PROJECTIVE AND CONFORMAL CLOSED MANIFOLDS WITH A HIGHER-RANK LATTICE ACTION

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Abstract. We prove global results about actions of cocompact lattices in higher-rank simple Lie groups on closed manifolds endowed with either a projective class of connections or a conformal class of pseudo-Riemannian metrics of signature (p, q), with $\min(p, q) \geqslant 2$. In the continuity of a recent article [Pec19], provided that such a structure is locally equivalent to its model $\mathbf{X}$, the main question treated here is the completeness of the associated $(G, \mathbf{X})$-structure. The similarities between the model spaces of non-Lorentzian conformal geometry and projective geometry make that lots of arguments are valid for both cases, and we expose the proofs in parallel. The conclusion is that in both cases, when the real-rank is maximal, the manifold is globally equivalent to either the model space $\mathbf{X}$ or its double cover.

Contents

1. Introduction

Zimmer's program suggests that actions of lattices in semi-simple Lie groups on closed manifolds have to be closely related to an homogeneous model. This voluntary vague formulation can be interpreted in various ways. We give in this article two geometric results that confirm this principle and are in the continuity of previous investigations for conformal actions [Pec19].

Let Γ be a lattice in a simple Lie group G of real-rank at least 2. Among all possible "geometric actions" $\rho : \Gamma \to \mathrm{Diff}(M, \mathcal{S})$ on a closed manifold M, we are especially interested in those for which the geometric structure $\mathcal{S}$ is *non-unimodular*. This is due to the fact that these structures do not naturally define a finite Γ-invariant measure, making more difficult the use of celebrated results such as Zimmer's cocycle super-rigidity. The new powerful tools about invariant measures, introduced in [BRHW16] and used in

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[BFH16] for proving Zimmer's conjectures, invite us to pay attention to these non-volume preserving dynamics.

Typical such structures are *parabolic Cartan geometries* ([ČS09]) which are curved versions of a given parabolic space $\mathbf{X} = G/P$, because on the model itself, there exists no finite Γ-invariant measure. We discuss in this article two cases of actions on parabolic geometries: those preserving a projective class $[\nabla]$ of linear connections and those preserving a conformal case $[g]$ of pseudo-Riemannian metrics.

We remind that two linear connections ∇ and ∇' on a same manifold M are said to be projectively equivalent if they define the same geodesics up to parametrization. For torsion free connections, this means that there exists a 1-form α such that $\nabla'_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X$ for all vector fields X, Y. A projective class $[\nabla]$ is an equivalence class of projectively equivalent linear connections, and the projective group $\mathrm{Proj}(M, [\nabla])$ is the group of diffeomorphisms that preserve this class. A projective structure on a manifold M^n is the same as the data of a Cartan geometry on M modeled on the projective space $\mathbf{X} = \mathbf{R}P^n$ ([KN64]). Two pseudo-Riemannian metrics g and g' on M are said to be conformal if there exists a smooth positive function $\varphi : M \to \mathbf{R}_{>0}$ such that $g' = \varphi g$. A conformal class $[g]$ is an equivalence class of conformal metrics and the conformal group $\mathrm{Conf}(M, [g])$ is the group of diffeomorphisms preserving this class. When $n = \dim M \geqslant 3$, a conformal class of signature (p, q) on M is the same as a normalized Cartan geometry on M modeled on $\mathbf{X} = \mathbf{Ein}^{p,q}$, the model space of conformal geometry discussed below in Section 2.

From [BFH16], we know that if Γ is cocompact and the action $\rho : \Gamma \to \mathrm{Diff}(M)$ has infinite image, then $\mathrm{Rk}_{\mathbf{R}}\, G \leqslant n = \dim M$ and that in the limit case $\mathrm{Rk}_{\mathbf{R}}\, G = n$, the restricted root-system of $\mathfrak{g}$ is A_n. It moreover follows from [Zha18] that $\mathfrak{g}$ is not isomorphic to $\mathfrak{sl}(n+1, \mathbf{C})$. Of course, the natural examples in this limit case are the restriction to Γ of the projective action of $\mathrm{SL}(n+1, \mathbf{R})$ on $\mathbf{S}^n$ or $\mathbf{R}P^n$, and conjecturally they are supposed to be the only examples. It is thus natural to start studying curved versions of these models, *i.e.* projective actions $\Gamma \to \mathrm{Proj}(M^n, [\nabla])$ with $\mathrm{Rk}_{\mathbf{R}}\, G = n$.

In [Pec19], we proved that if Γ is uniform and has an unbounded conformal action on a closed pseudo-Riemannian manifold (M, g) of signature (p, q), with $p + q \geqslant 3$, then $\mathrm{Rk}_{\mathbf{R}}\, G \leqslant \min(p, q) + 1$ and that (M, g) is conformally flat when $\mathrm{Rk}_{\mathbf{R}}\, G = \min(p, q) + 1$. This means that the conformal class $[g]$ defines a Γ-invariant atlas of $(\mathrm{Conf}(\mathbf{Ein}^{p,q}), \mathbf{Ein}^{p,q})$-structure on M, which we would like to understand. Projective flatness in the case of a projective action in maximal rank can be derived by the same kind of arguments (see Section 6).

So, in both projective and conformal case, if $\mathbf{X}$ denotes the model space and $G_{\mathbf{X}}$ its automorphisms group, it turns out that if $\rho(\Gamma)$ is unbounded, then $\mathrm{Rk}_{\mathbf{R}}\, G \leqslant \mathrm{Rk}_{\mathbf{R}}\, G_{\mathbf{X}}$ and that the structure is *flat* when equality holds. We see here a strong similarity with Theorem 5 of [BFM09] where semi-simple Lie groups actions on parabolic closed manifolds are considered. To obtain a similar conclusion for uniform lattices in such groups, the main problem here is thus to understand globally this Γ-invariant $(G_{\mathbf{X}}, \mathbf{X})$-structure on M. Even when Γ is large, this problem is interesting notably because its group structure may not be "visible" at a local scale, contrarily to the case of a Lie group action which gives rise to a Lie algebra of vector fields.

The model space of conformal geometry of signature (p, q) is $\mathbf{Ein}^{p,q} = (\mathbf{S}^p \times \mathbf{S}^q)/\{\pm \mathrm{id}\}$ endowed with the conformal class $[-g_{\mathbf{S}^p} \oplus g_{\mathbf{S}^q}]$, where $-\mathrm{id}$ acts via the products of the antipodal maps. When $\min(p, q) \neq 1$, the model spaces $\mathbf{R}P^n$ and $\mathbf{Ein}^{p,q}$ have very similar patterns, in particular they both are natural compactifications of an affine space, via affine charts and stereographic projections, and their universal cover is a 2-sheeted cover.

These similarities make that our approach works both closed $\mathbf{R}P^n$-manifolds and $\mathbf{Ein}^{p,q}$-manifolds, with *non-Lorentzian* signature $\min(p, q) \geqslant 2$. The Lorentzian model space $\mathbf{Ein}^{1,n-1}$ behaves a bit differently, due to the non-compactness of its universal cover that invalidates arguments used for proving the invectivity of the developing map. We leave its case for further investigations.

1.1. **Main results.** Combined with [Pec19], we obtain the following global conclusions for actions of uniform lattices of maximal real-rank.

Theorem 1. *Let G be a connected simple Lie group with finite center and real-rank $n \geqslant 2$, and let $\Gamma < G$ be a cocompact lattice. Let (M^n, ∇) be a closed manifold endowed with a linear connection ∇. Let $\rho : \Gamma \to \mathrm{Proj}(M, [\nabla])$ be a projective action.*

If $\rho(\Gamma)$ is infinite, then $(M, [\nabla])$ is projectively equivalent to either $\mathbf{S}^n$ or $\mathbf{R}P^n$ with their standard projective structures.

Thus the action is a group homomorphism into $\mathrm{SL}^{\pm}(n+1, \mathbf{R})$ or $\mathrm{PGL}(n+1, \mathbf{R})$ with infinite image. By Margulis' super-rigidity theorem, $\mathfrak{g} \simeq \mathfrak{sl}(n+1, \mathbf{R})$ and ρ extends to a locally faithful action of $\tilde{\mathrm{SL}}(n+1, \mathbf{R})$. This result can be viewed as a projective counterpart of a result of Zeghib [Zeg97] on affine, volume-preserving actions of lattices on closed manifolds, in which he improved a result of Goetze [Goe94]. See also [Zim86a, Fer92].

For conformal actions, we obtain a similar statement when the real-rank is maximal.

Theorem 2. *Let (M, g) be a closed pseudo-Riemannian manifold of signature (p, q), with $\min(p, q) \geqslant 2$, and $\Gamma < G$ be a uniform lattice in a simple Lie group of real-rank $\min(p, q) + 1$. Let $\rho : \Gamma \to \mathrm{Conf}(M, g)$ be a conformal action.*

If $\rho(\Gamma)$ is unbounded in $\mathrm{Conf}(M, g)$, then (M, g) is conformally equivalent to $\mathbf{Ein}^{p,q}$ or its double cover $\tilde{\mathbf{Ein}}^{p,q} = (\mathbf{S}^p \times \mathbf{S}^q, [-g_{\mathbf{S}^p} \oplus g_{\mathbf{S}^q}])$.

The action ρ is thus a group homorphism $\Gamma \to \mathrm{PO}(p+1, q+1)$ or $\Gamma \to O(p+1, q+1)$ whose image is unbounded. Let us say that $p \leqslant q$. Using Margulis super-rigidity, we deduce that $\mathfrak{g} \simeq \mathfrak{so}(p+1, k)$, with $p+1 \leqslant k \leqslant q+1$ and that the action extends to a Lie group action up to a "compact noise": up to passing to a finite cover of G and lifting Γ to it, there exists a compact Lie subgroup $K < \mathrm{Conf}(M, g)$, a Lie group homorphism $\overline{\rho} : G \to \mathrm{Conf}(M, g)$ with finite kernel and such that K centralized $\overline{\rho}(G)$, and $\rho_K : \Gamma \to K$ such that $\rho(\gamma) = \overline{\rho}(\gamma)\rho_K(\gamma)$ for all $\gamma \in \Gamma$.

Remark 1.1. It has to be noted that if it exists, a global conclusion for conformal actions of rank 2 uniform lattices on closed Lorentzian manifolds shall be a bit more complicated as it can be seen in the conclusions Theorem 3 of [FZ05] about semi-simple Lie groups actions.

1.2. **Structure of the proof: atlas of maximal charts.** Let $n \geqslant 2$ and (p, q) such that $\min(p, q) \geqslant 2$. Let $\mathbf{X}$ be either $\mathbf{R}P^n$ or $\mathbf{Ein}^{p,q}$ and let $G_{\mathbf{X}} = \mathrm{PGL}(n, \mathbf{R})$ or $\mathrm{PO}(p+1, q+1)$

accordingly. Let G be a simple Lie group with finite center, and let $\Gamma < G$ be a uniform lattice. We assume $\mathrm{Rk}_{\mathbf{R}}\, G = n$ if $\mathbf{X} = \mathbf{R}P^n$ and $\mathrm{Rk}_{\mathbf{R}}\, G = \min(p, q) + 1$ if $\mathbf{X} = \mathbf{Ein}^{p,q}$.

The dynamical starting point of our proof is the existence of sequences (γ_k) in Γ admitting a *uniformly contracting* dynamical behavior, which are used in [Pec19] for obtaining conformal flatness. With no substantially different arguments - and even less efforts -, we can also exhibit such sequences for projective actions of Γ, and projective flatness similarly follows by considering the associated Cartan connection. We explain this in the last Section 6, and start directly working with locally flat projective and conformal closed manifolds.

These sequences (γ_k) contract topologically an open set U to a point $x \in U$ and their derivatives are moreover Lyapunov regular with a uniform Lyapunov spectrum, see Section 4.1. The idea is to go backward and consider the $\gamma_k^{-1}U$. We show in Proposition 4.1 that at the limit, the sequence $(\gamma_k^{-1}U)$ gives rise to some maximal domain U_∞, which is a trivializing open set for the universal cover $\tilde{M} \to M$ and such that for any lift $\tilde{U}_\infty$, the developing map $\tilde{M} \to \mathbf{X}$ is injective in restriction to $\tilde{U}_\infty$ and sends it to an affine chart domain if $\mathbf{X} = \mathbf{R}P^n$ or a Minkowski patch if $\mathbf{X} = \mathbf{Ein}^{p,q}$. We call such domains U_∞ *maximal charts*, and Proposition 4.1 shows that any point of M is contained in a maximal chart.

So, once Proposition 4.1 is established, Theorem 1 and Theorem 2 follow from the above result.

Theorem 3. *Let M be a compact manifold endowed with a $(G_{\mathbf{X}}, \mathbf{X})$-structure. Let $\pi : \tilde{M} \to M$ be a universal cover and (D, ρ) be a developing pair. We assume the following:*

(H) Any point $\tilde{x} \in \tilde{M}$ has a neighborhood $\tilde{V}$ in restriction to which π and D are injective and such that $\tilde{V}$ is either projectively equivalent to $\mathbf{R}^n$ in the projective case or conformally equivalent to $\mathbf{R}^{p,q}$ in the conformal case.

Then, M is isomorphic, as a $(G_{\mathbf{X}}, \mathbf{X})$-manifold, to either $\mathbf{X}$ or $\tilde{\mathbf{X}}$.

Plan of the article. After reminding classic definitions of $\mathbf{Ein}^{p,q}$ and some properties of its stereographic projections in Section 2, we define in Section 3 maximal charts of $(G_{\mathbf{X}}, \mathbf{X})$-manifolds and establish useful properties of these charts that will be used later in the proof of the injectivity of the developing map. Section 4 is devoted to the proof of Proposition 4.1. Theorem 3 is proved in Section 5, which is easily reduced to the proof of the injectivity of the developing map $\mathfrak{D} : \tilde{M} \to \tilde{\mathbf{X}}$ under the assumption *(H)*. Finally, we give as announced in Section 6 the proof of projective flatness of n-dimensional manifolds $(M, [\nabla])$ admitting a non-trivial projective action of a cocompact lattice of rank n.

Convention and notations. We will note M a closed n-dimensional manifold, with $n \geqslant 2$. When M is endowed with a conformal structure, we assume $n \geqslant 3$. For signatures (p, q), with $p + q = n$, we fix the convention $p \leqslant q$ and we will only consider non-Riemannian signatures $p \geqslant 1$, as the conformal Riemannian case is completely understood with optimal assumptions [Fer71, Oba71]. As in the main theorems, G will always denote a non-compact simple Lie group with finite center and real-rank at least 2, and Γ a uniform lattice in G.

2. Stereographic projections and Minkowski patches of $\mathbf{Ein}^{p,q}$

We remind the convention $p \leqslant q$ and the notation $n = p+q \geqslant 3$. We will assume $p \geqslant 1$ for technical reasons. Quickly, we will only consider signatures such that $p \geqslant 2$. Let $(e_0, \dots, e_{n+1})$ be a basis of $\mathbf{R}^{p+1,q+1}$ in which the quadratic form reads $Q(u) = 2u_0u_{n+1} + \cdots + 2u_pu_{q+1} + u_{p+1}^2 + \cdots + u_q^2$. By definition, $\mathbf{Ein}^{p,q} \subset \mathbf{R}P^{n+1}$ is the smooth quadric defined by $\{Q = 0\}$, and its conformal structure is the one induced by the restriction of Q to the tangent spaces of the isotropic cone $\{Q = 0\}$. Its conformal group is then $\mathrm{Conf}(\mathbf{Ein}^{p,q}) = \mathrm{PO}(p+1, q+1)$.

We note $\mathbf{S}^{n+1}$ the standard Euclidean sphere in $\mathbf{R}^{p+1,q+1}$. The Einstein Universe $\mathbf{Ein}^{p,q}$ is doubly covered by $\{Q = 0\} \cap \mathbf{S}^{n+1}$, which is diffeomorphic to $\mathbf{S}^p \times \mathbf{S}^q$. Thus, it is its universal cover whenever $p \geqslant 2$, and when $p = 1$, its universal cover is diffeomorphic to $\mathbf{R} \times \mathbf{S}^{n-1}$. We fix once and for all a universal cover $p : \tilde{\mathbf{Ein}}^{p,q} \to \mathbf{Ein}^{p,q}$.

A celebrated result of conformal geometry in dimension at least 3 is the fact that local conformal maps of $\mathbf{Ein}^{p,q}$ are restrictions of global transformations. This was initially observed by Liouville in Riemannian signature.

