Characterizations and enumerations of classes of quasitrivial n-ary semigroups

Jimmy Devillet

University of Luxembourg
Luxembourg

in collaboraton with Miguel Couceiro

Part I: Quasitrivial semigroups

Quasitriviality

Definition

$G\colon X^2 \to X$ is said to be *quasitrivial* (or *conservative*) if

$$G(x,y) \ \in \ \{x,y\} \qquad x,y \in X$$

Example. $G = \max_{\leq}$ on $X = \{1,2,3\}$ endowed with the usual $\leq$

Quasitriviality

Definition

$G\colon X^2 \to X$ is said to be *quasitrivial* (or *conservative*) if

$$G(x,y) \in \{x,y\} \qquad x,y \in X$$

Example. $G = \max_{\leq}$ on $X = \{1,2,3\}$ endowed with the usual $\leq$

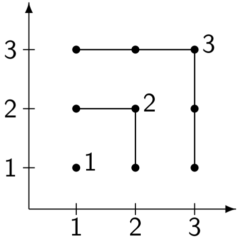
Quasitriviality

Definition

$G\colon X^2 \to X$ is said to be *quasitrivial* (or *conservative*) if

$$G(x,y) \ \in \ \{x,y\} \qquad x,y \in X$$

Example. $G = \max_{\leq}$ on $X = \{1,2,3\}$ endowed with the usual $\leq$



Projections

Definition.

The *projection operations* $\pi_1 \colon X^2 \to X$ and $\pi_2 \colon X^2 \to X$ are respectively defined by

$$\begin{aligned}\pi_1(x,y) &= x, \qquad x,y \in X\\ \pi_2(x,y) &= y, \qquad x,y \in X\end{aligned}$$

Quasitrivial semigroups

Theorem (Länger, 1980)

Let $G\colon X^2 \to X$.

$$G \text{ is associative and quasitrivial}$$

$$\Updownarrow$$

$$\exists \precsim_G \ : G|_{A\times B} = \begin{cases} \max_{\precsim_G}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \quad \forall A, B \in X/\sim_G$$

Quasitrivial semigroups

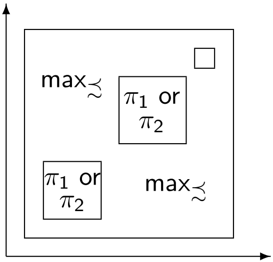
Theorem (Länger, 1980)

Let $G\colon X^2 \to X$.

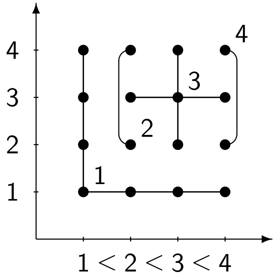
$$G \text{ is associative and quasitrivial}$$

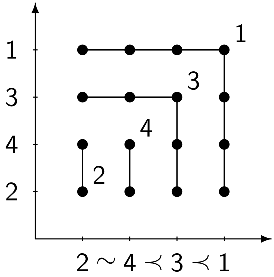
$$\Updownarrow$$

$$\exists \precsim_G \; : G|_{A\times B} = \begin{cases} \max_{\precsim_G}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \quad \forall A, B \in X/\sim_G$$



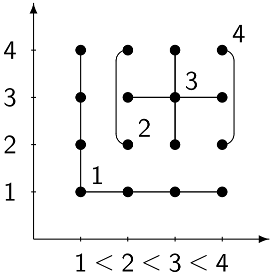
Quasitrivial semigroups

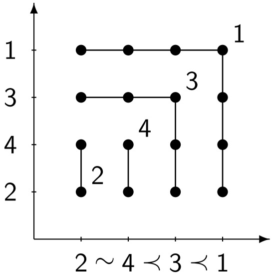




$$x \precsim y \iff |F^{-1}[x]| \leq |F^{-1}[y]|$$

Quasitrivial semigroups





$$x \precsim y \qquad \Longleftrightarrow \qquad |F^{-1}[x]| \leq |F^{-1}[y]|$$

Enumeration of associative and quasitrivial operations

$k \in \mathbb{N}$

$q(k)$: number of associative and quasitrivial operations on $\{1, \ldots, k\}$ (OEIS : A292932)

