HYPERIDEAL POLYHEDRA IN THE 3-DIMENSIONAL ANTI-DE SITTER SPACE

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ABSTRACT. We study hyperideal polyhedra in the 3-dimensional anti-de Sitter space $\mathbb{AdS}^3$, which are defined as the intersection of the projective model of $\mathbb{AdS}^3$ with a convex polyhedron in $\mathbb{RP}^3$ whose vertices are all outside of $\mathbb{AdS}^3$ and whose edges all meet $\mathbb{AdS}^3$. We show that hyperideal polyhedra in $\mathbb{AdS}^3$ are uniquely determined by their combinatorics and dihedral angles, as well as by the induced metric on their boundary together with an additional combinatorial data, and describe the possible dihedral angles and the possible induced metrics on the boundary.

Keywords: hyperideal polyhedra, anti-de Sitter, dihedral angle, induced metric.

1. INTRODUCTION

1.1. **Hyperbolic polyhedra.** The main motivation here is the beautiful theory of polyhedra in the 3-dimensional hyperbolic space $\mathbb{H}^3$. Using the Klein (projective) model of $\mathbb{H}^3$ as a ball in $\mathbb{R}^3$, it is possible to consider "compact" hyperbolic polyhedra that are fully contained in $\mathbb{H}^3$, but also "ideal" polyhedra — those with their vertices on the boundary of $\mathbb{H}^3$) — and "hyperideal" polyhedra (as an extension of ideal polyhedra)— with all their vertices outside of $\mathbb{H}^3$, but all edges intersecting $\mathbb{H}^3$.

Those hyperbolic polyhedra can be described in terms of two types of boundary quantities.

- The induced metric on the part of the boundary contained in $\mathbb{H}^3$. For compact polyhedra, this is a hyperbolic metric with cone singularities of angle less than 2π on the sphere, and Alexandrov [Ale05] proved that each such metric is obtained on a unique compact polyhedron. For hyperideal polyhedra, the induced metrics are complete hyperbolic metrics on punctured spheres, possibly of infinite area, and each such metric is obtained on a unique hyperideal polyhedron, see [Riv92, Sch98], and the metric has finite area if and only if the polyhedron is ideal.
- Ideal and hyperideal polyhedra are characterized by their combinatorics and dihedral angles, and the possible dihedral angles are described by a simple set of linear equations and inequalities [And71, Riv96, BB02, Rou04]. It is not known whether compact hyperbolic polyhedra are uniquely determined by their combinatorics and dihedral angles — this statement is known as the "hyperbolic Stoker conjecture", see [Sto68] — but this holds locally [MM11]. However those polyhedra are uniquely determined by the *dual metric*, the induced metric on the dual polyhedron in the de Sitter space [HR93].

Understanding hyperbolic polyhedra in terms of their dihedral angles is relevant in a number of areas of hyperbolic geometry, where polyhedra are used as "building blocks" to construct hyperbolic manifolds or orbifolds.

We are particularly interested here in hyperideal polyhedra, so we recall the result of Bao and Bonahon [BB02] on their dihedral angles. Let Γ be a 3-connected planar graph and let Γ^* denote the dual graph of Γ. For each edge $e \in E(\Gamma)$, we denote by $e^* \in E(\Gamma^*)$ the dual edge of e. Let $\theta : E(\Gamma) \to \mathbb{R}$ be a weight function on $E(\Gamma)$. It was shown in [BB02] that Γ is isomorphic to the 1-skeleton of a hyperideal polyhedron in $\mathbb{H}^3$ with exterior dihedral angle $\theta(e)$ at the edge $e \in E(\Gamma)$ if and only if the weight function $\theta : E(\Gamma) \to \mathbb{R}$ satisfy the following four conditions:

(1) $0 < \theta(e) < \pi$ for each edge of Γ.
(2) If $e_1^*, ..., e_k^*$ bound a face of Γ^*, then $\theta(e_1) + ... + \theta(e_k) \geq 2\pi$.
(3) If $e_1^*, ..., e_k^*$ form a simple circuit which does not bound a face of Γ^*, then $\theta(e_1) + ... + \theta(e_k) > 2\pi$.
(4) If $e_1^*, ..., e_k^*$ form a simple path starting and ending on the same face of Γ^*, but not contained in the boundary of that face, then $\theta(e_1) + ... + \theta(e_k) > \pi$.

This implies that Γ is isomorphic to the 1-skeleton of a hyperideal polyhedron in $\mathbb{H}^3$ if and only if there is a weight function $\theta : E(\Gamma) \to \mathbb{R}$ satisfying Conditions (1)-(4). Note that the equality in Condition (2) holds if and only if the vertex of Γ, dual to the face of Γ^* bounded by the edges $e_1^*, e_2^*, ..., e_k^*$, is ideal (see e.g. [BB02]).

1.2. **Anti-de Sitter geometry.** The anti-de Sitter (AdS) space can be considered as a Lorentzian cousin of hyperbolic space. We provide a more detailed description in Section 2.2 and just recall some key properties here.

The 3-dimensional AdS space can be defined as a quadric in $\mathbb{R}^{2,2}$, which is $\mathbb{R}^4$ equiped with a bilinear symmetric form $\langle \cdot, \cdot \rangle_{2,2}$ of signature $(2, 2)$:

$$\mathbb{AdS}^3 = \left\{ \mathrm{x} \in \mathbb{R}^{2,2} \mid \langle \mathrm{x}, \mathrm{x} \rangle_{2,2} = -1 \right\} .$$

It is a Lorentzian space of constant curvature -1. $\mathbb{AdS}^3$ is geodesically complete but not simply connected, its fundamental group is $\mathbb{Z}$. There is a deep analogy between quasifuchsian hyperbolic manifolds and the so-called globally hyperbolic maximal compact (GHMC) AdS spacetimes, see [Mes07, ABB+07].

There is a projective model of one half of $\mathbb{AdS}^3$, as the interior of a quadric $\mathbb{Q}_1^3$ of signature $(1,1)$ in $\mathbb{RP}^3$, where $\mathbb{Q}_1^3 = \{ [x] \in \mathbb{RP}^3 \mid \langle x, x \rangle_{2,2} = 0 \}$. A geodesic in $\mathbb{AdS}^3$ is space-like exactly when its image in the projective model intersects $\mathbb{Q}_1^3$ in two points, light-like when its image in the projective model is tangent to $\mathbb{Q}_1^3$, and time-like when its image in the projective model is disjoint from $\mathbb{Q}_1^3$. Similarly, a space-like plane in $\mathbb{AdS}^3$ is a plane whose image in the projective model intersects $\mathbb{Q}_1^3$ along an ellipse.

We consider $\mathbb{AdS}^3$ as an oriented and time-oriented space, so that there is at each point a notion of future and past time directions.

1.3. **Anti-de Sitter polyhedra.** We are interested in hyperideal polyhedra, given by following definition, which extends the notion of ideal polyhedra in $\mathbb{AdS}^3$. It uses the projective model (still denoted by $\mathbb{AdS}^3$ for simplicity) described above.

Note that the polyhedra we consider throughout the paper are *convex* in $\mathbb{RP}^3$ (i.e. they are contained in an affine chart of $\mathbb{RP}^3$ and are convex in that affine chart).

Definition 1.1. *A (convex) hyperideal polyhedron in $\mathbb{AdS}^3$ is the intersection of $\mathbb{AdS}^3$ with a convex polyhedron P' in $\mathbb{RP}^3$ whose vertices are all outside of $\mathbb{AdS}^3$ and whose edges all pass through $\mathbb{AdS}^3$.*

It is clear that ideal polyhedra are special cases of hyperideal polyhedra. Without causing confusion, when we talk about the vertices, edges, faces and 1-skeleton of a hyperideal polyhedron P, we always mean the corresponding vertices, edges, faces and 1-skeleton of the polyhedron P' in $\mathbb{RP}^3$ with $P' \cap \mathbb{AdS}^3 = \mathrm{P}$, and we will sometimes consider P' as a "hyperideal AdS polyhedron".

Let v be a vertex of a hyperideal polyhedron P in $\mathbb{AdS}^3$. We say that v is *ideal* if v lies on $\partial \mathbb{AdS}^3$. We say that v is *strictly hyperideal* if v lies outside of the closure of $\mathbb{AdS}^3$ in $\mathbb{RP}^3$. A hyperideal polyhedron P in $\mathbb{AdS}^3$ is said to be *ideal* (resp. *strictly hyperideal*) if all the vertices of P are ideal (resp. strictly hyperideal).

1.4. **Hyperideal polyhedra in $\mathbb{AdS}^3$.** Let P be a hyperideal polyhedron in $\mathbb{AdS}^3$. It follows from the definition that all the edges and faces of P are space-like. A face of P is called *future-directed* (resp. *past-directed*), if the outward-pointing unit normal vector is future-pointing (resp. past-pointing) at each point of the interior of that face. Therefore, the faces of P are sorted into two types (i.e. future-directed and past-directed), delimited on ∂P by a *Hamiltonian cycle*, which is a closed path following edges of P visiting each vertex of P exactly once. We call this Hamiltonian cycle the *equator* of P, see e.g. [DMS14].

Each hyperideal polyhedron P in $\mathbb{AdS}^3$ with N ($N \geq 4$) vertices is associated to a *marking*, which is an identification, up to isotopy, of the equator of P with the oriented N-cycle graph (whose vertices are labelled consecutively) so that the induced ordering of the vertices is positive with respect to the orientation and time-orientation of $\mathbb{AdS}^3$. A polyhedron in $\mathbb{RP}^3$ is *non-degenerate* if it bounds a non-empty (3-dimensional) domain in $\mathbb{RP}^3$, otherwise, it is *degenerate*. Let $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}_N$ be the set of all marked non-degenerate hyperideal polyhedra in $\mathbb{AdS}^3$ with N vertices. Note that the group $\mathrm{Isom}_0(\mathbb{AdS}^3)$ of orientation and time-orientation preserving isometries of $\mathbb{AdS}^3$ has a natural action on $\tilde{P}$. We denote by $\mathcal{P} = \mathcal{P}_N$ the quotient of $\tilde{\mathcal{P}}$ by this action of $\mathrm{Isom}_0(\mathbb{AdS}^3)$. For simplicity, we call both an element of $\mathcal{P}$ and its representative a hyperideal polyhedron in $\mathbb{AdS}^3$ henceforth.

Fix an orientation on $\Sigma_{0,N}$ (where $\Sigma_{0,N}$ is a 2-sphere with N marked points) and an oriented simple closed curve γ on $\Sigma_{0,N}$ which visits each marked point once, and then label the marked points in order along the path. Following the same notations as in [DMS14], we call the component of $\Sigma_{0,N} \setminus \gamma$ on the left (resp. right) side of γ the *top* (resp. *bottom*). It is natural to identify each hyperideal polyhedron in $\mathbb{AdS}^3$ with $\Sigma_{0,N}$ via the

isotopy class of the map taking each vertex to the corresponding marked point, the equator to γ, and the union of future (resp. past) faces to the top (resp. bottom) of $\Sigma_{0,N}$.

Let $\mathsf{Graph}(\Sigma_{0,N}, \gamma)$ be the set of *3-connected* graphs (a graph is 3-connected if it cannot be disconnected or reduced to a single point by removing 0, 1 or 2 vertices and their incident edges) embedded in $\Sigma_{0,N}$, such that the vertices are located at the marked points and the edge set contains the edges of γ, up to a homeomorphism isotopic to the identity (fixing each marked point). If two marked AdS hyperideal polyhedra $P_1, P_2 \in \tilde{P}$ are equivalent, then their 1-skeletons are isomorphic to the same graph in $\mathsf{Graph}(\Sigma_{0,N}, \gamma)$. Therefore, the 1-skeleton of an element P of $\mathcal{P}$ is always well-defined and is identified to a graph Γ in $\mathsf{Graph}(\Sigma_{0,N}, \gamma)$. We say that Γ is *realized* as the 1-skeleton of P, or that Γ is the 1-skeleton of P.

Let Σ be a piecewise totally geodesic space-like surface in $\mathbb{AdS}^3$, oriented by the orientation and time orientation of $\mathbb{AdS}^3$. Let H and H' be two (totally geodesic) faces of Σ meeting along a common space-like geodesic l and let U be a neighborhood of l in $\mathbb{AdS}^3$ intersecting only two faces H, H' of Σ. We say that Σ is *convex* (resp. *concave*) at l if the component of $U \setminus (H \cup H')$ lying on the opposite side of the time-like normal vectors (determined by the orientation of Σ) is geodesically convex (resp. concave). We define the *exterior dihedral angle* at l of Σ as follows. Consider the isometry of $\mathbb{AdS}^3$ that fixes the space-like geodesic l point-wise and maps the plane of H' to the plane of H, it is a hyperbolic rotation in the group $O(1,1)$ of a time-like plane orthogonal to l with the rotation amount, say θ, valued in $\mathbb{R}$ rather than in the circle S^1 (see e.g. [DMS14]). The *sign* of θ is defined as follows. If H and H' lie in the opposite space-like quadrants (locally divided by the light-cone of l in $\mathbb{AdS}^3$), we take θ to be non-negative (resp. positive) if the surface Σ is convex (resp. strictly convex) at l, and negative if Σ is strictly concave at l. If H and H' lie in the same space-like quadrants, we take θ to be non-positive (resp. negative) if the surface Σ is is convex (resp. strictly convex) at l, and positive if Σ is strictly concave at l. Let P be a convex polyhedron in $\mathbb{AdS}^3$. The boundary ∂P of P in $\mathbb{AdS}^3$ equipped with outward-pointing normal vectors is an oriented piecewise totally geodesic (possibly disconnected) space-like surface in $\mathbb{AdS}^3$. The *exterior dihedral angle* of P at an edge e is defined as the exterior dihedral angle at the edge e of the oriented surface ∂P.

In particular, if Σ is an oriented piecewise totally geodesic surface in $\mathbb{AdS}^3$ and H, H' are two totally geodesic space-like and time-like faces of Σ meeting orthogonally along a a common space-like geodesic l, we define the *exterior dihedral angle* θ at l between H and H' to be zero (see e.g. Definition 5.8 for Case (b)). Unless otherwise stated, the dihedral angles we consider throughout this paper are exterior dihedral angles.

1.5. **Main results.** We can now state the main results, describing the dihedral angles and induced metrics on hyperideal polyhedra in $\mathbb{AdS}^3$.

Definition 1.2. *Let $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$. We denote by Γ^* the dual graph of Γ and by $e^* \in E(\Gamma)$ the dual edge of e. We say that a function $\theta : E(\Gamma) \to \mathbb{R}$ is γ-admissible if it satisfies the following four conditions:*

  (i) *$\theta(e) < 0$ if e is an edge of the equator γ, and $\theta(e) > 0$ otherwise.*
 (ii) *If $e_1^*, ..., e_k^*$ bound a face of Γ^*, then $\theta(e_1) + ... + \theta(e_k) \geq 0$.*
(iii) *If $e_1^*, ..., e_k^*$ form a simple circuit which does not bound a face of Γ^*, and such that exactly two of the edges are dual to edges of γ, then $\theta(e_1) + ... + \theta(e_k) > 0$.*
(iv) *If $e_1^*, ..., e_k^*$ form a simple path starting and ending on the boundary of the same face of Γ^*, but not contained in the boundary of that face, and such that exactly one of the edges is dual to one edge of γ, then $\theta(e_1) + ... + \theta(e_k) > 0$.*

Theorem 1.3. *Let $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$ and let $\theta : E(\Gamma) \to \mathbb{R}$ be a function. Then θ can be realized as the exterior dihedral angles at the edges of an AdS hyperideal polyhedron with 1-skeleton Γ if and only if θ is γ-admissible. In that case, θ is realized on a unique hyperideal AdS polyhedron.*

The uniqueness stated here is of course up to global isometries of $\mathbb{AdS}^3$.

Let P be an AdS hyperideal polyhedron with 1-skeleton $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$ and equator γ and let $\theta : E(\Gamma) \to \mathbb{R}$ be the function assigning to each edge $e \in E(\Gamma)$ the exterior dihedral angle $\theta(e)$ at e of P. We will show in Section 3 that θ is γ-admissible. Moreover, the equality in (ii) holds if and only if the vertex of P, dual to the face of Γ^* bounded by $e_1^*, ..., e_k^*$, is ideal.

Conditions (i)-(iv) in Definition 1.2 can be viewed as modified versions of Conditions (1)-(4) in the angle conditions for hyperideal polyhedra in $\mathbb{H}^3$. It is worth mentioning that for a general graph $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$ Condition (iv) in Definition 1.2 does not follow from Conditions (i)-(iii). It follows only in some special cases (for instance, every vertex of Γ has degree 3).

Given a 3-connected planar graph Γ which admits a Hamiltonian cycle, we fix an orientation of the Hamiltonian cycle. There is an embedding of Γ into $\Sigma_{0,N}$ such that the vertices of Γ correspond to the marked points of $\Sigma_{0,N}$, and the Hamiltonian cycle of Γ corresponds to γ with the same orientation. This embedding is unique

up to isotopy among maps sending a fixed vertex of Γ to a fixed marked point of $\Sigma_{0,N}$. Therefore, we can identify this graph Γ with an element of $\mathsf{Graph}(\Sigma_{0,N}, \gamma)$. By Theorem 1.3, the existence of an AdS hyperideal polyhedron with 1-skeleton Γ is equivalent to the existence of a γ-admissible function on $E(\Gamma)$. Moreover, Definition 1.2 tells us the existence of a γ-admissible function on $E(\Gamma)$ is equivalent to the existence of the solution to a system of linear inequalities with finitely many variables (corresponding to the edges of Γ) given by Condition (i)-(iv). One can check that such a solution always exists by choosing the function on $E(\Gamma)$ such that it satisfies Condition (i) and has large enough angles at the non-equatorial edges (see Claim 6.1 for more details). We therefore have the following.

Corollary 1.4. *For any 3-connected planar graph Γ which admits a Hamiltonian cycle γ, there exists an AdS hyperideal polyhedron P whose 1-skeleton is Γ with equator γ.*

Theorem 1.5. *Let P be an AdS hyperideal (possibly degenerate) polyhedron in $\mathbb{RP}^3$ with N ($N \geq 3$). Then the induced metric on ∂P is a complete hyperbolic metric (of infinite area if at least one vertex is strictly hyperideal) on $\Sigma_{0,N}$. Conversely, each complete hyperbolic metric on $\Sigma_{0,N}$, possibly with infinite area, is induced on a unique (up to isometries) marked AdS hyperideal polyhedron.*

Note that the existence and uniqueness here is for the hyperbolic metric on $\Sigma_{0,N}$, considered up to isotopies. The hyperideal AdS polyhedron realizing a given hyperbolic metric h depends on the position of the "equator", and if two metrics h and h' on $\Sigma_{0,N}$ are isometric by an isometry not isotopic to the identity, and not preserving the equator, then the corresponding polyhedra P and P' might not be related by an isometry of $\mathbb{AdS}^3$. As it will be clear below, hyperideal polyhedra in $\mathbb{AdS}^3$ are uniquely determined by their induced metric together with the equator, which needs to be a simple closed curve visiting each vertex exactly once.

Theorems 1.3 and 1.5 extend to hyperideal polyhedra results obtained in [DMS14] for ideal polyhedra in $\mathbb{AdS}^3$.

1.6. **Outline of the proofs and organization.** Let $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$ and let $\mathcal{P}_\Gamma \subset \mathcal{P}$ be the set of all the elements of $\mathcal{P}$ whose 1-skeleton is Γ. It is clear that $\mathcal{P}$ is the disjoint union of $\mathcal{P}_\Gamma$ over all $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$, glued together along faces corresponding to common subgraphs. Let $\mathcal{A}_\Gamma$ be the set of γ-admissible weight functions $\theta \in \mathbb{R}^E$, where $E = E(\Gamma)$. We denote by $\mathcal{A}$ the disjoint union of $\mathcal{A}_\Gamma$ over all $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$, glued together along faces corresponding to common subgraphs.

Let $\Psi_\Gamma : \mathcal{P}_\Gamma \to \mathcal{A}_\Gamma$ be the map which assigns to each $P \in \mathcal{P}_\Gamma$ the exterior dihedral angle-weight function $\theta \in \mathbb{R}^E$. Now we consider the map $\Psi : \mathcal{P} \to \mathcal{A}$, which is defined by $\Psi(P) = \Psi_\Gamma(P)$ if $P \in \mathcal{P}_\Gamma$.

To prove Theorem 1.3, it suffices to show the following statement.

Theorem 1.6. *The map $\Psi : \mathcal{P} \to \mathcal{A}$ is a homeomorphism.*

To show this, we prove that Ψ is a well-defined (i.e. $\Psi(P) \in \mathcal{A}_\Gamma$ if $P \in \mathcal{P}_\Gamma$) proper local homeomorphism, and then argue by using some topological facts concerning $\mathcal{P}$ and $\mathcal{A}$. More precisely, we will prove the following propositions and lemmas:

Proposition 1.7. *For $N \geq 5$, the space $\mathcal{A}$ is connected and has real dimension $3N - 6$.*

Proposition 1.8. *$\mathcal{P}$ is a topological manifold of real dimension $3N - 6$ for $N \geq 4$.*

Proposition 1.9. *The map Ψ taking each $P \in \mathcal{P}_\Gamma$ to its dihedral angle θ has image in $\mathcal{A}_\Gamma$, that is, θ is γ-admissible (see Definition 1.2).*

Lemma 1.10. *The map $\Psi : \mathcal{P} \to \mathcal{A}$ is proper. In other words, if $(P_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{P}_\Gamma$ for $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$ which is divergent in $\mathcal{P}$ such that the dihedral angle-weight functions $\theta_n := \Psi_\Gamma(P_n) \in \mathcal{A}_\Gamma$ converge to $\theta_\infty \in \mathbb{R}^E$ (where E denotes the edge set of Γ), then θ_∞ is not γ-admissible.*

Lemma 1.11. *Let $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$ be a triangulation of $\Sigma_{0,N}$ and let E be the set of edges of Γ. Then for each $P \in \mathcal{P}$ whose 1-skeleton is a subgraph of Γ, $\Psi : \mathcal{P} \to \mathbb{R}^E$ is a local immersion near P.*

Proposition 1.7 and Proposition 1.8 are proved in Section 6. Proposition 1.9 is proved in Section 3. Lemma 1.10 and Lemma 1.11 are shown in Section 4 and Section 5, respectively. Combining these results we prove Theorem 1.6 in Section 6.3.