Theorem (Liouville). *Let $U, V \subset \mathbf{Ein}^{p,q}$ be two connected open subsets and $f : U \to V$ a conformal map. Then, there exists $\phi \in \mathrm{Conf}(\mathbf{Ein}^{p,q})$ such that $f = \phi|_U$.*

2.1. **Minkowski patches and stereographic projections.** Let $v \in \mathbf{R}^{p+1,q+1}$ be an isotropic vector and $x = [v] \in \mathbf{Ein}^{p,q}$. The **Minkowski patch** M_x associated to x is the intersection of $\mathbf{Ein}^{p,q}$ with the affine chart domain $\{[v'] \ : \ B(v, v') \neq 0\}$ where $B(.,.)$ denotes the scalar product of $\mathbf{R}^{p+1,q+1}$. The **light-cone** C_x of x is the complement of M_x in $\mathbf{Ein}^{p,q}$, *i.e.* $C_x = \{[v'] \in \mathbf{Ein}^{p,q} \ : \ B(v, v') = 0\}$. We will say that x is the vertex of M_x and C_x.

The light-cone C_x is a singular projective variety, with singularity $\{x\}$ and $C_x \setminus \{x\}$ is diffeomorphic to $\mathbf{R} \times \mathbf{Ein}^{p-1,q-1}$. The Minkowski patch M_x is an open-dense subset of $\mathbf{Ein}^{p,q}$ conformally equivalent to $\mathbf{R}^{p,q}$. This last statement is easily observed in the coordinates defined above and with $x = o$:

$$M_o = \{[-\frac{< v, v >_{p,q}}{2} : v : 1], \ v \in \mathbf{R}^{p,q}\}.$$

Let us note $s_o : M_o \to \mathbf{R}^{p,q}$ the inverse of the map $v \in \mathbf{R}^{p,q} \mapsto [-\frac{<v,v>_{p,q}}{2} : v : 1] \in M_o$.

Lemma 2.1. *An open subset $U \subset \mathbf{Ein}^{p,q}$ conformally equivalent to $\mathbf{R}^{p,q}$ is a Minkowski patch.*

Proof. Let $f : \mathbf{R}^{p,q} \to U$ be a conformal diffeomorphism. Then, $f \circ s_o : M_o \to U$ is a conformal diffeomorphism, which extends to a global conformal transformation $\phi \in \mathrm{Conf}(\mathbf{Ein}^{p,q})$ by Liouville's theorem. Thus, $U = \phi(M_o) = M_x$ is the Minkowski patch with vertex $x = \phi(o)$. □

Definition 2.2. We call *stereographic projection* any conformal diffeomorphism $s : M_x \to \mathbf{R}^{p,q}$, where $M_x \subset \mathbf{Ein}^{p,q}$ is a Minkowski patch.

It has to be noted that any stereographic projection $s : M_x \to \mathbf{R}^{p,q}$ can be uniquely written $s = s_o \circ \phi^{-1}$ where $\phi \in \mathrm{Conf}(\mathbf{Ein}^{p,q})$ is such that $\phi(o) = x$.

2.1.1. *Lifts to* $\tilde{\mathbf{Ein}}^{p,q}$. As any Minkowski patch $M_x \subset \mathbf{Ein}^{p,q}$ is simply connected, if $p : \tilde{\mathbf{Ein}}^{p,q} \to \mathbf{Ein}^{p,q}$ is the universal cover, then M_x is a trivializing open subset for p, and we define a Minkowski patch in $\tilde{\mathbf{Ein}}^{p,q}$ as being any connected component M'_x of $p^{-1}(M_x)$, where M_x is a Minkowski patch in $\mathbf{Ein}^{p,q}$.

Similarly:

Lemma 2.3. *An open subset $U \subset \tilde{\mathbf{Ein}}^{p,q}$ which is conformal to $\mathbf{R}^{p,q}$ is a Minkowski patch.*

Proof. Let M'_o be a Minkowski patch projecting to M_o. As M'_o is conformal to $\mathbf{R}^{p,q}$, we have a conformal diffeomorphism $f : M'_o \to U$. By Liouville's theorem, the map $p \circ f \circ (p|_{M'_o})^{-1} : M_o \to p(U)$ extends uniquely to a conformal map $\phi \in \mathrm{Conf}(\mathbf{Ein}^{p,q})$. Let $o' \in M'_o$ be the point projecting to o and let $\tilde{\phi} \in \mathrm{Conf}(\tilde{\mathbf{Ein}}^{p,q})$ be the lift of ϕ such that $\tilde{\phi}(o') = f(o')$. Since, $\tilde{\phi}$ and f coincide on a neighborhood of o', they coincide on M'_o by rigidity and connectedness of the latter.

Consequently, $U = \tilde{\phi}(M'_o)$ implying that $p(U) = \phi(M_o) = M_x$, so U is a Minkowski patch. □

Definition 2.4. A stereographic projection in $\tilde{\mathbf{Ein}}^{p,q}$ is a conformal diffeomorphism $\tilde{s} : M'_x \to \mathbf{R}^{p,q}$ where M'_x is a Minkowski patch.

Any such $\tilde{s}$ is of the form $s \circ p$, where $M_x = p(M'_x)$ and $s : M_x \to \mathbf{R}^{p,q}$ is a stereographic projection.

2.2. **Intersections of Minkowski patches.**

2.2.1. *Intersections in* $\mathbf{Ein}^{p,q}$. Let $M_x \subset \mathbf{Ein}^{p,q}$ be a Minkowski patch and $s : M_x \to \mathbf{R}^{p,q}$ a stereographic projection. Let $\mathcal{C} \subset \mathbf{R}^{p,q}$ denote the isotropic cone. Let $M_y \subset \mathbf{Ein}^{p,q}$ be another Minkowski patch, with $y \neq x$. There are two possible types for $M_x \cap M_y$:

- either $y \in M_x$, and in this situation $s(M_x \cap M_y) = \mathbf{R}^{p,q} \setminus (s(y) + \mathcal{C})$
- or $y \notin M_x$, and $s(M_x \cap M_y) = \mathbf{R}^{p,q} \setminus H_y$, where $H_y \subset \mathbf{R}^{p,q}$ is a degenerate affine hyperplane (of course, H_y depends on s).

It has to be noted that when $\min(p, q) \geqslant 2$, $M_x \cap M_y$ always has two connected components, whereas in Lorentzian signature, $M_x \cap M_y$ has three connected components in the first case.

Lemma 2.5. *Let M_x, M_y, M_z be three Minkowski patches in $\mathbf{Ein}^{p,q}$. If $M_x \cap M_z = M_y \cap M_z$, then $M_x = M_y$.*

Proof. If $M_x \cap M_z = M_z$, then $M_x = M_z = M_y$ by Liouville's theorem. So, let us assume that it is not the case. Then, by the previous paragraph, $x \in M_z$ if and only if $y \in M_z$, and in this case $x = y$ because they are sent by any stereographic projection of M_z to the singularity of a same light-cone.

In the other case, it is enough to observe - in suitable homogeneous coordinates - that when $x \notin M_z$, given a stereographic projection $s : M_z \to \mathbf{R}^{p,q}$, if $\Delta = v + \mathbf{R}.v_0$ is any affine isotropic line contained in the complement of $s(M_x \cap M_z)$, then $s^{-1}(v + tv_0) \to x$ as $t \to \pm\infty$. This shows that the data of $M_x \cap M_z$ determines x in this situation, and the lemma is proved. □

2.2.2. *Intersections in* $\tilde{\mathbf{Ein}}^{p,q}$. Let $M_1 \subset \tilde{\mathbf{Ein}}^{p,q}$ be a Minkowski patch, and let $s : M_1 \to \mathbf{R}^{p,q}$ be a stereographic projection. Let $M_2 \subset \tilde{\mathbf{Ein}}^{p,q}$ be another Minkowski patch such that $M_1 \cap M_2 \neq 0$. We note $\overline{s} : p(M_1) \to \mathbf{R}^{p,q}$ the stereographic projection such that $s = \overline{s} \circ p$.

Lemma 2.6. *$p(M_1 \cap M_2)$ is a connected component of $p(M_1) \cap p(M_2)$.*

Proof. Even though this lemma is valid for $\mathbf{Ein}^{1,n-1}$, we only give a proof for non-Lorentzian signatures $\min(p, q) > 1$ which is the case discussed in this article. We suppose $M_1 \neq M_2$, otherwise the statement is obvious.

Let us show that $p(M_1 \cap M_2)$ is closed in $p(M_1) \cap p(M_2)$. Let $\overline{x} \in (p(M_1) \cap p(M_2)) \setminus p(M_1 \cap M_2)$. Let $x \in M_1$ be such that $p(x) = \overline{x}$. By definition, $x \notin M_2$. Let $U_x \subset M_1$ be a connected neighborhood of x in restriction to which p is injective and such that $p(U_x) \subset p(M_2)$. Therefore, $U_x \cap M_2 = \emptyset$ because if not, Lemma 3.2 would imply $U_x \subset M_2$, contradicting $x \notin M_2$. By construction, $p(U_x) \cap p(M_1 \cap M_2) = \emptyset$ and we get as announced that $p(M_1 \cap M_2)$ is closed in $p(M_1) \cap p(M_2)$. Since it is also open, we get that $p(M_1 \cap M_2)$ is a union of connected components of $p(M_1) \cap p(M_2)$.

Because we assume $\min(p, q) \geqslant 2$, as observed above, $p(M_1) \cap p(M_2)$ has two connected components. And since we cannot have $p(M_1 \cap M_2) = p(M_1) \cap p(M_2)$ (otherwise $p|_{M_1 \cup M_2}$ would be injective), we get that $p(M_1 \cap M_2)$ must be a single connected component. $\square$

Thus, we deduce the following useful observation.

Observation 1. *When* $\min(p, q) \geqslant 2$, *given two distinct, non-antipodal Minkowski patches* $M_1, M_2 \subset \tilde{\mathbf{Ein}}^{p,q}$ *and a stereographic projection* $s : M_1 \to \mathbf{R}^{p,q}$, $s(M_1 \cap M_2)$ *is an open set of the form*

- $v_0 + U_S$, $v_0 \in \mathbf{R}^{p,q}$
- *or* $v_0 + U_T$, $v_0 \in \mathbf{R}^{p,q}$
- *or* $\{v \in \mathbf{R}^{p,q} \ : \ b(v, v_0) > \alpha\}$, *with* $v_0 \in \mathcal{C} \setminus \{0\}$ *and* $\alpha \in \mathbf{R}$,

where we note $b(v, w) = -v_1 w_1 - \cdots - v_p w_p + v_{p+1} w_{p+1} + \cdots + v_n w_n$, $q(v) = b(v, v)$, $\mathcal{C} = \{q = 0\}$, $U_S = \{q > 0\}$, $U_T = \{q < 0\}$. *For* $v_0 \in \mathcal{C} \setminus \{0\}$ *and* $\alpha \in \mathbf{R}$, *we will note* $H_{v_0,\alpha} = \{v \in \mathbf{R}^{p,q} \ : \ b(v_0, v) > \alpha\}$.

Definition 2.7. An open subset $U \subset \mathbf{R}^{p,q}$ of the form $v_0 + U_S$, $v_0 + U_T$ for any $v_0 \in \mathbf{R}^{p,q}$ or $H_{v_0,\alpha}$ for $v_0 \in \mathcal{C} \setminus \{0\}$ and $\alpha \in \mathbf{R}$ will be said of **intersection type**.

We will also use the fact that a Minkowski patch is determined by its intersection with another one.

Lemma 2.8. *Let* $M_1, M_2 \subset \tilde{\mathbf{Ein}}^{p,q}$ *be two Minkowski patches. Then,* $p(M_1 \cap \iota(M_2))$ *is the complement of* $p(M_1 \cap M_2)$ *in* $p(M_1) \cap p(M_2)$. *If* $M_1, M_2, M_3 \subset \tilde{\mathbf{Ein}}^{p,q}$ *are three Minkowski patches and if* $M_1 \cap M_3 = M_2 \cap M_3$, *then* $M_1 = M_2$.

Proof. If $M_1 \cap M_2 = \emptyset$, then $M_1 = \iota(M_2)$ and the claim follows directly. If $M_1 \cap M_2 = M_1$, then $M_1 = M_2$ by Observation 1. So, let us assume that $M_1 \cap M_2$ is a non-empty proper open subset of M_1 and M_2. Let $C_1 = p(M_1 \cap M_2) \subset p(M_1) \cap p(M_2)$, and let C_2 be the other connected component. Since $M_1 \cap \iota(M_2)$ is also a non-empty proper subset of M_1, it projects to either C_1 or C_2, and it must be on C_2 by injectivity of $p|_{M_1}$.

Let us assume $M_1 \cap M_3 = M_2 \cap M_3$. The conclusion is clear when these intersections are empty, or equal to all of M_3. Let us assume that it is not the case. Then, it follows that $p(M_1) \cap p(M_3)$ and $p(M_2) \cap p(M_3)$ have a common connected component, namely $p(M_1 \cap M_3)$. In all cases, the boundary of this component in $p(M_3)$ is the complement of $p(M_1) \cap p(M_3)$ in $p(M_3)$, as well as the complement of $p(M_2) \cap p(M_3)$ in $p(M_3)$. This proves $p(M_1) \cap p(M_3) = p(M_2) \cap p(M_3)$. By Lemma 2.5, it follows that $p(M_1) = p(M_2)$.

So, either $M_1 = M_2$ or $M_1 = \iota(M_2)$. But the last case is not possible because $M_1 \cap M_3 = M_2 \cap M_3$. □

Remark 2.9. Let $M_1, M_2 \subset \tilde{\mathbf{Ein}}^{p,q}$ be two distinct and non-antipodal Minkwoski patches. Let $s : M_1 \to \mathbf{R}^{p,q}$ be a stereographic projection, and let $s' = s \circ \iota : \iota(M_1) \to \mathbf{R}^{p,q}$. Since $s'(\iota(M_1) \cap M_2) = s(M_1 \cap \iota(M_2))$, it follows from the previous lemma that

- if $s(M_1 \cap M_2) = v + U_S$, then $s'(\iota(M_1) \cap M_2) = v + U_T$, and
- if $s(M_1 \cap M_2) = H_{v,\alpha}$, then $s'(\iota(M_1) \cap M_2) = H_{-v,-\alpha}$.

3. Maximal charts on $\tilde{M}$

Let $\mathbf{X}$ denote either $\mathbf{R}P^n$ or $\mathbf{Ein}^{p,q}$, with $\min(p, q) \geqslant 2$, and $G_{\mathbf{X}}$ its automorphisms group. Let M be a compact manifold endowed with a $(G_{\mathbf{X}}, \mathbf{X})$-structure. We fix $\pi : \tilde{M} \to M$ a universal cover and we pull back the geometric structure of M to $\tilde{M}$. The fundamental group of M identifies with a normal subgroup $\pi_1(M) \lhd \mathrm{Aut}(\tilde{M})$ such that $\mathrm{Aut}(M)$ is isomorphic to $\mathrm{Aut}(\tilde{M})/\pi_1(M)$.

We choose $(\mathfrak{D}, \tilde{\rho})$ a developing pair modeled on $\tilde{\mathbf{X}}$, *i.e.* a (projective or conformal) immersion $\mathfrak{D} : \tilde{M} \to \tilde{\mathbf{X}}$ and a homomorphism $\tilde{\rho} : \mathrm{Aut}(\tilde{M}) \to \mathrm{Aut}(\tilde{\mathbf{X}})$ such that $\mathfrak{D}$ is $\tilde{\rho}$-equivariant. We note $D = p \circ \mathfrak{D} : \tilde{M} \to \mathbf{X}$ and $\rho : \mathrm{Aut}(\tilde{M}) \to \mathrm{Aut}(\mathbf{X})$ the natural developing pair with model $\mathbf{X}$ associated to $(\mathfrak{D}, \tilde{\rho})$. The homomorphism ρ is $\tilde{\rho}$ followed by the natural projection $\mathrm{Aut}(\tilde{\mathbf{X}}) \to \mathrm{Aut}(\mathbf{X})$.

3.1. Definition of maximal charts and classic lemmas.

Definition 3.1. We call **maximal chart** an open subset $V \subset \tilde{M}$ in restriction to which π and $\mathfrak{D}$ are injective and such that $\mathfrak{D}(V) \subset \mathbf{S}^n$ is an hemisphere if $\mathbf{X} = \mathbf{R}P^n$ or $\mathfrak{D}(V) \subset \tilde{\mathbf{Ein}}^{p,q}$ is a Minkowski patch if $\mathbf{X} = \mathbf{Ein}^{p,q}$.

It has to be noted that in this definition, requiring that $\mathfrak{D}(V)$ is an hemisphere (resp. a Minkowski patch) is the same as asking that V is projectively equivalent to $\mathbf{R}^n$ (resp. conformally equivalent to $\mathbf{R}^{p,q}$) by Liouville's theorem.

We will use repeatedly the following classic results about local homeomorphisms. They are stated and proved in [Bar00], Section 2.1. Let M, N be two manifolds.