For any integers $0 \leq m \leq k$ the *Stirling number of the second kind* $\left\{ {k \atop m} \right\}$ is defined as

$$\left\{ {k \atop m} \right\} = \frac{1}{m!} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} i^k .$$

Enumeration of associative and quasitrivial operations

Theorem (C.,D.,Marichal, 2019)

We have the closed-form expression

$$q(k) = \sum_{i=0}^{k} 2^i \sum_{m=0}^{k-i} (-1)^m \binom{k}{m} \left\{ {k-m \atop i} \right\} (i+m)!\,, \qquad k \geq 0.$$

$q(0) = 1,\ q(1) = 1,\ q(2) = 4,\ q(3) = 20,\ q(4) = 138,\ \ldots$

Enumeration of associative and quasitrivial operations

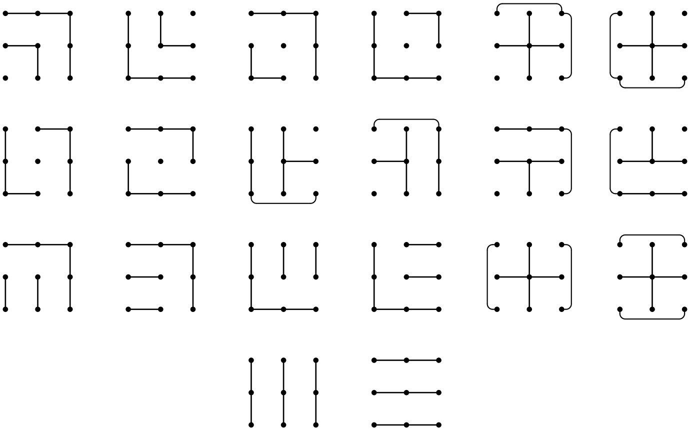
Theorem (C.,D.,Marichal, 2019)

We have the closed-form expression

$$q(k) \;=\; \sum_{i=0}^{k} 2^i \sum_{m=0}^{k-i} (-1)^m \binom{k}{m} \left\{ {k-m \atop i} \right\} (i+m)!\,, \qquad k \geq 0.$$

$q(0) = 1,\ q(1) = 1,\ q(2) = 4,\ q(3) = 20,\ q(4) = 138,\ \ldots$

Operations on $\{1, 2, 3\}$



Part II: Quasitrivial n-ary semigroups

Associativity and Quasitriviality

$n \in \mathbb{N}_{\geq 2}$

Definition

$F\colon X^n \to X$ is said to be

- *quasitrivial* if

$$F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\} \qquad x_1, \ldots, x_n \in X$$

- *associative* if

$$\begin{aligned} & F(x_1, \ldots, x_{i-1}, F(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{2n-1}) \\ = \quad & F(x_1, \ldots, x_i, F(x_{i+1}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1}) \end{aligned}$$

for all $x_1, \ldots, x_{2n-1} \in X$ and all $1 \leq i \leq n-1$.

Example. $F(x, y, z) = x + y + z \pmod 2$.

Associativity and Quasitriviality

$n \in \mathbb{N}_{\geq 2}$

Definition

$F\colon X^n \to X$ is said to be

- *quasitrivial* if

$$F(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\} \qquad x_1,\ldots,x_n \in X$$

- *associative* if

$$\begin{aligned} & F(x_1,\ldots,x_{i-1},F(x_i,\ldots,x_{i+n-1}),x_{i+n},\ldots,x_{2n-1}) \\ & \quad = \quad F(x_1,\ldots,x_i,F(x_{i+1},\ldots,x_{i+n}),x_{i+n+1},\ldots,x_{2n-1}) \end{aligned}$$

for all $x_1,\ldots,x_{2n-1} \in X$ and all $1 \leq i \leq n-1$.

Example. $F(x,y,z) = x+y+z \pmod 2$.

Associativity and Quasitriviality

$n \in \mathbb{N}_{\geq 2}$

Definition

$F \colon X^n \to X$ is said to be

- *quasitrivial* if

$$F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\} \qquad x_1, \ldots, x_n \in X$$

- *associative* if

$$F(x_1, \ldots, x_{i-1}, F(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{2n-1}) = F(x_1, \ldots, x_i, F(x_{i+1}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1})$$

  for all $x_1, \ldots, x_{2n-1} \in X$ and all $1 \leq i \leq n-1$.

