can be easily verified that the above map is the set-theoretic inverse of $\mathcal{H}$.

Tameness of $\mathcal{H}^{-1}$. Since $\|(2-\Delta)^{-1}\tau\|_{n+2} \le C_n\|\tau\|_n$ by elliptic regularity, the linear map $(2-\Delta)^{-1} : C^\infty(S,\mathbb{C}) \to C^\infty(S,\mathbb{C})$ is tame. As a consequence, $\mathcal{H}^{-1}$ is tame too. Being linear and tame, $\mathcal{H}^{-1}$ is automatically continuous. □

We will denote by $\mathcal{C}od_+ = \{\phi \in \mathcal{C}od \,|\, \Re(\phi) > 0\}$ and by

$$\mathcal{U} = \left\{(u,q,q') \in C^\infty(S,\mathbb{C}) \times Q_{\mathbb{C}} \;\middle|\; \Re(u\mathbb{1} - \mathrm{Hess}(u) + b_q + ib_{q'}) > 0\right\},$$

the corresponding subset of $C^\infty(S,\mathbb{C}) \times Q_{\mathbb{C}}$, so that $\mathcal{U} = \mathcal{H}^{-1}(\mathcal{C}od_+)$ is an open neighborhood of $\mathcal{H}^{-1}(\mathcal{D})$.

3.4. **Local structure of the space of immersion data.** We now turn to a more precise analysis of the space $\mathcal{D}$, with the goal of proving Theorem B stating that $\mathcal{D}$ is a finite-dimensional complex manifold.
As mentioned in the introduction, Proposition 3.9 allows for the definition of a tame $\mathbb{C}$-linear map

$$Q : \mathcal{C}od \to Q_{\mathbb{C}},$$

simply by composing $\mathcal{H}^{-1}$ with the projection onto $Q_{\mathbb{C}}$.
Recalling that $\mathcal{D} \subset \mathcal{C}od$, we are now fully equipped to state Theorem B, which we recall here.

Theorem B (Manifold structure on the space of minimizing maps). *Let (S,h) be a hyperbolic surface. The space $\mathcal{D}$ of immersion data is a complex submanifold of $\mathcal{C}od$ of complex dimension $6g-6$. Moreover, the restriction of Q over $\mathcal{D}$ is a local biholomorphism.*

The proof will use an additional map that extends the map Q defined above. We denote by Π the map

$$\begin{aligned} \Pi : \mathcal{C}od &\longrightarrow C^\infty(S,\mathbb{C}) \oplus Q_{\mathbb{C}} \\ \phi &\longmapsto \left(\Pi_1(\phi), Q(\phi)\right), \end{aligned}$$

where $\Pi_1 : \mathcal{C}od \to C^\infty(S,\mathbb{C})$ is defined as

$$\Pi_1(\phi) := \mathrm{tr}((J\phi)^2).$$

Note that Π_1 is induced by a smooth (and fiberwise holomorphic) morphism of vector bundles over S, and so the map Π_1 is smooth and tame (see Example

3.6.5 in part I of [12]). As a consequence, Π is a smooth, tame and it is induced by a fiberwise holomorphic morphism of vector bundles.
Since $\mathcal{D} = \Pi_1^{-1}(-2)$, Theorem B is an immediate consequence of the following proposition.

Proposition 3.10 (Π restricted to $\mathcal{Cod}_+$ locally biholomorphic)**.** *The restriction of* Π *to* $\mathcal{Cod}_+$ *is a local biholomorphism onto its image. As a consequence,* $\mathcal{D}$ *is a complex submanifold of* $\mathcal{Cod}$ *and the restriction of* Q *to* $\mathcal{D}$ *is a local diffeomorphism at* ϕ*.*

Proof. Recall that $\mathcal{H} : C^\infty(S,\mathbb{C}) \times Q_\mathbb{C} \to \mathcal{Cod}$ is a tame $\mathbb{C}$-linear isomorphism by Proposition 3.9, and consider

$$\widehat{\Pi} : \mathcal{U} \to C^\infty(S,\mathbb{C}) \oplus Q_\mathbb{C}.$$

obtained by restricting the tame (non-linear) map $\Pi \circ \mathcal{H}$ to the open subset $\mathcal{U}$.
We will show that $\widehat{\Pi}$ is holomorphic and locally invertible, and that every local inverse of $\widehat{\Pi}$ is smooth and tame. The conclusion will follow, since $\mathcal{H}(\mathcal{U})$ is a neighbourhood of $\mathcal{D}$.

Notice that $\widehat{\Pi}(u,q,q') = (\mathrm{tr}(J(u\mathbb{1} - \mathrm{Hess}(u) + b_q + ib_{q'}))^2, q, q')$, so $\widehat{\Pi}$ is holomorphic and its first component is a fully non-linear differential operator with respect to the variable u.
The fact that $\widehat{\Pi}$ is locally invertible and that every local inverse of $\widehat{\Pi}$ is smooth and tame will be a consequence of Nash-Moser theorem. In order to verify the hypotheses of that theorem for the map $\widehat{\Pi}$ (see Theorem 1.1.1 in Section III of [12]), we need to check two facts:

(a) the differential $D_{(u,q,q')}\widehat{\Pi} : C^\infty(S,\mathbb{C}) \oplus Q_\mathbb{C} \to C^\infty(S,\mathbb{C}) \oplus Q_\mathbb{C}$ is an isomorphism for every $(u,q,q') \in \mathcal{U}$;

(b) the map

$$\begin{aligned} G:\ \mathcal{U} \times (C^\infty(S,\mathbb{C}) \oplus Q_\mathbb{C}) &\longrightarrow C^\infty(S,\mathbb{C}) \oplus Q_\mathbb{C} \\ ((u,q,q'),(\dot{k},\dot{q},\dot{q}')) &\longmapsto (D_{(u,q,q')}\widehat{\Pi})^{-1}(\dot{k},\dot{q},\dot{q}') \end{aligned}$$

is smooth and tame.

(a) Notice that

$$D_{(u,q,q')}\widehat{\Pi}(\dot{u},\dot{q},\dot{q}') = (L_{\mathcal{H}(u,q,q')}(\dot{u}) + M_{\mathcal{H}(u,q,q')}(\dot{q},\dot{q}'), \dot{q}, \dot{q}')\ ,$$

where we have put $L_\phi(\dot{u}) = 2\mathrm{tr}(J\phi J((\dot{u}\mathbb{1} - \mathrm{Hess}(\dot{u})))$ and $M_\phi(\dot{q},\dot{q}') = 2\mathrm{tr}(J\phi J(b_{\dot{q}} + ib_{\dot{q}'}))$ for every $\phi \in \mathcal{Cod}_+$. So in order to prove that $D_{(u,q,q')}\widehat{\Pi}$ is an isomorphism for all $(u,q,q') \in \mathcal{U}$, it is sufficient to prove that $L_\phi : C^\infty(S,\mathbb{C}) \to C^\infty(S,\mathbb{C})$ is invertible for all $\phi \in \mathcal{Cod}_+$. This is exactly the content of Lemma 3.12.

(b) Recall now that the set Diff^2 of second-order linear differential operators on $C^\infty(S,\mathbb{C})$ can be identified to the set of smooth sections of the complex bundle $(\mathcal{J}^2)^*$, where $\mathcal{J}^2$ is the bundle of 2-jets of smooth complex-valued functions on S. In particular Diff^2 is a tame Fréchet space. Moreover, the subset $\mathrm{Diff}^2_{\mathrm{ell}}$ of invertible elliptic operators is open inside Diff^2, and the map $\mathrm{Diff}^2_{\mathrm{ell}} \times C^\infty(S,\mathbb{C}) \to C^\infty(S,\mathbb{C})$ given by $(L,\dot r) \mapsto L^{-1}(\dot r)$ is smooth and tame (this is a particular case of Theorem 3.3.1 in part II of [12]).

Now, for all $\phi \in \mathcal{C}od_+$ the operator L_ϕ is elliptic by Lemma 3.11 and invertible by Lemma 3.12, and so L_ϕ belongs to $\mathrm{Diff}^2_{\mathrm{ell}}$. Observe also that the map $\mathcal{U} \to \mathrm{Diff}^2_{\mathrm{ell}}$ given by $(u,q,q') \mapsto L_{\mathcal{H}(u,q,q')}$ is a differential operator, so it is smooth and tame. Putting together the above information, we conclude that the map $\mathcal{U} \times C^\infty(S,\mathbb{C}) \to C^\infty(S,\mathbb{C})$ given by $(u,q,q',\dot k) \mapsto (L_{\mathcal{H}(u,q,q')})^{-1}(\dot k)$ is smooth and tame. Now it is easy to see that

$$G((u,q,q'),(\dot k,\dot q,\dot q')) = (L^{-1}_{\mathcal{H}(u,q,q')}(\dot k - M_{\mathcal{H}(u,q,q')}(\dot q,\dot q')),\dot q,\dot q')$$

and so G can be regarded as a composition of smooth and tame maps, as M is manifestly smooth and tame. It follows that G is smooth and tame. $\square$

Lemma 3.11. *For every $\phi \in \mathcal{C}od_+$ the operator $L_\phi : C^\infty(S,\mathbb{C}) \to C^\infty(S,\mathbb{C})$, defined as $L_\phi(\dot u) = 2\mathrm{tr}(J\phi J((\dot u \mathbb{1} - \mathrm{Hess}(\dot u)))$, is elliptic.*

Proof. The principal symbol $\sigma(L_\phi)$ of L_ϕ is a complex-valued quadratic form on T^*S. We have to show that its real part is positive-definite at each point of S. It is thus enough to verify it in any chart.

Fix a chart. The operator L_ϕ will take the form $L_\phi(\dot u) = \sum_{i,j=1}^2 a_{ij}\partial_{ij}\dot u + \sum_{i=1}^2 b_i\partial_i \dot u + c\dot u$, where a_{ij}, b_i, c are complex valued functions. Its principal symbol can be written in the given chart as

$$\sigma(L_\phi)\psi = \textstyle\sum_{i,j=1}^2 a_{ij}\psi_i\psi_j$$

where ψ_1,ψ_2 are the coordinates of $\psi \in T^*S$ in the chart.