Let $\widetilde{\mathsf{polyg}} = \widetilde{\mathsf{polyg}}_N$ be the space of all marked (convex) hyperideal polygons in $\mathbb{H}^2$ with N ($N \geq 3$) vertices, and let $\mathsf{polyg} = \mathsf{polyg}_N$ be the quotient space of $\widetilde{\mathsf{polyg}}$ modulo by the group of orientation preserving isometries of $\mathbb{H}^2$. Note that the hyperbolic plane $\mathbb{H}^2$ can be isometrically embedded in $\mathbb{AdS}^3$ as a space-like hyperplane. Therefore any marked hyperideal polygon $P \in \widetilde{\mathsf{polyg}}$ can be viewed as a marked degenerate hyperideal polyhedron lying on a space-like plane in $\mathbb{AdS}^3$, which is two-sided (future-directed and past-directed) and the union of

whose edges form an oriented N-cycle graph, also called the equator of P. Let $\bar{\mathcal{P}} = \mathcal{P} \cup \mathsf{polyg}$ denote the space of all marked, non-degenerate and degenerate hyperideal polyhedra in $\mathbb{A}\mathrm{dS}^3$ with N vertices, up to isometries in $\mathrm{Isom}_0(\mathbb{A}\mathrm{dS}^3)$.

Recall that a *cusp* is a surface isometric to the quotient of the region $\{z = x + iy : y > a\}$ of the upper-half space model of hyperbolic plane, for some $a > 0$, by the isometry group generated by $z \to z + 1$. A *funnel* is a surface isometric to the quotient of a hyperbolic half-plane by an isometry of hyperbolic type whose axis is the boundary of that half-plane.

Let $\Sigma_{0,N}$ be the 2-sphere with N marked points $p_1, ..., p_N$ removed (with $N \geq 3$). We say that a hyperbolic metric on $\Sigma_{0,N}$ is *complete* if for each $1 \leq i \leq N$, there is a (small enough) region of $\Sigma_{0,N}$ near p_i (with the induced metric) isometric to either a cusp or a funnel. Note that a complete hyperbolic metric on $\Sigma_{0,N}$ might have infinite area. We assign to each p_i a number ϵ_i, which is defined to be 0 (resp. $+$) if that region near p_i is isometric to a cusp (resp. a funnel). In this way, every complete hyperbolic metric on $\Sigma_{0,N}$ is equipped with a *signature* $(\epsilon_1, ..., \epsilon_N) \in \{0, +\}^N$ at the puncture set $(p_1, ..., p_N)$.

For each $(\epsilon_1, ..., \epsilon_N) \in \{0, +\}^N$, we denote by $\mathcal{T}_{0,N}^{(\epsilon_1, ..., \epsilon_N)}$ the space of complete hyperbolic metrics on $\Sigma_{0,N}$ of signature $(\epsilon_1, ..., \epsilon_N)$, considered up to isotopy fixing each marked point and denote by $\mathcal{T}_{0,N}$ the space of complete hyperbolic metrics on $\Sigma_{0,N}$, possibly with infinite area, considered up to isotopy fixing each marked point. It is clear that $\mathcal{T}_{0,N}$ is the disjoint union of the spaces $\mathcal{T}_{0,N}^{(\epsilon_1, ..., \epsilon_N)}$ over all $(\epsilon_1, ..., \epsilon_N) \in \{0, +\}^N$, glued together along faces corresponding to common subsignatures. Here the *subsignature* of two signatures $(\epsilon_1, ..., \epsilon_N), (\epsilon'_1, ..., \epsilon'_N) \in \{0, +\}^N$ is defined to be $(\min\{\epsilon_1, \epsilon'_1\}, ..., \min\{\epsilon_N, \epsilon'_N\})$, where

$$\min\{0, 0\} = 0, \quad \min\{+, +\} = +, \quad \min\{0, +\} = 0.$$

Let $P \in \bar{\mathcal{P}}$ be an AdS hyperideal polyhedron with N vertices, say $v_1, ..., v_N$, corresponding to the punctures $p_1, ..., p_N$ of $\Sigma_{0,N}$, respectively. Recall that v_i is either ideal, or strictly hyperideal. For each $1 \leq i \leq N$, we associate to the vertex v_i a sign ϵ_i as follows. We take $\epsilon_i = 0$ if v_i is ideal, and take $\epsilon_i = +$ if v_i is strictly hyperideal. We denote by $\mathcal{P}^{(\epsilon_1, ..., \epsilon_N)} \subset \bar{\mathcal{P}}$ the space of all the elements of $\bar{\mathcal{P}}$ whose vertex v_i has the sign ϵ_i. It is clear that $\bar{\mathcal{P}}$ is the disjoint union of $\mathcal{P}^{(\epsilon_1, ..., \epsilon_N)}$ over all $(\epsilon_1, ..., \epsilon_N) \in \{0, +\}^N$, glued together along faces corresponding to common subsignatures.

Note that any space-like plane in $\mathbb{A}\mathrm{dS}^3$ is isometric to the hyperbolic plane $\mathbb{H}^2$ and each face of a hyperideal polyhedron in $\mathbb{A}\mathrm{dS}^3$ is isometric to a hyperideal polygon in the hyperbolic plane (i.e. the intersection of $\mathbb{H}^2$ with a polygon in $\mathbb{RP}^2$ whose vertices are all outside of $\mathbb{H}^2$ and whose edges all pass through $\mathbb{H}^2$). Therefore, the path metric induced on (the intersection with $\mathbb{A}\mathrm{dS}^3$ of) the boundary surface of $P \in \bar{\mathcal{P}}$ is a complete hyperbolic metric on $\Sigma_{0,N}$ with a cusp (resp. funnel) around p_i if the corresponding vertex v_i of P has sign $\epsilon_i = 0$ (resp. $\epsilon_i = +$). This determines a point (which is the corresponding isotopy class) in the Teichmüller space $\mathcal{T}_{0,N}^{(\epsilon_1, ..., \epsilon_N)}$. In particular, if $P \in \mathcal{P}^{(\epsilon_1, ..., \epsilon_N)}$ is a degenerate hyperideal polyhedron (i.e. a hyperideal polygon), then the induced metric on the boundary surface of P is obtained by doubling P, here the boundary surface is identified with the two copies of P glued along the equator.

Now we consider the map $\Phi : \bar{\mathcal{P}} \to \mathcal{T}_{0,N}$, which assigns to each hyperideal polyhedron P in $\bar{\mathcal{P}}$ the induced metric on the boundary surface of P. It follows from the aforementioned fact that for each $P \in \mathcal{P}^{(\epsilon_1, ..., \epsilon_N)}$, we have $\Phi(P) \in \mathcal{T}_{0,N}^{(\epsilon_1, ..., \epsilon_N)}$. Therefore, Φ has image in $\mathcal{T}_{0,N}$. To verify Theorem 1.5, it suffices to show the following theorem:

Theorem 1.12. *The map $\Phi : \bar{\mathcal{P}} \to \mathcal{T}_{0,N}$ is a homeomorphism.*

Using a similar idea as for Theorem 1.6, we need to show the following topological fact concerning $\bar{\mathcal{P}}$, the local parameterization and properness statements with respect to the induced metrics.

It is a basic fact in the Teichmüller theory of hyperbolic surfaces with cusps or geodesic boundary components that $\mathcal{T}_{0,N}^{(\epsilon_1, ..., \epsilon_N)}$ is a contractible manifold of real dimension $3N - 6 - \mathfrak{n}(\epsilon_1, ..., \epsilon_N)$, where $\mathfrak{n} : \{0, +\}^N \to \mathbb{R}$ is the counting function of the number of the signs 0 occurring in $(\epsilon_1, ..., \epsilon_N) \in \{0, +\}^N$. $\mathcal{T}_{0,N}$ is a contractible manifold with corners of real dimension $3N - 6$ (see e.g. [BKS11, FG07, BL07]), which can also be obtained from the enhanced Teichmüller space of hyperbolic surfaces with N boundary components of sign 0 (i.e. cusps), sign $+$ or $-$ by identifying the signs ($+$ and $-$) of the boundary components. It remains to check the topology of $\bar{\mathcal{P}}$. We will prove the following statements.

Proposition 1.13. *$\bar{\mathcal{P}}$ is a topological manifold of real dimension $3N - 6$ for all $N \geq 3$.*

Lemma 1.14. *The map $\Phi : \bar{\mathcal{P}} \to \mathcal{T}_{0,N}$ is proper.*

Lemma 1.15. *The map $\Phi : \bar{\mathcal{P}} \to \mathcal{T}_{0,N}$ is a local immersion.*

Proposition 1.13 is proved in Section 6. Lemma 1.14 and Lemma 1.15 are shown in Section 4 and Section 5, respectively. Combining these results we prove Theorem 1.12 in Section 6.4.

2. Background materials

This section contains definitions and notations needed elsewhere in the paper, as well as some background results.

2.1. **The hyperbolic-de Sitter space.** Let $\mathbb{R}_k^n$ be the real n-dimensional vector space $\mathbb{R}^n$ equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle_{n-k,k}$ of signature $(n-k,k)$ $(0 \leq k \leq n)$, where the bilinear form $\langle \cdot, \cdot \rangle_{n-k,k}$ is written as

$$\langle x, y \rangle_{n-k,k} = \sum_{i=1}^{n-k} x_i y_i - \sum_{j=n-k+1}^{n} x_j y_j,$$

for all $x, y \in \mathbb{R}_k^n$. For $0 \leq k \leq n$, we denote

$$H_k^n = \{x \in \mathbb{R}_{k+1}^{n+1} \mid \langle x, x \rangle_{n-k,k+1} = -1\}.$$

For $0 \leq k \leq n-1$, we denote

$$S_k^n = \{x \in \mathbb{R}_k^{n+1} \mid \langle x, x \rangle_{n-k+1,k} = 1\}.$$

The restriction of the bilinear form $\langle \cdot, \cdot \rangle_{n-k,k+1}$ (resp. $\langle \cdot, \cdot \rangle_{n-k+1,k}$) to the tangent space at each point of H_k^n (resp. S_k^n) induces a pseudo-Riemannian metric of signature $(n-k,k)$ (resp. $(n-k,k)$) of constant sectional curvature -1 (resp. $+1$). We denote by $\mathbb{H}_k^n$ (resp. $\mathbb{S}_k^n$) the projection of H_k^n (resp. S_k^n) to $\mathbb{RP}^n$. It is easy to see that H_k^n (resp. S_k^n) is a double cover of $\mathbb{H}_k^n$ (resp. $\mathbb{S}_k^n$), and $\mathbb{H}_k^n$ and $\mathbb{S}_k^n$ are naturally equipped with the metrics induced from the projection. For instance,

- $\mathbb{H}^n := \mathbb{H}_0^n$ is the projective model of n-dimensional *hyperbolic* space.
- $\mathrm{d}\mathbb{S}^n := \mathbb{S}_1^n$ is the projective model of n-dimensional *de Sitter* (dS) space.
- $\mathbb{A}\mathrm{d}\mathbb{S}^n := \mathbb{H}_1^n$ is the projective model of n-dimensional *anti-de Sitter* (AdS) space.

For more examples we refer to [IS10, Sch98].

To understand the induced geometric structures on the boundary of hyperideal AdS polyhedra, it is convenient to consider a notion of *HS space* expanding the hyperbolic, de Sitter or AdS spaces, see [Sch98]. It is based on the Hilbert distance, itself based on the cross-ratio of four points in $\mathbb{R}P^1$. Given four points x, y, a, b in a line, their *cross ratio* $(x, y; a, b)$ of is defined as

$$(x, y; a, b) = \frac{(x-a)(y-b)}{(y-a)(x-b)} \in \mathbb{R} \cup \{\infty\} \ .$$

It is independent of the choice of a linear coordinate on the line, and in fact invariant under projective transformations, so that it can be defined for four points on a line in $\mathbb{RP}^n$.

Definition 2.1. *Let $n \in N^*$, and let $p, q \in \{0, 1, ..., n\}$ with $p + q \leq n$. We denote by $\mathbb{Q}_{p,q}^n$ the quadric of $\mathbb{RP}^n$ of homogenous equation:*

$$\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q+1} x_j^2 = 0.$$

Using the cross-ratio, the complement of $\mathbb{Q}_{p,q}^n$ in $\mathbb{RP}^n$ can be equipped with a complex-valued "distance", the Hilbert distance with respect to the quadric $\mathbb{Q}_{p,q}^n$), which, restricted to each connected component ($\mathbb{H}_q^n$ or $\mathbb{S}_{q+1}^n$) is the distance induced from the aforementioned metric (up to multiplication by the factor i) of constant sectional curvature ± 1.

Definition 2.2. *For any $x, y \in \mathbb{RP}^n \setminus \mathbb{Q}_{p,q}^n$, the Hilbert distance between x and y is defined as*

$$d_H(x, y) = -\frac{1}{2} \log(x, y; a, b),$$

where $\log$ denotes the branch of the complex logarithm $\log : \mathbb{C} \setminus \{0\} \to \{z \in \mathbb{C} : \operatorname{Im} z \in (-\pi, \pi]\}$, while a, b are the intersections of the line $\Delta(x, y)$ passing through x and y in $\mathbb{RP}^n$ with the quadric $\mathbb{Q}_{p,q}^n$ (if the intersections are non-empty). Otherwise, a, b are the intersections of the complexified line $\bar{\Delta}(x, y)$ of $\Delta(x, y)$ with the the complexified quadric $\bar{\mathbb{Q}}_{p,q}^n$ of $\mathbb{Q}_{p,q}^n$. Moreover, a, b (if non-coincident) are ordered such that

- *if a, b are real, then the vectors $x - a$ and $b - a$ induce the same direction.*
- *if a, b are complex conjugate, then the vector $i(a - b)$ has the same direction with $y - x$.*

Definition 2.3. *We denote by $\mathbb{HS}_{p,q}^n$ the space $\mathbb{RP}^n$ equipped with the Hilbert distance with respect to the quadric $\mathbb{Q}_{p,q}^n$, defined only for points which are in the complement of $\mathbb{Q}_{p,q}^n$. For $k \in \{0,...,n\}$, we denote $\mathbb{HS}_k^n = \mathbb{HS}_{n-k,k}^n$ and $\mathbb{Q}_k^n = \mathbb{Q}_{n-k,k}^n$.*

The space $\mathbb{HS}_{p,q}^n$ is called the hyperbolic-de Sitter space of type (n,p,q).

For instance:

- $\mathbb{HS}_0^n = \mathbb{HS}_{n,0}^n = \mathbb{RP}^n \setminus \mathbb{Q}_0^n = \mathbb{H}^n \cup d\mathbb{S}^n$ is the hyperbolic-de Sitter space of dimension n.
- $\mathbb{HS}^n = \mathbb{HS}_{n-1,1}^n = \mathbb{RP}^n \setminus \mathbb{Q}_1^n = \mathbb{AdS}^n \cup \mathbb{S}_2^n$ is the hyperbolic-de Sitter space of type $(n, n-1, 1)$.

The isometry group $\mathrm{Isom}(\mathbb{HS}_{p,q}^n)$ of $\mathbb{HS}_{p,q}^n$ is isomorphic to the projective orthogonal group $\mathrm{PO}(p, q+1)$ with respect to the bilinear form $\langle \cdot, \cdot \rangle_{p,q+1}$. The identity component of $\mathrm{Isom}(\mathbb{HS}_{p,q}^n)$, denoted by $\mathrm{Isom}_0(\mathbb{HS}_{p,q}^n)$, is isomorphic to the identity component (denoted by $\mathrm{PO}_0(p,q)$) of $\mathrm{PO}(p,q)$, which consists of the isometries of $\mathbb{HS}_{p,q}^n$ preserving the overall orientation and the respective orientations on the p and $q+1$ dimensional subspaces on which the form $\langle \cdot, \cdot \rangle_{p,q+1}$ is definite respectively. For instance,

- $\mathrm{Isom}_0(\mathbb{HS}_0^3) = \mathrm{Isom}_0(\mathbb{H}^3) \cong \mathrm{PO}_0(3,1) \cong \mathrm{PSL}_2(\mathbb{C})$.
- $\mathrm{Isom}_0(\mathbb{HS}_1^3) = \mathrm{Isom}_0(\mathbb{AdS}^3) \cong \mathrm{PO}_0(2,2) \cong \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$.

The Hilbert metric d_H on $\mathbb{HS}_{p,q}^n$ is preserved by the isometry group $\mathrm{Isom}(\mathbb{HS}_{p,q}^n)$ and it provides a very convenient way to define the classical duality in $\mathbb{HS}_{p,q}^n$, see e.g. [Sch98].

2.2. The 3-dimensional AdS $\mathbb{AdS}^3$ in $\mathbb{HS}_1^3$. We are in particular interested in the hyperbolic-de Sitter space $\mathbb{HS}_1^3$ of type $(3,2,1)$, which contains the disjoint union of $\mathbb{AdS}^3$ and $\mathbb{AdS}^{3*} := \mathbb{S}_2^3$ (which is isometric to $\mathbb{AdS}^3$ with sign reversed). There are four types of complete geodesics (projective lines) in $\mathbb{RP}^3 \supset \mathbb{HS}_1^3$ (see e.g. [FS12,Sch98]) :

(a) the geodesics passing through two distinct points of the quadric $\mathbb{Q}_1^3$, whose intersection with $\mathbb{AdS}^3$ (resp. $\mathbb{AdS}^{3*}$) are space-like (resp. time-like),
(b) the geodesics contained in $\mathbb{AdS}^3$ and disjoint from $\mathbb{AdS}^{3*}$, which are time-like in $\mathbb{AdS}^3$,
(c) the geodesics contained in $\mathbb{AdS}^{3*}$ and disjoint from $\mathbb{AdS}^3$, which are space-like in $\mathbb{AdS}^{3*}$;
(d) the geodesics tangent to the quadric $\mathbb{Q}_1^3$, which are light-like.

For $x, y \in \mathbb{HS}_1^3$, let $\Delta(x,y)$ be the straight line which passes through x and y in an affine chart $\mathbb{R}^n$ of $\mathbb{RP}^n$. A direct computation show that (see e.g. [Sch98])

- if $\Delta(x,y)$ is of type (a),
    - for $x, y \in \mathbb{AdS}^3$, $d_H(x,y) \in \mathbb{R}^+$;
    - for $x, y \in \mathbb{AdS}^{3*}$ lying on the same component of $\Delta(x,y) \setminus \mathbb{AdS}^3$, $d_H(x,y) \in \mathbb{R}^-$;
    - for $x, y \in \mathbb{AdS}^{3*}$ lying on the different components of $\Delta(x,y) \setminus \mathbb{AdS}^3$, $d_H(x,y) = i\pi + r$ with $r \in \mathbb{R}$;
    - for $x \in \mathbb{AdS}^3$ and $y \in \mathbb{AdS}^{3*}$, $d_H(x,y) = i\frac{\pi}{2} + r$ with $r \in \mathbb{R}$.
- if $\Delta(x,y)$ is of type (b), $d_H(x,y) \in i(\mathbb{R}^+/\pi\mathbb{Z})$.
- if $\Delta(x,y)$ is of type (c), $d_H(x,y) \in i(\mathbb{R}^-/\pi\mathbb{Z})$.
- if $\Delta(x,y)$ is of type (d), $d_H(x,y) = 0$.

Recall that $\mathbb{AdS}^3$ is a 3-dimensional Lorentzian symmetric space of constant sectional curvature -1 diffeomorphic to a solid torus. The boundary in $\mathbb{RP}^3$ of $\mathbb{AdS}^3$ is exactly $\mathbb{Q}_1^3$ (we use the notion $\partial\mathbb{AdS}^3$ henceforth), which is a projective quadric foliated by two families of projective lines, called the left and right leaves, respectively.

Note that the coordinates of $\mathbb{AdS}^3$ in $\mathbb{R}^{2,2}$ satisfy that either $x_3 \neq 0$ or $x_4 \neq 0$. Since $\mathrm{Isom}_0(\mathbb{AdS}^3)$ acts transitively on $\mathbb{AdS}^3$, up to an element of $\mathrm{Isom}_0(\mathbb{AdS}^3)$, we can always assume that every (representative) polyhedron in $\bar{\mathcal{P}}$ is contained in the affine chart $x_4 = 1$. The intersection of $\mathbb{AdS}^3$ with the affine chart $x_4 = 1$ is the region $x_1^2 + x_2^2 - x_3^2 < 1$ in $\mathbb{R}^3$, bounded by the one-sheeted hyperboloid $x_1^2 + x_2^2 - x_3^2 = 1$ (i.e. the boundary $\partial\mathbb{AdS}^3$ in this affine chart).

It is important for some applications to notice that $\mathbb{AdS}^3$ is isometric to $SL(2, \mathbb{R})$ equiped with its Killing metric, see [Mes07]. This will however not play any role here.

2.3. Polyhedral HS structures. Let X be an embedded submanifold of $\mathbb{RP}^n$. We say X admits an *HS structure* if X has a maximal altas of charts with values onto the intersection of X with open sets of $\mathbb{HS}_{p,q}^n$ and any change of charts is given by the restriction (to each connected component of its domain of definition) of some element $g \in \mathrm{Isom}_0(\mathbb{HS}_{p,q}^n)$.