Lemma 3.2. *Let $f : M \to N$ be a local homeomorphism and let $U \subset M$ and $V \subset N$ be two open sets such that $f|_U$ is a homeomorphism onto V. If $W \subset M$ is a connected open subset such that $f(W) \subset V$ and $W \cap U \neq \emptyset$, then $W \subset U$.*

Proof. We prove that $W \cap U$ is closed in W, which will be enough as it is open and non-empty. Let $x \in W \setminus U$. We wish to prove that a neighborhood of x is contained in $W \setminus U$. This is immediate if $x \notin \partial U$. So let us assume $x \in \partial U$.

Since $x \in W$, $f(x) \in V$ and consequently there exists a unique $y \in U$ such that $f(y) = f(x)$. Since $x \neq y$, we can choose W_x, U_y disjoint open neighborhoods of x and y respectively such that $f|_{W_x}$ and $f|_{U_y}$ are homeomorphisms onto their images and such that $f(W_x) = f(U_y) \subset V$. Since $x \in \partial U$, $W_x \cap U \neq \emptyset$ and if $z \in W_x \cap U$, then $f(z) \in f(U_y)$ and it follows that $z \in U_y$ because f is injective on U. This contradicts $W_x \cap U_y = \emptyset$. □

Definition 3.3. A subset $X \subset M$ of a manifold M is said to be **locally connected relatively to** M if any point $x \in \overline{X}$ has a fundamental system of neighborhoods $\mathcal{V}_x$ such that for all $V \in \mathcal{V}_x$, $V \cap X$ is connected.

Typically, an affine chart domain is not locally connected relatively to $\mathbf{R}P^n$, while a hemisphere is locally connected relatively to $\mathbf{S}^n$.

Lemma 3.4. *Let $f : M \to N$ be a local homeomorphism. Let $U \subset M$ be an open subset in restriction to which f is injective. If $f(U)$ is locally connected relatively to N, then f is injective in restriction to $\overline{U}$.*

Proof. Let $x, y \in \partial U$ admitting a same image $z \in \partial f(U)$. By assumption, there are U_x, U_y two open neighborhoods of x, y respectively, in restriction to which f is injective, and such that $f(U_x) = f(U_y) =: V$ is such that $V \cap f(U)$ is connected. Consider now the open sets $U_{x,V} = U_x \cap f^{-1}(V \cap f(U))$ and $U_{y,V} = U_y \cap f^{-1}(V \cap f(U))$. Both are injectivity domains for f, and are sent homeomorphically onto $V \cap f(U)$, which is connected. So, both are connected and Lemma 3.2 gives $U_{x,V}, U_{y,V} \subset U$. In particular, a point $z \in V \cap f(U)$ has a preimage in $U_x \cap U$ and $U_y \cap U$, which must be the same by injectivity of $f|_U$. This proves $U_x \cap U_y \neq \emptyset$, and since they can be chosen arbitrarily small, we deduce $x = y$. □

Lemma 3.5. *Let $f : M \to N$ be a local homeomorphism. Let $V_1, V_2 \subset M$ be two open subsets such that $V_1 \cap V_2 \neq \emptyset$, $f|_{V_i}$ is injective for $i = 1, 2$, and if $U_i = f(V_i)$, such that $U_1 \cap U_2$ is connected. Then, $f(V_1 \cap V_2) = U_1 \cap U_2$. In particular, $f|_{V_1 \cup V_2}$ is injective.*

Proof. We note $U = U_1 \cap U_2$ and consider $W = V_1 \cap f^{-1}(U)$. Then, $f|_W$ is injective and $f(W) = U$ is connected. It implies that W is connected. Thus, as $W \cap V_2 \neq \emptyset$ and $f(W) \subset U_2 = f(V_2)$ we get $W \subset V_2$ by Lemma 3.2, implying $W = V_1 \cap V_2$.

Thus, $(f|_{V_1})^{-1}(U_1 \cap U_2) = V_1 \cap V_2$, and if $x \in V_1$ and $y \in V_2$ have same image, then $f(x) = f(y) \in U_1 \cap U_2$, implying $x \in V_1 \cap V_2$, and finally $x = y$ by injectivity of $f|_{V_2}$. □

3.2. **Relative compactness of maximal charts.**

Proposition 3.6. *Assume that $\tilde{M}$ is covered by maximal charts. Then, any maximal chart $\tilde{V}$ is relatively compact in $\tilde{M}$.*

Remark 3.7. The conclusion is still valid if we only assume M compact, however this statement is enough for the purpose of this article.

Proof. We assume to the contrary that $\tilde{V}$ contains a diverging sequence (x_k). By compactness of M, there exists a sequence $\gamma_k \in \pi_1(M)$ such that $\gamma_k.x_k \to x \in \tilde{M}$. Since x_k leaves any compact subset of $\tilde{M}$, we may assume the γ_k pairwise distinct.

The fact that $\pi|_{\tilde{V}}$ is injective means that for any $\gamma \in \pi_1(M)$, if $\gamma\tilde{V} \cap \tilde{V} \neq \emptyset$, then $\gamma = \mathrm{id}$. Consequently, the sequence $\tilde{V}_k := \gamma_k\tilde{V}$ is formed of pairwise disjoint open sets.

Let $U_k = \mathfrak{D}(\tilde{V}_k)$ and let $\tilde{V}_0 \ni x$ be a maximal chart containing x, and let $U_0 = \mathfrak{D}(\tilde{V}_0)$. We may assume that for all k, $\gamma_k x_k \in \tilde{V}_0$, implying that $\tilde{V}_k \cap \tilde{V}_0 \neq \emptyset$.

Lemma 3.8. *The subsets $U_k \cap U_0$ are pairwise disjoint.*

Proof. We have seen in Section 2.2.2 that when they intersect, two Minkwoski patches in $\tilde{\mathbf{Ein}}^{p,q}$ always have connected intersection. Consequently, the same being obvious for two hemispheres of $\mathbf{S}^n$, if k is such that $\tilde{V}_k \cap \tilde{V}_0 \neq \emptyset$, by Lemma 3.5, $\mathfrak{D}(\tilde{V}_k \cap \tilde{V}_0) = U_k \cap U_0$. The lemma now follows immediately, as the $\tilde{V}_k \cap \tilde{V}_0$ are pairwise disjoint. □

We finally get a contradiction with the following.

Lemma 3.9. *Let $H_0 \subset \mathbf{S}^n$ be a hemisphere. A family $(H_i)_{i\in I}$ of hemispheres such that the $H_0 \cap H_i$ are non-empty and pairwise disjoint has cardinality at most* 2.

Let $M_0 \subset \tilde{\mathbf{Ein}}^{p,q}$ be a Minkowski patch. A family $(M_i)_{i\in I}$ of Minkowski patches such that the $M_0 \cap M_i$, $i \in I$, are non-empty and pairwise disjoint has cardinality at most 2.

Proof. The first part is almost immediate: if $a : H_0 \to \mathbf{R}^n$ is an affine chart, then $a(H_0 \cap H_i)$ is either $\mathbf{R}^n$ if $H_0 = H_i$ or an open half-space in $\mathbf{R}^n$ if not.

We fix $s_0 : M_0 \to \mathbf{R}^{p,q}$ a stereographic projection. For all $i \in I$, we note $U_i = s_0(M_0 \cap M_i)$. They form a family of pairwise disjoint open sets of $\mathbf{R}^{p,q}$ and according to Observation 1, for all i, U_i is either a half-space with degenerate boundary, or a translate of U_S or U_T.

We make use of the following elementary considerations.

Fact 1. *If two half-spaces H_{v_1,α_1} and H_{v_2,α_2} are disjoint, then $v_2 = -v_1$ and $\alpha_1 \geqslant -\alpha_2$. A half-space $H_{v,\alpha}$ intersects any translate $v' + U_S$ and any translate of U_T. Two open sets $v_1 + U_S$ and $v_2 + U_S$ always intersect, as well as two translates of U_T. Moreover, $(v_1 + U_S) \cap (v_2 + U_T) = \emptyset$ if and only if $v_1 = v_2$.*

Proof. The first point is immediate. For the second, modifying α if necessary, we just have to consider $H_{v,\alpha} \cap U_S$ and $H_{v,\alpha} \cap U_T$. Since $v^\perp/\mathbf{R}.v$ has signature $(p-1, q-1)$, there exists $v_s \in U_S$ and $v_t \in U_T$ which are orthogonal to v_0. Thus, for any $v \in H_{v_0,\alpha}$, the lines $v + \mathbf{R}.v_s$ and $v + \mathbf{R}.v_t$ are contained in $H_{v_0,\alpha}$, and they intersect U_S and U_T respectively because the leading coefficient of $q(v+\lambda v_s)$ is positive and the one of $q(v+\lambda v_t)$ is negative. For the third point, we may only consider intersections $U_S \cap (v_0 + U_S)$, $U_T \cap (v_0 + U_T)$ and $U_S \cap (v_0 + U_T)$. If $q(v) > 0$, then for large enough λ, $v_0 + \lambda v \in U_S$ proving that $U_S \cap (v_0 + U_S)$ is always non-empty. The same argument works with $U_T \cap (v_0 + U_T)$. If $v_0 \in U_S$, we certainly have $U_S \cap (v_0 + U_T) \neq \emptyset$. If $v_0 \in U_T$, we choose $v \in v_0^\perp \cap U_S$ such that $q(v) < -q(v_0)$. We get $-v_0 + v \in U_T$ and $v_0 + (-v_0 + v) \in U_S$, proving $U_S \cap (v_0 + U_T) \neq \emptyset$. If $v_0 \in \mathcal{C} \setminus \{0\}$, then there is $v \in U_T$ such that $b(v_0, v) > 0$ (v_0 cannot be orthogonal to U_T and U_T is symmetric). As $q(v_0 + tv) = 2tb(v_0, v) + t^2 q(v)$, for small enough t, we have $v_0 + tv \in U_S$, proving $U_S \cap (v_0 + U_T) \neq \emptyset$. □

Let $U_1, U_2 \subset \mathbf{R}^{p,q}$ be two disjoint open subsets of intersection type. According to Fact 1, either $U_1 = H_{v_1,\alpha_1}$ and $U_2 = H_{v_2,\alpha_2}$ with v_1 isotropic, $v_2 = -v_1$ and $\alpha_1 \geqslant -\alpha_2$, or $U_1 = v + U_S$ and $U_2 = v + U_T$ with $v \in \mathbf{R}^{p,q}$. It is then clear that any third open subset U_3 of intersection type cannot be disjoint from U_1 and U_2. □

Thus, Lemma 3.9 is proved, contradicting the existence of $(U_k \cap U_0)_k$, and the proof of Proposition 3.6 is complete. □

3.3. **Thickenings.** A crucial point in the proof of the main results is the following.

Lemma 3.10. *If $V \subset \tilde{M}$ is a relatively compact maximal chart, then there is an open neighborhood $V' \supset \overline{V}$ of the closure of V in restriction to which $\mathfrak{D}$ is still injective.*

Proof. The first step is to prove that $\mathfrak{D}$ is injective in restriction to the closure $\overline{V}$. We simply have to verify that Lemma 3.4 applies. In the projective case, it is immediate that a hemisphere is locally connected relatively to $\mathbf{S}^n$. Let us see that it is also the case for a Minkowski patch $M_0 \subset \tilde{\mathbf{Ein}}^{p,q}$.

Let $x \in \partial M_0$, and let $\overline{x} = p(x) \in \partial p(M_0)$. Let $\overline{x}_0$ be the vertex of $p(M_0)$ and let $\overline{x}_1 \in p(M_0)$ be a point such that $\overline{x} \notin C(\overline{x}_1)$ and let $M_1 \subset \tilde{\mathbf{Ein}}^{p,q}$ be the Minkowski patch that projects to $\mathbf{Ein}^{p,q} \setminus C(\overline{x}_1)$ and that contains x. By construction, $\overline{x}_0 \in p(M_1)$. Let $x_0 \in p^{-1}(\overline{x_0})$ be the lift such that $x_0 \in M_1$. Finally, let $s_1 : M_1 \to \mathbf{R}^{p,q}$ be a stereographic projection such that $s_1(x_0) = 0$.

By Observation 1, $s_1(M_1 \cap M_0)$ is one of the connected components of $\mathbf{R}^{p,q} \setminus \mathcal{C}$, where $\mathcal{C}$ is the isotropic cone, *i.e.* either U_S or U_T.

Thus, x has a neighborhood M_1 with a chart $s_1 : M_1 \to \mathbf{R}^{p,q}$ in which M_0 is sent to one of the above open subsets. The problem being local, we are reduced to observe that U_S and U_T are locally connected relatively to $\mathbf{R}^{p,q}$. Since $\overline{U_S} \setminus \{0\}$ and $\overline{U_T} \setminus \{0\}$ are submanifolds with boundary of $\mathbf{R}^{p,q}$, this is obvious at the neighborhood of a non-zero vector in the boundary of these open sets. At the neighborhood of 0, noting $\pi^+ : \mathbf{R}^{p,q} \to \mathbf{R}^{p,q}/\mathbf{R}_{>0}$ the natural projection to the space of rays, it is enough to see that $\pi^+(U_S)$ and $\pi^+(U_T)$ are connected. The latter are diffeomorphic to $\{q = +1\}$ and $\{q = -1\}$ respectively, their connectedness is clear as $\min(p,q) \geqslant 2$.

Consequently, $\mathfrak{D}$ is injective in restriction to $\overline{V}$, which is compact by assumption. Assume to the contrary that $\mathfrak{D}$ is not injective in restriction to any neighborhood of $\overline{V}$. Considering a decreasing sequence $\{V_n\}$ such that $\overline{V} \subset V_n$ and $\overline{V} = \cap V_n$, we obtain two sequences $x_n, y_n \in V_n$ such that $\mathfrak{D}(x_n) = \mathfrak{D}(y_n)$ and $x_n \neq y_n$. By compactness of $\overline{V}$, we may assume that V_n is relatively compact, and up to an extraction, $(x_n) \to x \in \overline{V}$ and $(y_n) \to y \in \overline{V}$. Then $\mathfrak{D}(x) = \mathfrak{D}(y)$, implying $x = y$. Thus, (x_n) and (y_n) converge to a same limit x, contradicting the injectivity of $\mathfrak{D}$ on a neighborhood of x. $\square$

It has to be noted that $\mathfrak{D}(\overline{V}) = \overline{\mathfrak{D}(V)}$ by relative compactness of V. In particular, given any small enough neighborhood $\mathcal{V} \supset \overline{\mathfrak{D}(V)}$, there exists a neighborhood of $\overline{V}$ on which $\mathfrak{D}$ is injective and whose image is $\mathcal{V}$. This will be used in Section 5.

4. Atlas of maximal charts

We still consider a compact $(G_{\mathbf{X}}, \mathbf{X})$-manifold M, with universal cover $\pi : \tilde{M} \to M$. The aim of this section is to establish that in the dynamical context of a lattice action, $\tilde{M}$ is covered by maximal charts.

Proposition 4.1. *Let Γ be a cocompact lattice in a connected simple Lie group G with finite center.*

1. $\mathbf{X} = \mathbf{R}P^n$. *Assume that Γ acts projectively on M, with infinite image, and that* $\mathrm{Rk}_{\mathbf{R}}\, G = n$. *Then, any point of $\tilde{M}$ is contained in a maximal chart.*
2. $\mathbf{X} = \mathbf{Ein}^{p,q}$. *Assume that Γ acts conformally on M, with unbounded image, and that* $\mathrm{Rk}_{\mathbf{R}}\, G = p+1$. *Then, any point of $\tilde{M}$ is contained in a maximal chart.*

4.1. **Uniformly Lyapunov regular data.** This proposition relies on the dynamical phenomenon which is used in Section 6 for proving projective flatness, as well as in [Pec19] for proving conformal flatness. Namely:

Lemma 4.2. *In any compact, Γ-invariant subset of M, there is a point x such that there exist a sequence (γ_k) in Γ, a sequence of positive numbers $T_k \to \infty$, and a connected neighborhood V of x such that*

(1) *$\gamma_k V \to \{x\}$ for the Hausdorff topology,*
(2) *for all $v \in T_x M \setminus \{0\}$, $\frac{1}{T_k} \log \|D_x \gamma_k v\| \to -1$.*

In fact, we know more than this, but it is all what we need here.