Now, considering the Hessian as a differential operator $\mathrm{Hess} : C^\infty(S,\mathbb{C}) \to \mathrm{End}(T_{\mathbb{C}}S) = (T_{\mathbb{C}}S)^* \otimes (T_{\mathbb{C}}S)$, its associated principal symbol (which is a $\mathrm{End}(T_{\mathbb{C}}S)$-valued quadratic form on T^*S) is given by $\sigma(\mathrm{Hess})\psi = \psi \otimes \psi^\sharp$, where $\psi^\sharp \in TS$ is the vector that corresponds to $\psi \in T^*S$ through the isomorphism $T^*S \cong TS$ determined by the metric h. This fact can be easily checked using normal coordinates.

It follows that the principal symbol of L_ϕ is the complex-valued quadratic form on the cotangent space of S given by $\sigma(L_\phi)\psi = -2\mathrm{tr}(J\phi J(\psi\otimes\psi^\sharp))$, which can be also written as $\sigma(L_\phi)\psi = 2h(\phi J(\psi^\sharp), J(\psi^\sharp))$. Since $\Re(\phi)$ is a positive-definite self-adjoint operator, it follows that $\Re(\sigma(L_\phi))$ is positive-definite. $\square$

Lemma 3.12. *For all $\phi \in \mathcal{C}od_+$ the operator $L_\phi : C^\infty(S,\mathbb{C}) \to C^\infty(S,\mathbb{C})$ is invertible.*

Before proving Lemma 3.12, we will need the following computation.

Sublemma 3.13. *Let $\varphi : TS \to TS$ be an h-self-adjoint, invertible operator that satisfies the Codazzi equation. Then*

$$\int_S \dot u \cdot \mathrm{tr}((J\varphi J)\mathrm{Hess}(\dot u'))\omega_h = \int_S \det(\varphi)\langle \varphi^{-1}\,\mathrm{grad}(\dot u)\,, \mathrm{grad}(\dot u')\rangle\omega_h \tag{14}$$

for all $\dot u, \dot u' \in C^\infty(S,\mathbb{R})$.

Proof. The key observation is that, since φ is a real Codazzi operator from TS to TS, then the operator $J\varphi J$ is divergence-free for h, i.e. $\nabla^*(J\varphi J) = 0$. Indeed, if (e_1, e_2) is a local orthonormal frame, then

$$\begin{aligned}
\nabla^*(J\varphi J) &= J\nabla^*(\varphi J) \\
&= -J(\nabla_{e_1}(\varphi J e_1) + \nabla_{e_2}(\varphi J e_2) - \varphi J(\nabla_{e_1}e_1 + \nabla_{e_2}e_2)) \\
&= -J(\nabla_{e_1}(\varphi e_2) - \nabla_{e_2}(\varphi e_1) - \varphi(\nabla_{e_1}e_2 - \nabla_{e_2}e_1) \\
&= -J(d^\nabla\varphi)(e_1, e_2) \\
&= 0\ .
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\int_S \dot{u}\cdot \mathrm{tr}((J\varphi J)\mathrm{Hess}(\dot{u}'))\omega_h &= \int_S \langle \dot{u}(J\varphi J),\, \nabla\,\mathrm{grad}(\dot{u}')\rangle\omega_h \\
&= \int_S \langle \nabla^*(\dot{u}(J\varphi J)),\, \mathrm{grad}(\dot{u}')\rangle\omega_h \\
&= \int_S \langle -(J\varphi J)\,\mathrm{grad}(\dot{u}),\, \mathrm{grad}(\dot{u}')\rangle\omega_h\ ,
\end{aligned}$$

and the conclusion follows by observing that $J\varphi J = -\det(\varphi)\varphi^{-1}$. □

We now have all the ingredients to prove Lemma 3.12.

Proof of Lemma 3.12. We denote by $\langle\cdot,\cdot\rangle$ the L^2 scalar product on complex-valued functions on S, defined by

$$\langle u_1, u_2\rangle = \Re\left(\int_S u_1\overline{u}_2\,\omega_h\right)\ ,$$

and by $H^1(S)$ the Sobolev space of complex-valued L^2 functions with L^2 derivative on S. The operator L_ϕ extends to a second-order linear differential operator $L_\phi : H^1(S)\to H^{-1}(S)$ as $L_\phi = 2\mathrm{tr}(J\phi J\cdot \mathrm{cod}(\bullet))$.
We will show that $-L_\phi$ is positive, namely that that there exists a constant $c > 1$ such that for all $\dot{u}\in C^\infty(S,\mathbb{C})$,

$$\frac{1}{c}\|\dot{u}\|^2_{H^1} \le \langle -L_\phi(\dot{u}), \dot{u}\rangle \le c\|\dot{u}\|^2_{H^1}\ . \tag{15}$$

As a consequence, L_ϕ is invertible as a map from $H^1(S)$ to $H^{-1}(S)$. The fact that L_ϕ is invertible from $C^\infty(S,\mathbb{C})$ to itself then follows from standard elliptic regularity arguments (see, for instance, Theorem 3 in Section 6.3.1 of [9]).
Let $\dot{u} = \dot{u}_\Re + i\dot{u}_\Im$, where $\dot{u}_\Re$ and $\dot{u}_\Im$ are smooth real-valued functions on S, and let $\phi = \phi_\Re + i\phi_\Im$, where $\phi_\Re$ and $\phi_\Im$ are real operators from TS to TS. Then:

$$\begin{aligned}
\frac{1}{2}\langle L_\phi\dot{u}, \dot{u}\rangle &= \Re\int_S \overline{\dot{u}}\cdot\mathrm{tr}\Big(J\phi J\cdot\mathrm{cod}(\dot{u})\Big)\omega_h = \\
&= \Re\bigg(\int_S(\dot{u}_\Re - i\dot{u}_\Im)\mathrm{tr}\Big(\big[J(\phi_\Re + i\phi_\Im)J\big]\big[\big(\dot{u}_\Re\mathbb{1} - \mathrm{Hess}(\dot{u}_\Re)\big) + i\big(\dot{u}_\Im\mathbb{1} - \mathrm{Hess}(\dot{u}_\Im)\big)\big]\Big)\omega_h\bigg) = \\
&= \int_S \dot{u}_\Re\mathrm{tr}\Big[J\phi_\Re J\big(\dot{u}_\Re\mathbb{1} - \mathrm{Hess}(\dot{u}_\Re)\big)\Big]\omega_h + \int_S \dot{u}_\Im\mathrm{tr}\Big[J\phi_\Re J\big(\dot{u}_\Im\mathbb{1} - \mathrm{Hess}(\dot{u}_\Im)\big)\Big]\omega_h \\
&\quad - \int_S \dot{u}_\Re\mathrm{tr}\Big[J\phi_\Im J\big(\dot{u}_\Im\mathbb{1} - \mathrm{Hess}(\dot{u}_\Im)\big)\Big]\omega_h + \int_S \dot{u}_\Im\mathrm{tr}\Big[J\phi_\Im J\big(\dot{u}_\Re\mathbb{1} - \mathrm{Hess}(\dot{u}_\Re)\big)\Big]\omega_h\ .
\end{aligned}$$

Since the right-hand side in (14) is clearly symmetric in $\dot{u}$ and $\dot{u}'$, the last two summands above cancel out. Using Sublemma 3.13, we obtain that

$$\begin{aligned}\langle L_\phi \dot{u}, \dot{u}\rangle = 2\int_S \Big(\dot{u}_{\mathfrak{R}}^2 \mathrm{tr}(J\phi_{\mathfrak{R}} J) + \dot{u}_{\mathfrak{I}}^2 \mathrm{tr}(J\phi_{\mathfrak{R}} J)\Big)\, \omega_h \\ - 2\int_S \Big(\det(\phi_{\mathfrak{R}})\langle \phi_{\mathfrak{R}}^{-1}(\mathrm{grad}\, \dot{u}_{\mathfrak{R}}), \mathrm{grad}\, \dot{u}_{\mathfrak{R}}\rangle + \det(\phi_{\mathfrak{R}})\langle \phi_{\mathfrak{R}}^{-1}(\mathrm{grad}\, \dot{u}_{\mathfrak{I}}), \mathrm{grad}\, \dot{u}_{\mathfrak{I}}\rangle\Big)\, \omega_h\end{aligned}$$

so that

$$\begin{aligned}\langle -L_\phi \dot{u}, \dot{u}\rangle = 2\int_S \mathrm{tr}(\phi_{\mathfrak{R}})\big(\dot{u}_{\mathfrak{R}}^2 + \dot{u}_{\mathfrak{I}}^2\big)\, \omega_h \\ + 2\int_S \det(\phi_{\mathfrak{R}})\Big(\langle \phi_{\mathfrak{R}}^{-1}(\mathrm{grad}\, \dot{u}_{\mathfrak{R}}), \mathrm{grad}\, \dot{u}_{\mathfrak{R}}\rangle + \langle \phi_{\mathfrak{R}}^{-1}(\mathrm{grad}\, \dot{u}_{\mathfrak{I}}), \mathrm{grad}\, \dot{u}_{\mathfrak{I}}\rangle\Big)\, \omega_h\ .\end{aligned}$$

Equation (15) now follows from the fact that $\phi_{\mathfrak{R}} = \mathfrak{R}(\phi)$ is positive-definite: it is enough to take $c = \max\{4\lambda_+,\ 1/4\lambda_-\}$, where λ_+ and λ_- are the maximum and the minimum eigenvalue of $\mathfrak{R}(\phi)_p$ as p ranges over all points of S. □

4. Holomorphicity of the monodromy map

Thanks to Theorem A, the space $\mathcal{MI}$ of minimizing immersions in germs of hyperbolic manifolds can be identified to the space $\mathcal{D}$ of immersion data. The aim of this section is to show that, under this identification, the monodromy map $\mathsf{Mon} : \mathcal{D} \to \mathcal{X}$, that sends the datum $\phi \in \mathcal{D}$ corresponding to the $\mathrm{PSL}_2(\mathbb{C})$-class of an immersion $[\tilde{f}_\phi, \rho_\phi] \in \mathcal{MI}$ to the conjugacy class of its monodromy $[\rho_\phi] \in \mathcal{X}$, is a biholomorphism onto its (open) image and so to prove Theorem C.