The HS structure has a precise description via the natural local geometry near a point $x \in \mathbb{HS}_{p,q}^n$, that is, a *signed* distance on the projective tangent space $PT_x(\mathbb{HS}_{p,q}^n)$ at x (called also *HS sphere*), defined by using the Hilbert distance (see [Sch98, Section 3] for the general case and [DMS14, Section 6.1] for the AdS case). For $x, y \in \mathbb{H}_q^n$ (resp. $\mathbb{S}_{q+1}^n$) the *signed length* of the geodesic segment connecting x, y with respect to the HS

structure is equal to $d_H(x, y)$ (resp. $d_H(x, y)i$) if $d_H(x, y) \in \mathbb{R}$ (resp. if $d_H(x, y) \in i\mathbb{R}$), see [Sch98, Theorem 2.3].

Here we are interested in the HS structure, denoted by σ, on the the intersection with $\mathbb{HS}^3_k$ of the boundary surface ∂P of a polyhedron P in $\mathbb{RP}^3$, called the *polyhedral HS structure* induced on ∂P.

The following are some examples:

- Let P be an ideal polyhedron in $\mathbb{H}^3$ (resp. $\mathbb{AdS}^3$). Then σ is a complete hyperbolic metric on $\Sigma_{0,N}$ with cusps around the punctures, where $\partial P \cap \mathbb{H}^3$ (resp. $\partial P \cap \mathbb{AdS}^3$) are identified with $\Sigma_{0,N}$ and vertices of P are identified with punctures of $\Sigma_{0,N}$.
- Let P be a polyhedron in $\mathbb{RP}^3$ whose intersection with $\mathbb{H}^3$ (resp. $\mathbb{AdS}^3$) is a strictly hyperideal hyperbolic (resp. AdS) polyhedron. Then σ restricted to $\partial P \cap \mathbb{H}^3$ (resp. $\partial P \cap \mathbb{AdS}^3$) is a complete hyperbolic metric on $\Sigma_{0,N}$ with funnels around the punctures, where $\partial P \cap \mathbb{H}^3$ (resp. $\partial P \cap \mathbb{AdS}^3$) are identified with $\Sigma_{0,N}$ and neighborhoods of of $\partial P \cap \partial\mathbb{H}^3$ (resp. $\partial P \cap \partial\mathbb{AdS}^3$) in $\partial P \cap \mathbb{H}^3$ (resp. $\partial P \cap \mathbb{AdS}^3$) are identified with neighborhoods of punctures of $\Sigma_{0,N}$; while σ restricted to $\partial P \cap \mathrm{d}\mathbb{S}^3$ (resp. $\partial P \cap \mathbb{AdS}^{3*}$) is a disjoint union of complete $\mathrm{d}\mathbb{S}^2$ (resp. $\mathbb{AdS}^2$) disks with cone singularities at the vertices.

For simplicity, the polyhedral HS structure is called the HS structure or HS metric henceforth.

2.4. **The duality in $\mathbb{HS}^3_1$.** Let $X \subset \mathbb{HS}^n_k$ $(0 \leq k \leq n)$ be a totally geodesic subset, then the *dual* of X is defined as

$$X^\perp = \{x \in \mathbb{HS}^n_k : d_H(x, y) = i\pi/2 \text{ for all } y \in X\}.$$

In particular, for a point $x \in \mathbb{RP}^n$, if $x \in \mathbb{HS}^n_k$, by definition, the dual $x^\perp$ is a hyperplane of $\mathbb{HS}^n_k$; if $x \in \mathbb{Q}^n_k$, the dual $x^\perp$ is defined as the hyperplane of $\mathbb{RP}^n$ tangent to $\mathbb{Q}^n_k$ at x. For a hyperplane P of $\mathbb{HS}^n_k$, if P is not tangent to $\mathbb{Q}^n_k$, then the dual $P^\perp$ is the (only) intersection point in $\mathbb{HS}^n_k$ of all geodesics starting from P orthogonally at Hilbert distance $i\pi/2$; if P is tangent to $\mathbb{Q}^n_k$, then $P^\perp$ is defined to be the point of tangency in $\mathbb{Q}^n_k$.

We focus on the duality in $\mathbb{HS}^3_1$. Let P be a polyhedron in $\mathbb{RP}^3$. The dual polyhedron of P (with respect to the HS metric on $\mathbb{HS}^3_1$) is defined as the polyhedron P^* in $\mathbb{RP}^3$ such that for any $k = 0, 1, 2$,

- a k-face f^* of P^* is contained in the dual of a $(2-k)$-face f of P;
- a k-face f^* of P^* is contained in the boundary of a $(k+1)$-face $(f')^*$ if and only if the boundary of the corresponding $(2-k)$-face f of P contains the corresponding $(2-k-1)$-face f'.

We call the face f^* of P^* the *dual* of the face f of P. It is clear that the dual polyhedron P^* is uniquely determined by P, and furthermore, P^* is a convex if and only if P is convex. The boundary surface $\partial P \cap \mathbb{HS}^3_1$ is naturally equipped with an HS metric. Moreover, for an edge e of an AdS hyperideal polyhedron P, the (exterior) dihedral angle at e of P is equal to the signed length of the dual edge e^* of P^*.

3. NECESSITY

In this section, we show that the conditions in Theorem 1.3 are necessary by proving Proposition 1.9.

3.1. **Angle conditions.** Let P be an AdS hyperideal polyhedron with 1-skeleton $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$. Let $\theta : E(\Gamma) \to \mathbb{R}$ be a function assigning to each edge $e \in E(\Gamma)$ the (exterior) dihedral angle $\theta(e)$ at e of P. We prove in the following that θ is a γ-admissible function.

Proof of Proposition 1.9. Condition (i) follows from the sign convention for dihedral angles.

For Condition (ii), let v be the vertex of Γ dual to that face of Γ^*. If v is ideal, we use a similar argument as in [DMS14, Section 6]: the intersection of P with a small "horo-torus" centered at v is a convex polygon (with space-like edges) in the Minkowski plane. The exterior angle at each vertex of this polygon is equal to the exterior dihedral angle at the corresponding edge (containing that vertex) of P. It is known that the sum of the exterior angles at the vertices of a Minkowski polygon with space-like edges is zero, therefore, $\theta(e_1) + \ldots + \theta(e_k) = 0$.

If v is strictly hyperideal, consider the dual plane $v^\perp$ (which is isometric to $\mathbb{AdS}^2$), it is orthogonal to all the faces and edges of P adjacent to v and the intersection of $v^\perp$ with P is a convex compact polygon in $v^\perp \cong \mathbb{AdS}^2$ with space-like edges. Thus the exterior angle at each vertex of this polygon is equal to the exterior dihedral angle at the corresponding edge (containing that vertex) of P. Recall that the Gauss-Bonnet formula for Lorentzian polygons p with all edges space-like (see [Dza84] for more details) is the following:

$$(1) \qquad \iint_p K ds + \int_{\partial p} k_g dl + \sum_i \theta_i = 0,$$

where K, k_g, ds, dl and θ_i denote the Gaussian curvature, the geodesic curvature, the area form, the arc-length element and the exterior angle at the i-th vertex of p, respectively. This formula shows that $\theta(e_1) + ... + \theta(e_k) = \text{Area}(p) > 0$.

For Condition (iii), let $e_1^*, e_2^*, ..., e_k^*, e_{k+1}^* = e_1^*$ be a simple circuit which does not bound a face of Γ^*, and such that exactly two of the edges are dual to edges of γ. Denote the common endpoint of e_i^* and e_{i+1}^* by f_i^* for $i = 1, ..., k$, which is the vertex of Γ^* dual to the face f_i of P (equivalently, the boundary of f_i contains the edges e_i and e_{i+1}). To show that $\theta(e_1) + ... + \theta(e_k) > 0$, it suffices to consider the extension polyhedron Q (see e.g. [DMS14, Section 6]), which is a polyhedron obtained by extending the faces f_1, f_2, ..., f_k and forgetting about the other faces of P. Note that the analysis (see [DMS14, Section 6]) of the sum of the dihedral angles at the edges (containing e_1, e_2, ..., e_k) of the corresponding extension polyhedron in the case of ideal polyhedra in $\mathbb{A}d\mathbb{S}^3$ completely applies to the extension polyhedron Q here. This implies that θ satisfies Condition (iii).

The condition (iv) is an analogue of Condition (4) for the dihedral angles of hyperideal polyhedra in hyperbolic 3-space $\mathbb{H}^3$ [BB02, Theorem 1]. Let v be the vertex of Γ dual to that face, say f, of Γ^* on which the simple path $e_1^*, ..., e_k^*$ starts and ends.

We claim that the unique edge in the path $e_1^*, ..., e_k^*$ which is dual to one edge of γ is not contained in the boundary of the face f. Otherwise, combined with the assumption that $e_1^*, ..., e_k^*$ constitute a simple path starting and ending in the same boundary of f, it follows that the simple path $e_1^*, ..., e_k^*$ is contained in the boundary of f, which contradicts the assumption that the simple path $e_1^*, ..., e_k^*$ is not contained in the boundary of f. Therefore, it suffices to consider the case that the simple path $e_1^*, ..., e_k^*$ meets the boundary of f only at their two endpoints (which coincide with two vertices of Γ^*), since the dihedral angles at the edges of P which are dual to those in the intersection (if non-empty) of the simple path $e_1^*, ..., e_k^*$ with the boundary of f are positive.

If v is strictly hyperideal, we denote by X_v the half-space containing v bounded by the dual plane $v^\perp$. Let P' be the polyhedron obtained by doubling $P \setminus X_v$ via the reflection across $v^\perp$. Note that $v^\perp$ separates v from the other vertices of P and is orthogonal to the edges and faces of P that it meets, P' is a hyperideal polyhedron. Let Γ' be the 1-skeleton of P' and let Γ'^* be the dual graph of Γ'. It is clear that Γ'^* is obtained by gluing two copies of Γ^* along the the boundary of f. By the assumption, the path $e_1^*, ..., e_k^*$ and its images $e_1'^*, ..., e_k'^*$ in Γ'^* under the reflection form a simple circuit which does not bound a face of Γ'^*, and exactly two of the edges of the circuit are dual to edges of γ. Note that $\theta(e_i) = \theta(e_i')$ for $i = 1, ..., k$. Combined with the above result that θ satisfies Condition (iii), we have

$$\theta(e_1) + ... + \theta(e_k) + \theta(e_1') + ... + \theta(e_k') = 2\big(\theta(e_1) + ... + \theta(e_k)\big) > 0,$$

which implies θ satisfies Condition (iv).

If v is ideal, let δ be the simple circuit in Γ^* which bounds the face f of Γ^*. The endpoints of the simple path $e_1^*, ..., e_k^*$ decompose δ into two parts, say δ_1, δ_2. Since the path $e_1^*, ..., e_k^*$ contains exactly one edge dual to an edge of γ, then the endpoints of this path coincide with two vertices on δ of Γ^* which are dual to two faces of Γ on the left and right side of γ, respectively. Therefore, δ_1 (resp. δ_2) contains exactly one edge dual to an edge of γ. Without lost of generality, we assume that

$$(2) \qquad \sum_{e^* \subset \delta_1} \theta(e) \leq \sum_{(e')^* \subset \delta_2} \theta(e').$$

Since θ satisfies Condition (ii), then

$$(3) \qquad \sum_{e^* \subset \delta_1} \theta(e) + \sum_{(e')^* \subset \delta_2} \theta(e') = 0.$$

Combining the assumption (2) and the equality (3), we have

$$(4) \qquad \sum_{e^* \subset \delta_1} \theta(e) \leq 0.$$

The union of the simple path $e_1^*, ..., e_k^*$ and δ_1 forms a simple circuit in Γ^*, say ξ, which contains exactly two edges dual to edges of γ. If ξ does not bound a face of Γ^*, since θ satisfies Condition (iii), then

$$(5) \qquad \theta(e_1) + ... + \theta(e_k) + \sum_{e^* \subset \delta_1} \theta(e) > 0.$$

By the inequalities (4) and (5), we have $\theta(e_1) + ... + \theta(e_k) > 0$.

If ξ bounds a face f' of Γ^*, then $f' \neq f$ (since the path $e_1^*, ..., e_k^*$ is not contained in the boundary of f). Therefore δ_1 is contained in $\partial f \cap \partial f'$ and thus is an edge of Γ^*, say $\delta_1 := e_0^*$. Note that the dihedral angle along

each edge of P is non-zero (see also Condition (i)). Combined with the inequality (4), we have

$$(6) \qquad\qquad \sum_{e^* \subset \delta_1} \theta(e) = \theta(e_0) < 0.$$

By the equality (3) and the inequality (6), we have $\theta(e_1) + ... + \theta(e_k) > 0$.

Therefore, Condition (iv) follows.

$\square$

4. Properness

In this section, we prove Lemma 1.10 and Lemma 1.14, that is, the parameterization map Ψ (resp. Φ) of the space $\mathcal{P}$ (resp. $\bar{\mathcal{P}}$) of marked non-degenerate (resp. degenerate and non-degenerate) AdS hyperideal polyhedra up to elements of $\text{Isom}_0(\mathbb{AdS}^3)$ in terms of dihedral angles (resp. the induced metric on the boundary) is proper.

4.1. **Properness of Ψ.** In this part, we aim to show that the map $\Psi : \mathcal{P} \to \mathcal{A}$ (which takes each $P \in \mathcal{P}$ to the exterior dihedral angle-assignment on its edges) is proper (see Lemma 1.10).

4.1.1. *Definitions and notations.* We first clarify the topology of the space $\bar{\mathcal{P}}$.

Recall that each equator of a marked polyhedron or polygon in $\tilde{\mathcal{P}} \cup \widetilde{\text{polyg}}$ is identified with an oriented N-cycle graph. Let V be the set of vertices of this graph. Assume that we fix once and for all an affine chart $x_4 = 1$. For each $P \in \tilde{\mathcal{P}} \cup \widetilde{\text{polyg}}$, the convexity of P implies that P is the convex hull in $\mathbb{RP}^3$ of its N vertices. It is natural to endow the space $\tilde{\mathcal{P}} \cup \widetilde{\text{polyg}}$ with the topology induced by the embedding $\phi : \tilde{\mathcal{P}} \cup \widetilde{\text{polyg}} \to (\mathbb{RP}^3)^V$ which associates to each $P \in \tilde{\mathcal{P}} \cup \widetilde{\text{polyg}}$ its vertex set in $\mathbb{RP}^3$.

The space $\bar{\mathcal{P}} = (\tilde{\mathcal{P}} \cup \widetilde{\text{polyg}})/\text{Isom}_0(\mathbb{AdS}^3)$ therefore inherits the quotient topology from $\tilde{\mathcal{P}} \cup \widetilde{\text{polyg}}$. It is not hard to check that $\bar{\mathcal{P}}$ is Hausdorff (see e.g. [BB02, Lemma 8]). More precisely, whenever two sequences $(\tilde{P}_n)_{n \in \mathbb{N}}$ in $\tilde{\mathcal{P}} \cup \widetilde{\text{polyg}}$ and $(g_n)_{n \in \mathbb{N}}$ in $\text{Isom}_0(\mathbb{AdS}^3)$ are such that $(\tilde{P}_n)$ converges to some $\tilde{P}_\infty \in \tilde{\mathcal{P}} \cup \widetilde{\text{polyg}}$ and $(g_n \tilde{P}_n)$ converges to some $\tilde{P}'_\infty \in \tilde{\mathcal{P}} \cup \widetilde{\text{polyg}}$, there is a $g \in \text{Isom}_0(\mathbb{AdS}^3)$ such that $\tilde{P}'_\infty = g(\tilde{P}_\infty)$.

We say that a sequence $(P_n)_{n \in \mathbb{N}}$ in $\bar{\mathcal{P}}$ *converges* to $P_\infty \in \bar{\mathcal{P}}$ if there exists a sequence of representatives $\tilde{P}_n$ of P_n and a representative $\tilde{P}_\infty$ of P_∞ such that $(\tilde{P}_n)_{n \in \mathbb{N}}$ converges to $\tilde{P}_\infty$. The spaces $\mathcal{P}, \mathcal{P}_\Gamma, \mathcal{P}^{(\epsilon_1, ..., \epsilon_N)} \subset \bar{\mathcal{P}}$, endowed with the topology induced from that of $\bar{\mathcal{P}}$, are naturally Hausdorff. Similarly, we can define the convergence for each of them.

Let $(P_n)_{n \in \mathbb{N}}$ be a sequence of marked non-degenerate polyhedra (up to isometry) in $\mathcal{P}$. Note that the 1-skeleton of each P_n is identified to a graph Γ_n in $\text{Graph}(\Sigma_{0,N}, \gamma)$ and $\text{Graph}(\Sigma_{0,N}, \gamma)$ is finite, after passing to a subsequence, we can assume that all the P_n have the same combinatorics say $\Gamma \in \text{Graph}(\Sigma_{0,N}, \gamma)$. Let $\tilde{P}_n$ be a representative of P_n. Since all the $\tilde{P}_n$ lie in $\mathbb{RP}^3$ and $\mathbb{RP}^3$ is compact, after passing to a subsequence, each vertex of the $\tilde{P}_n$ converges to a point in $\mathbb{RP}^3$. Therefore, $(\tilde{P}_n)_{n \in \mathbb{N}}$ (we keep the notation for the convergent subsequence) converges to the convex hull in $\mathbb{RP}^3$ of the limit points, denoted by $\tilde{P}_\infty$, called the *limit* of $(\tilde{P}_n)_{n \in \mathbb{N}}$. In particular, we call the limit of a vertex (resp. an edge) of the $\tilde{P}_n$ a *limit vertex* (resp. *limit edge*).

It is easy to see that the limit $\tilde{P}_\infty$ is locally convex in $\mathbb{RP}^3$ (i.e. $\tilde{P}_\infty$ is convex in the chosen affine chart). However, $\tilde{P}_\infty$ is possibly not contained in the affine chart. It may not be convex, or not be hyperideal (for instance, with some edges tangent to $\partial\mathbb{AdS}^3$), or possibly have fewer vertices than the $\tilde{P}_n$. Moreover, vertices of the $\tilde{P}_n$ may converge to a point of $\tilde{P}_\infty$ which is not a vertex.

We will see from the following lemma that after choosing appropriate representatives say $\tilde{P}_n$ of P_n, the limit of $(\tilde{P}_n)_{n \in \mathbb{N}}$ has at least three non-colinear vertices.

4.1.2. *Some basic lemmas.*

Lemma 4.1. *Let $(P_n)_{n \in \mathbb{N}}$ be a sequence of marked non-degenerate polyhedra (considered up to isometry) in $\mathcal{P}_\Gamma$ with representatives $\tilde{P}_n$ converging to a limit in $\mathbb{RP}^3$. Then there exist a sequence $(g_n)_{n \in \mathbb{N}}$ of elements of $\text{Isom}_0(\mathbb{AdS}^3)$ such that $(g_n \tilde{P}_n)_{n \in \mathbb{N}}$ converges to a limit in $\mathbb{RP}^3$ with at least three non-colinear vertices.*

Proof. The proof uses the following steps. The idea is to normalize three points in the intersection of the $\tilde{P}_n$ with $\partial\mathbb{AdS}^3$, by applying a sequence of isometries.

Step 1. By extracting a subsequence, we can assume that for each vertex v of Γ, the corresponding vertex v_n of $\tilde{P}_n$ is either ideal for all n, or is strictly hyperideal for all n. We arbitrarily fix a face say f of Γ and consider the corresponding face f_n of the $\tilde{P}_n$. Recall that the boundary $\partial\mathbb{AdS}^3$ of $\mathbb{AdS}^3$ can be identified with the product $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ via the left and right projection to the boundary of a fixed totally geodesic space-like plane (which is the image of an embedding map $r_0 : \mathbb{H}^2 \to \mathbb{AdS}^3$ and is identified to $\mathbb{H}^2$ for convenience, see [BKS11, Remark

4.1]). Note that all the faces of the $\tilde{P}_n$ are space-like. For each face f_n of the $\tilde{P}_n$, there is an $h_n \in \text{Isom}_0(\mathbb{A}\text{d}\mathbb{S}^3)$ such that $h_n(f_n)$ lies on the same projective plane in $\mathbb{R}\mathbb{P}^3$ as the prescribed plane $r_0(\mathbb{H}^2)$.

Step 2. It is clear that $h_n(\tilde{P}_n)$ has the same combinatorics Γ as the $\tilde{P}_n$ for all n. We arbitrarily fix a triangulation say T of f and arbitrarily choose a triangle say Δ in the triangulation T of f. Then we consider the corresponding triangle Δ_n in the corresponding face $h_n(f_n)$ of the $h_n(\tilde{P}_n)$. It is clear that Δ_n falls into one of the following cases:

(1) all vertices of the Δ_n are ideal;
(2) exactly two vertices of the Δ_n are ideal;
(3) exactly one vertex of the Δ_n is ideal;
(4) all vertices of the Δ_n are strictly hyperideal.

Step 3. We consider the intersection set of the boundary of Δ_n with $\partial\mathbb{A}\text{d}\mathbb{S}^3$ and denote it by $\mathcal{I}_n$. It is clear that $\mathcal{I}_n$ has at least three points in Cases (1)-(4). Moreover, all the points of $\mathcal{I}_n$ lie on the boundary of the prescribed plane $r_0(\mathbb{H}^2)$. We arbitrarily choose three points from $\mathcal{I}_n$ in either case and they are of the form $p_n = (A_n, A_n)$, $q_n = (B_n, B_n)$, $r_n = (C_n, C_n)$, where A_n, B_n, $C_n \in \partial\mathbb{H}^2$ are distinct from each other. There exists a unique $S_n \in \text{PSL}_2(\mathbb{R})$ which takes A_n, B_n, C_n to 0, 1, ∞, respectively. Let $u_n = (S_n, S_n)$. Then $u_n \in \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \cong \text{Isom}_0(\mathbb{A}\text{d}\mathbb{S}^3)$ takes p_n, q_n, r_n to the three points $(0,0)$, $(1,1)$, (∞,∞) in $\partial\mathbb{A}\text{d}\mathbb{S}^3$.