Example. $F(x, y, z) = x + y + z \pmod 2$.

Associativity and Quasitriviality

$n \in \mathbb{N}_{\geq 2}$

Definition

$F \colon X^n \to X$ is said to be

- *quasitrivial* if

$$F(x_1, \ldots, x_n) \;\in\; \{x_1, \ldots, x_n\} \qquad x_1, \ldots, x_n \in X$$

- *associative* if

$$\begin{aligned} & F(x_1, \ldots, x_{i-1}, F(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{2n-1}) \\ & \quad = \quad F(x_1, \ldots, x_i, F(x_{i+1}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1}) \end{aligned}$$

  for all $x_1, \ldots, x_{2n-1} \in X$ and all $1 \leq i \leq n-1$.

Example. $F(x, y, z) = x + y + z \pmod 2$.

Reducibility

Definition

$F\colon X^n \to X$ and $G\colon X^2 \to X$ associative operations.

F is said to be *reducible to G* if

$$F(x_1,\ldots,x_n) = G(x_1, G(x_2, G(\ldots, G(x_{n-1},x_n)\ldots)))$$

Example.

$$F(x,y,z) = x+y+z \pmod 2$$

,

$$G(x,y) = x+y \pmod 2 \quad \text{and} \quad G'(x,y) = x+y+1 \pmod 2$$

.

Reducibility

Definition

$F\colon X^n \to X$ and $G\colon X^2 \to X$ associative operations.

F is said to be *reducible to G* if

$$F(x_1, \ldots, x_n) = G(x_1, G(x_2, G(\ldots, G(x_{n-1}, x_n)\ldots)))$$

Example.

$$F(x, y, z) = x + y + z \pmod 2$$

,

$$G(x, y) = x + y \pmod 2 \quad \text{and} \quad G'(x, y) = x + y + 1 \pmod 2$$

.

Reducibility

Definition

$F\colon X^n \to X$ and $G\colon X^2 \to X$ associative operations.

F is said to be *reducible to G* if

$$F(x_1, \ldots, x_n) = G(x_1, G(x_2, G(\ldots, G(x_{n-1}, x_n)\ldots)))$$

Example.

$$F(x, y, z) = x + y + z \pmod 2$$

,

$$G(x, y) = x + y \pmod 2 \quad \text{and} \quad G'(x, y) = x + y + 1 \pmod 2$$

.

Quasitrivial n-ary semigroups

Theorem (Ackerman 2011, Dudek and Mukhin 2006)

Every quasitrivial n-ary semigroup is reducible to a semigroup.

But the binary reduction is not necessarily quasitrivial nor unique.

Example.

$$F(x,y,z) = x + y + z \pmod 2$$

$$G(x,y) = x + y \pmod 2 \qquad G'(x,y) = x + y + 1 \pmod 2$$

Quasitrivial n-ary semigroups

Theorem (Ackerman 2011, Dudek and Mukhin 2006)

Every quasitrivial n-ary semigroup is reducible to a semigroup.

But the binary reduction is not necessarily quasitrivial nor unique.

Example.

$$F(x,y,z) = x + y + z \pmod 2$$

$$G(x,y) = x + y \pmod 2 \qquad G'(x,y) = x + y + 1 \pmod 2$$

Quasitrivial n-ary semigroups

Theorem (Ackerman 2011, Dudek and Mukhin 2006)

Every quasitrivial n-ary semigroup is reducible to a semigroup.

But the binary reduction is not necessarily quasitrivial nor unique.

Example.

$$F(x, y, z) = x + y + z \pmod 2$$

$$G(x, y) = x + y \pmod 2 \qquad G'(x, y) = x + y + 1 \pmod 2$$

Neutral elements

Definition

$e \in X$ is said to be a *neutral element for F* if

$$F(x, e, \ldots, e) = F(e, x, e, \ldots, e) = \ldots = F(e, \ldots, e, x) = x,$$

for all $x \in X$

Example. $F(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \pmod{n-1}$

Proposition

Every quasitrivial n-ary semigroup has at most two neutral elements.