In order to do that, we first provide a description of the tangent space to the space $\mathcal{I}$ of equivariant immersions in $\mathbb{H}^3$ (up to the action of $\mathrm{PSL}_2(\mathbb{C})$) and then of the locus $\mathcal{MI} \subset \mathcal{I}$ that is more suited to reveal its complex-linear nature, compared to the one given in Section 3. Then we show that $d\mathsf{Mon}$ is $\mathbb{C}$-linear.

Remark 4.1. Through the statement of Theorem C only deals with the map Mon between complex manifolds of finite dimension, our proof will involve the space $\widetilde{\mathcal{C}}$ and its open subset $\widetilde{\mathcal{I}}$, as well as $\mathcal{C}$ and its open subset $\mathcal{I}$. In Section 2 we defined smooth paths in $\widetilde{\mathcal{C}}$ (resp. in $\mathcal{C}$) and tangent spaces to $\widetilde{\mathcal{C}}$ (resp. to $\mathcal{C}$). If $\mathcal{M}, \mathcal{M}'$ are either an open subset of $\widetilde{\mathcal{C}}$ or $\mathcal{C}$, or a finite-dimensional manifold, we say that a map $\mathcal{F} : \mathcal{M} \to \mathcal{M}'$ is differentiable at $m \in \mathcal{M}$ if there exists a linear map $d\mathcal{F}_m : T_m\mathcal{M} \to T_{\mathcal{F}(m)}\mathcal{M}'$ such that $\mathcal{F}$ sends any germ of a smooth path at $(\mathcal{F}, m)$ with velocity v to the germ of a smooth path at $(\mathcal{M}', \mathcal{F}(m))$ with velocity $d\mathcal{F}_m(v)$. In this case we will say that $d\mathcal{F}_m$ is the differential of $\mathcal{F}$ at m.

4.1. **The bundle of local Killing vector fields on a hyperbolic manifold.**

We collect in this section some well-known facts that will be useful below.

Given a point $x \in \mathbb{H}^3$, we call *local Killing vector fields* the germs at x of Killing vector fields on $\mathbb{H}^3$ for the hyperbolic metric. The vector space $\mathbb{E}_x$ of such germs at x has a natural structure of Lie algebra, isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Definition 4.2 (Bundle of local Killing vector fields)**.** The *bundle of local Killing vector fields* in $\mathbb{H}^3$ is the bundle $\mathbb{E} \to \mathbb{H}^3$ whose fiber $\mathbb{E}_x$ at a point $x \in \mathbb{H}^3$ is the Lie algebra of local Killing vector fields at x.

Via the identification of $\mathfrak{sl}_2(\mathbb{C})$ with the space of global Killing vector fields on $\mathbb{H}^3$, the bundle $\mathbb{E}$ has a natural trivialization

$$\mathfrak{sl}_2(\mathbb{C}) \times \mathbb{H}^3 \longrightarrow \mathbb{E}$$

that sends a couple $(\check{\xi}, x)$ to the germ of $\check{\xi}$ at x. The natural flat connection on $\mathfrak{sl}_2(\mathbb{C}) \times \mathbb{H}^3$ then induces a flat connection $\mathbb{D}$ on $\mathbb{E}$: thus, global flat sections of $\mathbb{D}$ identify with global Killing vector fields on $\mathbb{H}^3$.
The action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{H}^3$ lifts to the product bundle $\mathfrak{sl}_2(\mathbb{C}) \times \mathbb{H}^3$ via the adjoint action on $\mathfrak{sl}_2(\mathbb{C})$. Similarly, it also naturally lifts to $\mathbb{E}$: if $g \in \mathrm{PSL}_2(\mathbb{C})$ and $\check{\xi} \in \mathbb{E}_x$ is a local Killing vector field at $x \in \mathbb{H}^3$, the image of $\check{\xi}$ in $\mathbb{E}_{g\cdot x}$ is the local Killing vector field $g_*\check{\xi}$ at $g \cdot x$. The above trivialization of $\mathbb{E}$ is equivariant with respect to such $\mathrm{PSL}_2(\mathbb{C})$-actions.

4.1.1. *Identification between* $\mathbb{E}$ *and* $T_{\mathbb{C}}\mathbb{H}^3$. There is a very natural *evaluation map* $\mathbb{E} \to T\mathbb{H}^3$ that, at $x \in \mathbb{H}^3$, sends a local Killing vector field $\check{\xi}(x) \in \mathbb{E}_x$ to its value at x.
Such evaluation map can be enriched so to include first-order derivatives of the local Killing vector field. Specifically in dimensione 3 there is an identification

$$\mathbb{E} \longrightarrow T_{\mathbb{C}}\mathbb{H}^3$$

that is defined as follows. Given $x \in \mathbb{H}^3$, it sends a local Killing vector field $\check{\xi}(x) \in \mathbb{E}_x$ to the unique complex tangent vector $\check{X}_{\check{\xi}}(x) + i\check{Y}_{\check{\xi}}(x) \in T_{\mathbb{C},x}\mathbb{H}^3$ that satisfies

- $\check{X}_{\check{\xi}}(x)$ equal to the value at x of $\check{\xi}(x)$, considered as a Killing vector field defined in the neighborhood of x,
- $\check{Y}_{\check{\xi}}(x)$ is defined by the condition that $\mathsf{S}^{\check{X}_{\check{\xi}}} = \check{Y}_{\check{\xi}} \times \bullet$ (see Lemma 2.7),

where we denoted by $\nabla^{\mathbb{H}^3}$ the Levi-Civita connection of the hyperbolic metric on $\mathbb{H}^3$, and by the same symbol its complexification on $T_{\mathbb{C}}\mathbb{H}^3 \cong (T\mathbb{H}^3) \otimes_{\mathbb{R}} \mathbb{C}$. Abusing notations a bit, we will still denote by $\mathbb{D}$ the flat connection on $T_{\mathbb{C}}\mathbb{H}^3$ obtained as the image of the connection $\mathbb{D}$ on $\mathbb{E}$ through the identification of $\mathbb{E}$ with $T_{\mathbb{C}}\mathbb{H}^3$.

Lemma 4.3 (Naturality of $\mathbb{E} \cong T_{\mathbb{C}}\mathbb{H}^3$)**.** *The identification*

$$\begin{aligned} \mathbb{E} &\longrightarrow T_{\mathbb{C}}\mathbb{H}^3 \\ \check{\xi} &\longmapsto \check{X}_{\check{\xi}} + i\check{Y}_{\check{\xi}} \end{aligned}$$

is $\mathbb{C}$*-linear and it is equivariant with respect to the natural action of* $\mathrm{PSL}_2(\mathbb{C})$ *on* $\mathbb{E}$ *and on* $T_{\mathbb{C}}\mathbb{H}^3$*. Moreover, the flat connection* $\mathbb{D}$ *on* $T_{\mathbb{C}}\mathbb{H}^3$ *can be expressed as*

$$\mathbb{D}_\bullet(\check{X} + i\check{Y}) = \nabla^{\mathbb{H}^3}_\bullet(\check{X} + i\check{Y}) + i(\check{X} + i\check{Y}) \times \bullet$$

in terms of $\nabla^{\mathbb{H}^3}$*.*

Proof. The $\mathbb{C}$-linearity is easy to check. The relation between $\mathbb{D}$ and $\nabla^{\mathbb{H}^3}$ is proven in [16]. ☐

4.1.2. *The case of equivariant maps.* Fix a universal cover $\widetilde{S} \to S$ and an equivariant map $\tilde{f} : \widetilde{S} \to \mathbb{H}^3$ with monodromy $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$.
The bundle $\mathbb{E}$ pulls-back via $\tilde{f}$ to the bundle $\widetilde{E}$ on $\widetilde{S}$ (isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \times \widetilde{S}$) endowed with a flat connection $\widetilde{D}$ and $\pi_1(S)$-action via ρ. Its quotient $E_\rho := \widetilde{E}/\pi_1(S)$ is a $\mathfrak{sl}_2(\mathbb{C})$-bundle on S and we denote by D its induced flat connection.
On the other hand, the bundle $\tilde{f}^* T_{\mathbb{C}}\mathbb{H}^3$ carries a connection still denoted by $\nabla^{\mathbb{H}^3}$, which is the pull-back via $\tilde{f}$ of the complexified Levi-Civita connection on $T_{\mathbb{C}}\mathbb{H}^3$.
As on $\mathbb{H}^3$, there is a natural evaluation map $\widetilde{E} \to \tilde{f}^* T\mathbb{H}^3$ that can be upgraded to an identification $\widetilde{E} \to \tilde{f}^* T_{\mathbb{C}}\mathbb{H}^3$ using Lemma 4.3. We denote by $\nabla^{\widetilde{E}}$ the connection on $\widetilde{E}$ corresponding to $\nabla^{\mathbb{H}^3}$ on $\tilde{f}^* T_{\mathbb{C}}\mathbb{H}^3$, and by ∇^E the induced one on E.

4.2. **The application $\tilde{\sigma}$.** Let $(\tilde{f}, \rho)$ be an equivariant immersion of $\widetilde{S}$ into $\mathbb{H}^3$. Define

$$\begin{aligned} \tilde{\sigma} :\ \Gamma(\tilde{f}^* T\mathbb{H}^3) &\longrightarrow \Gamma(\tilde{f}^* T_{\mathbb{C}}\mathbb{H}^3) \cong \Gamma(\widetilde{E}) \\ \tilde{X} &\longmapsto \tilde{\sigma}_{\tilde{X}} = \tilde{X} + i\tilde{X}' \end{aligned}$$

where $\tilde{X}'$ is the unique vector field that satisfies $\mathsf{S}^{\tilde{X}}_{\tilde{f}} = \tilde{X}' \times \bullet$ (see Definition 2.8). The main properties of $\tilde{\sigma}$ are collected in the following statement.