Proof. We summarize the ideas for the conformal case, which are easily transferable to the projective one, and refer to Section 6 of [Pec19] for more details. If $M^\alpha \to G/\Gamma$ is the suspension bundle and if $K \subset M$ is a compact Γ-invariant subset, then $K^\alpha := (K \times M)/\Gamma \subset M^\alpha$ is G-invariant. Let $A < G$ be a Cartan subspace. We pick a finite A-invariant, A-ergodic measure μ supported in K^α and that projects to the Haar measure of G/Γ. Super-rigidity of cocycles and the rigidity of the Γ-invariant geometric structure on M imply that μ cannot be G-invariant (see Proposition 4.1 of [Pec19]). We then consider its vertical Lyapunov exponents $\chi_1, \ldots, \chi_r \in \mathfrak{a}^*$. One of the key steps of the proof of the main result of [BFH16] then implies that there exists $X \in \mathfrak{a}$ such that $\chi_1(X) = \cdots = \chi_r(X) = -1$. Considering a recurrent point $x^\alpha \in K^\alpha$ and local stable manifolds of the corresponding flow on M^α, we get "pseudo-return" times T_k for ϕ_X^t. Translating this in terms of dynamics in M, we get the announced sequence (γ_k) (see Section 6.2 of [Pec19]). □

Given a Riemannian manifold with a differentiable action of Γ - or another group -, such a triple $(V, (\gamma_k), (T_k))$ is called a *uniformly Lyapunov regular data at x*. The choice of the Riemannian norm is arbitrary if the manifold is compact. So, we fix a Riemannian metric on $\mathbf{X}$ and M, and pull it back to $\tilde{\mathbf{X}}$ and $\tilde{M}$.

Our approach for establishing Proposition 4.1 consists in proving that a uniformly Lyapunov regular data at x gives rise to a maximal chart containing x.

Remark 4.3. If $\gamma_k.x = x$ for all k, then the second point means that the sequence of matrices $D_x\gamma_k \in \mathrm{GL}(T_x M)$ is uniformly (T_k)-Lyapunov regular, in the sense of Definition 6.9 of [Pec19].

Let $(V, (\gamma_k), (T_k))$ be a uniformly Lyapunov regular data at a point $x \in M$. Let $\tilde{x} \in \tilde{M}$ be a point over x. Reducing V if necessary, there is a neighborhood $\tilde{V}$ of $\tilde{x}$ such that $\pi : \tilde{V} \to V$ is a diffeomorphism. For k large enough, $\gamma_k V \subset V$ and there exists a unique $\tilde{\gamma}_k \in \mathrm{Aut}(\tilde{M})$ projecting to γ_k and such that $\tilde{\gamma}_k(\tilde{x}) \in \tilde{V}$. It follows that $\tilde{\gamma}_k \tilde{V} \subset \tilde{V}$ because $\pi(\tilde{\gamma}_k \tilde{V}) \subset V$. And since π conjugates smoothly the action of $\tilde{\gamma}_k$ on $\tilde{V}$ to that of γ_k on V, we get that $(\tilde{V}, (\tilde{\gamma}_k), (T_k))$ is a uniformly Lyapunov regular data at $\tilde{x}$.

Let $g_k = \rho(\tilde{\gamma}_k)$. If V is small enough, D realizes a diffeomorphism from $\tilde{V}$ onto its image $U \subset \mathbf{X}$. Then, g_k preserves U and has the same dynamical property as $\tilde{\gamma}_k|_{\tilde{V}}$, *i.e.* $(U, (g_k), (T_k))$ is a uniformly Lyapunov regular data at $x_0 := D(\tilde{x}) \in \mathbf{X}$.

4.2. **Uniformly Lyapunov regular data on X.** We now consider such dynamical data on the model space $\mathbf{X}$. We start with some notations.

For $\mathbf{X} = \mathbf{R}P^n$, we choose $x_0 = [1 : 0 : \ldots : 0]$ as an origin and note $P < G_{\mathbf{X}} = \mathrm{PGL}(n+1, \mathbf{R})$ its stabilizer. We note $\mathfrak{a} \subset \mathfrak{p}$ the Cartan subspace of $\mathfrak{g}_{\mathbf{X}}$ formed of traceless diagonal matrices. We note

$$\mathfrak{n}_- = \left\{ \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix},\ v \in \mathbf{R}^n \right\} \subset \mathfrak{sl}(n+1, \mathbf{R}) \text{ and } \mathfrak{p}_+ = \left\{ \begin{pmatrix} 0 & {}^t v \\ 0 & 0 \end{pmatrix},\ v \in \mathbf{R}^n \right\} \subset \mathfrak{p}.$$

For $\mathbf{X} = \mathbf{Ein}^{p,q}$, we use the coordinates of $\mathbf{R}^{p+1,q+1}$ introduced in Section 2, and we also note $x_0 = [1 : 0 : \ldots : 0]$ and $P < G_{\mathbf{X}} = \mathrm{PO}(p+1, q+1)$ its stabilizer. We note $\mathfrak{a} < \mathfrak{p}$ the Cartan subspace of $\mathfrak{g}_{\mathbf{X}}$ formed of diagonal matrices of the form

$$\begin{pmatrix} \mu_0 & & & & & & \\ & \ddots & & & & & \\ & & \mu_p & & & & \\ & & & 0 & & & \\ & & & & -\mu_p & & \\ & & & & & \ddots & \\ & & & & & & -\mu_0 \end{pmatrix} \text{ with } \mu_0, \ldots, \mu_p \in \mathbf{R} \text{ and the 0 of size } q-p.$$

in the coordinates introduced in Section 2. We also note

$$\mathfrak{n}_- = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ v & 0 & 0 \\ 0 & -{}^t v J_{p,q} & 0 \end{pmatrix},\ v \in \mathbf{R}^{p,q} \right\} \subset \mathfrak{so}(p+1, q+1)$$

and

$$\mathfrak{p}_+ = \left\{ \begin{pmatrix} 0 & -{}^t v J_{p,q} & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix},\ v \in \mathbf{R}^{p,q} \right\} \subset \mathfrak{p}.$$

In both cases, $\mathfrak{n}_-$ and $\mathfrak{p}$ are supplementary and $\mathfrak{p}_+$ is the nilradical of $\mathfrak{p}$. We note $P_+ = \exp(\mathfrak{p}+)$, and $G_0 < P$ the section of P/P_+ whose Lie algebra is

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} -\operatorname{Tr}(A) & \\ & A \end{pmatrix},\ A \in \mathfrak{gl}(n, \mathbf{R}) \right\} \text{ for } \mathbf{X} = \mathbf{R}P^n$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \mu_0 & & \\ & A & \\ & & -\mu_0 \end{pmatrix},\ A \in \mathfrak{so}(p, q),\ \mu_0 \in \mathbf{R} \} \right\} \text{ for } \mathbf{X} = \mathbf{Ein}^{p,q}.$$

Lemma 4.4. *Let (g_k) be a sequence in $G_{\mathbf{X}}$ and $x \in \mathbf{X}$ such that:*

(1) *for all sequence $x_k \to x$, we have $g_k.x_k \to x$,*
(2) *for all non-zero $v \in T_x\mathbf{X}$, $\frac{1}{T_k} \log \|D_x g_k v\| \to -1$.*

Then, up to passing to a subsequence, there exists $U_{\max} \ni x$ which is an affine chart domain if $\mathbf{X} = \mathbf{R}P^n$ or a Minkowski patch if $\mathbf{X} = \mathbf{Ein}^{p,q}$, and such that for all compact subset $K \subset U_{\max}$, $g_k K \to \{x\}$ for the Hausdorff topology.

Proof. By homogeneity, we may assume $x = x_0$. We first prove that there exists $X_k \in \mathfrak{n}_-$, with $(X_k) \to 0$, bounded sequences $(l_k), (l'_k)$ in P, and a sequence (a_k) in A such that

$$g_k = e^{X_k} l_k a_k l'_k \text{ for all } k. \tag{1}$$

This is in fact a basic case of Lemma 4.3 of [Fra12], we nonetheless explain how it works in this model situation. For k large enough, $g_k x_0 \in U_0$ and there exists a unique $X_k \in \mathfrak{n}_-$ such that $g_k x_0 = e^{X_k} x_0$, and $X_k \to 0$ since $g_k x_0 \to x_0$. Then, $p_k := e^{-X_k} g_k \in P$ satisfies the same properties as g_k. Indeed, if we note $g'_k = e^{-X_k}$, then g'_k, seen as diffeomorphisms of $\mathbf{X}$, are bounded in topology C^1 since $g'_k \to \mathrm{id}$ in the Lie group. Thus, there is $C > 0$ such that $\frac{1}{C}\|v\| \leqslant \|D_x g'_k v\| \leqslant C\|v\|$ for all $k \geqslant 0$ and (x, v) tangent vector of $\mathbf{X}$. The property on the exponential growth rate of $D_{x_0} p_k$ follows directly. Also, for all $x_k \to x_0$, we have $g_k x_k \to x_0$ by assumption, and there exists $h_k \in G$, with $h_k \to \mathrm{id}$ such that $g_k x_k = h_k x_0$, proving that $e^{-X_k} g_k x_k = e^{-X_k} h_k x_0 \to x_0$ since $e^{-X_k} h_k \to \mathrm{id}$.

We decompose $p_k = p_k^\ell e^{Y_k}$ where $p_k^\ell \in G_0$ and $Y_k \in \mathfrak{p}_+$ according to $P = G_0 \ltimes P_+$. The KAK decomposition of G_0 gives bounded sequences $(l_k), (m_k) \in G_0$, and a sequence $a_k \in A$ such that $p_k^\ell = l_k a_k m_k$. Thus, $p_k = l_k a_k e^{Y'_k} m_k$, where $Y'_k = \mathrm{Ad}(m_k) Y_k$. Let $p'_k = a_k e^{Y'_k}$. We claim that (p'_k) satisfies the same hypothesis as (g_k). Indeed, if $x_k \to x_0$, then writing $x_k = e^{X''_k} x_0$ for some $X''_k \in \mathfrak{n}_-$, such that $X''_k \to 0$, we get $(m_k)^{-1} x_k = e^{\mathrm{Ad}(m_k{}^{-1}) X''_k} x_0$ since $m_k \in P$, and $\mathrm{Ad}(m_k{}^{-1}) X''_k \to 0$ as $\mathrm{Ad}(m_k{}^{-1})$ is bounded. This proves that $m_k{}^{-1} x_k \to x_0$. Consequently, $p_k m_k{}^{-1} x_k \to x_0$, and finally $p'_k x_k = l_k^{-1} p_k m_k{}^{-1} x_k \to x_0$ by the same argument. The property on the exponential growth rate is also preserved because $D_{x_0} l_k$ and $D_{x_0} m_k$ are bounded sequences in $\mathrm{GL}(T_{x_0}\mathbf{X})$ (see Remark 4.3 and Lemma 6.10 of [Pec19]).

Using this property of p'_k, we prove now that Y'_k is a bounded sequence of $\mathfrak{p}_+$, which will establish (1).

- Case $\mathbf{X} = \mathbf{R}P^n$. We note

$$a_k = \begin{pmatrix} \lambda_0^{(k)} & & \\ & \ddots & \\ & & \lambda_n^{(k)} \end{pmatrix} \text{ and } e^{Y'_k} = \begin{pmatrix} 1 & v^{(k)} \\ 0 & \mathrm{id} \end{pmatrix}$$

where $\lambda_i^{(k)} > 0$ and $v^{(k)} \in \mathbf{R}^n$. We assume to the contrary that some component $v_i^{(k)}$ of $v^{(k)}$ is unbounded. Up to an extraction, we may assume $|v_i^{(k)}| \to \infty$. We get a contradiction with the first property of p'_k by considering its action on the projective line

$$p'_k[1 : 0 : \ldots : t : \ldots : 0] = [\lambda_0^{(k)}(1 + v_i^{(k)} t) : 0 : \ldots : \lambda_i^{(k)} : \ldots : 0],$$

where t stands at the $(i+1)$-th position. For k large enough, we can consider $x_k := [1 : 0 : \cdots : -\frac{1}{v_i^{(k)}} : \ldots : 0]$ and we get that $p'_k x_k = [0 : \ldots : 1 : \ldots : 0]$ does not converge to x_0, whereas $x_k \to x_0$ since $v_i^{(k)} \to \infty$, a contradiction.

- Case $\mathbf{X} = \mathbf{Ein}^{p,q}$. We note

$$a_k = \begin{pmatrix} \lambda_0^{(k)} & & \\ & \ddots & \\ & & \lambda_{n+1}^{(k)} \end{pmatrix} \text{ and } e^{Y'_k} = \begin{pmatrix} 1 & v^{(k)} & -\frac{1}{2} < v^{(k)}, v^{(k)} > \\ 0 & \mathrm{id} & -v^{(k)*} \\ 0 & 0 & 1 \end{pmatrix}$$

where $\lambda_i^{(k)} > 0$ for all i, and $v^{(k)} = (v_1^{(k)}, \dots, v_n^{(k)}) \in \mathbf{R}^n$. We remind the notation $v^* = J_{p,q}\,{}^t v$ for all line vector $v \in \mathbf{R}^n$ and $< v, v >= vJ_{p,q}\,{}^t v = vv^*$. The $\lambda_i^{(k)}$'s satisfy other relations that we will not use here.

Claim 1. *For all $i \in \{1, \dots, p\} \cup \{q+1, \dots, n\}$, the sequence $(v_i^{(k)})$ is bounded.*

Assume to the contrary that some $v_i^{(k)}$ is unbounded, for $i \in \{1, \dots, p\} \cup \{q+1, \dots, n\}$. Extracting if necessary, we may assume $v_i^{(k)} \neq 0$ and $|v_i^{(k)}| \to \infty$. From this we exhibit a sequence $x_k \in \mathbf{Ein}^{p,q}$ such that $x_k \to x_0$ but $p'_k x_k \nrightarrow x_0$ which will be a contradiction.

The plane spanned by e_0 and e_i in $\mathbf{R}^{p+1,q+1}$ is totally isotropic, and we can read the action of p'_k on the corresponding light-like circle of $\mathbf{Ein}^{p,q}$ in an affine chart:

$$p'_k.[1 : 0 : \dots : t : 0 : \dots : 0] = [\lambda_0^{(k)} + \lambda_0^{(k)} v_i^{(k)} t : 0 : \dots : \lambda_i^{(k)} t : 0 : \dots : 0].$$

where $t \in \mathbf{R}$ stands at the $(i+1)$-th coordinate. Letting $t_k := -\frac{1}{v_i^{(k)}}$, we get

$$p'_k.[1 : 0 : \dots : t_k : 0 : \dots : 0] = [0 : 0 : \dots : 1 : 0 : \dots : 0],$$

whereas $[1 : 0 : \dots : t_k : 0 : \dots : 0] \to x_0$ since $|v_i^{(k)}| \to \infty$. This proves the claim.

We can now prove that for all $i \in \{p+1, \dots, q\}$, the sequence $(v_i^{(k)})$ is also bounded. Let us assume to the contrary that for some such i it is not the case. Up to an extraction, we may assume $|v_i^{(k)}| \to \infty$. Let us consider the action of p'_k on a point of the form $x_t := [1 : t : 0 : \dots : t : \dots : 0 : -\frac{t}{2} : 0] \in \mathbf{Ein}^{p,q}$ with $t \in \mathbf{R}$ and its second occurrence standing at the $(i+1)$-th coordinate. We get

$$p'_k x_t = [\lambda_0^{(k)}(1 + tv_1^{(k)} + tv_i^{(k)} - \frac{t}{2} v_n^{(k)}) : \lambda_1^{(k)} t : 0 : \dots : t : \dots : -\frac{t}{2} \lambda_n^{(k)} : 0]$$

It has to be noted that $\lambda_{i+1}^{(k)} = 1$ as $p+1 \leqslant i \leqslant q$ and that $\lambda_1^{(k)} \lambda_n^{(k)} = 1$. By assumption, $|v_i^{(k)}| \to \infty$, and by Claim 1, $v_1^{(k)}$ and $v_n^{(k)}$ are bounded. So, for k large enough, we can define $t_k := -1/(v_1^{(k)} + v_i^{(k)} - \frac{v_n^{(k)}}{2})$ and $t_k \to 0$. Then, we get

$$p'_k x_{t_k} = [0 : \lambda_1^{(k)} : 0 : \dots : 1 : \dots : -\frac{\lambda_n^{(k)}}{2} : 0],$$

proving that $p'_k x_{t_k}$ cannot converge to $x_0 = [1 : 0 : \dots : 0]$, a contradiction.

Finally, we have proved that $e^{Y'_k} \in P$ is bounded in both cases, and if we set $l'_k = e^{Y'_k} m_k$, we get as announced

$$g_k = e^{X_k} p_k = e^{X_k} l_k p'_k m_k = e^{X_k} l_k a_k l'_k$$

where $X_k \in \mathfrak{n}_-$ goes to 0, $a_k \in A$, and $l_k, l'_k \in P$ are bounded sequences.