We conclude that after applying $g_n = u_n \circ h_n \in \text{Isom}_0(\mathbb{A}\text{d}\mathbb{S}^3)$ to each $\tilde{P}_n$, the limit contains an ideal triangle on $r_0(\mathbb{H}^2)$ with vertices $(0,0)$, $(1,1)$, (∞,∞), and therefore has at least three non-colinear vertices. $\qquad\square$

Remark 4.2. *Using Lemma 4.1, henceforth, for any sequence $(P_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_\Gamma$ satisfying the assumptions in this lemma, we always choose representatives $\tilde{P}_n$ of P_n such that the sequence $(\tilde{P}_n)_{n\in\mathbb{N}}$ have a limit with at least three non-colinear vertices.*

The following gives some basic properties about the limit of a convergent sequence of polyhedra in $\tilde{\mathcal{P}}$.

Lemma 4.3. *Let $(\tilde{P}_n)_{n\in\mathbb{N}}$ be a sequence of marked non-degenerate AdS hyperideal polyhedra in $\tilde{\mathcal{P}}$ which converge to the limit $\tilde{P}_\infty \in \tilde{\mathcal{P}}$. Then the following statements holds:*

(1) If at least two vertices of the $\tilde{P}_n$ converge to a limit point say w of $\tilde{P}_\infty$, then w lies on $\partial\mathbb{A}\text{d}\mathbb{S}^3$.
(2) If w' is a vertex of $\tilde{P}_\infty$ lying outside of the closure of $\mathbb{A}\text{d}\mathbb{S}^3$, then there is exactly one vertex of the $\tilde{P}_n$ converging to w'.

Proof. It is clear that Statement (1) holds, since all the edges of the $\tilde{P}_n$ must pass through $\mathbb{A}\text{d}\mathbb{S}^3$. Now we show Statement (2). Suppose by contradiction that there are at least two vertices of the $\tilde{P}_n$ converging to w'. Then the edge connecting those two vertices of the $\tilde{P}_n$ does not pass through $\mathbb{A}\text{d}\mathbb{S}^3$ for n sufficiently large. This leads to contradiction. $\qquad\square$

4.1.3. *The dynamics of pure AdS translations.* The argument of Lemma 1.10 is based on the dynamics of a particular class of AdS isometries acting on $\mathbb{R}\mathbb{P}^3$.

More precisely, we consider the action on $\mathbb{R}\mathbb{P}^3$ of a pure translation along a complete space-like geodesic ℓ in $\mathbb{A}\text{d}\mathbb{S}^3$, say u. Let v_-, $v_+ \in \partial\mathbb{A}\text{d}\mathbb{S}^3$ denote the endpoints at infinity of ℓ such that the translation direction of u along ℓ is from v_- to v_+. Recall that $\text{Isom}_0(\mathbb{A}\text{d}\mathbb{S}^3) \cong \text{PO}_0(2,2)$. The isometry u is indeed a linear transformation of $\mathbb{R}^{2,2}$ and its action on $\mathbb{R}^{2,2}$ is deduced from its action on eigenspaces. Let P_-, P_+ denote the planes tangent to $\partial\mathbb{A}\text{d}\mathbb{S}^3$ at v_-, v_+. Note that u preserves v_-, v_+ and $\partial\mathbb{A}\text{d}\mathbb{S}^3$. One can check that

- v_- corresponds to an eigenvector of u with eigenvalue less than 1;
- v_+ corresponds to an eigenvector of u with eigenvalue greater than 1;
- the circle at infinity (in the affine chart $\mathbb{R}^3$ of $\mathbb{A}\text{d}\mathbb{S}^3$) of the tangent plane P_- or P_+ corresponds to a 2-dimensional eigenspace of u with eigenvalue 1.

As a consequence, the action of u in the affine chart $\mathbb{R}^3$ has the following description:

- u preserves the tangent planes P_- and P_+ respectively, and acts as an expanding (resp. contracting) homothety on P_- (resp. P_+) with fixed point v_- (resp. v_+);
- on the region between P_- and P_+, u acts with repulsive fixed point at v_- and attractive fixed point at v_+ (this can be viewed as a "north-south type" dynamics);
- on the complement in $\mathbb{R}^3$ of the region between P_- and P_+, the dynamics of u is similar, with each trajectory away from P_- and towards P_+.

Recall that the action on $\partial\mathbb{A}\text{d}\mathbb{S}^3$ (identified with $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$) of AdS isometries induces an identification between $\text{Isom}_0(\mathbb{A}\text{d}\mathbb{S}^3)$ and $\text{PSL}(2,\mathbb{R}) \times \text{PSL}(2,\mathbb{R})$. The isometry u has a precise formulation viewing from its action on $\partial\mathbb{A}\text{d}\mathbb{S}^3$ (see [BKS11, Lemma 5.4]). Indeed, up to conjugation, we can assume that the geodesic line ℓ has endpoints $v_- = (0,0)$ and $v_+ = (\infty,\infty)$, then u can be written as $(e^\lambda z, e^\lambda z)$, where λ is the translation

length of u. The dynamics of u has a clear picture by looking at its action on each totally geodesic plane through ℓ. In particular, u acts on the space-like plane through ℓ (identified with $\mathbb{H}^2$ in the parameterization of $\partial \mathrm{AdS}^3$ by $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$) as the hyperbolic element $z \mapsto e^\lambda z$ acting on $\mathbb{H}^2$.

4.1.4. *The proof.* We are ready to prove the properness. Recall that the gluing principle for the complexes $\mathcal{A}$ and $\mathcal{P}$ are the same: for any two graphs Γ_1, $\Gamma_2 \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$, the faces $\mathcal{A}_{\Gamma_1}$ and $\mathcal{A}_{\Gamma_2}$ (resp. $\mathcal{P}_{\Gamma_1}$ and $\mathcal{P}_{\Gamma_2}$) are glued along the face $\mathcal{A}_{\Gamma_1 \cap \Gamma_2}$ (resp. $\mathcal{P}_{\Gamma_1 \cap \Gamma_2}$) if and only if the subgraph $\Gamma_1 \cap \Gamma_2 \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$. By the definition of the topology of the complexes $\mathcal{A}$ and $\mathcal{P}$, to show the properness of the map $\Psi : \mathcal{P} \to \mathcal{A}$, it suffices to show the properness of $\Psi_\Gamma : \mathcal{P}_\Gamma \to \mathcal{A}_\Gamma$ for each $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$, where Ψ_Γ is the restriction of Ψ to $\mathcal{P}_\Gamma$.

Proof of Lemma 1.10. Let $(P_n)_{n \in \mathbb{N}}$ be a sequence of marked polyhedra in $\mathcal{P}_\Gamma$ with representatives $\tilde{P}_n$ converging to a limit say $\tilde{P}_\infty$ in $\mathbb{RP}^3$, and with the angle-assignments $\theta_n := \Psi(P_n)$ converging to a function $\theta_\infty : E(\Gamma) \to \mathbb{R}$. We need to show that if θ_∞ is γ-admissible, then the equivalence class of $\tilde{P}_\infty$ lies in $\mathcal{P}_\Gamma$ (namely, $\tilde{P}_\infty$ is non-degenerate convex hyperideal with the 1-skeleton Γ). Assume that θ_∞ is γ-admissible. We prove the lemma using the following steps.

Step 1. *We claim that $\tilde{P}_\infty$ has the same number of vertices as the $\tilde{P}_n$.*

Note that all the $\tilde{P}_n$ are convex hyperideal with each vertex passed through by a Hamiltonian cycle which separates the boundary $\partial\tilde{P}_n$ into future and past faces, the limit in $\tilde{P}_\infty$ of any vertex of the $\tilde{P}_n$ cannot lie in the interior of any face of $\tilde{P}_\infty$ (here $\tilde{P}_\infty$ has at least three non-colinear vertices by Lemma 4.1 and Remark 4.2). Suppose by contradiction that $\tilde{P}_\infty$ has fewer vertices. Then it falls into one of the following two cases:

(a) There are at least two vertices of the $\tilde{P}_n$ converging to a point say v_∞ of $\tilde{P}_\infty$.

(b) Case (a) does not occur, but there is a vertex of the $\tilde{P}_n$ whose limit lies in the interior of an edge of $\tilde{P}_\infty$.

We first discuss Case (a). By Statement (1) of Lemma 4.3, v_∞ lies on $\partial\mathrm{AdS}^3$. Moreover, v_∞ is either a vertex of $\tilde{P}_\infty$ or a point lying in the interior of an edge of $\tilde{P}_\infty$. Note that an edge of $\tilde{P}_\infty$ either passes through AdS^3 or is tangent to $\partial\mathrm{AdS}^3$, we split the discussion into the following several claims according to the possible cases of the limit edges adjacent to v_∞.

The trick is applying the dynamics of AdS pure translations (along a space-like geodesic in AdS^3) to "zoom in" the domain near v_∞ of $\tilde{P}_\infty$ so that we can reveal the asymptotically degenerate behavior of the dihedral angles at some edges (for instance, the edges adjacent to those vertices converging to v_∞) of $\tilde{P}_n$ from the degenerate geometric phenomenon near v_∞ of $\tilde{P}_\infty$. Indeed, we fix arbitrarily a complete space-like geodesic say l in AdS^3 with one endpoint at infinity v_∞, and denote by w_∞ the other endpoint at infinity of l. Let P_v (resp. P_w) be the tangent plane of $\partial\mathrm{AdS}^3$ at v_∞ (resp. w_∞). We consider a pure AdS translation say u_n in $\mathrm{Isom}_0(\mathrm{AdS}^3)$ along l with translation length $L_n > 0$, attractive fixed point w_∞ and repulsive fixed point v_∞. The dynamics of the u_n acting on $\mathbb{RP}^3$ (see Section 4.1.3) is a technical tool throughout the following proof.

Claim 1. *If all the limit edges adjacent to v_∞ pass through AdS^3, then θ_∞ does not satisfy Condition (iii).*

By carefully choosing the translation length L_n of u_n, we can control the approaching speed towards w_∞ of the images under u_n of the points disjoint from P_v in the affine chart. In particular, we choose L_n such that under the action of u_n, those vertices of the $\tilde{P}_n$ converging to v_∞ keep converging to v_∞, while all the other vertices (note that the number of the other vertices is at least two, since $\tilde{P}_\infty$ has at least two more vertices apart from v_∞ by Lemma 4.1) of $\tilde{P}_n$ go towards w_∞ in the limit. This is realizable since the number of those vertices of the $\tilde{P}_n$ converging to v_∞ is finite and all the other vertices of $\tilde{P}_n$ are disjoint from P_v, we can choose L_n with $L_n \to +\infty$ and such that the convergence speed of those vertices of the $\tilde{P}_n$ converging to v_∞ is faster than their expanding speed (towards w_∞) under u_n as n tends to infinity.

Therefore, the limit of $u_n(\tilde{P}_n)$ is a space-like line in AdS^3 with endpoints v_∞, w_∞. This divides the vertices of the Γ into two groups: one group consisting of the vertices whose correspondence in $u_n(\tilde{P}_n)$ converge to v_∞ and the other group for w_∞. We consider the closed geodesic say α_n (with respect to the induced path metric) on the boundary of $u_n(\tilde{P}_n)$ which separates the corresponding two groups of vertices of $u_n(\tilde{P}_n)$. It is clear that the length of α_n tends to zero as n tends to infinity. Up to isometries, this can be viewed as an "ideal" vertex appearing in the limit in the middle of those edges of the $u_n(\tilde{P}_n)$ intersecting α_n (see Figure 1 for instance). Let c be the simple circuit in the dual graph Γ^* that separates the two group of vertices of Γ. Similarly as the proof of Condition (ii) in Proposition 1.9, the sum of the dihedral angles of the $\tilde{P}_n$ (note that u_n preserves the marking and dihedral angles) at the edges dual to those in c tends to zero in the limit. Note that each of the two aforementioned groups separated by c consists of at least two vertices and $\tilde{P}_\infty$ is locally convex, c is a simple circuit which does not bound any face of Γ^* and exactly two of the edges in c are dual to edges of γ. This implies that the limit function θ_∞ fails to satisfy Condition (iii), which contradicts our assumption.
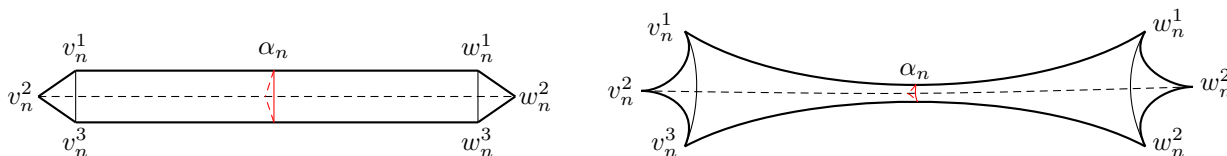
FIGURE 1. An example of $u_n(\tilde{P}_n)$ in Claim 1, with vertices v_n^1, v_n^2, v_n^3 (resp. w_n^1, w_n^2, w_n^3) ideal and converging to v_∞ (resp. w_∞). The bold lines represent the equator. The left figure is drawn in the projective model, while the right figure is a visualization of the induced (path) metric on $\partial u_n(\tilde{P}_n)$. The length of the shortest closed geodesic α_n (separating $\{v_n^1, v_n^2, v_n^3\}$) and $\{w_n^1, w_n^2, w_n^3\}$) tends to zero.

Claim 2. *If exactly one limit edge adjacent to v_∞ is tangent to $\partial \mathbb{AdS}^3$ at v_∞, then θ_∞ does not satisfy Condition (iv) and possibly Condition (i).*

Let v'_∞ denote the other endpoint of the (limit) edge tangent to $\partial \mathbb{AdS}^3$ at v_∞. It is clear that v'_∞ lies outside of the closure of $\mathbb{AdS}^3$. By Statement (2) of Lemma 4.3, exactly one vertex of the $\tilde{P}_n$ converges to v'_∞. Since $v'_\infty \neq v_\infty$ lies on the tangent plane P_v of v_∞, the dynamics of u_n (see Section 4.1.3) shows that $u_n(v'_\infty)$ will tend to a point in $\mathbb{RP}^3$, say v''_∞, at infinity in the affine chart along the direction from v_∞ to v'_∞ as the translation length $L_n \to +\infty$. Moreover, a diagonal extraction argument shows that the vertex of the $\tilde{P}_n$ that converges to v'_∞ will tend to v''_∞ under u_n whenever $L_n \to +\infty$. In particular, we choose u_n with $L_n \to +\infty$ such that under the action of u_n, those vertices of the $\tilde{P}_n$ converging to v_∞ keep converging to v_∞, the vertex of the $\tilde{P}_n$ converging to v'_∞ tends to the point v''_∞ at infinity, while all the other vertices (note that the number of the other vertices is at least one, since $\tilde{P}_\infty$ has at least one more vertex apart from v_∞ and v'_∞ by Lemma 4.1) of the $\tilde{P}_n$ go towards w_∞. Combined with the discussion in Claim 1, this is realizable, observing that all the vertices of the $\tilde{P}_n$ apart from those converging to v_∞ and v'_∞ are disjoint from P_v. Therefore, the limit of $u_n(\tilde{P}_n)$ is a half-infinite strip with one edge connecting v_∞ and w_∞, and two edges tangent to $\partial \mathbb{AdS}^3$ at v_∞ and w_∞ on one side and sharing a common vertex v''_∞ at infinity on the other side.

Up to reordering indices, let v_0 be the vertex of Γ whose correspondence in the $\tilde{P}_n$ (resp. $u_n(\tilde{P}_n)$) converges to v'_∞ (resp. v''_∞). Let $\mathcal{V}_v$ (resp. $\mathcal{V}_w$) denote the set consisting of the vertices of Γ whose correspondence in the $u_n(\tilde{P}_n)$ converge to v_∞ (resp. w_∞). We consider the edges of Γ connecting one point in $\mathcal{V}_v$ and the other point in $\mathcal{V}_w$. In particular, exactly two of those edges (say e_1, e_2 up to reordering indices) in Γ are adjacent to the two faces (say f_1, f_2 respectively) containing the vertex v_0. Let β_n denote the shortest path (with respect to the induced path metric) on the boundary of $u_n(\tilde{P}_n)$ with endpoints on the edges of the $u_n(\tilde{P}_n)$ corresponding to e_1 and e_2.

It is clear that the length of β_n tends to zero as n tends to infinity. Moreover, the two faces of the $u_n(\tilde{P}_n)$ corresponding to f_1, f_2 tend to be tangent at the limits (coincident) of the two paths (say η_n^1, η_n^2) ending in the vertex of the $u_n(\tilde{P}_n)$ corresponding to v_0 and the endpoints of β_n respectively (see Figure 2 for instance). Therefore, the endpoints of β_n tends to coincide in the limit, which implies that β_n tends to be a closed geodesic of length zero. Up to isometries, this can be viewed as an "ideal" vertex appearing in the limit in the middle of those edges of the $u_n(\tilde{P}_n)$ intersecting β_n (see Figure 2 for instance). Let c be the simple path in the dual graph Γ^* separating $\mathcal{V}_v$ and $\mathcal{V}_w$ in Γ and with endpoints dual to f_1 and f_2 respectively. Similarly as the proof of Condition (ii) in Proposition 1.9, the sum of the dihedral angles of the $u_n(\tilde{P}_n)$ (also $\tilde{P}_n$) at the edges dual to those in c tends to zero in the limit.

Note that $\mathcal{V}_v$ consists of at least two vertices by assumption. $\mathcal{V}_w$ consists of at least one vertices from the above discussion. Combined with the fact that $\tilde{P}_\infty$ is locally convex, c is a simple path which starts and ends in the boundary of the same face v_0^* (dual to v_0) in Γ^* but not contained in the boundary of v_0^*, and exactly one of the edges in c is dual to an edge of γ. Combined with the above angle result along c, the limit function θ_∞ fails to satisfy Condition (iv), which contradicts our assumption.

In particular, if $\mathcal{V}_w$ consists of only one vertex, the faces f_1 and f_2 share a common equatorial edge, say e_3 (up to reordering indices), connecting v_0 and the only vertex in $\mathcal{V}_w$. Note that the two faces of the $u_n(\tilde{P}_n)$ corresponding to f_1 and f_2 tend to be tangent at the limit of the edges of the $u_n(\tilde{P}_n)$ corresponding to e_3 (see Figure 3 for instance), which implies that the dihedral angle of the $u_n(\tilde{P}_n)$ (also $\tilde{P}_n$) at e_3 tends to zero. This implies that the limit function θ_∞ fails to satisfy Condition (i) for some equatorial edge of Γ, which again contradicts our assumption.

Claim 3. *If more than one limit edge adjacent to v_∞ are tangent to $\partial \mathbb{AdS}^3$ at v_∞, then θ_∞ does not satisfy Condition (i).*
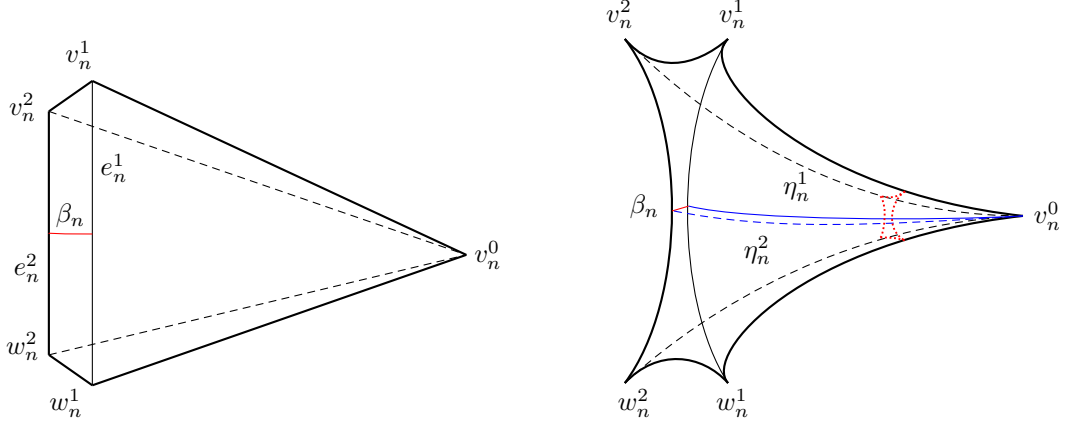
FIGURE 2. An example of $u_n(\tilde{P}_n)$ in Claim 2, with ideal vertices except for v_n^0 (which is strictly hyperideal and converges to v_∞''). The bold lines represent the equator. $\mathcal{V}_v = \{v_n^1, v_n^2\}$, $\mathcal{V}_w = \{w_n^1, w_n^2\}$. The left figure is drawn in the projective model, while the right figure is a visualization of the induced (path) metric on $\partial u_n(\tilde{P}_n)$, where the metric goes to infinity at the (dotted) circuit around v_n^0 (which represents the intersection of $\partial u_n(\tilde{P}_n)$ with $\partial \mathbb{AdS}^3$). The paths η_n^1, η_n^2 have the same limit.
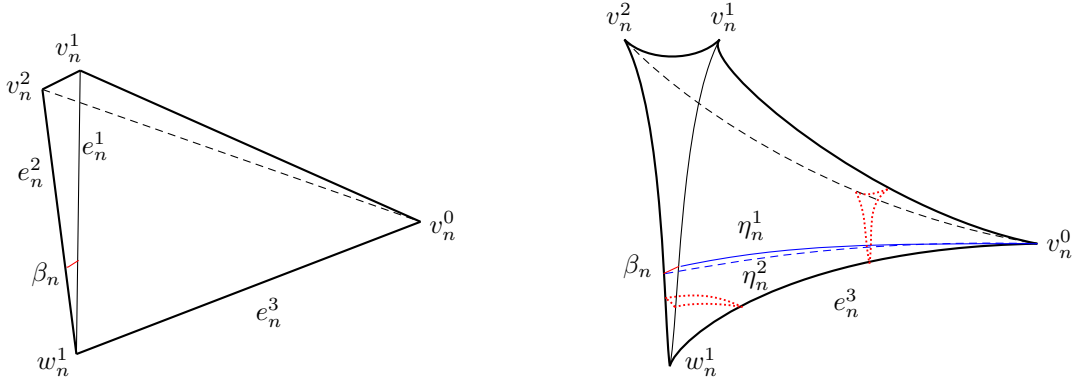

FIGURE 3. An example of $u_n(\tilde{P}_n)$ in the particular case of Claim 2, with ideal vertices except for v_n^0 and w_n^1 (which are strictly hyperideal and converge to v_∞'' and w_∞ respectively). The bold lines represent the equator. $\mathcal{V}_v = \{v_n^1, v_n^2\}$, $\mathcal{V}_w = \{w_n^1\}$. The left figure is drawn in the projective model, while the right figure is a visualization of the induced (path) metric on $\partial u_n(\tilde{P}_n)$, where the metric goes to infinity at the (dotted) circuits around v_n^0 and w_n^1 respectively (whose union are the intersection of $\partial u_n(\tilde{P}_n)$ with $\partial \mathbb{AdS}^3$). The paths η_n^1, η_n^2 and the edge e_n^3 have the same limit.