Neutral elements

Definition

$e \in X$ is said to be a *neutral element for F* if

$$F(x,e,\ldots,e) = F(e,x,e,\ldots,e) = \ldots = F(e,\ldots,e,x) = x,$$

for all $x \in X$

Example. $F(x_1,\ldots,x_n) = \sum_{i=1}^{n} x_i \pmod{n-1}$

Proposition

Every quasitrivial n-ary semigroup has at most two neutral elements.

Neutral elements

Definition

$e \in X$ is said to be a *neutral element for F* if

$$F(x, e, \ldots, e) = F(e, x, e, \ldots, e) = \ldots = F(e, \ldots, e, x) = x,$$

for all $x \in X$

Example. $F(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \pmod{n-1}$

Proposition

Every quasitrivial n-ary semigroup has at most two neutral elements.

Uniqueness of a reduction

Proposition

Every quasitrivial n-ary semigroup has at most two neutral elements.

Theorem

$F\colon X^n \to X$ associative and quasitrivial. TFAE

(i) Any binary reduction of F is idempotent

(ii) Any binary reduction of F is quasitrivial

(iii) F has at most one binary reduction

(iv) F has at most one neutral element

Uniqueness of a reduction

Proposition

Every quasitrivial n-ary semigroup has at most two neutral elements.

Theorem

$F\colon X^n \to X$ associative and quasitrivial. TFAE

(i) Any binary reduction of F is idempotent

(ii) Any binary reduction of F is quasitrivial

(iii) F has at most one binary reduction

(iv) F has at most one neutral element

Characterization of quasitrivial n-ary semigroups

Theorem

Let $F\colon X^n \to X$.

F is associative, quasitrivial, and has at most one neutral element

$$\Updownarrow$$

$\exists$ a binary reduction $G\colon X^2 \to X$ of F and $\precsim_G$ such that

$$G|_{A\times B} = \begin{cases} \max_{\precsim_G}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \qquad \forall A, B \in X/\sim_G$$

Enumeration of quasitrivial n-ary semigroups

$$q(k) \;=\; \sum_{i=0}^{k} 2^i \sum_{m=0}^{k-i} (-1)^m \binom{k}{m} \left\{ {k-m \atop i} \right\} (i+m)!\,, \qquad k \geq 0.$$

$q^n(k)$: number of associative and quasitrivial n-ary operations that have at most one neutral element on $\{1,\ldots,k\}$

Corollary

We have

$$q^n(k) \;=\; q(k), \qquad k \geq 0.$$

Enumeration of quasitrivial n-ary semigroups

$$q(k) \;=\; \sum_{i=0}^{k} 2^i \sum_{m=0}^{k-i} (-1)^m \binom{k}{m} \left\{ {k-m \atop i} \right\} (i+m)!\,, \qquad k \geq 0.$$

$q^n(k)$: number of associative and quasitrivial n-ary operations that have at most one neutral element on $\{1,\ldots,k\}$

Corollary

We have

$$q^n(k) \quad = \quad q(k), \qquad k \geq 0.$$

Final remarks

In http://orbilu.uni.lu/handle/10993/39337

1. Characterization of the class of quasitrivial n-ary semigroups that have exactly two neutral elements

2. New integer sequences (http://www.oeis.org)
   - Number of quasitrivial n-ary semigroups that have no neutral element: A308352
   - Number of quasitrivial n-ary semigroups that have exactly two neutral elements: A308354
   - Number of quasitrivial n-ary semigroups: A308362 & A292932

Some references

N. L. Ackerman.
A characterization of quasitrivial n-semigroups.
To appear in *Algebra Universalis*.

M. Couceiro, J. Devillet, and J.-L. Marichal.
Quasitrivial semigroups: characterizations and enumerations.
Semigroup Forum, 98(3):472–498, 2019.

M. Couceiro and J. Devillet.
Every quasitrivial n-ary semigroup is reducible to a semigroup.
http://orbilu.uni.lu/handle/10993/39337.

W. A. Dudek and V. V. Mukhin.
On n-ary semigroups with adjoint neutral element.
Quasigroups and Related Systems, 14:163–168, 2006.

H. Länger.
The free algebra in the variety generated by quasi-trivial semigroups.
Semigroup Forum, 20:151–156, 1980.

N. J. A. Sloane (editor).
The On-Line Encyclopedia of Integer Sequences.
http://www.oeis.org