We note $\rho : P \to \mathrm{GL}(\mathfrak{g}_{\mathbf{X}}/\mathfrak{p})$ the map obtained by inducing the adjoint representation of P on $\mathfrak{g}_{\mathbf{X}}/\mathfrak{p}$. We remind that ρ is conjugate to the isotropy representation $P \to \mathrm{GL}(T_{x_0}\mathbf{X})$ via the identification $T_{x_0}\mathbf{X} \simeq \mathfrak{g}_{\mathbf{X}}/\mathfrak{p}$ given by the orbital map at x_0.

Claim 2. *The sequence $\rho(a_k) \in \mathrm{GL}(\mathfrak{g}_{\mathbf{X}}/\mathfrak{p})$ is (T_k)-uniformly Lyapunov regular (see Remark 4.3).*

By Lemma 6.10 of [Pec19], it is the same as saying that $\rho(l_k a_k l_k') = \rho(p_k)$ is uniformly Lyapunov regular. And this was observed at the beginning of the proof, proving this claim.

The action of $\rho(a_k)$ on $\mathfrak{g}_{\mathbf{X}}/\mathfrak{p}$ is the same as $\mathrm{Ad}(a_k)$ on $\mathfrak{n}_-$. Writing

$$\mathrm{Ad}(a_k)|_{\mathfrak{n}_-} = \mathrm{diag}(\mu_1^{(k)}, \dots, \mu_n^{(k)}),$$

the previous claim means $\frac{1}{T_k}\log \mu_i^{(k)} \to -1$ for all $i \in \{1,\dots,n\}$. This implies that for any compact subset $\mathcal{K} \subset \mathfrak{n}_-$, $\mathrm{Ad}(a_k)\mathcal{K} \to \{0\}$ for the Hausdorff topology, because for k large enough, $\mu_i^{(k)} \leqslant e^{-T_k/2}$.

Up to an extraction, we may assume $l_k' \to l' \in P$. Let $U_{\max} := l'^{-1}\exp(\mathfrak{n}_-).x_0$. We prove now that for all compact subset $K \subset U_{\max}$, $g_k K \to \{x_0\}$ for the Hausdorff topology.

Let $V \ni x_0$ be a neighborhood of x_0. As $X_k \to 0$, there is k_0 and another neighborhood $V_0 \ni x_0$ such that for all $k \geqslant k_0$, $e^{X_k}V_0 \subset V$. Reducing V_0 if necessary, we may assume $V_0 = \exp(\mathcal{V}_0).x_0$, for some $\mathcal{V}_0 \subset \mathfrak{g}'$ neighborhood of 0. Since l_k is relatively compact in P, we can choose a smaller neighborhood $\mathcal{V}_1$ such that for all k, $\mathrm{Ad}(l_k)\mathcal{V}_1 \subset \mathcal{V}_0$. Hence, $(l_k\exp(\mathcal{V}_1)).x_0 = \exp(\mathrm{Ad}(l_k)\mathcal{V}_1).x_0 \subset V_0$. Let $V_1 = \exp(\mathcal{V}_1).x_0$.

Let $K' = l'K \subset \exp(\mathfrak{n}_-).x_0$. Let $K'' \subset \exp(\mathfrak{n}_-).x_0$ be a compact subset and k_1 be such that for all $k \geqslant k_1$, $l_k'K = (l_k' l'^{-1})K' \subset K''$. Let $\mathcal{K}'' \subset \mathfrak{n}_-$ be such that $K'' = \exp(\mathcal{K}'')x_0$ and let k_2 such that $\mathrm{Ad}(a_k)\mathcal{K}'' \subset \mathcal{V}_1$ for all $k \geqslant k_2$, so $a_k K'' \subset V_1$.

For $k \geqslant \max(k_0,k_1,k_2)$, we get $g_k.K = e^{X_k} l_k a_k l_k' K \subset e^{X_k} l_k a_k K'' \subset e^{X_k} l_k V_1 \subset V$. $\square$

4.3. **Conclusion.** We remind that we are considering a uniformly regular Lyapunov data $(\tilde{V}, (\tilde{\gamma}_k), (T_k))$ at a point $\tilde{x} \in \tilde{M}$ and that we note $x_0 = D(\tilde{x})$, $g_k = \rho(\tilde{\gamma}_k)$ and $U = D(\tilde{V})$. Since $(U,(g_k),(T_k))$ is a uniformly regular Lyapunov data at x_0, we consider $U_{\max} \subset \mathbf{X}$ the open set given by Lemma 4.4. Restricting $\tilde{V}$ if necessary, we assume $\overline{U} \subset U_{\max}$.

Consider for $k \geqslant 0$ the open neighborhood $\tilde{V}_k = \tilde{\gamma}_k^{-1}\tilde{V} \subset \tilde{M}$ of $\tilde{x}$. By equivariance, D is injective in restriction to $\tilde{V}_k$. Also, since $\{\tilde{\gamma}_k \tilde{V}\} \to \{\tilde{x}\}$, we may assume $\tilde{\gamma}_k\tilde{V} \subset \tilde{V}$ for all k, and then $\tilde{V} \subset \tilde{V}_k$ for all k. We introduce now

$$\tilde{V}_\infty = D^{-1}(U_{\max}) \cap \bigcup_{k \geqslant 0} \bigcap_{l \geqslant k} \tilde{V}_l.$$

Claim 3. *$\tilde{V}_\infty$ is a maximal chart containing $\tilde{x}$ and such that $D(\tilde{V}_\infty) = U_{\max}$.*

The injectivity of D in restriction to $\tilde{V}_\infty$ is immediate as for any two points in $\tilde{V}_\infty$, there is $k \geqslant 0$ such that they both belong to $\tilde{V}_k$.

To see that it is open, let us prove that for all $k \geqslant 0$, every $\tilde{y} \in D^{-1}(U_{\max}) \cap \bigcap_{l \geqslant k} \tilde{V}_l$ admits a neighborhood contained in $D^{-1}(U_{\max}) \cap \bigcap_{l \geqslant k'} \tilde{V}_l$, for some k'. Let $y_0 = D(\tilde{y})$.

By definition, $y_0 \in U_{\max}$ and for all $l \geqslant k$, $\tilde{\gamma}_l \tilde{y} \in \tilde{V}$. We then choose a connected open neighborhood V_0 of $\tilde{y}$ such that $\overline{D(V_0)} \subset U_{\max}$. By Lemma 4.4, there is k' such that for all $l \geqslant k'$, $g_l \overline{D(V_0)} \subset U$. Consequently, $D(\tilde{\gamma}_l V_0) \subset U$ for $l \geqslant k'$ and $\tilde{\gamma}_l V_0 \cap \tilde{V} \neq \emptyset$ if $l \geqslant k$.

Since D is injective on $\tilde{V}$, Lemma 3.2 implies that for $l \geqslant \max(k,k')$, we have $\tilde{\gamma}_l V_0 \subset \tilde{V}$, *i.e.* $V_0 \subset \bigcap_{l \geqslant \max(k,k')} \tilde{V}_l$, and then $V_0 \subset \tilde{V}_\infty$ proving that the latter is open.

Let us prove now that $D(\tilde{V}_\infty) = U_{\max}$. Let $W \subset U_{\max}$ a connected open subset such that $\overline{W} \subset U_{\max}$ and $U \subset W$. There exits $k_0 \geqslant 0$ such that $g_k.\overline{W} \subset U$ for all $k \geqslant k_0$. If $k \geqslant k_0$, then $g_k \overline{W} \subset U$, and then $W \subset D(\tilde{V}_k)$. Consider now $\tilde{V}_{k,W} = (D|_{\tilde{V}_k})^{-1}(W)$ which is well defined since D is injective in restriction to $\tilde{V}_k$. Note that $\tilde{V}_{k,W}$ is connected. We claim that $\tilde{V}_{k,W} = \tilde{V}_{l,W}$ for all $k,l \geqslant k_0$. Indeed, $D(\tilde{\gamma}_k \tilde{V}_{l,W}) = g_k W \subset U$. By Lemma 3.2, we get $\tilde{\gamma}_k \tilde{V}_{l,W} \subset \tilde{V}$ because $\tilde{V} \subset \tilde{V}_{l,W}$ implies $\tilde{V} \cap \tilde{\gamma}_k \tilde{V}_{l,W} \neq \emptyset$, and then $\tilde{V}_{l,W} \subset \tilde{V}_k$. Thus, $\tilde{V}_{l,W} = \tilde{V}_{k,W}$ since $D(\tilde{V}_{l,W}) = W$.

Consequently, $\tilde{V}_{k_0,W} \subset \bigcap_{k \geqslant k_0} \tilde{V}_k$ and $D(\tilde{V}_{k_0,W}) = W$. Thus, $W \subset D(\tilde{V}_\infty)$, and this for all connected, relatively compact, open subset $W \subset U_{\max}$, proving $U_{\max} \subset D(\tilde{V}_\infty)$.

Finally, for all k, the projection $\pi : \tilde{M} \to M$ is injective in restriction to $\tilde{V}_k$ since $\pi(\tilde{\gamma}_k^{-1}\tilde{y}) = \pi(\tilde{\gamma}_k^{-1}\tilde{z})$ implies $\gamma_k \pi(\tilde{y}) = \gamma_k \pi(\tilde{z})$ and since $\pi|_{\tilde{V}}$ is injective. The same argument as for the injectivity of $D|_{\tilde{V}_\infty}$ then applies, proving Claim 3.

We can conclude the proof of Proposition 4.1. Let $\tilde{y} \in \tilde{M}$ and $y = \pi(\tilde{y})$. Applying Lemma 4.2 to $\overline{\Gamma.y}$ yields a Lyapunov regular data $(V, (\gamma_k), (T_k))$ at a point $x \in \overline{\Gamma.y}$. This section has proved that any $\tilde{x} \in \pi^{-1}(x)$ is contained in a maximal chart $\tilde{V}_\infty$. Let $\gamma \in \Gamma$ be such that $\gamma.y \in \pi(\tilde{V}_\infty)$. Let $\tilde{\gamma} \in \mathrm{Aut}(\tilde{M})$ be the element that projects to γ and such that $\tilde{\gamma}.\tilde{y} \in \tilde{V}_\infty$. Then, $\tilde{\gamma}^{-1}\tilde{V}_\infty$ is a maximal chart containing $\tilde{y}$ and the proof of Proposition 4.1 is complete.

5. Injectivity of the developing map

In this section, we prove Theorem 3. As explained in the introduction, combined with Proposition 4.1, Proposition 6.1 below and Theorem 1 of [Pec19], this will conclude the proof of Theorem 1 and Theorem 2. We remind that we have fixed $\mathfrak{D} : \tilde{M} \to \tilde{\mathbf{X}}$ a developing map, with holonomy $\rho : \mathrm{Aut}(\tilde{M}) \to \mathrm{Aut}(\tilde{X})$. Assuming that every point of $\tilde{M}$ is contained in a maximal chart, we claim that is enough to prove that $\mathfrak{D}$ is injective to get the conclusion.

Indeed, if $V \subset \tilde{M}$ is a maximal chart, then $\gamma V \cap V = \emptyset$ for any non-trivial $\gamma \in \pi_1(M)$ by definition. By injectivity of ρ and $\mathfrak{D}$, the $\rho(\gamma)\mathfrak{D}(V)$, $\gamma \in \pi_1(M)$ are pairwise disjoint. By definition, $\mathfrak{D}(V)$ is an hemisphere of $\mathbf{S}^n$ in the projective case, and a Minkowski patch of $\mathbf{Ein}^{p,q}$ in the conformal one. Consequently $|\pi_1(M)| \leqslant 2$, $\tilde{M}$ is compact and $\mathfrak{D}$ is a diffeomorphism. The conclusion follows directly.

So, Theorem 3 is reduced to the proof of the injectivity of $\mathfrak{D}$, which we establish in this section.

5.1. **Common principle.** Let (V_m) be a covering of $\tilde{M}$ by pairwise distinct maximal charts such that for all $m \geqslant 1$, V_{m+1} intersects $\cup_{k \leqslant m} V_k$. We remind that the V_i's are relatively compact in $\tilde{M}$ by Proposition 3.6. If for all $m \geqslant 1$, $\mathfrak{D}(V_{m+1}) \cap (\mathfrak{D}(V_1) \cup \cdots \cup$

$\mathfrak{D}(V_m))$ is connected, then, using Lemma 3.5, we get by induction that $\mathfrak{D}$ is injective in restriction to $V_1 \cup \cdots \cup V_m$ for all m, *i.e.* that $\mathfrak{D}$ is injective.

So, let us assume that there exists m such that $\mathfrak{D}(V_{m+1}) \cap (\mathfrak{D}(V_1) \cup \cdots \cup \mathfrak{D}(V_m))$ is not connected, and let us choose the smallest one. Then, by the same argument as above, we get that $\mathfrak{D}$ is injective in restriction to $V_1 \cup \cdots \cup V_m$. Note that $m \geqslant 2$ by construction.

We pick a chart $\varphi : \mathfrak{D}(V_{m+1}) \to \mathbf{R}^n$ which is either an affine chart in the projective case, or a stereographic projection in the conformal case, and we note for $1 \leqslant i \leqslant m$, $U_i = \varphi(\mathfrak{D}(V_i) \cap \mathfrak{D}(V_{m+1}))$.

Claim 4. *The U_i's are pairwise distinct.*

Indeed, if $U_i = U_j$, then $\mathfrak{D}(V_i) = \mathfrak{D}(V_j)$ because an hemisphere (resp. a Minkowski patch) is determined by its intersection with a given hemisphere (resp. Minkowski patch). By injectivity of $\mathfrak{D}$ on $V_1 \cup \ldots \cup V_m$, this implies $V_i = V_j$ and then $i = j$ by choice of (V_i).

We then classify the configurations in which a family of such open subsets of $\mathbf{R}^n$ can have non-connected union. Finally we prove that in such configurations, if one of the V_i's is thickened (see Section 3.3), then Lemma 3.5 applies and yields an open set $U \subset \tilde{M}$ in restriction to which $\mathfrak{D}$ is injective and such that $\mathfrak{D}(U) = \tilde{\mathbf{X}}$, proving that $U = \tilde{M}$, and completing the proof of the injectivity of $\mathfrak{D}$ in this *a priori* problematic situation.

5.2. **Projective case.** For $\mathbf{X} = \mathbf{R}P^n$, $\mathfrak{D}$ sends maximal charts onto hemispheres of $\mathbf{S}^n$. We will use the basic facts recalled below. Let $\iota : \mathbf{S}^n \to \mathbf{S}^n$ be the antipodal map.

5.2.1. *Some conventions and facts on hemispheres.* We embed $\mathbf{S}^n \subset \mathbf{R}^{n+1}$ in the standard way. A hemisphere $H \subset \mathbf{S}^n$ is the data of a half-line $\mathbf{R}_{>0}\ell$, for $\ell \in (\mathbf{R}^{n+1})^*$ such that $H = \mathbf{S}^n \cap \{\ell > 0\}$.

Given two hemispheres H_0 and H_1, and an affine chart $\varphi : H_0 \to \mathbf{R}^n$, if H_0 and H_1 are not equal or antipodal, then $\varphi(H_0 \cap H_1)$ is an affine half-space of $\mathbf{R}^n$. This gives a bijection between the set of hemisphere minus $\{H_0, \iota(H_0)\}$ and the set of affine half-spaces of $\mathbf{R}^n$.

We will use below the following facts which can be easily observed in coordinates.

Fact 2. *Let H_0, H_1, H_2 be three hemispheres such that $H_1 \neq \iota(H_2)$. If $H_0 \cap H_1$ and $H_0 \cap H_2$ are disjoint, then $\iota(H_0) \subset H_1 \cup H_2$.*

Fact 3. *Let H_0, H_1, H_2 be three hemispheres and let $\varphi : H_0 \to \mathbf{R}^n$ be an affine chart. Assume that $\varphi(H_0 \cap H_1)$ and $\varphi(H_0 \cap H_2)$ are parallel and in the same direction, that is there exists $\phi \in (\mathbf{R}^n)^*$, $\alpha_1, \alpha_2 \in \mathbf{R}$ such that $\varphi(H_0 \cap H_i) = \{\phi > \alpha_i\}$ for $i = 1, 2$. Then, $H_1 \cap \partial H_0 = H_2 \cap \partial H_0$.*

Let $V \subset \tilde{M}$ be a relatively compact maximal chart, and let $H = \mathfrak{D}(V)$. We have seen in Section 3.3 that $\mathfrak{D}$ is still injective on small enough neighborhoods of $\overline{V}$. In particular, if $\varepsilon > 0$ is small enough, there is a neighborhood V^ε of $\overline{V}$ on which $\mathfrak{D}$ is injective and such that $\mathfrak{D}(V^\varepsilon) = H^\varepsilon := \mathbf{S}^n \cap \{\ell > -\varepsilon\}$ where ℓ is such that $H = \mathbf{S}^n \cap \{\ell > 0\}$.