Let v_∞^l, v_∞^r be the endpoints (other than v_∞) of two limit edges of $\tilde{P}_\infty$ tangent to $\partial \mathbb{AdS}^3$ at v_∞. One can check that v_∞^l, v_∞^r lie on different sides of v_∞ and share the same line with v_∞. Otherwise, if they lie on the same side, then the edge connecting them lies outside of the closure of $\mathbb{AdS}^3$. If v_∞^l, v_∞^r and v_∞ are not colinear, then the face of $\tilde{P}_\infty$ containing the two tangent edges lies in a light-like plane. This contradicts the assumption that the limit of the angle-assignments θ_n exists on each edge of the $\tilde{P}_n$. Therefore, there are exactly two limit edges adjacent to v_∞ and they are contained in an edge of $\tilde{P}_\infty$ tangent to $\partial \mathbb{AdS}^3$ at v_∞.

Similar to the argument in Claim 2, using the dynamics of u_n with the translation length $L_n \to +\infty$, the (only) vertex of the $\tilde{P}_n$ converging to v_∞^l (resp. v_∞^r) converges to the same point say z_∞ in $\mathbb{RP}^3$ at infinity in the affine chart along the direction from v_∞ to v_∞^l (resp. v_∞^r), noting that v_∞^l and v_∞^r lie on the same line tangent to $\partial \mathbb{AdS}^3$ at v_∞. In particular, we choose u_n with $L_n \to +\infty$ such that under the action of u_n, the vertices of the $\tilde{P}_n$ converging to v_∞ keep converging to v_∞, the vertices of the $\tilde{P}_n$ converging to v_∞^l or v_∞^r tend to z_∞, while all the other vertices (note that the number of the other vertices is at least one, since $\tilde{P}_\infty$ has at least one more vertex in addition to v_∞^l and v_∞^r by Lemma 4.1) of the $\tilde{P}_n$ converge to w_∞. As discussed before, this is realizable, observing that all the vertices of the $\tilde{P}_n$ apart from those converging to v_∞, v_∞^l and v_∞^r are disjoint from P_v. Therefore, the limit of $u_n(\tilde{P}_n)$ is a bi-infinite strip with two infinite edges tangent to $\partial \mathbb{AdS}^3$ at v_∞ and w_∞ and sharing a common vertex z_∞ at infinity.
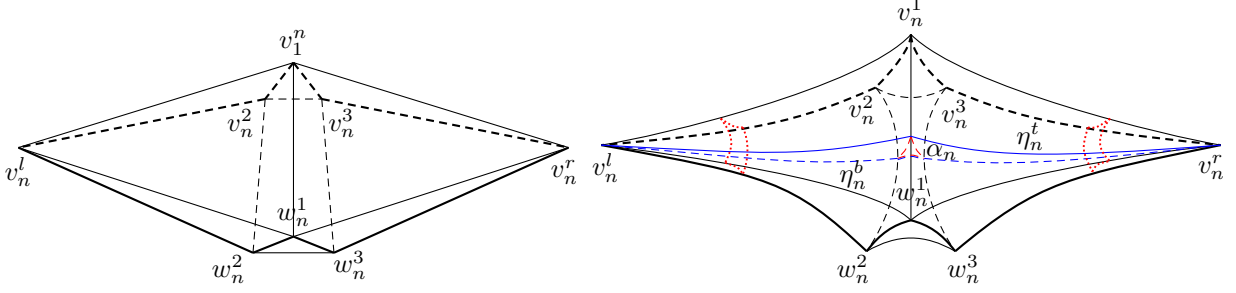
FIGURE 4. An example of $u_n(\tilde{P}_n)$ in Claim 3, with ideal vertices except for v_n^l and v_n^r (which are strictly hyperideal and converge to v_∞^l and v_∞^r). The bold lines represent the equator. $\mathcal{V}_v = \{v_n^1, v_n^2, v_n^3\}$, $\mathcal{V}_w = \{w_n^1, w_n^2, w_n^3\}$. The left figure is drawn in the projective model, while the right figure is a visualization of the induced (path) metric on $\partial u_n(\tilde{P}_n)$, where the metric goes to infinity at the (dotted) circuits around v_n^l and v_n^r respectively (whose union are the intersection of $\partial u_n(\tilde{P}_n)$ with $\partial \mathbb{A}d\mathbb{S}^3$). The paths η_n^t, η_n^b have the same limit.

We keep the notation $\mathcal{V}_v$ and $\mathcal{V}_w$ as in Claim 2 and consider the edges of Γ with one endpoint in $\mathcal{V}_v$ and the other in $\mathcal{V}_w$, collected by the set E_{vw}. From the discussion above, we have that $\mathcal{V}_v$, $\mathcal{V}_w$ are non-empty and hence E_{vw} is non-empty. For each edge in E_{vw}, the correspondence in the $u_n(\tilde{P}_n)$ is a non-equatorial edge contained either in the top or in the bottom (delimited by the equator) of the boundary of $u_n(\tilde{P}_n)$. Moreover, we denote by S_n^t (resp. S_n^b) the future-directed (resp. past-directed) pleated space-like surface which is the union of the faces of $u_n(\tilde{P}_n)$ containing an edge corresponding to an element in E_{vw} on the top (resp. bottom). Let v_n^l (resp. v_n^r) denote the vertex of $u_n(P_n)$ converging to v_∞^l (resp. v_∞^r).

Note that $u_n(\tilde{P}_n)$ can be decomposed into three hyperideal polyhedra: one (say $\tilde{P}_n^m$) has the vertex set $\mathcal{V}_v \cup \mathcal{V}_w$, the other two (say $\tilde{P}_n^l$ and $\tilde{P}_n^r$) are the polyhedra in the complement in $u_n(\tilde{P}_n)$ of $\tilde{P}_n^m$ containing the vertex v_n^l and v_n^r, respectively. The analysis for $\tilde{P}_n^m$ is exactly the same as in Claim 1, while the analysis for $\tilde{P}_n^l$ and $\tilde{P}_n^r$ are exactly the same as in Claim 2. Indeed, let α_n be the shortest closed geodesic on $\partial \tilde{P}_n^m$ separating the vertices of $\tilde{P}_n^m$ corresponding to those in $\mathcal{V}_v$ and $\mathcal{V}_w$ (note that α_n is therefore orthogonal to the edges of $\tilde{P}_n^m$ corresponding to those in E_{vw}). As shown in Claim 1, the length of α_n tends to zero in the limit. Besides, let $\eta_n^{l,t}$ (resp. $\eta_n^{l,b}$) be the path on $\tilde{P}_n^l$ connecting the vertex v_n^l and the vertices of α_n lying on $S_n^t \cap \tilde{P}_n^l$ (resp. $S_n^b \cap \tilde{P}_n^l$). As shown in Claim 2, $S_n^t \cap \tilde{P}_n^l$ (resp. $S_n^b \cap \tilde{P}_n^l$) tend to be tangent along the limits (coincident) of the two paths $\eta_n^{l,t}$ and $\eta_n^{l,b}$. Similarly, we have the analogous result for $\tilde{P}_n^r$. Now we consider the path η_n^t (resp. η_n^b) on the S_n^t (resp. S_n^b) of $u_n(\tilde{P}_n)$ connecting v_n^l and v_n^r, and passing through the vertices of α_n lying on S_n^t (resp. S_n^b). Combining the above analysis on $\tilde{P}_n^m$, $\tilde{P}_n^l$ and $\tilde{P}_n^r$, the surfaces S_n^t and S_n^b tend to be tangent along the limits (coincident) of the paths η_n^t and η_n^b (see Figure 4 for instance). This implies that the dihedral angles of $u_n(\tilde{P}_n)$ (also $\tilde{P}_n$) at the edges corresponding to elements in E_{vw} tend to zero. As a consequence, the limit function θ_∞ fails to satisfy Condition (i) for some non-equatorial edge of Γ, which contradicts our assumption.

Now we discuss Case (b). Assume that a sequence $(v_n)_{n \in \mathbb{N}}$ of vertices of $\tilde{P}_n$ converge to a point v_∞ lying in the interior of an edge (say e_∞) of the $\tilde{P}_\infty$. It is not hard to see that e_∞ is tangent to $\partial \mathbb{A}d\mathbb{S}^3$ at v_∞. Moreover, there are two edges of $\tilde{P}_n$ adjacent to v_n converging to the half-edges contained in e_∞ lying on the left and right-hand sides of v_∞. The main idea for Claim 3 is still valid in this case. Indeed, it suffices to find a non-empty edge set E_{vw}. Namely, we need to show that the corresponding sets $\mathcal{V}_v$ and $\mathcal{V}_w$ are non-empty. This is true by Lemma 4.1 and an analogous analysis. Hence, θ_∞ does not satisfy Condition (i) for some non-equatorial edge of Γ. which contradicts our assumption.

Step 2. $\tilde{P}_\infty$ *is non-degenerate, convex and hyperideal.*

By assumption, θ_∞ is γ-admissible. In particular, it satisfies Condition (i), which implies that $\tilde{P}_\infty$ is non-degenerate. By Step 1, $\tilde{P}_\infty$ contains more than one vertex, combined with the local convexity, it is therefore convex. It remains to show that $\tilde{P}_\infty$ is hyperideal. By Lemma 4.3, each vertex of $\tilde{P}_\infty$ lies either outside of the closure of $\mathbb{A}d\mathbb{S}^3$ or on $\partial \mathbb{A}d\mathbb{S}^3$. It suffices to show that each edge of P_∞ passes through $\mathbb{A}d\mathbb{S}^3$. Suppose that $\tilde{P}_\infty$ has a limit vertex say v_∞ at which at least one adjacent edge is tangent to $\partial \mathbb{A}d\mathbb{S}^3$. Similar to the Claim 2 and Claim 3 in Step 1, we split the discussion into one of the following two cases:

- Exactly one limit edge adjacent to v_∞ is tangent to $\partial \mathbb{A}d\mathbb{S}^3$ at v_∞. Note that the main idea for Claim 2 still works. Indeed, in this case the vertex set $\mathcal{V}_v$ consists of only one vertex, while $\mathcal{V}_w$ consists of at

least two vertices. By switching the notation $\mathcal{V}_v$ and $\mathcal{V}_w$ in the current case, it falls exactly into the particular case in Claim 2. As a consequence, θ_∞ fails to satisfy Condition (iv) and Condition (i) for some equatorial edge of Γ.
- More than one limit edge adjacent to v_∞ are tangent to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ at v_∞. Similarly as the Case (b) above, using an analogous argument of Claim 3, it follows that θ_∞ does not satisfy Condition (i) for some non-equatorial edge of Γ.

Combining the above two cases, we conclude that no edge of $\tilde{P}_\infty$ is tangent to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$. As a consequence, $\tilde{P}_\infty$ is hyperideal. The Step 2 is done.

Step 3. $\tilde{P}_\infty$ *has the same combinatorics* Γ *as the* P_n.

Combining Step 1 and Step 2, no limit edge in $\tilde{P}_\infty$ is reduced to a single point, no limit face in $\tilde{P}_\infty$ is collapsed to a point or an edge, and no limit face in $\tilde{P}_\infty$ is tangent to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$. Moreover, the assumption ensures that the limit of the dihedral angle of $\tilde{P}_n$ at any edge is non-zero and preserves the sign (either positive or negative). Therefore, two vertices v, w of Γ are connected by an edge if and only if the corresponding vertices v_∞, w_∞ of $\tilde{P}_\infty$ are connected by a limit edge. Furthermore, an edge of Γ is contained in the γ (resp. top, bottom) if and only if the corresponding limit edge in $\tilde{P}_\infty$ lies on the equator (resp. the future-directed faces, past-directed faces). The Step 3 is done.

Combining Steps 1 to 3, Lemma 1.10 follows. $\qquad\square$

Remark 4.4. *The "dynamical" argument also works in the hyperbolic hyperideal setting (see [BB02, Proposition 21]). Using the dynamics of a pure hyperbolic translation (along a complete geodesic in the projective model of $\mathbb{H}^3$) acting on $\mathbb{RP}^3$, we provide a new perspective to detect the contradictions against the assumptions (i.e. Conditions (1),(3),(4) in Section 1.5) in the argument of the properness of the parameterization map of non-degenerate hyperbolic hyperideal polyhedra (up to isometries) in terms of dihedral angles. Besides, the method also works for the AdS ideal setting (see [DMS14, Lemma 1.14]), in which the contradiction is against Condition (iii), corresponding exactly to Claim 1 in the above argument.*

4.2. **Properness of Φ.** In this subsection, we show that the map $\Phi : \bar{\mathcal{P}} \to \mathcal{T}_{0,N}$ which assigns to each $P \in \bar{\mathcal{P}}$ the induced metric on ∂P is proper.

Recall that the complexes $\bar{\mathcal{P}}$ and $\mathcal{T}_{0,N}$ share the same gluing principle: for any two signatures $(\epsilon_1,...,\epsilon_N)$, $(\epsilon'_1,...,\epsilon'_N) \in \{0,+\}^N$ of marked vertices (resp. marked points on $\Sigma_{0,N}$), the faces $\mathcal{P}^{(\epsilon_1,...,\epsilon_N)}$ and $\mathcal{P}^{(\epsilon'_1,...,\epsilon'_N)}$ (resp. $\mathcal{T}_{0,N}^{(\epsilon_1,...,\epsilon_N)}$ and $\mathcal{T}_{0,N}^{(\epsilon'_1,...,\epsilon'_N)}$) are glued along the face corresponding to the common subsignature of $(\epsilon_1,...,\epsilon_N)$ and $(\epsilon'_1,...,\epsilon'_N)$. By the definition of the topology of the complexes $\bar{\mathcal{P}}$ and $\mathcal{T}_{0,N}$, to show the properness of the map $\Phi : \bar{\mathcal{P}} \to \mathcal{T}_{0,N}$, it is enough to show the properness of $\Phi^{(\epsilon_1,...,\epsilon_N)} : \bar{\mathcal{P}}^{(\epsilon_1,...,\epsilon_N)} \to \mathcal{T}_{0,N}^{(\epsilon_1,...,\epsilon_N)}$ for each $(\epsilon_1,...,\epsilon_N) \in \{0,+\}^N$, where $\Phi^{(\epsilon_1,...,\epsilon_N)}$ is the restriction of Φ to $\mathcal{P}^{(\epsilon_1,...,\epsilon_N)}$.

Proof of Lemma 1.14. Let (P_n) be a sequence of marked polyhedra in $\mathcal{P}^{(\epsilon_1,...,\epsilon_N)}$ with representatives $\tilde{P}_n$ converging to a limit say $\tilde{P}_\infty$ in $\mathbb{RP}^3$, and with the induced metric $h_n := \Phi(P_n)$ on $\partial\tilde{P}_n \cap \mathbb{A}\mathrm{d}\mathbb{S}^3$ converging to a metric h_∞ on $\partial\tilde{P}_\infty \cap \mathbb{A}\mathrm{d}\mathbb{S}^3$. We need to show that if $h_\infty \in \mathcal{T}_{0,N}^{(\epsilon_1,...,\epsilon_N)}$, then the equivalence class of $\tilde{P}_\infty$ lies in $\mathcal{P}^{(\epsilon_1,...,\epsilon_N)}$ (namely, $\tilde{P}_\infty$ is convex hyperideal with vertex signature $(\epsilon_1,...,\epsilon_N)$). Assume that $h_\infty \in \mathcal{T}_{0,N}^{(\epsilon_1,...,\epsilon_N)}$, we prove this lemma using the following steps.

Step 1. P_∞ *is convex in* $\mathbb{RP}^3$.

As discussed before, $\tilde{P}_\infty$ is a locally convex (possibly degenerate) polyhedron in $\mathbb{RP}^3$. Suppose by contradiction that P_∞ is not convex, then it is projective equivalent to an infinite prism (or strip) in an affine chart $\mathbb{R}^3 \subset \mathbb{RP}^3$. Then the boundary surface $\partial\tilde{P}_\infty \cap \mathbb{A}\mathrm{d}\mathbb{S}^3$ with the induced metric is isometric to a complete hyperbolic annulus or a bi-infinite strip, and therefore is not an element of $\mathcal{T}_{0,N}$ for $N \geq 3$, which leads to contradiction.

Step 2. P_∞ *is hyperideal.*

It is clear that any vertex of $\tilde{P}_\infty$ is disjoint from $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Suppose by contradiction that there is at least one edge of $\tilde{P}_\infty$ which does not pass through $\mathbb{A}\mathrm{d}\mathbb{S}^3$, then this edge must be tangent to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ at a point, say q. One can check that any (small) closed region in (the intersection with $\mathbb{A}\mathrm{d}\mathbb{S}^3$ of) $\partial\tilde{P}_\infty$ near q (with the induced metric) is not compact and it is neither isometric to a cusp nor isometric to a funnel. This implies that $h_\infty \notin \mathcal{T}_{0,N}$ for $N \geq 3$, which contradicts the assumption.

Step 3. $\tilde{P}_\infty$ *has the same number of vertices as the* $\tilde{P}_n$ *and its vertex signature is* $(\epsilon_1,...,\epsilon_N)$.

It is easy to check that the signature assigned to an AdS hyperideal polyhedron P at a vertex v_i is the same as the signature assigned to the induced metric on $\partial P \cap \mathbb{A}d\mathbb{S}^3$ (which is a complete hyperbolic metric on $\Sigma_{0,N}$) at the puncture p_i corresponding to v_i for $1 \le i \le N$, where N is equal to the number of vertices of P. Combined with Step 1 and Step 2, $\tilde{P}_\infty$ has N vertices and the vertex signature is $(\epsilon_1, ..., \epsilon_N)$, equal to the signature of the induced metric h_∞.

Combining the above three steps, the lemma follows.   □

5. Rigidity

In this section we prove Lemma 1.11 and Lemma 1.15.

5.1. **The infinitesimal Pogorelov map.** We first recall the definition of the infinitesimal Pogorelov map and its key properties (see [Sch98, Definition 5.6, Proposition 5.7] and [Fil11, Section 3.3] in particular for the related proofs, and also [Izm09, LS00, Sch01] for relevant references). As an adaption of the infinitesimal version of a remarkable map introduced by Pogorelov [Pog73] to solve rigidity questions in spaces of constant curvature, this turns out to be an important tool that translates infinitesimal rigidity questions for polyhedra (or submanifolds) in constant curvature pseudo-Riemannian space-forms to those in flat spaces (see e.g. [DMS14, IS10]).

Choose an affine chart $x_4 = 1$ and denote by H_∞ the projective plane in $\mathbb{RP}^3$ which contains the totally geodesic space-like hyperplane in $\mathbb{A}d\mathbb{S}^3$ at infinity (with respect to the chosen affine chart). Then the dual of H_∞, defined as $x_0 := H_\infty^\perp = \{[y] \in \mathbb{RP}^3 : \langle y, x \rangle_{2,2} = 0 \text{ for all } x \in H_\infty\}$ is contained in $\mathbb{A}d\mathbb{S}^3$ and it is exactly the origin in this affine chart $\mathbb{R}^3$. Let $\mathbb{R}^{2,1}$ denote the 3-dimensional Minkowski space, which is the vector space $\mathbb{R}^3$ endowed with the metric induced from the bilinear form $\langle x, y \rangle_{2,1} = x_1 y_1 + x_2 y_2 - x_3 y_3$. Recall that an affine chart $\mathbb{R}^3$ can be equipped with the HS metric, the Euclidean metric and the Minkowski metric.

Let $C(x_0)$ denote the union of all light-like geodesic rays in $\mathbb{A}d\mathbb{S}^3$ starting from x_0 (not containing x_0). We call it the *light cone* at x_0. Let $U = \mathbb{RP}^3 \setminus H_\infty$. Then U is exactly the above-mentioned affine chart $\mathbb{R}^3$ of the projective space $\mathbb{RP}^3$. Let $\iota : U \to \mathbb{R}^{2,1}$ be an inclusion of U into $\mathbb{R}^{2,1}$, which sends x_0 to the origin 0 of $\mathbb{R}^{2,1}$. It is clear that ι is an isometry at the tangent space to x_0 and it sends $C(x_0)$ to the light cone at 0 of $\mathbb{R}^{2,1}$. For any $x \in U \setminus C(x_0)$ and any vector $v \in T_x U$, write $v = v_r + v_\perp$, where v_r is tangent to the radial geodesic passing through x_0 and x, and $v_\perp$ is orthogonal to this radial geodesic. Let $\Upsilon : T(U \setminus C(x_0)) \to T\mathbb{R}^{2,1}$ be the bundle map over the inclusion ι, which is defined in the following way:

$$\Upsilon(v) = d\iota(v)$$

for all $v \in T_{x_0} U$, and

$$(7) \qquad \Upsilon(v) = \sqrt{\frac{\|\hat{x}\|_{HS}^2}{\|d\iota(\hat{x})\|_{2,1}^2}} d\iota(v_r) + d\iota(v_\perp),$$

for all $v \in T_x U$ with $x \in U \setminus C(x_0)$, where $\hat{x}$ is a unit radial vector of v_r such that $\langle \hat{x}, \hat{x} \rangle_{2,1} = \pm 1$, and $\| \cdot \|_{HS}^2$ is the squared norm with respect to the HS metric. It is worth mentioning that ι sends a radial geodesic of U (passing through x_0) to a radial geodesic in $\mathbb{R}^{2,1}$ (passing through the origin 0) of the same type (space-like, time-like and light-like), with the induced length (if non-zero) changed. This implies that the map in (7) is always well-defined.