Lemma 4.4 (Properties of $\tilde{\sigma}$)**.** *The map $\tilde{\sigma}$ is $\mathbb{R}$-linear and it satisfies the following properties:*

(i) $\tilde{\sigma}_{\gamma_* \tilde{X}}(\gamma(\tilde{p})) = \mathrm{Ad}_{\rho(\gamma)} \tilde{\sigma}_{\tilde{X}}(\tilde{p})$ *for all* $\tilde{p} \in \widetilde{S}$*;*
(ii) $\tilde{X} \in \Gamma(\tilde{f}^* T\mathbb{H}^3)$ *is the evaluation of a global Killing vector field* $\tilde{\xi} \in \Gamma(\widetilde{E})$ *if and only if* $\tilde{\sigma}_{\tilde{X}}$ *is* $\tilde{D}$*-parallel (and in this case* $\tilde{\sigma}_{\tilde{X}} = \tilde{\xi}$*);*
(iii) $\tilde{\sigma}_{\tilde{X}} \in \Gamma(\tilde{f}^* T_{\mathbb{C}}\mathbb{H}^3)$ *is* $\tilde{D}$*-parallel if and only if* $\mathsf{A}^{\tilde{X}}_{\tilde{f}} = 0$*;*

for every $\tilde{X} \in \Gamma(\tilde{f}^* T\mathbb{H}^3)$.

Proof. The $\mathbb{R}$-linearity of $\tilde{\sigma}$ follows directly from the definition.

As for (i), we compute $\tilde{\sigma}_{\gamma_* \tilde{X}}(\gamma(\tilde{p})) = \gamma_* \tilde{X}(\gamma(\tilde{p})) + i(\gamma_* \tilde{X})'(\gamma(\tilde{p}))$. Since $\pi_1(S)$ acts on $\tilde{f}^* T\mathbb{H}^3$ isometrically, it preserves the cross-product and $\nabla^{\mathbb{H}^3}$. Hence, $\mathsf{S}^{\gamma_* \tilde{X}}_{\tilde{f}} \circ \gamma_* = \gamma_* \circ \mathsf{S}^{\tilde{X}}_{\tilde{f}}$ and so $(\gamma_* \tilde{X})' = \gamma_* \tilde{X}'$. We have then that $\tilde{\sigma}_{\gamma_* \tilde{X}}(\gamma(\tilde{p})) = \gamma_* \big((\tilde{X} + i\tilde{X}')(\tilde{p})\big)$. The conclusion then follows because the identification $\widetilde{E} \cong \tilde{f}^* T_{\mathbb{C}}\mathbb{H}^3$ is equivariant with respect to the action of $\pi_1(S)$.

About (ii), $\tilde{\sigma}_{\tilde{X}}$ is $D^{\tilde{E}}$-parallel if and only if there exists a global Killing field $\tilde{\xi}$ such that $\tilde{\sigma}_{\tilde{X}} = \tilde{\xi}$. This clearly happens if and only if $\tilde{X}$ is the evaluation of such global Killing vector field $\tilde{\xi}$.

Concerning (iii), suppose first that $\tilde{\sigma}_{\tilde{X}}$ is $\tilde{D}$-parallel. By (ii), the vector field $\tilde{X}$ is the evaluation of a global Killing vector field and so $\mathsf{A}^{\tilde{X}}_{\tilde{f}} = 0$.

Vice versa, suppose that $\mathsf{A}_{\tilde f}^{\tilde X} = 0$, and so $\nabla^{\mathbb{H}^3}\tilde X = \mathsf{S}_{\tilde f}^{\tilde X}$. Let $\tilde X', \tilde X''$ be the sections of $\tilde f^* T\mathbb{H}^3$ defined by $\mathsf{S}_{\tilde f}^{\tilde X} = \tilde X' \times \bullet$ and $\mathsf{S}_{\tilde f}^{\tilde X'} = \tilde X'' \times \bullet$. Now

$$\Re(\tilde D \tilde\sigma_{\tilde X}) = \Re(\tilde D(\tilde X + i\tilde X')) = \nabla^{\mathbb{H}^3}\tilde X - (\tilde X' \times \bullet) = 0.$$

On the other hand, the imaginary part of $\tilde D\tilde\sigma_{\tilde X}$ is given by

$$\Im(\tilde D \tilde\sigma_{\tilde X}) = \Im(\tilde D(\tilde X + i\tilde X')) = \nabla^{\mathbb{H}^3}\tilde X' + (\tilde X \times \bullet) = \mathsf{A}_{\tilde f}^{\tilde X'} + \big((\tilde X'' + \tilde X) \times \bullet\big).$$

By [3, Lemma 5.7], we have that

$$J^{\tilde I}\mathsf{A}_{\tilde f}^{\tilde X'} + \langle \tilde X + \tilde X'', \tilde N\rangle \mathbb{1} = 0$$

where $\tilde N$ is the positive unit vector normal to the immersion.
Since $\mathsf{A}_{\tilde f}^{\tilde X'}$ is $\tilde I$-self-adjoint, $J^{\tilde I}\mathsf{A}_{\tilde f}^{\tilde X'}$ is traceless and so

$$\mathsf{A}_{\tilde f}^{\tilde X'} = 0\ . \tag{16}$$

Moreover, $\langle \tilde X + \tilde X'', \tilde N\rangle \mathbb{1} = 0$ so that $\tilde X + \tilde X''$ is tangent to the immersion. On the other hand, by the curvature properties of $\mathbb{H}^3$ we have

$$R(e_1, e_2)\tilde X = (e_1 \times e_2) \times \tilde X \tag{17}$$

for any local frame (e_1, e_2) on $\widetilde S$. Since $\mathsf{A}_{\tilde f}^{\tilde X'} = 0$, we have $\nabla^{\mathbb{H}^3}\tilde X' = (\tilde X'' \times \bullet)$ and so

(18)

$$R(e_1,e_2)\tilde X = \nabla^{\mathbb{H}^3}_{e_1}\tilde X' \times e_2 - \nabla^{\mathbb{H}^3}_{e_2}\tilde X' \times e_1 = (\tilde X'' \times e_1)\times e_2 - (\tilde X'' \times e_2) \times e_1 = -(e_1 \times e_2)\times \tilde X''.$$

Comparing (17) and (18), we deduce that $(e_1 \times e_2) \times (\tilde X + \tilde X'') = 0$ and so the tangential part of $\tilde X + \tilde X''$ vanishes. Since we have seen above that $\tilde X + \tilde X''$ is tangent to the image of $\tilde f$, we conclude that $\tilde X + \tilde X'' = 0$. This identity and (16) together prove that $\Im(\tilde D \tilde\sigma_{\tilde X}) = 0$. □

4.3. **A complex viewpoint on first-order deformations of immersions.** Let $\tilde f$ be an immersion of $\widetilde S$ into $\mathbb{H}^3$. We recall that a deformation $\tilde{\boldsymbol f} = (\tilde f_t)_{t\in(-\epsilon,\epsilon)}$ of $\tilde f$ determines a variational field $\tilde X \in \Gamma(\tilde f^* T\mathbb{H}^3)$ by Corollary 2.36. Moreover, the deformation $\tilde{\boldsymbol f}$ is tangent to a $\mathrm{PSL}_2(\mathbb{C})$-orbit if and only if $\tilde X$ is the evalution of a global Killing vector field.
The above considerations can be rephrased in terms of the complex $C^\bullet_{\tilde D}(\widetilde E)$ of $\widetilde E$-valued differential forms on $\widetilde S$ with differential induced by $\tilde D$.

Lemma 4.5 (First-order deformations of immersions and $\widetilde E$-valued forms)**.** *The sequence*

$$0 \longrightarrow Z^0_{\tilde D}(\widetilde E) \longrightarrow \Gamma(\tilde f^* T\mathbb{H}^3) \xrightarrow{\tilde\theta} Z^1_{\tilde D}(\widetilde E)$$

induced by the evaluation map $\Gamma(\widetilde E) \to \Gamma(\tilde f^* T\mathbb{H}^3)$ and by $\tilde\theta$ defined as $\tilde\theta_{\tilde X} := \tilde D\tilde\sigma_{\tilde X}$ is exact. Moreover, first-order deformations of $\tilde f$ "up to the action of $\mathrm{PSL}_2(\mathbb{C})$*" identify to the image of $\tilde\theta$ inside $Z^1_{\tilde D}(\widetilde E)$.*

Remark 4.6 (Non-surjectivity of $\tilde{\theta}$). It is not true that the map $\tilde{\theta}$ is surjective, since elements of $Z^1_{\tilde{D}}(\widetilde{E})$ of type $\tilde{\sigma}_{\tilde{X}}$ are determined by their real part.

4.3.1. *The equivariant case.* Suppose now that $(\tilde{f},\rho)$ is an equivariant immersion of $\widetilde{S}$ into $\mathbb{H}^3$ and that $(\tilde{\boldsymbol{f}},\boldsymbol{\rho})$ is a deformation of $(\tilde{f},\rho)$. By Lemma 2.32 and Lemma 2.34

- $(\tilde{X},\varsigma)\in\mathcal{Z}^1_{(\tilde{f},\rho)}$, i.e. $\gamma_*\tilde{X}=\tilde{X}-\varsigma_\gamma\big|_{\widetilde{S}}$, and
- $(\tilde{\boldsymbol{f}},\boldsymbol{\rho})$ is tangent to a $\mathrm{PSL}_2(\mathbb{C})$-orbit if and only if $(\tilde{X},\varsigma)\in\mathcal{B}^1_{(\tilde{f},\rho)}$, i.e. there exists a global Killing vector field $\tilde{\xi}\in\Gamma(\widetilde{E})$ such that $\tilde{X}$ is the evaluation of $\tilde{\xi}$ and $\varsigma_\gamma=\tilde{\xi}-\mathrm{Ad}_{\rho(\gamma)}\tilde{\xi}$ for all $\gamma\in\pi_1(S)$.