Affine charts are not well adapted to these thickenings H^ε, it is more relevant to use stereographic projections even though no conformal structure is involved.

Notably, if $x \notin H^\varepsilon$, and if $s : \mathbf{S}^n \setminus \{x\} \to \mathbf{R}^n$ is a stereographic projection, then $s(H^\varepsilon)$ is a ball. The following fact is then clear.

Fact 4. *Let H_1, H_2 be two hemispheres. If $\varepsilon > 0$ is small enough, then $H_1^\varepsilon \cap H_2$ is connected.*

Proof. If $H_2 = \iota(H_1)$, then we pick $x \notin H_1^\varepsilon$ and fix a stereographic projection $s : \mathbf{S}^n \setminus \{x\} \to \mathbf{R}^n$. Then, $s(H_2 \setminus \{x\})$ is the complement of a closed ball $\overline{B}_1$ and $s(H_1^\varepsilon)$ is another open ball B_2, that contains $\overline{B}_1$. So, $H_1^\varepsilon \cap H_2$ is diffeomorphic to $B_2 \setminus \overline{B}_1$, which is connected.

If $H_2 \neq \iota(H_1)$, then for $\varepsilon > 0$ small enough, we can choose $x \notin H_1^\varepsilon \cup H_2$. A stereographic projection defined on $\mathbf{S}^n \setminus \{x\}$ then sends $H_1^\varepsilon \cap H_2$ onto the intersection of two balls in $\mathbf{R}^n$, which is connected. □

Finally, we will make use of the following.

Fact 5. *Let H_0, H_1, H_2 be three hemispheres, with H_1 and H_2 not antipodal. Assume that $H_0 \cap (H_1 \cup H_2)$ is not connected. Then, for small enough ε, $H_0^\varepsilon \cap (H_1 \cup H_2)$ is connected. See Figure 1.*

Proof. By Fact 4, it is enough to prove that $H_0^\varepsilon \cap H_1 \cap H_2$ is non-empty for small enough ε. By assumption, $H_0 \cap H_1$ and $H_0 \cap H_2$ are disjoint. Let ℓ_0, ℓ_1, ℓ_2 be linear forms defining H_0, H_1, H_2 respectively. Our assumption means that ℓ_1 and ℓ_2 are non-colinear and $-\ell_0 \in \mathrm{Conv}(\ell_1, \ell_2)$ (the open convex hull).

In particular, there is a point $x \in \cap_i \partial H_i$. Since $x \in H_0^\varepsilon$, it is enough to observe that H_1 and H_2 intersect arbitrarily close to x. To see it, we pick $s : \mathbf{S}^n \setminus \{-x\} \to \mathbf{R}^n$ a stereographic projection such that $s(x) = 0$. Then, H_1 and H_2 are sent to half-spaces delimited by two distinct linear hyperplanes. It is then immediate that they intersect arbitrarily close to 0. □

5.2.2. *Configurations where the induction fails.* For all $i \leqslant m+1$, we note $\mathfrak{D}(V_i) = H_i \subset \mathbf{S}^n$. As announced above, we consider the smallest integer m such that $H_{m+1} \cap (H_1 \cup \dots H_m)$ is not connected. Let $\varphi : H_{m+1} \to \mathbf{R}^n$ be an affine chart. For all $1 \leqslant i \leqslant m$, $U_i = \varphi(H_i \cap H_{m+1}) \subset \mathbf{R}^n$ is either empty, a half-space, or $\mathbf{R}^n$.

We remind that $U_1, \dots, U_m$ are pairwise distinct, in particular at most one of them is empty. Thus, there is $l \in \{m-1, m\}$, with $l \geqslant 2$, an injective map $\sigma : \{1, \dots, l\} \to \{1, \dots, m\}$, $\phi \in (\mathbf{R}^n)^*$, $k_0 \in \{1, \dots, l-1\}$ and $\alpha_1 < \cdots < \alpha_{k_0} \leqslant \alpha_{k_0+1} < \cdots < \alpha_l$ such that $U_{\sigma(k)} = \{\phi > \alpha_k\}$ for all k, and if $l = m-1$ and i is the unique element not in the range of σ, $U_i = \emptyset$. We note $i_0 = \sigma(k_0)$ and $j_0 = \sigma(k_0+1)$.

5.2.3. *Case $\alpha_{i_0} = \alpha_{j_0}$.* In this situation, $H_{i_0} = \iota(H_{j_0})$. Because $H_1 \cup \cdots \cup H_m$ is connected, it contains a point $y_0 \in \partial H_{i_0} = \partial H_{j_0}$. Let $y \in V_1 \cup \cdots \cup V_m$ be its preimage. Then, $y \in \partial V_{i_0} \cap \partial V_{j_0}$, showing $\partial V_{i_0} \cap \partial V_{j_0} \neq \emptyset$.

Now, let $V_{i_0}^\varepsilon \supset \overline{V_{i_0}}$ be a neighborhood in restriction to which $\mathfrak{D}$ is injective and such that $\mathfrak{D}(V_{i_0}^\varepsilon) = H_{i_0}^\varepsilon$ for some $\varepsilon > 0$. Then $V_{i_0}^\varepsilon \cap V_{j_0} \neq \emptyset$ and $H_{i_0}^\varepsilon \cap H_{j_0}$ is connected by Fact 4. By Lemma 3.5, we get that $\mathfrak{D}$ is injective in restriction to $V_{i_0}^\varepsilon \cup V_{j_0}$, and $\mathfrak{D}(V_{i_0}^\varepsilon \cup V_{j_0}) = \mathbf{S}^n$.

Thus, we get that $V_{i_0}^\varepsilon \cup V_{j_0} = \tilde{M}$ and that $\mathfrak{D}$ is a diffeomorphism onto $\mathbf{S}^n$.

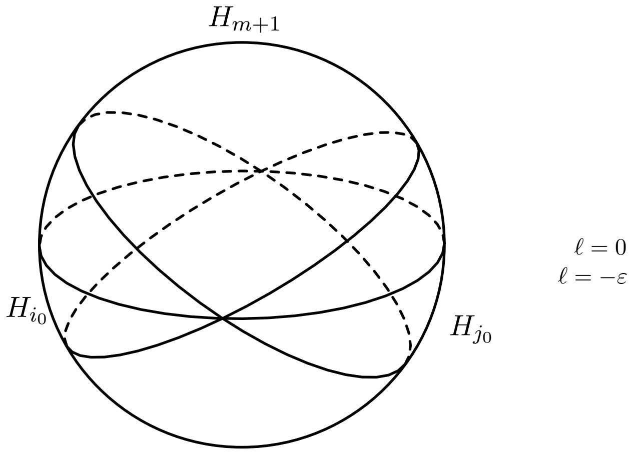

FIGURE 1. Configuration for $\alpha_{i_0} < \alpha_{j_0}$

5.2.4. *Case* $\alpha_{i_0} < \alpha_{j_0}$. In this situation, we claim that $H_1 \cup \cdots \cup H_m = H_{i_0} \cup H_{j_0}$. Indeed, considering the partition $\mathbf{S}^n = H_{m+1} \cup \partial H_{m+1} \cup \iota(H_{m+1})$, we see first that $H_{m+1} \cap (H_1 \cup \cdots \cup H_m)$ and $H_{m+1} \cap (H_{i_0} \cup H_{j_0})$ coincide by assumption and choice of k_0. Then, by Fact 2, we get $\iota(H_{m+1}) \subset H_{i_0} \cup H_{j_0}$, proving in particular that $\iota(H_{m+1}) \cap (H_1 \cup \cdots \cup H_m)$ and $\iota(H_{m+1}) \cap (H_{i_0} \cup H_{j_0})$ also coincide. Finally, by Fact 3, we have $H_{\sigma(1)} \cap \partial H_{m+1} = \cdots = H_{\sigma(k_0)} \cap \partial H_{m+1}$ and $H_{\sigma(k_0+1)} \cap \partial H_{m+1} = \cdots = H_{\sigma(l)} \cap \partial H_{m+1}$, proving that $\partial H_{m+1} \cap (H_{i_0} \cup H_{j_0}) = \partial H_{m+1} \cap (H_1 \cup \cdots \cup H_m)$.

By injectivity of $\mathfrak{D}$ in restriction to $V_1 \cup \cdots \cup V_m$, we have that $V_{i_0} \cup V_{j_0} = V_1 \cup \cdots \cup V_m$. Therefore, $V_{i_0} \cap V_{j_0} \neq \emptyset$ and $V_{m+1} \cap (V_{i_0} \cup V_{j_0}) \neq \emptyset$ and $H_{m+1} \cap (H_{i_0} \cup H_{j_0})$ is not connected.

Therefore, Fact 5 implies that $\overline{V_{m+1}}$ admits a neighborhood V^{ε}_{m+1} in restriction to which $\mathfrak{D}$ is injective and such that $\mathfrak{D}(V^{\varepsilon}_{m+1}) \cap (\mathfrak{D}(V_{i_0} \cup \mathfrak{D}(V_{j_0}))$ is connected. Thus, we can apply Lemma 3.5 to obtain first that $\mathfrak{D}$ is injective in restriction to $V_{i_0} \cup V_{j_0}$, and then in restriction to $V^{\varepsilon}_{m+1} \cup V_{i_0} \cup V_{j_0}$. The image of this open subset of $\tilde{M}$ is $\mathbf{S}^n$, proving that $\tilde{M} = V^{\varepsilon}_{m+1} \cup V_{i_0} \cup V_{j_0}$ and that $\mathfrak{D} : \tilde{M} \to \mathbf{S}^n$ is a diffeomorphism.

5.3. **Conformal case.** For $\mathbf{X} = \mathbf{Ein}^{p,q}$, $\mathfrak{D}$ sends maximal chart to Minkowski patches of $\tilde{\mathbf{Ein}}^{p,q}$. For $1 \leqslant i \leqslant m+1$, we note $M_i = \mathfrak{D}(V_i)$. We remind that m is assumed to be the smallest integer such that $M_{m+1} \cap (M_1 \cup \cdots \cup M_m)$ is not connected. We note $\iota : \tilde{\mathbf{Ein}}^{p,q} = \mathbf{S}^p \times \mathbf{S}^q \to \tilde{\mathbf{Ein}}^{p,q}$ the product of the antipodal maps.

5.3.1. *Facts about Minkowski patches.*

Fact 6. *Let M_0, M_1, M_2 be three Minkowski patches such that $M_0 \cap M_1$ and $M_0 \cap M_2$ are degenerate half-spaces of M_0, and $M_1 \neq \iota(M_2)$. If $M_0 \cap M_1$ and $M_0 \cap M_2$ are disjoint, then $\iota(M_0) \subset M_1 \cup M_2$.*

Proof. Let $s : M_0 \to \mathbf{R}^{p,q}$ be a stereographic projection. By assumption, there is $v \in \mathbf{R}^{p,q}$ isotropic and $\alpha, \beta \in \mathbf{R}$ such that $s(M_0 \cap M_1) = H_{v,\alpha}$ and $s(M_0 \cap M_2) = H_{-v,\beta}$, and $\alpha > -\beta$ (cf. Definition 2.7). Let $s' := s \circ \iota : \iota(M_0) \to \mathbf{R}^{p,q}$.

By Lemma 2.8, noting $p : \tilde{\mathbf{Ein}}^{p,q} \to \mathbf{Ein}^{p,q}$ the projection, $p(\iota(M_0) \cap M_i))$ is the complement of $p(\iota(M_0) \cap \iota(M_i))$ in $p(M_0) \cap p(M_i)$, for $i = 1, 2$. Thus, we get that $s'(\iota(M_0) \cap M_1) = H_{-v,\alpha}$ and $s'(\iota(M_0) \cap M_2) = H_{v,\beta}$, showing $\iota(M_0) \cap (M_1 \cup M_2) = \iota(M_0)$ □

Lemma 5.1. *Let M_0, M_1, M_2 be three Minkowski patches and $s : M_0 \to \mathbf{R}^{p,q}$ a stereographic projection. Assume that $s(M_0 \cap M_1) = H_{v,\alpha}$ and $s(M_0 \cap M_2) = H_{v,\beta}$ for $v \in \mathbf{R}^{p,q}$ isotropic, and $\alpha, \beta \in \mathbf{R}$. Then, $\partial M_0 \cap M_1 = \partial M_0 \cap M_2$.*

Remark 5.2. The condition on the intersection of the Minkowski patches is independent of the choice of s.

Proof. Let $M_x = p(M_0)$, $M_y = p(M_1)$ and $M_z = p(M_2)$ be their projections in $\mathbf{Ein}^{p,q}$, with vertices $x, y, z \in \mathbf{Ein}^{p,q}$. By hypothesis, x, y, z lie on a same light-like geodesic $\Delta \subset \mathbf{Ein}^{p,q}$. By hypothesis, $y \neq x$ and $z \neq x$, because M_1 and M_2 cannot be equal to M_0 or antipodal to M_0 by assumption.

As $\mathrm{PO}(p+1, q+1)$ acts transitively on the set of pointed light-like projective lines of $\mathbf{Ein}^{p,q}$, we may assume $x = [1 : 0 : \cdots : 0]$ and $\Delta = \{[s_0 : s_1 : 0 : \cdots : 0],\ (s_0, s_1) \neq (0, 0)\}$ in the coordinates introduced in Section 2. Applying ι if necessary, we may also assume

$$M_0 = \left\{ s^{-1}(u) := \frac{1}{\|(-\frac{q(u)}{2}, u, 1)\|} \left(-\frac{q(u)}{2}, u, 1 \right),\ u \in \mathbf{R}^{p,q} \right\}$$

where $\|.\|$ denotes the usual Euclidean norm on $\mathbf{R}^n$ and q the quadratic form on $\mathbf{R}^{p,q}$ induced by our choice of coordinates. Note that this means that $(-1, 0, \ldots, 0)$ is the space-like vertex of M_0 and $(1, 0, \ldots, 0)$ its time-like vertex.

Now, there is $t \in \mathbf{R}$ such that $y = [t : 1 : 0 : \cdots : 0]$. Noting $v_y = (t, 1, 0, \ldots, 0) \in \mathbf{R}^{p+1,q+1}$, M_1 is one of the two connected components of $\tilde{\mathbf{Ein}}^{p,q} \backslash v_y^{\perp}$, where the orthogonal is taken relatively to the inner product of $\mathbf{R}^{p+1,q+1}$. That is:

$$\begin{aligned} \text{either } M_1 &= \{(s_0, \ldots, s_{n+1}) \in \tilde{\mathbf{Ein}}^{p,q} \ :\ s_n + t s_{n+1} > 0\} \\ \text{or } M_1 &= \{(s_0, \ldots, s_{n+1}) \in \tilde{\mathbf{Ein}}^{p,q} \ :\ s_n + t s_{n+1} < 0\}. \end{aligned}$$

In each case, we get $s(M_0 \cap M_1) = \{u \in \mathbf{R}^{p,q} \ :\ u_n > -t\} = H_{e_1,-t}$ or $s(M_0 \cap M_1) = \{u \in \mathbf{R}^{p,q} \ :\ u_n < -t\} = H_{-e_1,t}$. Let us say that we are in the first case.

We consider now $C_0 = \partial M_0 = \tilde{\mathbf{Ein}}^{p,q} \cap e_0^{\perp}$. We get

$$\begin{aligned} C_0 = &\left\{ \frac{1}{\|(1, x, 0)\|}(1, x, 0),\ x \in C^{p,q} \right\} \cup \left\{ \frac{1}{\|(1, x, 0)\|}(-1, x, 0),\ x \in C^{p,q} \right\} \\ &\cup \left\{ (0, x, 0),\ x \in \tilde{\mathbf{Ein}}^{p-1,q-1} \right\}, \end{aligned}$$

where $C^{p,q}$ denotes $\{x \in \mathbf{R}^{p,q} \ : \ q(x) = 0\}$. So, $C_0 \cap M_1$ is simply the same union, with the additional requirement that $x_n > 0$, where x_n is the last coordinate of x. Thus, the parameter t defining the position of y on Δ does not appear any longer.

This finishes the proof. Indeed, there is $t' \in \mathbf{R}$ such that $z = [t' : 1 : 0 : \cdots : 0]$. Necessarily, we will have

$$M_2 = \{(s_0, \dots, s_{n+1}) \in \tilde{\mathbf{Ein}}^{p,q} \ : \ s_n + t's_{n+1} > 0\}$$

because $s(M_0 \cap M_1)$ and $s(M_0 \cap M_2)$ are assumed to be "oriented" by the same isotropic vector. Consequently, $M_2 \cap C_0 = M_1 \cap C_0$ as announced. □

Similarly to the projective case, we will use the fact that $\mathfrak{D}$ is still injective on some neighborhood of the closure of the V_i's. We will consider neighborhoods of closures of maximal charts which are developed to the following type of neighborhoods of closures of Minkowski patches.