By an adaption of the proof in [Fil11, Lemma 11], we obtain the following property of the bundle map Υ.

Lemma 5.1. *Let Z be a vector field on $U \setminus C(x_0) \subset \mathbb{R}^3$. Then Z is a Killing vector field for the HS metric if and only if $\Upsilon(Z)$ is a Killing vector field for the Minkowski metric on $\mathbb{R}^{2,1}$.*

Indeed, it follows from this lemma that the bundle map Υ, which so far is defined over $U \setminus C(x_0)$, has a continuous extension for all of U. We call this extended bundle map, still denoted by Υ, an *infinitesimal Pogorelov map*.

Now we translate the infinitesimal rigidity questions in the Minkowski 3-space $\mathbb{R}^{2,1}$ to those in the Euclidean 3-space $\mathbb{E}^3$, by considering the bundle map over the identity:

$$\Xi : T\mathbb{R}^{2,1} \to T\mathbb{E}^3,$$

which simply changes the sign of the last coordinate of a given tangent vector. It sends a Killing vector field on $\mathbb{R}^{2,1}$ to a Killing vector field on $\mathbb{E}^3$. Denote $\Pi = \Xi \circ \Upsilon : TU \to T\mathbb{E}^3$. Then Π is a bundle map over the inclusion $\tau : U \hookrightarrow \mathbb{E}^3$ and it has the following property (see e.g. [Sch98, Proposition 5.7]):

Lemma 5.2. *Let Z be a vector field on U. Then Z is a Killing vector field for the HS metric if and only if $\Pi(Z)$ is a Killing vector field for the Euclidean metric on $\mathbb{E}^3$.*

The bundle map Π is also called an *infinitesimal Pogorelov map*, which is a key tool to solve the infinitesimal rigidity problem in our case, together with the following classical theorem of Alexandrov [Ale05], which provides a strong version of the infinitesimal rigidity of convex Euclidean polyhedra (see e.g. [DMS14, IS10]).

Theorem 5.3. *Let P be a convex polyhedron in $\mathbb{E}^3$ and let V be an infinitesimal deformation of P which does not change the induced metric on ∂P at first order (possibly changing the combinatorics), then V is the restriction to the vertices of P of a global Euclidean Killing vector field.*

5.2. **Rigidity with respect to induced metrics.** This part is dedicated to proving Lemma 1.15, which states that the map $\Phi : \mathcal{P} \to \mathcal{T}_{0,N}$ is a local immersion.

Fix an affine chart of $\mathbb{RP}^3$ and let P be a polyhedron in $\mathbb{HS}^3_1$ contained in that affine chart. An *infinitesimal deformation* of P, denoted by $\dot{P}$, is the assignment to P a tangent vector in $T(\mathbb{R}^3)^V$ at the coordinate $\Theta(P) \in (\mathbb{R}^3)^V$ of its vertex set V. We say that $\dot{P}$ is a *first-order isometric deformation* of a polyhedron P if it does not change the HS structure induced on ∂P at first order or, equivalently, if there is a triangulation of the polyhedral surface ∂P which is given by a triangulation of each face (without adding new vertices) of P and a Killing vector field (for the HS metric) on each face of this triangulation, such that two Killing vector fields on two adjacent triangles coincide on the common vertices and edges, and moreover the restriction of the Killing vector field to a vertex is exactly the restriction of $\dot{P}$ to the corresponding vertex. We say an infinitesimal deformation of P is *trivial* if it is the restriction to the vertex set of P of a global Killing vector field of $\mathbb{HS}^3_1$.

To show Lemma 1.15, it suffices to consider the infinitesimal rigidity question for a (possibly degenerate) hyperideal AdS polyhedron P with respect to the induced metric, which asks whether any first-order isometric deformation of P is trivial.

We first prove the following:

Proposition 5.4. *Let P be a polyhedron in $\mathbb{HS}^3_1$ and $\dot{P}$ be an infinitesimal deformation of P. If $\dot{P}$ is a first-order isometric deformation with respect to the induced HS metric, then $\dot{P}$ is trivial.*

Proof. By assumption, there is a triangulation, say T, of the boundary surface ∂P, given by a triangulation of each face (without adding new vertices) of P and there is a Killing vector field κ_f (with respect to the HS metric) on each face f of this triangulation T, such that the restriction of κ_f to each vertex of f is equal to the restriction of $\dot{P}$ to the same vertex, and for any two faces f_1 and f_2 of T with a common edge e, the Killing vector fields κ_{f_1} and κ_{f_2} agree on the edge e. By Lemma 5.2, $\Pi(\kappa_f)$ is the restriction of a Killing vector field of $\mathbb{E}^3$ to the face $\tau(f)$ of the triangulation $\tau(T)$ of the boundary surface $\partial \tau(P)$, and for any two faces $\tau(f_1)$, $\tau(f_2)$ of $\tau(T)$ sharing an edge $\tau(e)$, $\Pi(\kappa_{f_1})$ and $\Pi(\kappa_{f_2})$ agree on the edge $\tau(e)$. Moreover, for each face $\tau(f)$ of $\tau(P)$, the restriction of $\Pi(\kappa_f)$ to the vertices of $\tau(P)$ coincides with the restriction of $\Pi(\dot{P})$ to the corresponding vertices. This implies that the infinitesimal deformation $\Pi(\dot{P})$ of $\tau(P)$ does not change the induced Euclidean metric on $\partial \tau(P)$ at first order. Combined with Theorem 5.3, $\Pi(\dot{P})$ is the restriction to the vertices of $\tau(P)$ of a global Euclidean Killing vector field Y. Using Lemma 5.2 again, $\dot{P}$ is the restriction to the vertices of P of a global Killing vector field $\Pi^{-1}(Y)$ of $\mathbb{HS}^3_1$. This implies that $\dot{P}$ is trivial. The lemma follows. $\qquad\square$

Proof of Lemma 1.15. Let $P \in \bar{\mathcal{P}}$ be a hyperideal AdS polyhedron, then the induced HS metric restricted to $\partial P \cap \mathbb{AdS}^3$ is exactly a complete hyperbolic metric on $\Sigma_{0,N}$, which determines a point in $\mathcal{T}_{0,N}$. This complete hyperbolic metric uniquely determines the induced HS metric on $\partial P \cap \mathbb{HS}^3_1$. If an infinitesimal deformation $\dot{P}$ of P preserves the hyperbolic metric on $\partial P \cap \mathbb{AdS}^3$ at first order, then $\dot{P}$ preserves the induced HS metric at first order. Combining this with Proposition 5.4, Lemma 1.15 follows. $\qquad\square$

5.3. **Rigidity with respect to dihedral angles.** In this part we will prove Lemma 1.11, which states that the map $\Psi : \mathcal{P} \to \mathbb{R}^E$ is a local immersion near a non-degenerate AdS hyperideal polyhedron P whose 1-skeleton is a subgraph of a triangulation Γ of $\Sigma_{0,N}$, where $E = E(\Gamma)$. To show this, it suffices to consider the infinitesimal rigidity question of P with respect to dihedral angles, which asks whether any infinitesimal deformation of P that preserves the dihedral angle (at each edge of P) at first order is trivial.

5.3.1. *An alternative version of the rigidity question.* Instead of directly considering the infinitesimal rigidity question with respect to dihedral angles, we translate it into an alternative version.

For each non-degenerate AdS hyperideal polyhedron P, we construct a polyhedron, say P_0, in the following way: if each vertex of P is ideal, we let $P_0 = P$; otherwise, for each strictly hyperideal vertex v, we denote by $v^\perp$ the dual plane of v and by H_v the half-space which is delimited by $v^\perp$ and does not contain v. Then we define P_0 to be the intersection of P with H_v over all strictly hyperideal vertices v. We call such P_0 the *truncated polyhedron* of P.

Indeed, the dihedral angle at each new edge of P_0 obtained from the truncations is always orthogonal (which is defined to be zero by our convention), since the dual plane (serving as a truncated plane) of a strictly hyperideal vertex v of P is orthogonal to all the faces adjacent to v. Therefore, the infinitesimal rigidity question of P with respect to dihedral angles is equivalent to the infinitesimal rigidity question of P_0 with respect to dihedral angles.

Now we consider the dual polyhedron, say P_0^*, of P_0. Recall that an edge e of P_0 connecting two vertices v, v' is dual to an edge e^* of P_0^* between the two faces v^*, $(v')^*$ dual to v, v' respectively. Therefore the (exterior) dihedral angle at each edge e of P_0 is equal to the signed length of the dual edge e^* of P_0^* (see Section 2.4). Therefore, the infinitesimal rigidity of P_0 with respect to dihedral angles is equivalent to the infinitesimal rigidity question of P_0^* with respect to the edge lengths.

The following is a key property of a first-order deformation (with respect to dihedral angles) $\dot{P}$ of P with at least one ideal vertex.

Proposition 5.5. *Let P be an AdS hyperideal polyhedron with at least one ideal vertex v and let $\dot{P}$ be an infinitesimal deformation of P. If $\dot{P}$ preserves the dihedral angle at each edge of P at first order, then the restriction of $\dot{P}$ to the vertex v is tangent to the boundary $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$.*

Proof. Suppose by contradiction that the restriction of $\dot{P}$ to the vertex v is not tangent to the boundary $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$, for instance that the vector $\dot{P}|_v$ is towards the exterior of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$, and we will show that the sum of angles at the edges adjacent to v then strictly increases (at first order). The result will clearly follow.

Since this sum of angles remains constant when $\dot{P}$ at v is tangent to the boundary, we can assume that $\dot{v}$ is orthogonal to the boundary and of unit norm in the Euclidean metric, and that $\dot{P}$ vanishes at all the other vertices.

Let $(P_t)_{t\in[0,\epsilon)}$ be a family of convex AdS hyperideal polyhedra that realize the infinitesimal deformation $\dot{P}$ of P at $t = 0$. More precisely, $P_0 = P$, the vertices v_t of P_t (corresponding to v of P) is such that $dv_t/dt_{|t=0} = \dot{v}$, and $v'_t = v'$ for any vertex $v' \neq v$ of P. Let e_1, e_2, ..., e_k denote the edges of P adjacent to v and let e_i^t denote the edges of P_t corresponding to e_i.

Since $v \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^3$,

$$(8) \qquad \sum_{i=1}^{k} \theta(e_i) = 0 \ .$$

We claim that there exists a constant $c > 0$ (depending onn P) such that for $t > 0$ small enough,

$$(9) \qquad \sum_{i=1}^{k} \theta(e_i^t) \geq ct \ .$$

Indeed, as v_t moves away from $v = v_0$ at velocity 1 (for the Euclidean metric), its dual plane v_t^* also moves towards the opposite direction at velocity 1, and therefore $v_t^* \cap P_t$ contains a disk D_t of (Euclidean) radius $c't$ centered on the line through v and parallel to $\dot{v}$, for some $c' > 0$. However the intersection $v_t^* \cap \mathbb{A}\mathrm{d}\mathbb{S}^3$ scales as $\sqrt{t}$. Therefore, the size of D_t for the AdS metric scales as $t/\sqrt{t} = \sqrt{t}$, and it follows that the area of D_t is bounded from below from ct for $t > 0$ small enough, for another constant $c > 0$ depending on c'.

By the Gauss-Bonnet formula, $\sum_{i=1}^{k} \theta(e_i^t)$ is equal to the area of $v_t^* \cap P_t$, (9) follows. The proof of the proposition then follows from this lower bound. $\qquad\square$

As a consequence, we obtain the following property of an infinitesimal deformation $\dot{P}_0^*$ of P_0^* with at least one face contained in a light-like plane.

Corollary 5.6. *Let P_0 be an AdS hyperideal polyhedron with at least one ideal vertex v and let $\dot{P}_0^*$ be an infinitesimal deformation of P_0^*. If $\dot{P}_0^*$ preserves the edge lengths of P_0^* at first order, then for any family $\{(P_0^*)_t\}_{t\in(-\epsilon,\epsilon)}$ of polyhedra with derivative $\dot{P}_0^*$ at $t = 0$, the face f_v^t of $(P_0^*)_t$ corresponding to the face f_v (dual to v) of P_0^* remains a light-like plane tangent to v at first order at $t = 0$.*

We now state a simple but crucial relation between the variation of edge lengths of a polyhedron P in $\mathbb{R}\mathbb{P}^3$ (with all vertices disjoint from $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$) and the deformation of the induced HS metric, which will be used to show Lemma 1.11.

Lemma 5.7. *Let P be a polyhedron in $\mathbb{H}\mathbb{S}_1^3$ whose vertices are all disjoint from $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$. Assume that P admits a triangulation T given by subdivising (without adding new vertices) each face. Then an infinitesimal deformation $\dot{P}$ of P that does not change the edge lengths of T at first order (with respect to the induced HS metric) is a first-order isometric deformation of P.*

Proof. Note that all the vertices of P are disjoint from $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$, therefore, for each edge e of the triangulation T, the length of e with respect to the induced HS metric is finite.

Therefore, for each edge e of T, the restriction of an infinitesimal deformation of P to the two endpoints of e completely determines the infinitesimal deformation of e. By assumption, $\dot{P}$ does not change the edge lengths of T at first order, so that, for each face f of the triangulation T, there is a unique Killing vector field κ_f on f such that the restriction of κ_f to each vertex of f is equal to the restriction of $\dot{P}$ to the same vertex. Moreover, for any two faces f_1 and f_2 of T with a common edge e, the Killing vector fields κ_{f_1} and κ_{f_2} agree on the endpoints of e and hence on e. Therefore, $\dot{P}$ is a first-order isometric deformation of P. $\qquad\square$

5.3.2. *Degenerate metrics induced on light-like planes.* In the case that P has at least one ideal vertex v, to prove the infinitesimal rigidity of P_0^* with respect to edge lengths, we need some basic facts on the degenerate metric induced on the plane v^* dual to v.

Note that v^* is a totally geodesic light-like plane tangent to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ that intersects the boundary $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ along two light-like lines meeting at v. The HS metric induced on $v^* \setminus \partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ is degenerate. Indeed, for each geodesic line ℓ contained in $v^* \setminus \partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ passing through v, the induced (pseudo) distance, say d_{HS}, between any two points p, q in ℓ is zero (see Section 2.2).

It remains to determine the pseudo-distance d_{HS} for any two points p_1 and p_2 distinct from v and lying on two distinct geodesic rays $\overline{vp_1}$ and $\overline{vp_2}$ in $v^* \setminus \partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ with the initial point v. It can be seen later that in this case it is more natural to first extend d_{HS} to a signed directed measure, say m_{HS}, that is, $m_{HS}(p_1, p_2)$ is allowed to be negative and different from $m_{HS}(p_2, p_1)$, and then define d_{HS} by determining the choice of the direction.

To clarify this, we identify the plane v^* with the real 2-dimensional vector space $\mathbb{R}^2$ (where v is identified with the origin 0) equipped with the push-forward degenerate HS metric. Let ℓ_L and ℓ_R be two lines in $\mathbb{R}^2$ through 0 with slopes 1 and -1 respectively, which are identified with the left and right leaves at the intersection of v^* with $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$. These two lines divides $\mathbb{R}^2$ into four regions, say I, II, III and IV, consecutively in the anticlockwise direction with $I := \{(x, y) \in \mathbb{R}^2 : x > |y| > 0\}$. A geodesic ray in $v^* \setminus \partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ starting from v (not containing v) is thus identified with a ray from 0 (not containing 0), called a *non-singular* ray, which is exactly contained in one of the four regions.

For any two rays ℓ_1, ℓ_2 in $\mathbb{R}^2$, we denote by $\ell_1 0 \ell_2$ a *directed angle* with the initial direction ℓ_1, the terminal direction ℓ_2, and a specified region bounded by ℓ_1 and ℓ_2 with Euclidean angle at 0 not greater than π. It is clear that the choice of this region is unique in the case that ℓ_1 and ℓ_2 are not opposite to each other. Otherwise, we arbitrarily fix one of the two regions.

We say the directed angle $\ell_1 0 \ell_2$ is *non-singular* if both ℓ_1 and ℓ_2 are non-singular. A non-singular directed angle $\ell_1 0 \ell_2$ is called a *fundamental* angle if either

(a) ℓ_1 and ℓ_2 are contained in the same region of the four, or

(b) the specified region of $\ell_1 0 \ell_2$ intersects exactly one of the two lines ℓ_L, ℓ_R and ℓ_1 is orthogonal to ℓ_2 with respect to the Minkowski metric $\langle \cdot, \cdot \rangle_{1,1}$ on $\mathbb{R}^2$.

We first define the *non-directed sectorial measure* with respect to $\langle \cdot, \cdot \rangle_{1,1}$ on $\mathbb{R}^2$ of the non-directed angle between the above two rays ℓ_1, ℓ_2, denoted by $\angle(\ell_1, \ell_2)$, in the following (see [Dza84, Definition 2] for an alternative version).

Definition 5.8. *Let ℓ_1 and ℓ_2 be two non-singular rays lying in either Case (a) or Case (b).*

- *in Case (a), if ℓ_1 and ℓ_2 lie in the same region I or III (resp. II or IV), then the measure $\angle(\ell_1, \ell_2)$ is defined to be non-positive (resp. non-negative) and satisfies that*

$$\cosh \angle(\ell_1, \ell_2) = |\langle w_1 . w_2 \rangle|,$$

  where w_i is the unit vector in the direction ℓ_i with respect to $\langle \cdot, \cdot \rangle_{1,1}$ for $i = 1, 2$.
- *in Case (b), the measure $\angle(\ell_1, \ell_2)$ is defined to be zero.*

We then define the *directed sectorial measure*, denoted by $\angle \ell_1 0 \ell_2$, of the non-singular directed fundamental angle $\ell_1 0 \ell_2$ to be $\angle \ell_1 0 \ell_2 := \angle(\ell_1, \ell_2)$ if the specified region of $\ell_1 0 \ell_2$ is obtained by an anticlockwise rotation from ℓ_1 to ℓ_2, otherwise, we define it as $\angle \ell_1 0 \ell_2 := -\angle(\ell_1, \ell_2)$. The following gives a natural way to extend the directed sectorial measure to a general non-singular directed angle $\ell_1 0 \ell_2$ (see e.g. Definition 7 in [Dza84]).

Definition 5.9. *Let $\ell_1 0 \ell_2$ be a general non-singular directed angle. The directed sectorial measure $\angle \ell_1 0 \ell_2$ of $\ell_1 0 \ell_2$ is defined by splitting the angle $\ell_1 0 \ell_2$ into successive non-overlapping non-singular directed fundamental angles and then summing the directed sectorial measures of these fundamental angles. One can directly check that this definition is independent of the choices of splittings.*

It is clear that $\angle\ell_1 0\ell_2$ is zero whenever ℓ_1 and ℓ_2 are the same, opposite or orthogonal to each other. This also explains why the choice of the specified region for the directed angle $\ell_1 0\ell_2$ can be arbitrary in the case that ℓ_1 and ℓ_2 are opposite.

Now we are ready to extend d_{HS} to a signed directed measure, say m_{HS}, for any ordered pairs of points (p_i, p_j) with p_i and p_j lying on two non-singular rays ℓ_i and ℓ_j in v^* respectively.

Definition 5.10. *Let ℓ_1, ℓ_2 be two non-singular rays in $\mathbb{R}^2$ and let p_1, p_2 be any two points lying on ℓ_1, ℓ_2. The (signed) length of the directed segment $\overline{p_1 p_2}$ in $\mathbb{R}^2$ connecting p_1 to p_2 is defined to be $m_{HS}(p_1, p_2) := \angle\ell_1 0\ell_2$.*

By definition, $m_{HS}(p_1, p_2) = -m_{HS}(p_2, p_1)$. Recall that in this subsection P has at least one ideal vertex v and let f_v be the face of P_0^* dual to the vertex v of the truncated polyhedron P_0. By duality, for each edge e of P_0 adjacent to v, the (exterior) dihedral angle of e is equal to the signed length (with respect to the HS metric induced on $\partial P_0^* \cap \mathbb{HS}_1^3$) of e^* (which is an edge of f_v). Indeed, from our convention, the dihedral angle at the edge e adjacent to two faces f_1 and f_2 of P_0 can be viewed as the directed sectorial measure of the directed angle $\ell_1 0\ell_2$ (resp. $\ell_2 0\ell_1$) with ℓ_1 and ℓ_2 the outwards-pointing rays (based on one interior point of e) orthogonal to f_1 and f_2 respectively if ℓ_2 (resp. ℓ_1) is obtained from ℓ_1 (resp. ℓ_2) by an anticlockwise rotation with Euclidean angle less than π. This directed angle induces a natural orientation of the dual edge e^* in the following way: e^* is directed from the endpoints f_2^* to f_1^* (resp. f_1^* to f_2^*) if the aforementioned directed angle at e is $\ell_1 0\ell_2$ (resp. $\ell_2 0\ell_1$).

Therefore, each edge of the face $f_v \subset v^*$ is implicitly equipped with a direction. One can check that the union of the directions over the edges of f_v indeed determine an orientation of the face f_v, which is exactly an anticlockwise rotation around ∂f_v. The signed length of e^* with respect to the induced degenerate HS metric on $\partial P_0^* \cap \mathbb{HS}_1^3$ is exactly the signed length with respect to m_{HS} of the directed edge e^* (see Definition 5.10).

The directed measure m_{HS} has a convenient property, stated here for completeness.

Claim 5.11. *Let Q be an oriented convex polygon in the light-like plane v^* with all vertices disjoint from $\partial\mathbb{AdS}^3$. Then the sum of the signed lengths (with respect to m_{HS}) over all the directed edges of Q is 0.*

Proof. Without lost of generality, we assume that the boundary of Q is anticlockwise oriented. By assumption, the ray starting from v through any vertex of Q is non-singular. Therefore, the signed length of each directed edge of Q is well-defined (see Definition 5.10). We discuss Q in the following two cases:

- Q contains the vertex v in its boundary or in its interior.
- The closure of Q is disjoint from v.