Applying Lemma 4.4 and using that $\gamma_*\tilde{X}=\tilde{X}-\varsigma_\gamma\Big|_{\widetilde{S}}$, we have that

$$\tilde{\sigma}_{\tilde{X}}(\gamma(\tilde{p}))-\varsigma_\gamma(\tilde{p})=\mathrm{Ad}_{\rho(\gamma)}\tilde{\sigma}_{\tilde{X}}(\tilde{p})\,. \tag{19}$$

Applying $\tilde{D}$ to (19) and remembering that ς_γ is $\tilde{D}$-parallel, we obtain

$$\tilde{\theta}_{\tilde{X}}(\gamma(\tilde{p}))\circ(d\gamma)_{\tilde{p}}=\mathrm{Ad}_{\rho(\gamma)}\circ\tilde{\theta}_{\tilde{X}}(\tilde{p}). \tag{20}$$

Condition (20) is in fact equivalent to the fact that $\tilde{\theta}$ is the lift of a D-closed E-valued 1-form on S. Thus, an infinitesimal deformation $(\widetilde{X},\varsigma)$ corresponds to a deformation through a family of equivariant maps if and only if $\tilde{\theta}_{\tilde{X}}$ is the lift of an E-valued D-closed 1-form $\theta_{\tilde{X}}$, whose periods then correspond to the infinitesimal variation of the monodromy. In particular, if $\tilde{X}$ is a ρ-invariant section of $\tilde{f}^*T\mathbb{H}^3$, then the couple $(\tilde{X},\varsigma=0)$ corresponds to an infinitesimal isomonodromic deformation and $\tilde{\theta}_{\tilde{X}}$ is the lift of a D-exact E-valued 1-form on S, i.e. the section $\tilde{\sigma}_{\tilde{X}}$ is the lift of a section of E.
Using the complex $C^\bullet_D(E)$ is E-valued differential forms on S with differential induced by D, we can condense the above observations in the following lemma.

Lemma 4.7 (First-order deformations of equivariant immersions and E-valued forms)**.** *First-order deformations of the equivariant immersion* $(\tilde{f},\rho)$ *correspond to those elements* $\theta\in Z^1_D(E)$ *that are induced by* $\tilde{\theta}_{\tilde{X}}$ *for some* $\tilde{X}$. *In particular, elements in* $B^1_D(E)$ *correspond to first-order deformations that fix the conjugacy class of the monodromy. Moreover, first-order deformations of* $\rho\in\mathcal{X}$ *correspond to elements of* $H^1_D(E)$*, whose periods give the infinitesimal deformation of the monodromy.*

Remark 4.8. Note that ς does not have any role in the definition of $\tilde{\theta}_{\tilde{X}}$. Indeed, $\tilde{\theta}_{\tilde{X}}$ determines $\tilde{X}$ up to adding a vector field obtained by evaluation a global Killing field, and ς can be recovered from $\varsigma_\gamma\Big|_{\widetilde{S}}=\tilde{X}-\gamma_*\tilde{X}$, since $\tilde{f}$ is assumed to be an immersion.

If $(\tilde{f}_\phi,\rho_\phi)$ corresponds to the immersion datum $\phi\in\mathcal{D}$, then the conclusions drawn in the above lemma can be also visually synthetized into the following commutative diagram

$$\begin{tikzcd}
 & T_{(\tilde f_\phi,\rho_\phi)}\widetilde{\mathcal{MI}} \arrow[r, hook] \arrow[d] & T_{(\tilde f_\phi,\rho_\phi)}\widetilde{\mathcal{I}} \arrow[r, "\tilde\sigma"] \arrow[d] \arrow[dr, "\Theta"] & \{\tilde\sigma \in \Gamma(\widetilde{E}) \,|\, \tilde D\tilde\sigma \text{ is } \rho_\phi\text{-invariant}\} \arrow[d, "\tilde D"] \\
T_\phi\mathcal{D} \arrow[r, leftrightarrow] \arrow[drr, bend right, "d\mathsf{Mon}_\phi"] & T_{[\tilde f_\phi,\rho_\phi]}\mathcal{MI} \arrow[r, hook] & T_{[\tilde f_\phi,\rho_\phi]}\mathcal{I} \arrow[r] \arrow[d] & Z^1_D(E) \arrow[d] \\
 & & T_{[\rho_\phi]}\mathcal{X} \arrow[r, "\cong"] & H^1_D(E)
\end{tikzcd}$$

in which the horizontal arrow in the top-right corner sends $\tilde X$ to $\tilde\sigma_{\tilde X}$, and Θ sends $\tilde X$ to $\theta_{\tilde X}$.
We remark that the vector spaces in the above diagram that are endowed with a complex structure are $T_\phi\mathcal{D} \cong T_{[\tilde f_\phi,\rho_\phi]}\mathcal{MI}$ and the ones in the right column. As a consequence, $T_{(\tilde f_\phi,\rho_\phi)}\widetilde{\mathcal{MI}}$ is a complex vector space too. We will see below that the map $T_\phi\mathcal{D} \to Z^1_D(E)$ between complex vector spaces that sends $\dot\phi$ associated to a variational field $\tilde X$ to $\theta_{\tilde X}$ is <u>not</u> $\mathbb{C}$-linear in general.

4.4. **Complex-linearity of $d\mathsf{Mon}$.** Fix an immersion $(\tilde f_\phi,\rho_\phi)$ with corresponding datum $\phi \in \mathcal{D}$ and let $\widetilde{E} \to \widetilde{S}$ and $E \to S$ be the associated bundles of local Killing vector fields.
Denote by $\tilde N$ the section of $\tilde f_\phi^* T\mathbb{H}^3$ representing the positively-oriented unit vector field normal to the image of $\tilde f_\phi$. Viewing $\tilde N$ as a section of $\widetilde{E}$, it is ρ_ϕ-invariant and so it descends to a section $N \in \Gamma(E)$.
Consider a tangent vector $\dot\phi \in T_\phi\mathcal{D}$. By Theorem A, there is a germ of path $t \mapsto \phi + t\cdot\dot\phi + o(t)$ of immersion data which is realized by a deformation $(\tilde{\boldsymbol{f}},\boldsymbol{\rho})$ of $(\tilde f_\phi,\rho_\phi)$. Denote by $\tilde X$ the variational field associated to $\tilde{\boldsymbol{f}}$ and by $\theta_1 \in Z^1_D(E)$ the 1-cocycle $\theta_{\tilde X}$ associated to $\tilde X$. Similarly, the path $t \mapsto \phi + t\cdot(i\dot\phi) + o(t)$ is realized by a family of immersions corresponding to the 1-cocycle $\theta_i \in Z^1_D(E)$.
The holomorphicity of Mon will then be a consequence of the following result.

Theorem 4.9 (Relation between θ_1 and θ_i)**.** *There exists a smooth function $\nu : S \to \mathbb{C}$ such that $i\theta_1 - \theta_i + D(\nu N) = 0$. As a consequence, $[\theta_i] = i[\theta_1] \in H^1_D(E)$.*

Remark 4.10. Since the function ν can be nonzero, the map $T_\phi\mathcal{D} \to Z^1_D(E)$ is not $\mathbb{C}$-linear in general.

Since $\tilde f$ is an immersion, $T\widetilde{S}$ is a subbundle of $\tilde f^* T\mathbb{H}^3$. Having identified $\tilde f^* T_{\mathbb{C}}\mathbb{H}^3$ to $\widetilde{E}$, it makes sense to decompose θ_1 into a component θ_1^T tangent to the surface and a normal component. More explicitly, separating real and imaginary parts as $\theta_1 = \Re(\theta_1) + i\Im(\theta_1)$, we have $\theta_1^T = \Re(\theta_1)^T + i\Im(\theta_1)^T$. Indeed, θ_1^T is a $T_{\mathbb{C}}S$-valued 1-form on S.
The following proposition relates $\dot\phi$ with θ_1^T and will be the key point to prove Theorem 4.9.

Proposition 4.11 (First-order variation of immersion data and 1-cocycles)**.** *There exists a smooth function $\eta : S \to \mathbb{R}$ such that*

$$\dot{\phi} = b\theta_1^T + \eta J\phi.$$

Proof. Recall that $\tilde{\theta}_1 = \tilde{D}\tilde{\sigma}_{\tilde{X}} = \tilde{D}(\tilde{X} + i\tilde{X}')$, where $\tilde{X}'$ is defined as above in Section 4.2. Since $\tilde{f}$ is fixed, we denote the $\tilde{I}$-self-adjoint derivative $\mathsf{A}_{\tilde{f}}$ and the $\tilde{I}$-skew-self-adjoint derivative $\mathsf{S}_{\tilde{f}}$ just by A and S.
As in Lemma 5.5 of [3], we have

$$\dot{\tilde{I}} = 2\tilde{I}(\mathsf{A}^{\tilde{X}}\bullet, \bullet).$$

Since

$$\dot{\tilde{I}} = \tilde{h}(\dot{\tilde{b}}, \tilde{b}) + \tilde{h}(\tilde{b}, \dot{\tilde{b}}) = \tilde{I}(\tilde{b}^{-1}\dot{\tilde{b}}\bullet, \bullet) + \tilde{I}(\bullet, \tilde{b}^{-1}\dot{\tilde{b}}\bullet) ,$$

the operators $\mathsf{A}^{\tilde{X}}$ and $b^{-1}\dot{b}$ have the same $\tilde{I}$-self-adjoint component, so

$$\dot{\tilde{b}} = \tilde{b}\mathsf{A}^{\tilde{X}} + \tilde{\eta}\tilde{J}\tilde{b} \tag{21}$$

for some smooth function $\tilde{\eta} : \widetilde{S} \to \mathbb{R}$. On the other hand, by Lemma 5.6 of [3] we have

$$\dot{\tilde{a}} = J^{\tilde{I}}\mathsf{A}^{\tilde{X}'} - \langle \tilde{X} + \tilde{X}'', \tilde{N}\rangle \mathbb{I} - \mathsf{A}^{\tilde{X}}\tilde{a} .$$

Since $\frac{d}{dt}\left(\tilde{b}\tilde{a}\right) = \tilde{b}\dot{\tilde{a}} + \dot{\tilde{b}}\tilde{a}$, we get

$$\frac{d}{dt}\left(\tilde{b}\tilde{a}\right) = \tilde{b}J^{\tilde{I}}\mathsf{A}^{\tilde{X}'} - \langle \tilde{X} + \tilde{X}'', \tilde{N}\rangle\tilde{b} - \tilde{b}\mathsf{A}^{\tilde{X}}\tilde{a} + \tilde{b}\mathsf{A}^{\tilde{X}}\tilde{a} + \tilde{\eta}\tilde{J}\tilde{b}\tilde{a} \tag{22}$$

$$= \tilde{J}\tilde{b}\mathsf{A}^{\tilde{X}'} - \langle \tilde{X} + \tilde{X}'', \tilde{N}\rangle\tilde{b} + \tilde{\eta}\tilde{J}\tilde{b}\tilde{a} . \tag{23}$$