Definition 5.3. Let $M_0 \subset \tilde{\mathbf{Ein}}^{p,q}$ be a Minkowski patch and $s : M_0 \to \mathbf{R}^{p,q}$ a stereographic projection. For all $\varepsilon > 0$, we define $M_0^{s,\varepsilon} = \tilde{\mathbf{Ein}}^{p,q} \setminus (s \circ \iota)^{-1}(B(0, \frac{1}{\varepsilon}))$, where $B(0, R) = \{v \in \mathbf{R}^{p,q} \ : \ v_1^2 + \cdots + v_n^2 \leqslant R^2\}$ for $R > 0$.

Lemma 5.4. *For any open neighborhood $\mathcal{V} \supset \overline{M_0}$ and any stereographic projection $s : M_0 \to \mathbf{R}^{p,q}$, there exists $\varepsilon > 0$ such that $M_0^{s,\varepsilon} \subset \mathcal{V}$.*

Proof. Let $v_0 \in \mathbf{R}^{p+1,q+1}$ be an isotropic vector such that $M_0 = \tilde{\mathbf{Ein}}^{p,q} \cap \{v \in \mathbf{S}^{n+1} \ : \ B(v, v_0) > 0\}$ where $B(.,.)$ denotes the scalar product on $\mathbf{R}^{p+1,q+1}$. It is enough to observe that for all $\delta > 0$, there is $\varepsilon > 0$ such that

$$M_0^{s,\varepsilon} \subset \{v \in \mathbf{S}^{n+1} \ : \ B(v, v_0) > -\delta\}.$$

By homogeneity, we may assume $v_0 = (1, 0, \dots, 0)$, so that

$$\iota(M_0) = \left\{ \frac{1}{\|(\frac{q(u)}{2}, u, -1)\|} \left(\frac{q(u)}{2}, u, -1 \right), \ u \in \mathbf{R}^{p,q} \right\}$$

and there is $\phi \in \mathrm{Conf}(\mathbf{R}^{p,q}) = \mathrm{CO}(p, q) \ltimes \mathbf{R}^n$ such that for all $u \in \mathbf{R}^{p,q}$,

$$(s \circ \iota)^{-1}(u) = \frac{1}{\|(\frac{q(\phi(u))}{2}, \phi(u), -1)\|} \left(\frac{q(\phi(u))}{2}, \phi(u), -1 \right)$$

It follows that

$$\begin{aligned} &(s \circ \iota)(\iota(M_0) \cap \{v \in \mathbf{S}^{n+1} \ : \ B(v, v_0) > -\delta\}) \\ &\qquad\qquad = \left\{ u \in \mathbf{R}^{p,q} \ : \ 1 + \|\phi(u)\|^2 + \frac{q(\phi(u))^2}{4} > \frac{1}{\delta^2} \right\} \end{aligned}$$

Thus, any $\varepsilon > 0$ such that $\|u\| > \frac{1}{\varepsilon} \Rightarrow \|\phi(u)\| > \frac{1}{\delta}$ will be convenient. □

Thus, if $V \subset \tilde{M}$ is a relatively compact maximal chart and $s : \mathfrak{D}(V) \to \mathbf{R}^{p,q}$ a stereographic projection, for small enough $\varepsilon > 0$, there exists a neighborhood $V^{s,\varepsilon}$ of $\overline{V}$ on which $\mathfrak{D}$ is still injective and such that $\mathfrak{D}(V^{s,\varepsilon}) = \mathfrak{D}(V)^{s,\varepsilon}$.

Lemma 5.5. *Let M_0, M_1 be two Minkowski patches in $\tilde{\mathbf{Ein}}^{p,q}$. Then, for all $\varepsilon > 0$ and stereographic projection $s : M_0 \to \mathbf{R}^{p,q}$, $M_0^{s,\varepsilon} \cap M_1$ is a non-empty, connected open set.*

Proof. The lemma is clear if $M_1 = M_0$ or $M_1 = \iota(M_0)$. We note $s' := s \circ \iota$. According to the partition $\widetilde{\mathbf{Ein}}^{p,q} = \overline{M_0} \cup \iota(M_0)$, because $M_0^{s,\varepsilon}$ is a neighborhood of the closure of $\overline{M_0}$, we have $M_0^{s,\varepsilon} \cap M_1 = (M_1 \cap \overline{M_0}) \cup ((M_1 \cap \iota(M_0)) \setminus s'^{-1}(B(0, \frac{1}{\varepsilon})))$.

Because $M_1 = (M_1 \cap \overline{M_0}) \cup (M_1 \cap \iota(M_0))$ is connected, it is enough to observe that $(M_1 \cap \iota(M_0)) \setminus s'^{-1}(B(0, \frac{1}{\varepsilon}))$ is connected. In the stereographic projection $s' : \iota(M_0) \to \mathbf{R}^{p,q}$, this open set is sent to

- either $(v + U_S) \setminus B(0, \frac{1}{\varepsilon})$ for some $v \in \mathbf{R}^{p,q}$,
- or $(v + U_T) \setminus B(0, \frac{1}{\varepsilon})$ for some $v \in \mathbf{R}^{p,q}$,
- or $H_{v,\alpha} \setminus B(0, \frac{1}{\varepsilon})$ for some isotropic $v \in \mathbf{R}^{p,q}$ and $\alpha \in \mathbf{R}$.

All of them are always connected, proving the lemma. □

5.3.2. *Configurations where the induction fails.* Let $s : M_{m+1} \to \mathbf{R}^{p,q}$ be a stereographic projection and $U_i = s(M_i \cap M_{m+1})$ for $1 \leqslant i \leqslant m$, so that U_i is either empty or of intersection type (Definition 2.7).

Lemma 5.6. *Let $W_1, \dots, W_l \subset \mathbf{R}^{p,q}$ be a finite family of pairwise distinct, non-empty open sets of intersection type. Then, $W_1 \cup \dots \cup W_l$ is not connected if and only if*

(1) *either there exist $v \in \mathcal{C} \setminus \{0\}$, $\alpha_1, \dots, \alpha_l \in \mathbf{R}$ and $1 \leqslant k_0 \leqslant l-1$ such that up to permutation, $\alpha_1 > \dots > \alpha_{k_0} \geqslant -\alpha_{k_0+1} > \dots > -\alpha_l$ and $W_i = H_{v,\alpha_i}$ for all $i \leqslant k_0$ and $W_i = H_{-v,\alpha_i}$ for all $i > k_0$.*

(2) *or $l = 2$, and up to permutation, $U_1 = v + U_S$ and $U_2 = v + U_T$ for $v \in \mathbf{R}^{p,q}$.*

Proof. We use repeatedly Fact 1. Let us assume that $U := W_1 \cup \dots \cup W_l$ is not connected.

Case 1: There exists i such that $W_i = H_{v_i,\alpha_i}$ for $v_i \in \mathbf{R}^{p,q}$ isotropic and $\alpha_i \in \mathbf{R}$.

In this situation, necessarily for all j, W_j is also of the form $W_j = H_{v_j,\alpha_j}$. Indeed, let us assume for instance that, to the contrary, there exists j such that $W_j = v_j + U_S$. Then, for all $1 \leqslant k \leqslant l$, W_k intersects $W_i \cup W_j$. The latter is connected because $W_i \cap W_j \neq \emptyset$. Thus, given any $1 \leqslant k, k' \leqslant n$, $W_k \cup W_i \cup W_j \cup W_{k'}$ is connected, proving that U is connected, a contradiction.

Moreover, by similar arguments, all the vectors v_j must lie on a same isotropic line. If we rescale them, we get that up to a permutation of $\{1, \dots, l\}$, there is $k_0 \in \{1, \dots, l-1\}$ and $v \in \mathbf{R}^{p,q}$ isotropic such that for all $k \leqslant k_0$, $W_k = H_{v,\alpha_k}$ and for all $k \geqslant k_0 + 1$, $W_k = H_{-v,\alpha_k}$ and with $\alpha_1 > \dots > \alpha_{k_0}$ and $-\alpha_{k_0+1} > \dots > -\alpha_l$. For all $k \leqslant k_0$, we have $W_k \subset W_{k_0}$ and for all $k \geqslant k_0 + 1$, $W_k \subset W_{k_0+1}$. So, $U = W_{k_0} \cup W_{k_0+1}$, and necessarily this union is disjoint, *i.e.* $\alpha_{k_0} \geqslant -\alpha_{k_0+1}$.

Case 2: For all i, W_i is of the form $v_i + U_S$ or $v_i + U_T$.

Case 2.a: All W_i's are of the same type. Then, they intersect pairwise and U is connected, a contradiction.

Case 2.b: There exist i, j such that $W_i = v_i + U_S$ and $W_j = v_j + U_T$. If we had $v_i \neq v_j$, then $W_i \cup W_j$ would be connected, and since any other W_k would intersect it, we would get as before that U is connected. Moreover, if there exists a third open subset W_k (distinct from W_i and W_j), then W_k intersects W_i and W_j, and it follows that $W_i \cup W_j \cup W_k$ is connected and dense in $\mathbf{R}^{p,q}$. In particular, it intersects any other $W_{k'}$, proving that U is connected.

Finally, in Case 2, we must have $l = 2$ and $W_1 = v + U_S$ and $W_2 = v + U_T$ as announced. □

5.3.3. *Case of a family of half-spaces.* We assume here that if we remove the eventual U_i which is empty, the remaining ones are in the first configuration of Lemma 5.6. We then have $l \in \{m-1, m\}$, $W_1, \dots, W_l \subset \mathbf{R}^n$, $1 \leqslant k_0 \leqslant l-1$, $\alpha_1 > \cdots > \alpha_{k_0} \geqslant -\alpha_{k_0+1} > \cdots > -\alpha_l$ such that $W_k = H_{v,\alpha_k}$ for $k \leqslant k_0$ and $W_k = H_{-v,\alpha_k}$ for $k > k_0$, and an injective map $\sigma : \{1, \dots, l\} \to \{1, \dots, m\}$ such that $W_k = U_{\sigma(k)}$ for all k and if $l = m-1$ and i is the unique element not in the range of σ, $U_i = \emptyset$. Let i_0, j_0 be such that $U_{i_0} = W_{k_0}$ and $U_{j_0} = W_{k_0+1}$.

Case $\alpha_{k_0} = \alpha_{k_0+1}$.

In this situation $M_{j_0} = \iota(M_{i_0})$. Let $x \in V_1 \cup \cdots \cup V_m$ be a point in ∂V_{i_0}, which exists by connectedness of $V_1 \cup \cdots \cup V_m$. For all open neighborhood $x \in U \subset V_1 \cup \cdots \cup V_m$, we have $U \cap V_{j_0} \neq \emptyset$ because if not, we would have $\mathfrak{D}(U) \subset \tilde{\mathbf{Ein}}^{p,q} \setminus \iota(M_{i_0})$ by injectivity of $\mathfrak{D}$ in restriction to $V_1 \cup \cdots \cup V_m$, implying $\mathfrak{D}(x) \in M_{i_0}$. So, $x \in \partial V_{i_0} \cap \partial V_{j_0}$ proving that the latter is non-empty.

Consequently, if $s_0 : M_{i_0} \to \mathbf{R}^{p,q}$ is any stereographic projection and if $\varepsilon > 0$ is small enough, such that there exists a neighborhood $V_{i_0}^{s_0,\varepsilon}$ of $\overline{V}_{i_0}$ in restriction to which $\mathfrak{D}$ is injective and such that $\mathfrak{D}(V_{i_0}^{s_0,\varepsilon}) = M_{i_0}^{s_0,\varepsilon}$, then $V_{i_0}^{s_0,\varepsilon} \cap V_{j_0} \neq \emptyset$. The intersection $M_{i_0}^{s_0,\varepsilon} \cap M_{j_0}$ is homeomorphic to the complement of a ball in $\mathbf{R}^{p,q}$, thus connected. We conclude by Lemma 3.5 that $\mathfrak{D}$ is injective on $V_{i_0}^{s_0,\varepsilon} \cup V_{j_0}$, and the image of the latter is $\tilde{\mathbf{Ein}}^{p,q}$.

Case $\alpha_{k_0} < \alpha_{k_0+1}$.

Claim 5. *In this situation, $M_1 \cup \ldots \cup M_m = M_{i_0} \cup M_{j_0}$.*

Proof. We prove the non-obvious inclusion by observing that the traces of $M_1 \cup \ldots \cup M_m$ on the partition $\tilde{\mathbf{Ein}}^{p,q} = M_{m+1} \cup \partial M_{m+1} \cup \iota(M_{m+1})$ are included in $M_{i_0} \cup M_{j_0}$. By Fact 6 and by the choice of i_0, j_0, we have $\iota(M_{m+1}) \subset M_{i_0} \cup M_{j_0}$. Applying s, it is immediate by construction that $M_{m+1} \cap M_i \subset M_{i_0} \cup M_{j_0}$ for all i. Finally, let $i \in \{1, \dots, m\}$.

(1) If i is not in the range of σ, then it means that $M_i = \iota(M_{m+1})$, and then $\partial M_{m+1} \cap M_i = \emptyset$.
(2) If $i = \sigma(k)$ for $k \leqslant k_0$, then we get $\partial M_{m+1} \cap M_i = \partial M_{m+1} \cap M_{i_0}$ by Lemma 5.1.
(3) If $i = \sigma(k)$ for $k > k_0$, then we get $\partial M_{m+1} \cap M_i = \partial M_{m+1} \cap M_{j_0}$ by Lemma 5.1.

In all cases, we have $\partial M_{m+1} \cap M_i \subset M_{i_0} \cup M_{j_0}$, and the claim is proved. $\square$

As in the projective case, by injectivity of $\mathfrak{D}$ in restriction to $V_1 \cup \cdots \cup V_m$, it follows that $V_1 \cup \cdots \cup V_m = V_{i_0} \cup V_{j_0}$. So, $V_{i_0} \cap V_{j_0} \neq \emptyset$, $V_{m+1} \cap (V_{i_0} \cup V_{j_0}) \neq \emptyset$. Let $\varepsilon > 0$ and $V_{m+1}^{s,\varepsilon} \supset \overline{V_{m+1}}$ be an open neighborhood of $\overline{V_{m+1}}$ in restriction to which $\mathfrak{D}$ is injective and that develops onto $M_{m+1}^{s,\varepsilon}$.

By Lemma 5.5, $\mathfrak{D}(V_{m+1}^{s,\varepsilon}) \cap \mathfrak{D}(V_{i_0})$ and $\mathfrak{D}(V_{m+1}^{s,\varepsilon}) \cap \mathfrak{D}(V_{j_0})$ are connected. Let us prove that they intersect. Let $s' := s \circ \iota$. Then, we have $s'(\iota(M_{m+1}) \cap M_{i_0}) = H_{-v,-\alpha_{k_0}}$ and $s'(\iota(M_{m+1}) \cap M_{j_0}) = H_{v,-\alpha_{k_0+1}}$ according to Remark 2.9. Since $\alpha_{k_0} > -\alpha_{k_0+1}$, $H_{-v,-\alpha_{k_0}} \cap H_{v,-\alpha_{k_0+1}} = \{w \in \mathbf{R}^{p,q} \ : \ \alpha_{k_0} > b(w,v) > -\alpha_{k_0+1}\}$ is a non-empty strip. This shows that $s'(\iota(M_{m+1}) \cap M_{i_0} \cap M_{j_0})$ contains vectors with arbitrary large Euclidean norm, so $M_{m+1}^{s,\varepsilon} \cap M_{i_0} \cap M_{j_0} \neq \emptyset$. Consequently, $\mathfrak{D}(V_{m+1}^{s,\varepsilon}) \cap (\mathfrak{D}(V_{i_0}) \cup \mathfrak{D}(V_{j_0}))$ is connected, and Lemma 3.5 implies that $\mathfrak{D}$ is injective in restriction to $V_{m+1}^{\varepsilon} \cup V_{i_0} \cup V_{j_0}$. Finally, since $M^{\varepsilon}_{m+1} \cup M_{i_0} \cup M_{j_0} = \tilde{\mathbf{Ein}}^{p,q}$, we obtain that $\mathfrak{D}$ is a diffeomorphism onto $\tilde{\mathbf{Ein}}^{p,q}$ similarly as before.

5.3.4. *Case of space/time open sets.* We finally assume that if we remove the eventual U_i which is empty, the remaining ones are in the second configuration of Lemma 5.6. Thus, there is $v \in \mathbf{R}^{p,q}$ such that for all i, either $U_i = \emptyset$, or $U_i = v + U_S$, or $U_i = v + U_T$.