In the first case, let v_1, v_2, ..., v_k be the consecutive vertices of Q in the anticlockwise order. Denote the ray starting from v through v_i by ℓ_i for $i = 1, ..., k$. Since the vertices of Q are all disjoint from $\partial\mathbb{AdS}^3$, the rays $\ell_1, \ell_2, ..., \ell_k$ then split the whole plane v^* into k successive non-overlapping non-singular angles $\ell_1 0\ell_2$, $\ell_2 0\ell_3$, ..., $\ell_k 0\ell_1$. By Definition 5.10, the signed length of the directed edge with endpoints v_i and v_{i+1} is $\angle\ell_i 0\ell_{i+1}$ for $i = 1, ..., k$ (where $v_{k+1} = v_1$). Let ℓ_1^* deonte the ray opposite to ℓ_1. It follows from Definition 5.9 that the sum of $\angle\ell_i 0\ell_{i+1}$ over $i = 1, ..., k$ (where $\ell_{k+1} = \ell_1$) is equal to the sum of $\angle\ell_1 0\ell_1^*$ and $\angle\ell_1^* 0\ell_1$, since they share a common finer splitting of the plane v^*. Note that $\angle\ell_1 0\ell_1^* + \angle\ell_1^* 0\ell_1 = \angle\ell_1 0\ell_1^* - \angle\ell_1 0\ell_1^* = 0$. We obtain the desired result.

In the second case, we order the vertices of Q and thus the corresponding rays successively in the anticlockwise order, say $\ell_1, \ell_2, ..., \ell_k$, such that the anticlockwise rotation from ℓ_1 to ℓ_k crosses all the other rays $\ell_2, \ell_3, ...,$ ℓ_{k-1}. It is easy to check that $\angle\ell_1 0\ell_2 + \angle\ell_2 0\ell_3 + ... + \angle\ell_{k-1} 0\ell_k = \angle\ell_1 0\ell_k = -\angle\ell_k 0\ell_1$. By definition, the sum of the signed lengths over all the directed edges of Q is equal to $\angle\ell_1 0\ell_2 + \angle\ell_2 0\ell_3 + ... + \angle\ell_{k-1} 0\ell_k + \angle\ell_k 0\ell_1$ and is thus zero. $\square$

Claim 5.12. *Let v be an ideal vertex of P and let f_v be the face of P_0^* contained in the dual plane v^*. Let $T = \{T_j\}_{j=1}^m$ be a triangulation of ∂P_0^* given by a triangulation (without adding new vertices) of f_v. If an infinitesimal deformation $\dot{P}_0^*$ of P_0^* preserves the edge lengths of f_v at first order, then $\dot{P}_0^*$ preserves the lengths of all the directed edges of T_j at first order for $1 \leq j \leq m$.*

Proof. Let $v_1, ..., v_k$ be the consecutive vertices of the face f_v in the anticlockwise order and let ℓ_i be the ray starting from v through v_i for $i = 1, ..., k$. Since v_i is dual to the space-like face in $\mathbb{AdS}^3$ of P_0, v_i is contained in $\mathbb{AdS}^3$ and thus contained in $v^* \setminus \partial\mathbb{AdS}^3$. This implies that ℓ_i is a non-singular ray for all $i = 1, ..., k$. Moreover, each edge of f_v connecting v_i and v_{i+1} is equipped with an orientation from v_i to v_{i+1} (where $v_{k+1} = v_1$), compatible with the sign of the dihedral angle at its dual edge. For each T_j, we orient ∂T_j anticlockwise. This is compatible with the orientation of ∂f_v at the intersection.

Step 0. First we claim that if $\dot{P}_0^*$ preserves the (signed) lengths of two directed edges, say e_1, e_2 of a triangle say T_j at first order, then $\dot{P}_0^*$ also preserves the (signed) length of the third directed edge, say e_3, of T_j at first

order. Let v_1, v_2, v_3 denote the three vertices of T_j. By definition and our convention, the sum of the (signed) lengths of directed edges of T_j is either equal or opposite to the sum of dihedral angles at the edges of the (unbounded) polyhedron (say P_j^*) bounded by the three dual (space-like) planes $v_1^\perp$, $v_2^\perp$, $v_3^\perp$ (intersecting at a common point which is exactly the ideal vertex v) of v_1, v_2, v_3, where P_j^* is oriented with outward-pointing time-like vectors. Note that $\dot{P}_0^*$ preserves the edge lengths of f_v at first order, then the corresponding infinitesimal deformation, denoted by $\dot{P}$, of P preserves the dihedral angle at each edge of P at first order. Combined with Proposition 5.5, the restriction of $\dot{P}$ to the ideal vertex v is tangent to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$, hence $\dot{P}$ preserves the sum of dihedral angles at the edges of the (unbounded) polyhedron bounded by at least three planes (containing a face of P adjacent to v respectively) at first order. This implies that $\dot{P}$ preserves the sum of dihedral angles at the edges adjacent to the vertex v of P_j^* at first order. As a consequence, $\dot{P}_0^*$ preserves the sum of the (signed) lengths of directed edges of T_j at first order and thus preserves the (signed) length of e_3 at first order. The claim follows.

If f_v is a triangle, Claim 5.12 follows. Otherwise, we do the following procedure:

Step 1. We consider the triangles in T with exactly two edges contained in ∂f_v, denoted by $T_{n_1}, ..., T_{n_{k_1}}$. For each T_j with $j = n_1, ..., n_{k_1}$, note that $\dot{P}_0^*$ preserves the (signed) lengths of the two directed edges of T_j contained in ∂f_v at first order. By Step 0, $\dot{P}_0^*$ also preserves the (signed) length of the third directed edge of T_j at first order. On the other hand, for each edge e in the triangulation T, if $\dot{P}_0^*$ preserves the (signed) length of a directed edge underlying e at first order, then $\dot{P}_0^*$ also preserves the (signed) length of the reversely directed edge underlying e at first order.

Step 2. If $\{T_j\}_{j=1}^m \setminus \{T_{n_1}, ..., T_{n_{k_1}}\}$ is empty or the remaining triangles are those with edges either contained in ∂f_v or belonging to some triangles in $\{T_{n_1}, ..., T_{n_{k_1}}\}$, we are done. Otherwise, we consider the triangles, collected by $\{T_{n_{k_1+1}}, ..., T_{n_{k_2}}\}$, with exactly one edge neither contained in ∂f_v nor belonging to any triangle in $\{T_{n_1}, ..., T_{n_{k_1}}\}$. By the assumption of $\dot{P}_0^*$ and applying Step 0 again as above to the triangles $T_{n_{k_1+1}}, ..., T_{n_{k_2}}$, then $\dot{P}_0^*$ preserves the lengths of the third (directed) edges of those triangles at first order.

Step 3. Repeat the same procedure as Step 2, by replacing $\{T_{n_1}, ..., T_{n_{k_1}}\}$ with $\{T_{n_1}, ..., T_{n_{k_1}}, T_{n_{k_1+1}}, ..., T_{n_{k_2}}\}$. After finitely many steps, we have that $\{T_j\}_{j=1}^m \setminus \{T_{n_1}, ..., T_{n_{k_1}}, ..., T_{n_{k_N}}\}$ is either empty or the remaining triangles are those with edges either contained in ∂f_v or belonging to some triangles in $\{T_{n_1}, ..., T_{n_{k_1}}, ..., T_{n_{k_N}}\}$ for some $N \in \mathbb{N}^+$. This concludes the proof that $\dot{P}_0^*$ preserves the (signed) lengths of all the directed edges of T_j at first order for $1 \le j \le m$. $\square$

Proposition 5.13. *Let $P \in \mathcal{P}$ and let P_0^* be the dual polyhedron of the truncated polyhedron P_0 of P. Let $\dot{P}_0^*$ be an infinitesimal deformation of P_0^*. If $\dot{P}_0^*$ does not change the edge lengths of P_0^* at first order, then $\dot{P}_0^*$ is trivial.*

Proof. We discuss P according to the positions of its vertices with respect to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$.

Case 1. All the vertices of P are strictly hyperideal.

In this case, each vertex of P_0 has degree three. Therefore, each face of the polyhedron P_0^* dual to P_0 is a triangle. Moreover, all the vertices of P_0^* are disjoint from $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$. Since $\dot{P}_0^*$ does not change the edge lengths of P_0^* at first order and all the faces of P_0^* constitute a triangulation of the boundary surface ∂P_0^*, by Lemma 5.7, $\dot{P}_0^*$ is a first-order isometric deformation of P_0^*. Combined with Proposition 5.4, $\dot{P}_0$ is trivial.

Case 2. At least one vertex of P is ideal.

In this case, we still consider the polyhedron P_0^* dual to the truncated polyhedron P_0 (which coincides with P if and only if all the vertices of P are ideal). For each ideal vertex v, we denote by f_v the face of P_0^* dual to v, which is contained in a light-like plane v^*. In particular, all the edges of f_v are space-like (resp. time-like) in the intersection with $\mathbb{A}\mathrm{d}\mathbb{S}^3$ (resp. $\mathbb{A}\mathrm{d}\mathbb{S}^{3*}$) and all vertices are contained in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ and thus disjoint from $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$.

Note that f_v is not necessarily a triangle. Moreover, if P_0^* has a non-triangular face, then it must be the dual face of an ideal vertex of P_0. After taking a triangulation (without adding new vertices) of each non-triangular face of P_0^*, we obtain a triangulation of ∂P_0^*, say T. By the assumption of $\dot{P}_0^*$ and Claim 5.12, $\dot{P}_0^*$ preserves the edge lengths of the triangulation T at first order. Applying Lemma 5.7 and Proposition 5.4 again, we conclude that $\dot{P}_0^*$ is trivial. $\square$

Proof of Lemma 1.11. This follows immediately from Proposition 5.13 and the equivalence between the infinitesimal rigidity of P with respect to dihedral angles and that of P_0^* with respect to edge lengths. $\square$

This also provides an alternative method to prove the infinitesimal rigidity with respect to dihedral angles at the edges of hyperbolic hyperideal polyhedra (see [BB02]) and the infinitesimal rigidity with respect to dihedral angles at the edges of AdS ideal polyhedra (see [DMS14]).

6. Topology

In this section, we prove Proposition 1.7, and Proposition 1.8 and Proposition 1.13, concerning the topology of $\mathcal{A}$ and the dimensions of the spaces $\mathcal{P}$ and $\bar{\mathcal{P}}$. We then proceed to give the proofs of Theorem 1.6 and Theorem 1.12.

6.1. **The topology of $\mathcal{A}$.** Recall that $\mathcal{A}$ is the disjoint union of $\mathcal{A}_\Gamma$ over all $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$, glued together along faces corresponding to common subgraphs, where $\mathcal{A}_\Gamma$ is the set of γ-admissible functions on $E(\Gamma)$. Let $\mathcal{G}_N$ denote the set of the graphs in $\mathsf{Graph}(\Sigma_{0,N}, \gamma)$ which form triangulations of $\Sigma_{0,N}$. To understand the topology of $\mathcal{A}$, we first consider the topology of the space $\mathcal{A}_\Gamma$ for each $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$.

Claim 6.1. *For each $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$, $\mathcal{A}_\Gamma$ is a non-empty and contractible topological manifold of (real) dimension $|E(\Gamma)|$. As a consequence, the maximum (real) dimension $3N - 6$ is attained exactly when $\Gamma \in \mathcal{G}_N$, that is, when Γ is the 1-skeleton of a triangulation.*

Proof. Note that $\mathcal{A}_\Gamma$ can be described as the space of the solutions of a system of linear equations and inequalities with variables (corresponding to the edges of Γ) determined by Conditions (i)-(iv) in Definition 1.2. By the definition of Conditions (i)-(iv), the solution space is given by the intersection of finitely many linear half-spaces in $\mathbb{R}^{|E(\Gamma)|}$ (whose boundary contains the origin) corresponding to those conditions.

Moreover, the compatibility among Conditions (i)–(iv) ensures that this intersection is non-empty and indeed has dimension $|E(\Gamma)|$: to find a solution of Conditions (i)–(iv), one can first choose the value of the angles on the equator, and then choose any set of large enough angles on the non-equatorial edges.

Observe that if θ is such a solution, then for any $t > 0$, $t\theta$ is also a solution. Therefore, $\mathcal{A}_\Gamma$ is a $|E(\Gamma)|$-dimensional convex cone (without containing the cone point zero) in $\mathbb{R}^{|E(\Gamma)|}$. The lemma follows. $\square$

We call a triangulation of $\Sigma_{0,N}$ (with vertices located at the marked points of $\Sigma_{0,N}$) *admissible* if it belongs to $\mathcal{G}_N$. Using an elementary move (the so-called flip) among the triangulations in $\mathcal{G}_N$ (see e.g. [HN99], [Pen87, Section 7]), we define a graph of admissible triangulations of $\Sigma_{0,N}$ as follows.

Definition 6.2. *Let $G_T(\Sigma_{0,N})$ denote the graph of admissible triangulations on $\Sigma_{0,N}$, with vertex set $\mathcal{G}_N$ and an edge connecting two vertices, say Γ_1 and Γ_2, if they differ by a single flip: Γ_2 is obtained from Γ_1 by replacing one (non-equatorial) edge say e of Γ_1 by the other diagonal in the quadrilateral which is the union of the two triangles adjacent to e.*

We first discuss the connectivity of the graph $G_T(\Sigma_{0,N})$, which is closely related to the connectivity of $\mathcal{A}$. To see this, we need some preliminary lemmas.

For convenience, we divide the non-equatorial edges of a graph $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$ into two classes: *top edges* and *bottom edges*, in which "top" (resp. "bottom") means that the edges lie on the left (resp. right) side of the equator (considered as an oriented curve) of $\Sigma_{0,N}$. We denote by Γ^t (resp. Γ^b) the restriction of $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$ to the union of the top (resp. bottom) and the equator of $\Sigma_{0,N}$, called the *top subgraph* (resp. *bottom subgraph*) of Γ. Note that by our convention a top (or bottom) edge of a graph $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$ is always non-equatorial, while the top and bottom subgraphs Γ^t, Γ^b of a graph $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$ always contain the equator of Γ.

In particular, the top (resp. bottom) subgraph of an admissible triangulation $\Gamma \in \mathcal{G}_N$ is a triangulation of the closure of the top (resp. bottom) of $\Sigma_{0,N}$. Conversely, given a triangulation T_1 of the closure of the top of $\Sigma_{0,N}$ and a triangulation T_2 of the closure of the bottom of $\Sigma_{0,N}$, the union $T_1 \cup T_2$ is naturally a triangulation of $\Sigma_{0,N}$. However, $T_1 \cup T_2$ is not necessarily admissible.

The following lemma gives a criterion for determining whether a given triangulation of $\Sigma_{0,N}$ with vertices located at the marked points and the edge set containing all the equatorial edges is admissible.

Lemma 6.3. *Let Γ be a triangulation of $\Sigma_{0,N}$ ($N \geq 4$) with vertices located at the marked points and the edge set containing all the equatorial edges. Then Γ is admissible if and only if there is no pair of vertices (on the equator) that are connected by both a top edge and a bottom edge of Γ.*

Proof. We first show the necessity. Suppose by contradiction that there are two vertices say v_1, v_2 that are connected by a top edge and a bottom edge of Γ. After removing v_1, v_2 and their adjacent edges, the resulting graph is divided into two components. Then Γ is not 3-connected and is thus not admissible. The necessity follows.

Now we show the sufficiency. Assume that there is no pair of vertices (on the equator) that are connected by both a top edge and a bottom edge of Γ. We need to show that Γ is admissible. Note that Γ is obviously planar. It suffices to show that Γ is 3-connected. Let $\Gamma_{v_1 v_2}$ denote the graph obtained from Γ by removing the vertices v_1, v_2 of Γ and their adjacent edges. We divide the discussion into the following two cases:

(A) No top edge of Γ connects v_1 to $\gamma_1 \sqcup \gamma_2$.   (B) $w_1 = v_{21}$.   (C) $w_1 \neq v_{21}$
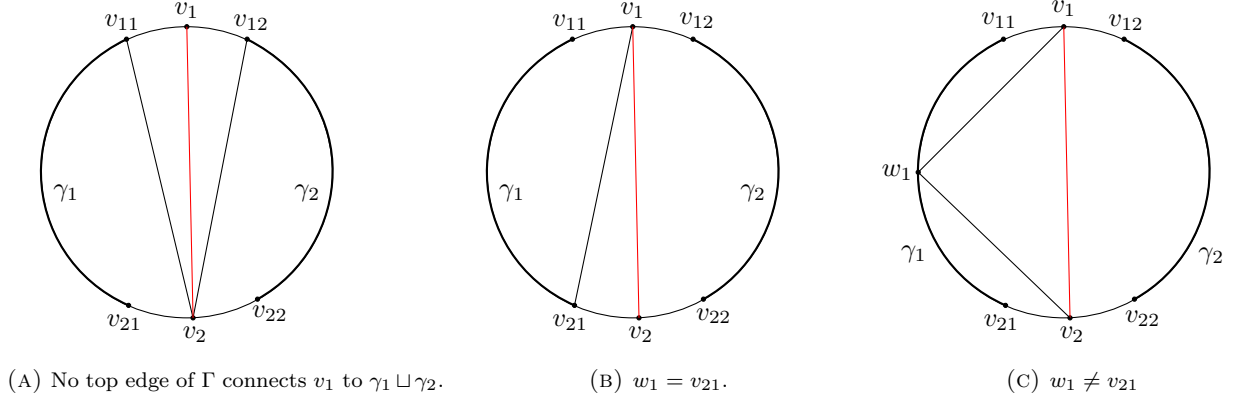
FIGURE 5. The examples of the top subgraph of Γ in Case 2 of Lemma 6.3 (partially drawn, with the other vertices and edges hidden), where the circle represents the equator of $\Sigma_{0,N}$, the bold lines represent the two components γ_1, γ_2 of the restriction of $\Gamma_{v_1v_2}$ to the equator.

Case 1. If v_1, v_2 are adjacent along the equator. We claim that $\Gamma_{v_1v_2}$ is connected. Indeed, noting that $N \geq 4$, the restriction (say γ_0) of $\Gamma_{v_1v_2}$ to the equator remains non-empty and connected, so that the top and bottom subgraphs of $\Gamma_{v_1v_2}$ can be connected through γ_0. The claim follows.

Case 2. If v_1, v_2 are not adjacent along the equator, then, again by the hypothesis that $N \geq 4$, the restriction of $\Gamma_{v_1v_2}$ to the equator has two components, say γ_1 and γ_2. We claim that there must be a top edge or a bottom edge of $\Gamma_{v_1v_2}$ connecting γ_1 and γ_2. This will imply directly that $\Gamma_{v_1v_2}$ is connected. We just need to prove the claim.

Suppose by contradiction that there is neither top edge nor bottom edge of $\Gamma_{v_1v_2}$ connecting γ_1 and γ_2. We are going to show that there must be a top edge and a bottom edge of Γ connecting v_1 to v_2.

We first show that there is a top edge of Γ connecting v_1 to v_2. Let v_{i1}. v_{i2} denote the two vertices on γ_1, γ_2 adjacent to v_i for $i = 1, 2$. Note that v_{11} and v_{21} (resp. v_{12} and v_{22}) might coincide. By the assumption above, there is no top edge of $\Gamma_{v_1v_2}$ connecting γ_1 and γ_2. Therefore, there is at least one top edge of Γ connecting v_1 to some vertex of $\gamma_1 \sqcup \gamma_2 \sqcup \{v_2\}$. Otherwise, since Γ is a triangulation, then there is a top edge of Γ connecting v_{11} to v_{12} (thus γ_1 to γ_2). It is clear that this top edge also belongs to $\Gamma_{v_1v_2}$, which contradicts the assumption.

If there is no top edge of Γ connecting v_1 to $\gamma_1 \sqcup \gamma_2$, then v_1 is connected to v_2 by a top edge of Γ (see Figure 5a for instance). Otherwise, there is at least one top edge of Γ connecting v_1 to $\gamma_1 \cup \gamma_2$. Without lost of generality, we assume that there is at least one top edge of Γ connecting v_1 to γ_1. Let w_1 denote the vertex on γ_1 which is connected to v_1 by a top edge of Γ and is closest to v_2 along γ_1 away from v_1.

If $w_1 = v_{21}$, note that by assumption $w_1 \in \gamma_1$ cannot be connected to any vertex of γ_2 and also that Γ is a triangulation, then v_1 is connected to v_2 by a top edge of Γ (as shown in Figure 5b).

Otherwise $w_1 \neq v_{21}$. Note that by assumption there is no top edge connecting γ_1 to γ_2 and there is no top edge connecting v_1 to any vertex (distinct from w_1) in γ_1 between w_1 and v_2 away from v_1. Since Γ is a triangulation, then there must be a top edge connecting v_1 to v_2 (as indicated in Figure 5c).

This concludes that there is a top edge of Γ connecting v_1 to v_2. Applying an analogous argument, we can show that there is also a bottom edge of Γ connecting v_1 to v_2. This contradicts our assumption at the beginning of Case 2. Therefore the sufficiency follows. $\square$

With this criterion, we have the following results, which are crucial to prove the connectedness of $G_T(\Sigma_{0,N})$.