Since $\tilde{\phi} = \tilde{b} - i\tilde{J}\tilde{b}\tilde{a}$, using (21) and (23), we obtain that

$$\begin{aligned} \dot{\tilde{\phi}} &= \tilde{b}\mathsf{A}^{\tilde{X}} + \tilde{u}\tilde{J}\tilde{b} + i(\tilde{b}(\mathsf{A}^{\tilde{X}'} + \tilde{\eta}\tilde{a}) + \langle \tilde{X} + \tilde{X}'', \tilde{N}\rangle\tilde{J}\tilde{b}) \\ &= \tilde{b}\mathsf{A}^{\tilde{X}} + \tilde{\eta}\tilde{J}\tilde{b} + i\tilde{b}(\mathsf{A}^{\tilde{X}'} + \tilde{\eta}\tilde{a} - \langle \tilde{X} + \tilde{X}'', \tilde{N}\rangle\tilde{b}^{-1}\tilde{J}\tilde{b}) \\ &= \tilde{b}(\mathsf{A}^{\tilde{X}} + i(\mathsf{A}^{\tilde{X}'} + \langle \tilde{X} + \tilde{X}'', \tilde{N}\rangle J^{\tilde{I}})) + \tilde{\eta}\tilde{J}\tilde{b} + i\tilde{\eta}\tilde{b}\tilde{a} . \end{aligned}$$

Notice that

$$\begin{aligned} \tilde{\theta}_1 &= \tilde{D}(\tilde{X} + i\tilde{X}') \\ &= (\nabla\tilde{X} - \tilde{X}' \times \bullet) + i(\nabla\tilde{X}' + \tilde{X} \times \bullet) \\ &= (\mathsf{A}^{\tilde{X}} + \mathsf{S}^{\tilde{X}} - \tilde{X}' \times \bullet) + i(\mathsf{A}^{\tilde{X}'} + \mathsf{S}^{\tilde{X}'} + \tilde{X} \times \bullet) \\ &= \mathsf{A}^{\tilde{X}} + i(\mathsf{A}^{\tilde{X}'} + (\tilde{X} + \tilde{X}'') \times \bullet) , \end{aligned}$$

and so $\tilde{\theta}_1^T = \mathsf{A}^{\tilde{X}} + i(\mathsf{A}^{\tilde{X}'} + \langle \tilde{X} + \tilde{X}'', \tilde{N}\rangle J^{\tilde{I}})$. It follows that

$$\dot{\tilde{\phi}} = \tilde{b}\tilde{\theta}_1^T + \tilde{\eta}\tilde{J}(\tilde{b} - i\tilde{J}\tilde{b}\tilde{a}) = \tilde{b}\tilde{\theta}_1^T + \tilde{\eta}\tilde{J}\tilde{\phi}.$$

Since all the other tensors are invariant, $\tilde{\eta}$ must come from a function $\eta : S \to \mathbb{R}$ and the result follows. □

As a consequence of the above proposition, we obtain a relation between the tangent components of θ_1 and θ_i.

Corollary 4.12 (Tangent components of θ_1 and θ_i)**.** *There exists a smooth complex valued function $\nu : S \to \mathbb{C}$ such that $i\theta_1^T - \theta_i^T + \nu \cdot DN = 0$.*

Proof. By Proposition 4.11 there exist smooth functions $\eta_1, \eta_i : S \to \mathbb{R}$ such that

$$\dot{\phi} = b\theta_1^T + \eta_1 J\phi$$
$$i\dot{\phi} = b\theta_i^T + \eta_i J\phi$$

so we deduce that $b(i\theta_1 - \theta_i)^T + i\nu \cdot J\phi = 0$, where $\nu := \eta_1 + i\eta_i$. Recalling that $DN = -a + iJ^I$, or equivalently $J\phi = -ib \cdot DN$, we obtain that

$$b \cdot (i\theta_1^T - \theta_i^T + \nu \cdot DN) = 0\,.$$

Since b is invertible, the result follows. □

The relation obtained in the above corollary is almost the wished one. In order to take care of the normal component of θ_1 and θ_i, we will need the following.

Lemma 4.13 (Vanishing 1-cocycles are detected by their tangent component)**.** *Let $\tau \in Z^1_D(E)$ be any smooth 1-cocycle. Then $\tau = 0$ if and only if $\tau^T = 0$.*

Proof. Clearly, if $\tau = 0$, then its tangent component vanishes.
Conversely, suppose that $\tau^T = 0$, so that $\tau = \zeta \otimes N$, where ζ is a (complex-valued) 1-form on S. We want to prove that $\zeta = 0$.
Since $0 = D\tau = d\zeta \otimes N + \zeta \wedge DN$, and since DN takes values in $T_{\mathbb{C}}S$, we deduce that $d\zeta = 0$ and $\zeta \wedge DN = 0$.
Fix any point of S and take a I-orthonormal basis (e_1, e_2) of TS at that point, formed by eigenvectors for a. Then imposing that $(\zeta \wedge DN)(e_1, e_2) = 0$, and using that $DN = -a + iJ^I$, one gets

$$\zeta \wedge DN = \left(-(\Re\zeta) \wedge a - (\Im\zeta) \wedge J^I\right) + i\left((\Re\zeta) \wedge J^I - (\Im\zeta) \wedge a\right)$$

and so

$$0 = 2\Re(\zeta \wedge DN)(e_1, e_2) = -\left((\Re\zeta)(e_1)\, a(e_2) + (\Im\zeta)(e_1)\, J^I e_2\right) \;+\; \left((\Re\zeta)(e_2)\, a(e_1) + (\Im\zeta)(e_2)\, J^I e_1\right)\;,$$
$$0 = 2\Im(\zeta \wedge DN)(e_1, e_2) = \left((\Re\zeta)(e_1)\, J^I e_2 - (\Im\zeta)(e_1)\, a(e_2)\right) \;-\; \left((\Re\zeta)(e_2)\, J^I e_1 - (\Im\zeta)(e_2)\, a(e_1)\right)\;.$$

Putting $a(e_i) = \lambda_i e_i$, by the first equation it follows that

$$\begin{aligned} \lambda_2(\Re\zeta)(e_1) &= (\Im\zeta)(e_2)\;, \\ -(\Im\zeta)(e_1) &= \lambda_1(\Re\zeta)(e_2)\;. \end{aligned} \tag{24}$$

By the second equation

$$\begin{aligned} (\Re\zeta)(e_1) &= \lambda_1(\Im\zeta)(e_2)\;, \\ -\lambda_2(\Im\zeta)(e_1) &= (\Re\zeta)(e_2)\;. \end{aligned} \tag{25}$$

Suppose by contradiction that $\zeta \neq 0$. Then Equations (24) and (25) imply that $\det(a) = \lambda_1\lambda_2 = 1$. But for a critical immersion, Equations (7) and (9) together show that $\det(a) \leq 0$. This contradiction proves the lemma. □

The $\mathbb{C}$-linearity of $d\mathsf{Mon}$ is then readily obtained.

Proof of Theorem 4.9. Let ν be the function given in Corollary 4.12 and consider the 1-cocycle $\tau := i\theta_1 - \theta_i + D(\nu N) \in Z^1_D(E)$. Since

$$\tau^T = i\theta_1^T - \theta_i^T + \nu \cdot DN,$$

Corollary 4.12 implies that $\tau^T = 0$. Hence, we conclude that $\tau = 0$ by Lemma 4.13. □

We can now give a complete proof of Theorem C.

Proof of Theorem C. By Lemma 2.39, a minimal immersion corresponding to $\phi \in \mathcal{D}$ has non-elementary monodromy. Moreover, the map Mon is holomorphic by Theorem 4.9 and it is injective by Corollary 2.38. Since $\mathcal{D}$ and $\mathcal{X}$ are complex manifolds of the same dimension, Mon is a biholomorphism onto its image, which is in fact an open subset of $\mathcal{X}$. Note finally that such open subset $\mathsf{Mon}(\mathcal{D})$ contains the Fuchsian locus by Theorem 2.19. □

4.5. **The complexified functional.** As in the introduction, define now the functional $\mathsf{F} : \mathcal{X} \to \mathbb{R}_{\geq 0}$ as

$$\mathsf{F}(\rho) \coloneqq \inf \left\{ F(\tilde{f}) \;\middle|\; [\rho, \tilde{f}] \in \mathcal{I} \right\}.$$

The uniqueness proven in Corollary 2.38 implies that the map Mon that sends the immersion datum ϕ to the class $[\rho_\phi]$ of the monodromy representation of the immersion corresponding to ϕ is injective.
Thus, we can identify $\mathcal{D}$ to $\mathsf{Mon}(\mathcal{D}) \subset \mathcal{X}$, so that

$$\mathsf{F}(\rho_\phi) = F(\phi) = \int_S \mathrm{tr}(\Re(\phi))\omega_h = \Re \int_S \mathrm{tr}(\phi)\omega_h$$

for every $\phi \in \mathcal{D}$, by the minimality property of Corollary 2.38. By the complex nature of $\mathcal{D}$, such F can be viewed as the real part of $\mathsf{F}_{\mathbb{C}} : \mathsf{Mon}(\mathcal{D}) \to \mathbb{C}$ defined as

$$\mathsf{F}_{\mathbb{C}}([\rho]) \coloneqq \int_S \mathrm{tr}(\mathsf{Mon}^{-1}(\rho))\omega_h \,.$$

We can now prove the last main result of our paper.

Proof of Theorem D. In view of the above discussion, we are only left to show that $\mathsf{F}_{\mathbb{C}}$ is a holomorphic function. Note that the map $\mathcal{D} \to \mathbb{C}$ defined as $\phi \mapsto \int_S \mathrm{tr}(\phi)\omega_h$ is clearly holomorphic. Thus the conclusion follows, since $\mathsf{Mon} : \mathcal{D} \to \mathsf{Mon}(\mathcal{D})$ is a biholomorphism by Theorem C. □

5. Questions and applications

Let $\mathcal{T}(S)$ be the Teichmüller space of hyperbolic metrics on S, $\mathcal{ML}(S)$ be the space of measured laminations on S and let $\mathcal{QF}(S)$ be the quasi-Fuchsian space of S, i.e. the space of quasi-Fuchsian metrics on $M = S \times \mathbb{R}$. The Fuchsian locus in $\mathcal{QF}(S)$ consists of metrics for which $S \times \{0\}$ is a totally geodesic hyperbolic surface. Consider only maps $f : S \to M$ that are homotopy equivalences.