Necessarily, $U_1 = \emptyset$ or $U_2 = \emptyset$. Indeed, if both are non-empty, then up to a permutation, $U_1 = v + U_S$ and $U_2 = v + U_T$, for $v \in \mathbf{R}^{p,q}$. It implies that $M_1 = \iota(M_2)$ and in particular $M_1 \cap M_2 = \emptyset$, contradicting $V_1 \cap V_2 \neq \emptyset$. So, $m = 3$ and exchanging V_1 and V_2 if necessary, we may assume $U_1 = \emptyset$ and $U_2 = v + U_S$ or $U_2 = v + U_T$. Let us assume $U_2 = v + U_S$, the other case being similar. The U_i's being pairwise distinct, we must have $U_3 = v + U_T$, implying as above that $M_3 = \iota(M_2)$.

Finally, exchanging V_1 and V_2 if necessary, we have $m = 3$ and $M_1 = \iota(M_4)$, M_2 such that $s(M_2 \cap M_4) = v + U_S$ and $M_3 = \iota(M_2)$.

Thus, the same reasoning as in the case $\alpha_{k_0} = \alpha_{k_0+1}$ of Section 5.3.3 applies if V_2, V_3 play the role of V_{i_0}, V_{j_0}: both are included in a connected injectivity domain of $\mathfrak{D}$, and they develop to antipodal Minkowski patches. We thus obtain that $\mathfrak{D}$ is also a diffeomorphism in this last situation, completing the proof of Theorem 3.

6. Projective flatness

In this section, we prove as announced the following proposition.

Proposition 6.1. *Let Γ be a cocompact lattice in a connected simple Lie group G of $\mathbf{R}$-rank $n \geqslant 2$, and let (M^n, ∇) be a closed n-manifold endowed with a linear connection. Let $\alpha : \Gamma \to \mathrm{Proj}(M, \nabla)$ be a projective action. If $\alpha(\Gamma)$ is infinite, then ∇ is projectively flat.*

Throughout this section, $\mathbf{X} = \mathbf{R}P^n$ and $\mathfrak{g}_{\mathbf{X}} = \mathfrak{sl}(n+1, \mathbf{R})$.

6.1. **Associated Cartan geometry modeled on $\mathbf{R}P^n$.** We note $P < \mathrm{PGL}(n+1, \mathbf{R})$ the stabilizer of a line.

Theorem ([KN64]). *Let $(M^n, [\nabla])$ be a manifold with a projective class of linear connections. There exist a P-principal bundle $\pi_B : B \to M$ and a 1-form $\omega \in \Omega^1(B, \mathfrak{g}_{\mathbf{X}})$ satisfying the following properties:*

1. *for all $b \in B$, $\omega_b : T_b B \to \mathfrak{g}_{\mathbf{X}}$ is a linear isomorphism,*
2. *for all $A \in \mathfrak{p}$, $\omega(A^*) = A$,*
3. *for all $p \in P$, $(R_p)^*\omega = \mathrm{Ad}(p^{-1})\omega$,*

where R_p stands for the right action of p on B and A^* denotes the fundamental vertical vector field associated to A, and such that $\mathrm{Proj}(M, [\nabla])$ is exactly the set of diffeomorphisms $f : M \to M$ that can be lifted to bundle morphisms $F : B \to B$ satisfying $F^*\omega = \omega$.

The triple (M, B, ω) is called the Cartan geometry associated to $(M, [\nabla])$, $\pi_B : B \to M$ its Cartan bundle and ω its Cartan connection. The first property implies that the action of $\mathrm{Proj}(M, [\nabla])$ on B is free, and its Lie group structure is - by definition - such that its action on B is moreover proper.

6.2. **Uniform Lyapunov spectrum.** We reuse some of the notations of [BFH16], which we recalled in Section 2.1 of [Pec19]. We note $M^\alpha \to G/\Gamma$ the suspension fiber bundle. We fix $A < G$ a Cartan subspace. Let μ be any A-invariant A-ergodic measure on M^α, which projects to the Haar measure on G/Γ. Let $\chi_1, \ldots, \chi_r \in \mathfrak{a}^*$ be its Lyapunov functionals. Similarly to Proposition 4.1 of [Pec19], we have:

Lemma 6.2. *Such a measure μ cannot be G-invariant.*

Proof. Let us assume to the contrary that μ is G-invariant. Then, we get a Γ-invariant finite measure ν on M. Considering the action of Γ on the Cartan bundle $B \to M$ associated to $[\nabla]$, super-rigidity implies that the cocycle $\Gamma \times M \to P$ is measurably cohomologous to a compact valued cocyle, as there is no non-trivial homomorphism $\mathfrak{g} \to \mathfrak{gl}(n, \mathbf{R}) \ltimes \mathbf{R}^n$. By the same arguments as in the proof of Lemma 4.4 of [Pec19], this implies that there exists a finite Γ-invariant measure ν_B on B. Since $\operatorname{Proj}(M, \nabla)$ acts freely and properly on B, it follows that the action $\alpha : \Gamma \to \operatorname{Proj}(M, \nabla)$ has relatively compact image (see Lemma 4.3 of [Pec19]). In particular, the action preserves a Riemannian metric on M, implying that α takes values in a compact Lie group of dimension at most $n(n+1)/2$, hence that $\alpha(\Gamma)$ is finite (see Section 7 of [BFH16]), a contradiction. □

Since $\dim M = n = \operatorname{Rk}_{\mathbf{R}} G$, we have $r \leqslant n$. On the other hand, if $\chi_1, \ldots, \chi_r$ spanned a space of dimension strictly less than n, we would get a direction $X \in \mathfrak{a}$ on which all the χ_i's vanish. By Proposition 4.7 of [Pec19] - which is a citation of a central property of the work of [BFH16] -, it would imply that μ is G-invariant, a contradiction.

Thus, $r = n$ and $\chi_1, \ldots, \chi_n$ are linearly independent. So, they define a line in $\mathfrak{a}$ in restriction to which they all coincide, and similarly to Section 6.2 of [Pec19], there exists $X \in \mathfrak{a}$ such that $\chi_1(X) = \cdots = \chi_n(X) = -1$. The proof of Proposition 6.1 of [Pec19] applies - no conformal geometry is involved in this proposition - and we obtain $g \in G$ and $x \in M$ such that $[(g, x)] \in \operatorname{Supp} \mu$, a sequence (γ_k) in Γ, $(T_k) \to \infty$ and an open neighborhood U of x such that

1. $\gamma_k U \to \{x\}$ for the Hausdorff topology,
2. $\frac{1}{T_k} \log |D_x \gamma_k v| \to -1$ for all non-zero $v \in T_x M$
3. $\frac{1}{T_k} \log |\det \operatorname{Jac}_x \gamma_k| \to -n$.

6.3. **Holonomy sequences associated to γ_k.** Let $\pi_B : B \to M$ be the Cartan bundle corresponding to $[\nabla]$, with structural group $P \simeq \mathrm{GL}(n, \mathbf{R}) \ltimes \mathbf{R}^n$. Let $A_{\mathbf{X}} < P$ be the Cartan subspace formed of diagonal matrices with positive entries.

Proposition 6.3. *Reducing U if necessary, there is a sequence (a_k) in $A_{\mathbf{X}}$ such that for all $y \in U$, there exists a bounded sequence $b_k \in \pi^{-1}(y)$ such that the sequence $\gamma_k b_k a_k^{-1}$ is bounded. Moreover, if $A_k \in \mathfrak{a}_{\mathbf{X}}$ is such that $a_k = \exp(A_k)$, we have*

$$\frac{1}{T_k} A_k \to \operatorname{diag}\left(\frac{n}{n+1}, -\frac{1}{n+1}, \ldots, -\frac{1}{n+1}\right).$$

Proof. As $\gamma_k x \to x$, if $b \in \pi_B^{-1}(x)$, we can choose $p_k' \in P$ such that $\gamma_k b p_k'^{-1}$ is bounded (a *holonomy sequence* for γ_k in the terminology introduced by Frances). If we decompose p_k' according to $P = G_0 \ltimes \exp(\mathfrak{p}^+)$ and if we use the Cartan decomposition of G_0, we can write $p_k' = l_k a_k l_k' \tau_k$, with $a_k \in A_{\mathbf{X}}$, $l_k, l_k' \in G_0$ bounded and $\tau_k \in \exp(\mathfrak{p}^+)$. So, if

$b_k := bl_k'^{-1}$ and if τ_k is replaced by $l_k'\tau_k l_k'^{-1} \in \exp(\mathfrak{p}^+)$, the we get that $\gamma_k b_k(a_k\tau_k)^{-1}$ is bounded, with $b_k \in \pi^{-1}(x)$ bounded. Let us note $p_k = a_k\tau_k$.

Let $\rho : P \to \mathrm{GL}(\mathfrak{g}_{\mathbf{X}}/\mathfrak{p})$ be the representation induced by the adjoint map. Similarly to Lemma 6.11 of [Pec19], we have

Lemma 6.4. *Let (f_k) be a sequence of projective maps $(M, [\nabla])$ and $x \in M$ such that $(f_k(x)) \to x_\infty$. The following are equivalent.*

1. *(f_k) is Lyapunov regular at x, with Lyapunov exponents χ_i of multiplicity d_i.*
2. *For any b in the fiber of x and any sequence (p_k) in P such that $f_k(b).p_k^{-1} \to b_\infty$, for some b_∞ in the fiber of x_∞, the sequence $\rho(p_k)$ is Lyapunov regular with Lyapunov exponents χ_i and multiplicity d_i.*

In our situation, γ_k is Lyapunov regular at x with a non-zero Lyapunov exponent of multiplicity n. Since $\gamma_k.b.(p_k l_k')^{-1}$ is bounded by construction, up to an extraction, it follows that $\rho(p_k l_k')$ is a Lyapunov regular sequence with a non-zero Lyapunov exponent of multiplicity n. From Lemma 6.10 of [Pec19], we deduce that $\rho(p_k)$ has the same property. Moreover, since $\exp(\mathfrak{p}^+)$ is in the kernel of ρ, if we note

$$a_k = \begin{pmatrix} \lambda_0^{(k)} & & \\ & \ddots & \\ & & \lambda_n^{(k)} \end{pmatrix},$$

we get that $\rho(p_k) = \rho(a_k)$ is conjugate to the diagonal matrix $\mathrm{diag}(\lambda_1^{(k)}{\lambda_0^{(k)}}^{-1}, \ldots, \lambda_n^{(k)}{\lambda_0^{(k)}}^{-1})$. If $A_k \in \mathfrak{a}_{\mathbf{X}}$ is such that $a_k = \exp(A_k)$, the property of $\rho(p_k)$ means

$$\frac{1}{T_k} A_k \to \mathrm{diag}\left(\frac{n}{n+1}, -\frac{1}{n+1}, \ldots, -\frac{1}{n+1}\right).$$

We claim now that (τ_k) is bounded. This can be observed by adapting almost directly the proof of Fait 4.4 of [Fra12].

Indeed, let us write $\tau_k = \exp(T_k)$ with $T_k = (T_1^{(k)}, \ldots, T_n^{(k)})$ (see Section 4.2), and assume to the contrary that a sequence $(T_i^{(k)})$ is unbounded. Up to an extraction, we may assume that $|T_i^{(k)}| \to \infty$. Then, p_k preserves the projectivization of $\mathrm{Span}(e_1, e_i) \subset \mathbf{R}P^n$, and acts on it via the matrix

$$\begin{pmatrix} \lambda_0^{(k)} & \lambda_0^{(k)} T_i^{(k)} \\ 0 & \lambda_i^{(k)} \end{pmatrix}$$

Then, the same argumentation as in page 17 of [Fra12] applies literally and gives a sequence of points $x_k \to x$ such that $\gamma_k x_k \to y \neq x$, contradicting the fact that $\gamma_k U \to \{x\}$ for the Hausdorff topology.

So, (τ_k) is bounded and consequently, if we replace b_k by $b_k\tau_k^{-1}$ which is still bounded, the announced property is valid at x with this choice of a_k. Let $\mathcal{U} \subset \mathfrak{n}_-$ be a neighborhood of the origin on which the exponential map of the Cartan geometry (see [Sha97], Ch. 5) is defined at every b_k, which exists because $\{b_k\}$ is a relatively compact subset of the fiber $\pi^{-1}(x)$. Given the asymptotic properties of $\mathrm{Ad}(a_k)|_{\mathfrak{n}_-}$, we may assume that $\mathrm{Ad}(a_k)$ preserves $\mathcal{U}$, and we have

$$\forall X \in \mathcal{U},\ \gamma_k \exp(b_k, X) a_k^{-1} = \exp(\gamma_k b_k a_k^{-1}, \mathrm{Ad}(a_k)X).$$

The fact that $\{b_k\}$ is relatively compact implies that $\cap_{k\geqslant 0}\pi_B(\exp(b_k, \mathcal{U}))$ is a neighborhood of x. If y is in this neighborhood, there is $X_k \in \mathcal{U}$ such that $\pi_B(\exp(b_k, X_k)) = y$ and the formula above implies that $\exp(b_k, X_k)$ is a convenient sequence for y since $\mathrm{Ad}(a_k)X_k$ goes to 0. □

6.4. **Vanishing of the curvature map near x.** We refer to [Sha97], Definition 3.22, Ch. 5, for the definition of the curvature map of a Cartan geometry. We note it $\kappa : B \to \mathrm{Hom}(\Lambda^2(\mathfrak{g}_{\mathbf{X}}/\mathfrak{p}), \mathfrak{g}_{\mathbf{X}})$. It is $\mathrm{Aut}(M, B, \omega)$-invariant and P-equivariant for the right action of P on $\mathrm{Hom}(\Lambda^2(\mathfrak{g}_{\mathbf{X}}/\mathfrak{p}), \mathfrak{g}_{\mathbf{X}})$ given by $(p.w)(u, v) = \mathrm{Ad}(p^{-1})w(\mathrm{Ad}(p)u, \mathrm{Ad}(p)v)$ for all $w \in \mathrm{Hom}(\Lambda^2(\mathfrak{g}_{\mathbf{X}}/\mathfrak{p}), \mathfrak{g}_{\mathbf{X}})$ and $u, v \in \mathfrak{g}_{\mathbf{X}}/\mathfrak{p}$. In particular, if κ vanishes at one point $b \in B$, then it vanishes on all of the fiber $b.P$.

Let $y \in U$ and $b_k \in \pi_B^{-1}(y)$ a bounded sequence such that $b_k' := \gamma_k b_k a_k^{-1}$ is bounded. We prove by contradiction that κ vanishes in restriction to $\pi_B^{-1}(y)$. So, we assume that it is non-zero at every point of this fiber. Up to an extraction, $b_k \to b_\infty$, and in a basis of $\mathfrak{g}_{\mathbf{X}}/\mathfrak{p}$ that diagonalizes $\mathrm{Ad}(a_k)$, we pick two vectors u_i, u_j such that $\kappa(b_\infty)(u_i, u_j) \neq 0$. By equivariance, we have

$$\begin{aligned}\mathrm{Ad}(a_k)^{-1}\kappa(b_k')(u_i, u_j) &= \kappa(b_k)(\mathrm{Ad}(a_k)^{-1}u_i, \mathrm{Ad}(a_k)^{-1}u_j)\\ &= {\lambda_0^{(k)}}^2 {\lambda_i^{(k)}}^{-1} {\lambda_j^{(k)}}^{-1}\kappa(b_k)(u_i, u_j).\end{aligned}$$

This proves that

$$\frac{1}{T_k}\log|\mathrm{Ad}(a_k)^{-1}\kappa(b_k')(u_i, u_j)| \to 2.$$

This is a contradiction because $\kappa(b_k')(u_i, u_j)$ is a bounded sequence of $\mathfrak{g}_{\mathbf{X}}$ and for all $\varepsilon > 0$, $\mathrm{Ad}(a_k)$ acts diagonally on $\mathfrak{g}_{\mathbf{X}}$ with all its eigenvalues of modulus at most $e^{(1+\varepsilon)T_k}$ for k large enough.

This proves that κ vanishes on all of $\pi_B^{-1}(U)$.

6.5. **Conclusion.** We have proved that κ vanishes near every point b that projects to a point $x \in M$ such that there exists $g \in G$ such that $[(g, x)] \in \mathrm{Supp}\,\mu$, for any A-invariant, A-ergodic finite measure μ on M^α that projects to the Haar measure of G/Γ.

Similarly to Section 6.6 of [Pec19], we deduce that for all Γ-invariant compact $K \subset M$, there is $b \in \pi_B^{-1}(K)$ such that κ vanishes on a neighborhood of b. Applying this to any orbit closure $\overline{\Gamma.x} \subset M$, we obtain that κ vanishes on all of B, whence $(M, [\nabla])$ is projectively flat.

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