5.1. **Existence of smooth minimizing maps.** Once a hyperbolic metric $h \in \mathcal{T}(S)$ on S is fixed, we have seen (Theorem C) that there exists a neighborhood Ω_h of the Fuchsian locus in $\mathcal{QF}(S)$ consisting of quasi-Fuchsian structures g on M for which there exists a smooth minimizing map $f : (S, h) \to (M, h_M)$. This smooth minimizing map is unique by Theorem B.
However, we do not know how large this neighbourhood Ω_h is. Moreover, a priori, Ω_h might depend on h.
Does Ω_h coincide with the whole $\mathcal{QF}(S)$? Does Ω_h at least contain all "almost Fuchsian" structures (i.e. metrics h_M for which (M, h_M) contains an embedded minimal surface with principal curvatures in $(-1, 1)$)?

The following less ambitious statement asks whether the neighbourhood Ω_h can be chosen to be independent of the hyperbolic metric h.

Question 5.1. Is there a neighborhood Ω of the Fuchsian locus in $\mathcal{QF}(S)$ such that, for all $h_M \in \Omega$ and all $h \in \mathcal{T}(S)$, there exists a smooth minimizing map from (S,h) to (M,h_M)?

We believe that given any quasi-Fuchsian structure h_M on M, there exists a mapping $f:(S,h) \to (M,h_M)$ in the BV class which is minimizing in a weak sense (but which might be not smooth) – we believe that the existence of minimizing BV maps can be obtained by relatively standard methods.

5.2. **Uniform convexity and uniqueness among non-smooth maps.** Once $h \in \mathcal{T}(S)$ and $h_M \in \mathcal{QF}(S)$ are fixed, one can ask whether a (possibly non-smooth) minimizing map $f:(S,h) \to (M,h_M)$ is unique. Note that such question is equivalent to the uniqueness of the ρ-equivariant minimizing $\tilde{f}:(\widetilde{S},\tilde{h}) \to \mathbb{H}^3$, where $\rho:\pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ is the monodromy representation associated to $M \cong \mathbb{H}^3/\rho(\pi_1(S))$.
Since our arguments for the uniqueness of minimizing maps require some regularity, we believe that uniqueness can be proven among maps of class C^1. One can ask whether uniqueness holds among continuous maps which are minimizing in the weak sense. This question can be related to the convexity of the functional F over the space of maps of lower regularity from (S,h) to (M,h_M), if (M,h_M) is a quasi-Fuchsian (or more generally a complete hyperbolic) 3-dimensional manifold. It is even less clear whether uniqueness of the ρ-equivariant minimizing map can be proven for maps $\widetilde{S} \to \mathbb{H}^3$ in the BV class.

5.3. **Relation between $\mathsf{F}_{\mathbb{C}}$ and the complex length.** When considering diffeomorphisms between hyperbolic surfaces, the 1-energy F is closely related to the hyperbolic length of measured laminations. Specifically, let $h^\star \in \mathcal{T}_S$ be a hyperbolic metric with monodromy ρ, and let $(h_n)_{n\in\mathbb{N}}$ be a sequence of hyperbolic metrics such that $t_n \cdot h_n \to \lambda$ in the sense of convergence of the length spectrum, where $\lambda \in \mathcal{ML}(S)$ and $t_n \to 0$. Call $f_n:(S,h_n) \to (S,h^\star)$ the unique minimal Lagrangian map homotopic to the identity, which can be in fact viewed as a minimizing embedding inside the Fuchsian 3-manifold (M,h_M) associated to $h^\star$, and denote by $\mathsf{F}_{h_n}([\rho])$ the 1-enegy $F(f_n)$.
In [3] we proved that $t_n \cdot \mathsf{F}_{h_n}([\rho]) \to \ell_\lambda([\rho])$, where $\ell_\lambda([\rho])$ is the length of the lamination λ in $(S,h^\star)$, or equivalently in (M,h_M). Such result can be rephrased by saying that, if ρ is the (Fuchsian) monodromy representation of $(S,h^\star)$, then $\mathsf{F}_\bullet([\rho])$ defines a continuous function

$$\mathsf{F}_\bullet([\rho]):(\mathbb{R}_+ \times \mathcal{T}(S)) \cup \mathcal{ML}(S) \longrightarrow \mathbb{R}$$

that restricts to $\ell_\bullet(h^\star)$ on $\mathcal{ML}(S)$.

Suppose now that (M,h_M) is a fixed quasi-Fuchsian manifold with monodromy ρ and that (h_n) is a sequence of hyperbolic metrics on S with $t_n \cdot h_n \to \lambda$ and $t_n \to 0$ as above. Assume that for all n there exists a minimizing map $f_n:(S,h_n) \to (M,h_M)$ with associated immersion datum ϕ_n (which would follow, for example, if the answer to Question 5.1 was positive). Denote by $\mathsf{F}_{\mathbb{C},h_n}([\rho])$ the complex number $\int_S \mathrm{tr}(\phi_n)\omega_{h_n}$ associated to the minimizing map f_n.

Question 5.2. Does $t_n \cdot \mathsf{F}_{\mathbb{C},h_n}([\rho]) \to \ell_{\mathbb{C},\lambda}([\rho])$, where $\ell_{\mathbb{C},\lambda}([\rho])$ is the complex length of the lamination λ in (M, h_M)?

More ambitiously, fixed a quasi-Fuchsian manifold (M, h_M) with monodromy ρ, one could ask whether the complex valued functional $\mathsf{F}_{\mathbb{C},\bullet}([\rho])$ can be extended so to define a continuous function

$$\mathsf{F}_{\mathbb{C},\bullet}([\rho]) : (\mathbb{R}_+ \times \mathcal{T}(S)) \cup \mathcal{ML}(S) \longrightarrow \mathbb{C}$$

that restricts to the complex length function $\ell_{\mathbb{C},\bullet}(M, h_M)$ on $\mathcal{ML}(S)$.

5.4. **Non-quasi-Fuchsian targets.** This above questions are not necessarily restricted to quasi-Fuchsian manifolds – given a closed 3-dimensional hyperbolic manifold (M, h_M) and a homotopy class of maps from (S, h) into (M, h_M) that induce an injection $\pi_1(S) \hookrightarrow \pi_1(M)$, one can ask whether it contains a smooth minimizing immersion, or whether uniqueness holds among minimizing maps of lower regularity. The arguments used to prove uniqueness of smooth minimizing maps in quasi-Fuchsian manifolds also work in this setting.

Appendix A. On the 1-Schatten norm of linear maps

In this section we recall properties of the 1-Schatten norm on the space of linear homomorphisms between vector space of finite dimension endowed with a positive-definite scalar product.

A.1. **Definition and basic properties.** Let V and W be finitely generated vector spaces, equipped with positive-definite scalar products, and assume that $\dim V \leq \dim W$. Any linear map $L : V \to W$ can be factorized as the composition $L = \sigma_L \circ b_L$, where $b_L = \sqrt{L^T \circ L}$ is a non-negative g-self-adjoint endomorphism of V, the map $\sigma_L : V \to W$ is an isometric linear embedding and $L^T : W \to V$ denotes the adjoint of L.

Remark A.1 (Polar decomposition). While b_L is always well-defined, σ_L is uniquely determined provided that L is injective (or equivalently that $\det b_L \neq 0$). In this case we refer to the decomposition $L = \sigma_L \circ b_L$ as the *polar decomposition* of L.

Definition A.2 (1-Schatten norm of a linear map)**.** The 1-*Schatten norm of a linear map* $L : V \to W$ is $\|L\|_1 := \mathrm{tr}(b_L)$, where $b_L := \sqrt{L^T \circ L}$.

Lemma A.3 (Lipschitz nature of 1-Schatten norm on a Hom-space)**.** *The function* $\|\cdot\|_1 : \mathrm{Hom}(V, W) \to \mathbb{R}$ *is a norm on the vector space* $\mathrm{Hom}(V, W)$*. Moreover, it is Lipschitz as a function from* $\mathrm{Hom}(V, W)$ *to* $\mathbb{R}$ *(say, with respect to the natural Riemannian metric on* $\mathrm{Hom}(V, W)$*). It is smooth at homomorphism of maximal rank, but not* C^1 *at homomorphisms of non-maximal rank.*

Proof. The only non-obvious point is that the Schatten norm is a norm. This is proved in [1, Section IV.2]. ☐

Lemma A.4 (1-Schatten norm and Lipschitz linear maps). *If A is a linear endomorphism of W which is C-Lipschitz, then $\|AL\|_1 \le C \cdot \|L\|_1$.*

Proof. This easily follows from the fact that $\|ALv\|_W \le C \cdot \|Lv\|_W$ for all $v \in V$, which is a particular case of [1, Prop IV.2.4], together with the fact that the Schatten norm is unitarily invariant, see [1, Theorem IV.2.1]. □

A.2. **Convexity.** In order to study the convexity properties of the 1-Schatten norm, we consider a smooth perturbation of it. For brevity, our treatment is limited to the special case we are interested in. There result we want to prove is the following.

Proposition A.5 (Convexity of 1-Schatten norm along paths with positive acceleration). *Let V and W be vector spaces of dimension 2 and 3 respectively and let $A : [0,1] \to \mathrm{Hom}(W,W)$ be a smooth path of positive self-adjoint operators on W. Consider a smooth path $T : [0,1] \to \mathrm{Hom}(V,W)$ of linear operators such that $\ddot{T}(s) = A(s) \circ T(s)$. Then the function $u : [0,1] \to \mathbb{R}$ defined as $u(s) := \|T(s)\|_1$ is convex. Moreover, if the rank of $T(s)$ is 2 and $A(s) \circ T(s) \neq 0$, then $\ddot{u}(s) > 0$.*

Before proving Proposition A.5, let us introduce suitable regularized versions of the 1-Schatten norm and study their properties.

Let V, W be real vector spaces endowed with positive-definite scalar products, of dimension 2 and 3 respectively. For any $\epsilon \ge 0$ and for $L \in \mathrm{Hom}(V,W)$ the *ϵ-regularized 1-Schatten norm* of L is

$$q_\epsilon(L) := \mathrm{tr}\sqrt{\epsilon^2 \mathbb{1} + L^* L}$$

where L^* is the adjoint of L. Notice that q_0 coincides with the 1-Schatten norm.

Lemma A.6 (Basic properties of regularized 1-Schatten norms). *The ϵ-regularized 1-Schatten norm $q_\epsilon : \mathrm{Hom}(V,W) \to \mathbb{R}$ satisfies the following properties.*

(a) *For any $L \in \mathrm{Hom}(V,W)$,*

$$q_\epsilon(L) = \sqrt{\mathrm{tr}(\epsilon^2 \mathbb{1} + L^* L) + 2\sqrt{\det(\epsilon^2 \mathbb{1} + L^* L)}}\ . \tag{26}$$

As a consequence, q_ϵ is smooth for $\epsilon > 0$.

(b) *The function q_ϵ is convex for any $\epsilon \ge 0$.*

(c) *Let $L : V \to W$ be a linear map, $A : W \to W$ be nonnegative self-adjoint and let $\epsilon \ge 0$. In the case $\epsilon = 0$, suppose furthermore that L has rank 2. Then*

$$\frac{d}{dt} q_\epsilon((1+tA)L)\Big|_{t=0} \ge 0\ .$$

Moreover the strict inequality holds if $A \circ L \neq 0$.

Before proving the lemma, we mention the following observation.

Sublemma A.7. *Let $\epsilon \ge 0$. Consider the planar domain $\Omega = \{\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2 | t_2 \ge t_1 \ge 0\}$ and define $n_\epsilon : \Omega \to \mathbb{R}$ by $n_\epsilon(t_1,t_2) := \sqrt{\epsilon^2 + t_1^2} + \sqrt{\epsilon^2 + t_2^2}$. Then, n_ϵ is convex. Moreover, given $(t_1,t_2), (t_1', t_2') \in \Omega$ such that $t_2 \le 2_1'$ and $t_1 + t_2 \le t_1' + t_2'$, we have $n_\epsilon(t_1,t_2) \le n_\epsilon(t_1', t_2')$.*

Proof. Clearly, the function $t \mapsto \sqrt{\epsilon^2 + t^2}$ is increasing and convex and so n_ϵ is convex too. It follows that $n_\epsilon(t_1', t_2') \geq n_\epsilon(t_1, t_2) + (\partial_{t_1} n_\epsilon)_{\boldsymbol{t}}(t_1' - t_1) + (\partial_{t_2} n_\epsilon)_{\boldsymbol{t}}(t_2' - t_2)$. Now, the simple and key remark is that $(\partial_{t_2} n_\epsilon)_{\boldsymbol{t}} \geq (\partial_{t_1} n_\epsilon)_{\boldsymbol{t}} \geq 0$ for every $\boldsymbol{t} \in \Omega$. Thus, $(\partial_{t_1} n_\epsilon)_{\boldsymbol{t}}(t_1' - t_1) + (\partial_{t_2} n_\epsilon)_{\boldsymbol{t}}(t_2' - t_2) \geq (\partial_{t_1} n_\epsilon)_{\boldsymbol{t}}(t_1' - t_1 + t_2' - t_2) \geq 0$ and the conclusion follows. □

The relevance of the above sublemma relies on the fact that, given $L \in \mathrm{Hom}(V, W)$, we have that $q_\epsilon(L) = n_\epsilon(\lambda_1, \lambda_2)$, where $\lambda_1 \leq \lambda_2$ are the singular values of L.

Proof of Lemma A.6. If λ_1, λ_2 are the singular values of L, that is, the eigenvalues of $b_L = \sqrt{L^* L}$, then

$$q_\epsilon(L) = \sqrt{\epsilon^2 + \lambda_1^2} + \sqrt{\epsilon^2 + \lambda_2^2}\,.$$

So $q_\epsilon(L)^2 = 2\epsilon^2 + \lambda_1^2 + \lambda_2^2 + 2\sqrt{(\epsilon^2 + \lambda_1^2)(\epsilon^2 + \lambda_2^2)}$. As the eigenvalues of $\epsilon^2 \mathbb{1} + L^* L$ are $\epsilon^2 + \lambda_1^2$ and $\epsilon^2 + \lambda_2^2$, the identity (26) is immediately verified. Moreover, $q_\epsilon(L)^2$ can be written as $q_\epsilon(L)^2 = 2\epsilon^2 + \mathrm{tr}(L^* L) + 2\sqrt{\epsilon^4 + 2\epsilon^2 \mathrm{tr}(L^* L) + \det(L^* L)}$, which shows that q_ϵ is smooth for $\epsilon > 0$. This proves (a).

In order to prove (b), note that q_ϵ is continuous and so it is enough to show that $2q_\epsilon(L + L') \leq q_\epsilon(2L) + q_\epsilon(2L')$ for all $L, L' \in \mathrm{Hom}(V, W)$. Consider then $L, L' \in \mathrm{Hom}(V, W)$ and denote by $\lambda_1 \leq \lambda_2$ the singular values of L, by $\lambda_1' \leq \lambda_2'$ the singular values of L' and by $\mu_1 \leq \mu_2$ the singular values of $L + L'$. By Theorem 2 of [22] there are linear isometries $P, Q : V \to V$ such that

$$b_{L+L'} \leq P^* b_L P + Q^* b_{L'} Q$$

where $\leq$ means that the difference is a nonnegative self-adjoint matrix.

If $\hat{\mu}_1 \leq \hat{\mu}_2$ are the eigenvalues of $P^* b_L P + Q^* b_{L'} Q$, we deduce that $\mu_i \leq \hat{\mu}_i$ for $i = 1, 2$. Thus $q_\epsilon(L + L') \leq n_\epsilon(\hat{\mu}_1, \hat{\mu}_2)$. Moreover, $\hat{\mu}_1 + \hat{\mu}_2 = \mathrm{tr}(b_{L+L'}) \leq \mathrm{tr}(P^* b_L P + Q^* b_{L'} Q) = \mathrm{tr}(b_L) + \mathrm{tr}(b_{L'}) = \lambda_1 + \lambda_1' + \lambda_2 + \lambda_2'$. On the other hand, by the classical Weyl Theorem, $\hat{\mu}_2 \leq \lambda_2 + \lambda_2'$. Hence, using Sublemma A.7, we get

$$\begin{aligned} 2q_\epsilon(L + L') \leq 2n_\epsilon(\hat{\mu}_1, \hat{\mu}_2) \leq 2n_\epsilon(\lambda_1 + \lambda_1', \lambda_2 + \lambda_2') \leq \\ \leq n_\epsilon(2\lambda_1, 2\lambda_2) + n_\epsilon(2\lambda_1', 2\lambda_2') = q_\epsilon(2L) + q_\epsilon(2L')\,, \end{aligned}$$

which shows that q_ϵ is convex.

As for (c), note that $t \mapsto q_\epsilon((1 + tA)L)$ is a smooth function near $t = 0$. For $\epsilon > 0$ this is clear, because q_ϵ is smooth. For $\epsilon = 0$ this depends on the fact that L has rank 2 and so has $(1 + tA)L$ for $|t|$ small. A straightforward computation using (26) shows that

(27)
$$\frac{d}{dt} q_\epsilon((1 + tA)L)\Big|_{t=0} = \frac{1}{2q_\epsilon(L)} \left(\mathrm{tr}(\hat{A}) + \sqrt{\det(\epsilon^2 \mathbb{1} + L^* L)}\, \mathrm{tr}((\epsilon^2 \mathbb{1} + L^* L)^{-1} \hat{A}) \right)\,,$$

where we have put

$$\hat{A} := \frac{d}{dt} \Big(((\mathbb{1} + tA)L)^* (\mathbb{1} + tA)L \Big)\Big|_{t=0} = 2L^* A L\,.$$

Now, $\hat{A}$ and $(\epsilon^2 \mathbb{1} + L^* L)$ are respectively non-negative and positive self-adjoint operators of V, so the derivative in (27) is non-negative. Finally, if $A \circ L \neq 0$, then $\hat{A} \neq 0$, and this implies the strict positivity of the derivative. $\square$

After such preparation, we can now prove the main statement of this section.

Proof of Proposition A.5. For $\epsilon \geq 0$ let $u_\epsilon(s) = q_\epsilon(T(s))$, so that $u_0 = u$. We remark that for $\epsilon > 0$ the function u_ϵ is smooth at every $s \in [0,1]$, whereas for $\epsilon = 0$ it is smooth at those $s \in [0,1]$ such that $T(s)$ has rank 2. At those points we have

$$\ddot{u}_\epsilon(s) = (d^2 q_\epsilon)_{T(s)}(\dot{T}(s), \dot{T}(s)) + (dq_\epsilon)_{T(s)}(\ddot{T}(s))$$

where $(d^2 q_\epsilon) : V \times V \to W$ is the Hessian of q_ϵ and we view $\ddot{u}_\epsilon(s)$, $\dot{T}(s)$ and $\ddot{T}(s)$ as elements of $\mathrm{Hom}(V, W)$.

As q_ϵ is convex, $(d^2 q_\epsilon)_{T(s)}(\dot{T}(s), \dot{T}(s)) \geq 0$. Moreover, $(dq_\epsilon)_{T(s)}(\ddot{T}(s)) = \frac{d}{dt} q_\epsilon\Big((1 + tA(s))T(s)\Big)\Big|_{t=0}$. Hence, by Lemma A.6(c) the term $(dq_\epsilon)_{T(s)}(\ddot{T}(s))$ is non-negative, and in fact strictly positive if $A(s) \circ T(s) \neq 0$.

For $\epsilon = 0$ this shows the positivity of $\ddot{u}$ at those $s \in [0,1]$ such that the rank of $T(s)$ is 2 and $A(s) \circ T(s) \neq 0$. On the other hand, for $\epsilon > 0$ the function u_ϵ turns out to be convex. The convexity of u follows, since $u = \lim_{\epsilon \to 0} u_\epsilon$. $\square$

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