Observation 6.4. *Let $\Gamma \in \mathcal{G}_N$ be an admissible triangulation of $\Sigma_{0,N}$ and T be a triangulation of the closure of the top of $\Sigma_{0,N}$ with $N \geq 4$. Assume that T and the top subgraph Γ^t of Γ differ by one flip. Then there is at most one pair of vertices (on the equator) connected by both an edge of T and a bottom edge of Γ.*

Proof. Note that since Γ is admissible, by Lemma 6.3, no top edge and bottom edge of Γ share two common endpoints. Note also that T is obtained from Γ^t by a single flip, thus is distinct from Γ^t only at the edge say e' which is obtained from Γ^t by a flip. Therefore, the endpoints of the edge e' of T are the only possible pair of vertices on the equator that are also shared with a bottom edge of Γ. The desired result follows. $\square$

Observation 6.5. *Let $\Gamma \in \mathcal{G}_N$ be an admissible triangulation of $\Sigma_{0,N}$ and T be a triangulation of the closure of the top of $\Sigma_{0,N}$ with $N \geq 5$. Assume that T and Γ^t differ by one flip, and that there is a pair of vertices (on the equator) connected by both a bottom edge (say e) of Γ and an edge (say e') of T. Denote by $(\Gamma^b)_e$*
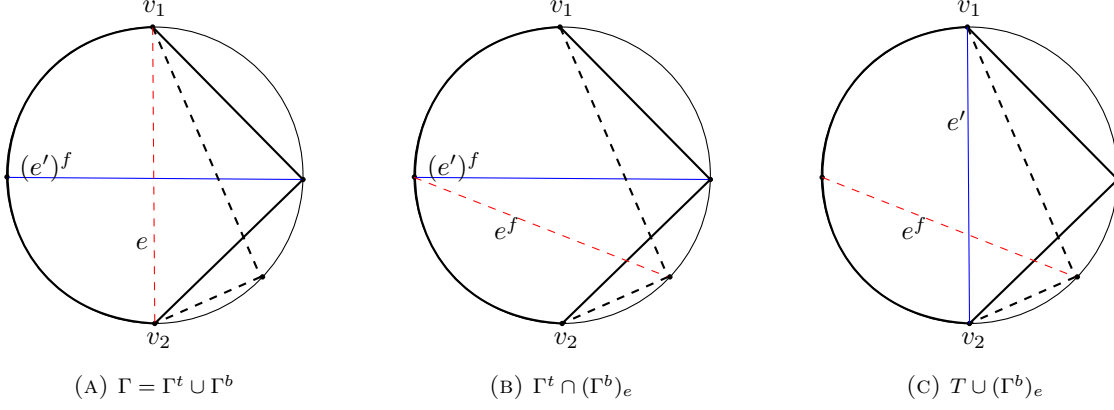
FIGURE 6. The examples of three relevant triangulations of $\Sigma_{0,N}$ in the proof Observation 6.5 (partially drawn, with the other vertices and edges hidden), where the circle represents the equator of $\Sigma_{0,N}$ and the left semicircle has exactly one vertex between v_1 and v_2, the solid (resp. dashed) lines represent the top (resp. bottom) edges, the region bounded by the left semicircle and bold solid (resp. dashed) lines represents the quadrilateral q^t (resp. q^b).

the triangulation obtained from Γ^b by a flip at e. Then the triangulations $T \cup (\Gamma^b)_e$ and $\Gamma^t \cup (\Gamma^b)_e$ are both admissible.

Proof. By our assumption and the argument of Observation 6.4, the endpoints (say v_1, v_2) of the edges e and e' are the only pair of vertices (on the equator) connected by both a bottom edge of Γ and an edge of T. Moreover, the edge e' of T is exactly the edge obtained from Γ^t by a flip. Let $(e')^f$ denote the edge of Γ^t corresponding to e' by a flip and e^f denote the edge of $(\Gamma^b)_e$ corresponding to e by a flip. It is clear that $(e')^f$ and e^f are both non-equatorial.

Note that since $(e')^f \subset \Gamma^t$ and $e' \subset T$ (resp. $e \subset \Gamma^b$ and $e^f \subset (\Gamma^b)_e$) differ by a flip, they share the same quadrilateral say q^t (resp. q^b). It is clear that $q^t \subset \Gamma^t \cap T$ and $q^b \subset \Gamma^b \cap (\Gamma^b)_e$. We claim that the quadrilaterals q^t and q^b have at most one more common vertex other than v_1, v_2 (see Figure 6a for instance). Otherwise, at least one non-equatorial edge of $q^t \subset \Gamma$ shares common two endpoints with a non-equatorial edge of $q^b \subset \Gamma$, since $N \geq 5$. This will imply that $\Gamma^t \cup \Gamma^b = \Gamma$ is not admissible by Lemma 6.3, which leads to contradiction.

As a consequence, the edge on the top of $\Sigma_{0,N}$ with the same endpoints as e^f must cross the quadrilateral $q^t \subset \Gamma^t \cap T$. This implies that e^f shares no common two endpoints with any edges of Γ^t and T. Moreover, by Lemma 6.3 and $\Gamma \in \mathcal{G}_N$, no non-equatorial edge of $(\Gamma^b)_e \setminus \{e^f\} = \Gamma^b \setminus \{e\}$ shares common two endpoints with any edge of Γ^t and T. Using Lemma 6.3 again, it follows that both $\Gamma^t \cup (\Gamma^b)_e$ and $T \cup (\Gamma^b)_e$ are admissible (see Figure 6b and Figure 6c for instance). This concludes the observation. $\square$

The following is a known result (see [HN99, Theorem 4.2]) about the (vertex)-connectivity of the graph of triangulations of a convex polygon with N vertices, which will be used to show the connectedness of $G_T(\Sigma_{0,N})$.

Lemma 6.6. *Let p_N be a convex polygon with N vertices $(N \geq 4)$. Then the graph $G_T(p_N)$ of triangulations of p_N is (N-3)-connected. In particular, $G_T(p_N)$ is connected.*

We are now ready to show the connectedness of the graph $G_T(\Sigma_{0,N})$.

Proposition 6.7. *For $N \geq 5$, the graph $G_T(\Sigma_{0,N})$ is connected.*

Proof. We prove this result in the following three steps:

Step 1. We show that for any admissible triangulation $\Gamma \in \mathcal{G}_N$ and any triangulation T of the closure of the top of $\Sigma_{0,N}$, there is an admissible triangulation $\Gamma' \in \mathcal{G}_N$ whose top subgraph coincides with T and is connected to Γ by a finite number of flips.

Indeed, by Lemma 6.6, the top subgraph Γ^t of Γ can be connected to T by a finite number of flips, say $\Gamma^t := T_0, T_1, ..., T_n := T$. Denote $\Gamma_0 := \Gamma$. We first show that Γ_0 can be connected to an admissible triangulation, say Γ_1, whose top subgraph is T_1.

If there is no pair of vertices (on the equator of $\Sigma_{0,N}$) connected by both a bottom edge of Γ and an edge of T_1, then, by Lemma 6.3, the union $T_1 \cup \Gamma^b$ is an admissible triangulation. We let $\Gamma_1 = T_1 \cup \Gamma^b$. Then $\Gamma_0 = \Gamma$ is connected to Γ_1 by an edge of $G_T(\Sigma_{0,N})$.
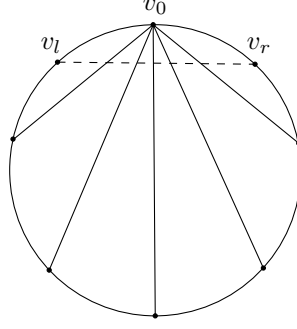
FIGURE 7. An example of Γ_1 (or Γ_2) in Step 2 of the proof of Proposition 6.7 (with only one bottom edge drawn), whose top subgraph (shown in solid lines) is a fan and bottom graph (shown in dashed lines) contains an edge connecting v_l and v_r.

Otherwise, by Observation 6.4, there is exactly one edge say e of Γ^b and one edge e' (which is the new one obtained from Γ^t by a flip) of T_1 connecting the same vertices. Let $(\Gamma^b)_e$ be the triangulation obtained from Γ^b by a flip at e. By Observation 6.5, the triangulations $T_1 \cup (\Gamma^b)_e$ and $\Gamma^t \cup (\Gamma^b)_e$ are both admissible. We let $\Gamma_1 = T_1 \cup (\Gamma^b)_e$. Then $\Gamma_0 = \Gamma$ is connected to Γ_1 by a path of $G_T(\Sigma_{0,N})$ through $\Gamma^t \cup (\Gamma^b)_e$.

Using the same argument, we can show that there is always an admissible triangulation Γ_{i+1} (whose top subgraph is T_{i+1}) connected to an admissible triangulation Γ_i (whose top subgraph is T_i) by a path of $G_T(\Sigma_{0,N})$ for $i = 1, ..., n-1$. This concludes Step 1.

Step 2. Let T_0 be a special triangulation (called *fan*) of the closure of the top of $\Sigma_{0,N}$, whose non-equatorial edge set is obtained by connecting one vertex (say v_0) to all the others non-adjacent to v_0 along the equator. We show that for any two admissible triangulations $\Gamma_1, \Gamma_2 \in \mathcal{G}_N$ with $\Gamma_1^t = T_0 = \Gamma_2^t$, there is a path in $G_T(\Sigma_{0,N})$ connecting Γ_1 to Γ_2.

Indeed, let v_l, v_r denote the two vertices adjacent to v_0 on the equator of $\Sigma_{0,N}$. By Lemma 6.3 and the assumption that Γ_1, Γ_2 are both admissible triangulations of $\Sigma_{0,N}$, the bottom subgraphs Γ_1^b and Γ_2^b of Γ_1 and Γ_2 both contain an edge connecting v_l and v_r (see Figure 7 for example).

Let $(\Gamma_1^b)_{v_0}$ (resp. $(\Gamma_2^b)_{v_0}$) denote the graph obtained from Γ_1^b (resp. Γ_2^b) by removing the vertex v_0 and its adjacent edges. Then $(\Gamma_1^b)_{v_0}$ and $(\Gamma_2^b)_{v_0}$ can be identified with two triangulations of a convex polygon with $N-1$ vertices. Note that $N-1 \geq 4$. By Lemma 6.6, $(\Gamma_1^b)_{v_0}$ and $(\Gamma_2^b)_{v_0}$ differ by a finite number of flips, say $(\Gamma_1^b)_{v_0} := G_0, G_1, ..., G_m := (\Gamma_2^b)_{v_0}$. It is clear that the union of G_i and the two equatorial edges adjacent to v_0 becomes again a triangulation, say G_i', of the closure the bottom of $\Sigma_{0,N}$ for $i = 0, ..., m$. Moreover, G_i' and G_{i+1}' differ by a flip for $i = 0, ..., m-1$. We consider the triangulations $T_0 \cup G_i'$ of $\Sigma_{0,N}$ for $i = 1, ..., m$. Note that the endpoints of each bottom edge of G_i' are distinct from v_0 for $i = 0, ..., m$. Using Lemma 6.3 again, $T_0 \cup G_i'$ is admissible for $i = 1, ..., m-1$. This gives a path in $G_T(\Sigma_{0,N})$ connecting $\Gamma_1 = T_0 \cup G_0'$ and $\Gamma_2 = T_0 \cup G_m'$. Step 2 is complete.

Step 3. Given any two admissible triangulations $\Gamma_1, \Gamma_2 \in \mathcal{G}_N$, we first connect Γ_1 (resp. Γ_2) to an admissible triangulation Γ_1' (resp. Γ_2') whose top subgraph is a given fan T_0 (i.e. $(\Gamma_1')^t = (\Gamma_2')^t = T_0$) by a finite number of flips. By Step 1, this is doable. Then we connect Γ_1' and Γ_2' by a finite number of flips. By Step 2, this is also doable. Therefore, Γ_1 can be connected to Γ_2 by a finite number of flips. This concludes the proof.   □

Using the connectedness of $G_T(\Sigma_{0,N})$, we prove the following:

Claim 6.8. *For $N \geq 5$, the space $\mathcal{A}$ is connected.*

Proof. Note that the space $\mathcal{A}_\Gamma$ is non-empty and contractible for all $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$ with $N \geq 4$ (see Claim 6.1). Assume that $N \geq 5$. We prove the claim in the following three steps:

Step 1. We show that for any two graphs $\Gamma_1, \Gamma_2 \in \mathcal{G}_N$ which differ by a flip, the space $\mathcal{A}_{\Gamma_1}$ is connected to $\mathcal{A}_{\Gamma_2}$ in $\mathcal{A}$. Indeed, the graph $\Gamma_1 \cap \Gamma_2$, as a subgraph of both Γ_1 and Γ_2, is still 3-connected and contains the equator, thus belongs to $\mathsf{Graph}(\Sigma_{0,N}, \gamma)$. Step 1 follows by the definition of $\mathcal{A}$, which says that the spaces $\mathcal{A}_{\Gamma_1}$ and $\mathcal{A}_{\Gamma_2}$ are glued together along the space $\mathcal{A}_{\Gamma_1 \cap \Gamma_2}$ as long as Γ_1, Γ_2 and $\Gamma_1 \cap \Gamma_2$ belong to $\mathsf{Graph}(\Sigma_{0,N}, \gamma)$.

Step 2. We show that for any two graphs in $\mathcal{G}_N$, one can be obtained from the other by a finite number of flips. This follows directly from Proposition 6.7 that any two graphs $\Gamma_1, \Gamma_2 \in \mathcal{G}_N$ can be connected by a path in $G_T(\Sigma_{0,N})$, of which every two adjacent graphs differ by a flip.

Step 3. It remains to show that for any graph $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma) \setminus \mathcal{G}_N$, $\mathcal{A}_\Gamma$ is connected to $\mathcal{A}_{\Gamma_0}$ for some triangulation $\Gamma_0 \in \mathcal{G}_N$ of which Γ is a subgraph (note that such a triangulation Γ_0 always exists by triangulating

the non-triangular faces of Γ in a way such that no top edge and bottom edge share common two endpoints). Similar to Step 1, the definition of $\mathcal{A}$ states that $\mathcal{A}_\Gamma$ and $\mathcal{A}_{\Gamma_0}$ are glued together along $\mathcal{A}_\Gamma$. This concludes the proof of Step 3.

Combining the above three steps and Claim 6.1, this claim follows. $\qquad\square$

Proof of Proposition 1.7. Combining Claim 6.1 and Claim 6.8, this proposition follows directly. $\qquad\square$

6.2. **The dimensions of $\mathcal{P}$ and $\bar{\mathcal{P}}$.** Recall that $\mathcal{P}$ is the disjoint union of $\mathcal{P}_\Gamma$ over all $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$, glued together along faces corresponding to common subgraphs, where $\mathcal{P}_\Gamma$ is the subspace of $\mathcal{P}$ consisting of hyperideal polyhedra with 1-skeleton Γ. Recall that $\bar{\mathcal{P}} = \mathcal{P} \cup \mathsf{polyg}$, where $\mathsf{polyg} = \mathsf{polyg}_N$ is the space of all marked convex hyperideal polygons in $\mathbb{H}^2$ with N vertices, considered up to orientation preserving isometries of $\mathbb{H}^2$. We first prove the following.

Claim 6.9. *For each $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$, the space $\mathcal{P}_\Gamma$ is a topological manifold of (real) dimension $|E(\Gamma)|$. In particular, for each $\Gamma \in \mathcal{G}_N$, the space $\mathcal{P}_\Gamma$ realizes the highest (real) dimension $3N - 6$.*

Proof. Let $P \in \mathcal{P}_\Gamma$ and let $v_1, ..., v_N$ denote the vertices of P. As before, we fix a representative for P and fix an affine chart $\mathbb{R}^3$.

If Γ is a triangulation, then for each vertex v_i, there exists a sufficiently small neighborhood U_i in $\mathbb{R}^3$ of v_i, such that for any tuple $(v_1', \cdots, v_N') \in (U_1 \times \cdots \times U_N) \setminus (\mathbb{A}d\mathbb{S}^3)^N$, the convex hull of $v_1', ..., v_N'$ is a hyperideal polyhedron with the same combinatorics (i.e. 1-skeleton) as P, and thus lies in $\mathcal{P}_\Gamma$. Since each vertex of P has three degree of freedom in $U_i \subset \mathbb{R}^3$ and such polyhedra are identified up to elements of $\mathrm{Isom}_0(\mathbb{A}d\mathbb{S}^3) = \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ (which has dimension 6), hence $\mathcal{P}_\Gamma$ has dimension $3N - 6 = |E(\Gamma)|$.

If Γ is not a triangulation, then Γ has at least one non-triangular face. We first consider the perturbation of the vertices of the non-triangular faces. Note that the deformation in a small neighborhood in $\mathcal{P}_\Gamma$ of P keeps the combinatorics of P, it never creates new edges in the non-triangular faces. In other words, the vertices of each non-triangular face continue lying in the same plane.

Note that the perturbation of three arbitrary vertices of a non-triangular face determines a plane in which the other vertices of the same face have to be located. This implies that the deformation of every other vertex (whose number is equal to that of the edges needed to add to form a triangulation of that non-triangular face) has only two degrees of freedom, one fewer than that in the case of triangulations. Therefore, this deformation of P has in total $3N - 6 - |E(\Gamma)|$ fewer degrees of freedom, compared to the above case (when the combinatoric of P is a triangulation). As a result, $\mathcal{P}_\Gamma$ has dimension $3N - 6 - (3N - 6 - |E(\Gamma)|) = |E(\Gamma)|$.

Combining these two cases, $\mathcal{P}_\Gamma$ has dimension $|E(\Gamma)|$ for all $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$. $\qquad\square$

Proof of Proposition 1.8 and Proposition 1.13. By Claim 6.9, Proposition 1.8 follows. Note that $\bar{\mathcal{P}} = \mathcal{P} \cup \mathsf{polyg}$ and it is well-known that the dimension of the space $\mathsf{polyg} = \mathsf{polyg}_N$ is $2N - 3$ for $N \geq 3$, which is less than or equal to $3N - 6$ (with equality if and only if $N = 3$) for all $N \geq 3$. Proposition 1.13 follows. $\qquad\square$

6.3. **The proof of Theorem 1.6.** We first recall a known result of [DMS14] for the ideal case, which concerns the parameterization of the space $\mathcal{P}'$ of all marked non-degenerate convex ideal polyhedra in $\mathbb{A}d\mathbb{S}^3$ with N vertices (up to isometries) in terms of dihedral angles.

Let $\mathcal{A}'$ denote the disjoint union of the spaces $\mathcal{A}'_\Gamma$ of weight functions (satisfying certain angle conditions) on the edges of Γ over $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$, glued together along faces corresponding to common subgraphs. In this case, the angle conditions are a modification of Conditions (i)-(iv) in Definition 1.2 obtained by replacing the inequalities $\theta(e_1) + ... + \theta(e_k) \geq 0$ in Condition (ii) for each vertex by the equality $\theta(e_1) + ... + \theta(e_k) = 0$ (see also [DMS14, Definition 1.3]).

It was proved in [DMS14, Theorem 1.4] that the map $\Psi' : \mathcal{P}' \to \mathcal{A}'$ which assigns to a polyhedron $P' \in \mathcal{P}'$ its dihedral angle at each edge is a homeomorphism.

6.3.1. *The case $N \geq 5$.* It follows from Lemma 1.11 that for each triangulation $\Gamma \in \mathcal{G}_N$, the angle-assignation map $\Psi : \mathcal{P} \to \mathbb{R}^{E(\Gamma)}$ is a local immersion near each $P \in \mathcal{P}$ whose 1-skeleton is a subgraph of the Γ. By Claim 6.1 and Claim 6.9, the spaces $\mathcal{A}_\Gamma$ and $\mathcal{P}_\Gamma$ have the same dimension $3N - 6$ for a triangulation $\Gamma \in \mathcal{G}_N$. Therefore, the map $\Psi : \mathcal{P} \to \mathbb{R}^{E(\Gamma)}$ is a local homeomorphism near each $P \in \mathcal{P}$ whose 1-skeleton is a subgraph of the Γ. Note that the map Ψ, pieced together from Ψ_Γ over all $\Gamma \in \mathsf{Graph}(\Sigma_{0,N}, \gamma)$, is an open map by the definition of the topology of the complex $\mathcal{A}$. As a consequence the map $\Psi : \mathcal{P} \to \mathcal{A}$ is a local homeomorphism in a neighbourhood of any point P in the closure of the stratum $\mathcal{P}_\Gamma$ of $\mathcal{P}$ for each $\Gamma \in \mathcal{G}_N$. Therefore, Ψ is a local homeomorphism. Combined with the properness (see Lemma 1.10), Ψ is a covering. Moreover, Ψ is an m-sheeted covering for a positive integer m, since $\mathcal{A}$ is connected (see Proposition 1.7).

It suffices to show that $m = 1$. Indeed, let $\theta \in \mathcal{A}'$ be an angle assignment. We claim that θ has a unique preimage in $\mathcal{P}$. Recall the known result (see [DMS14, Theorem 1.4]) that the parameterization map $\mathcal{P}' \to \mathcal{A}'$, which is exactly the restriction of Ψ to the space $\mathcal{P}'$, is a homeomorphism. Therefore, θ has a unique preimage in $\mathcal{P}'$. It remains to show that θ has no preimage in $\mathcal{P} \setminus \mathcal{P}'$. Otherwise, there is a polyhedron say $P \in \mathcal{P} \setminus \mathcal{P}'$ whose dihedral angle-assignment is θ. However, P has at least one strictly hyperideal vertex say v, by the necessity (see Proposition 1.9 for the proof of Condition (ii)), the sum of the dihedral angles at the edges adjacent to v is greater than 0. This implies that $\theta \notin \mathcal{A}'$, which leads to contradiction. As a consequence, Ψ is a one-sheeted covering and is therefore a homeomorphism.

6.3.2. *The case $N = 4$.* In this case, $\mathsf{Graph}(\Sigma_{0,N}, \gamma) = \mathcal{G}_N$ and it consists of exactly two graphs, say Γ_1, Γ_2, which are both triangulations of $\Sigma_{0,N}$. According to the gluing construction of $\mathcal{P}_\Gamma$ (resp. $\mathcal{A}_\Gamma$) for $\mathcal{P}$ (resp. $\mathcal{A}$), $\mathcal{P}$ (resp. $\mathcal{A}$) has exactly two connected components $\mathcal{P}_{\Gamma_1}$ and $\mathcal{P}_{\Gamma_2}$ (resp. $\mathcal{A}_{\Gamma_1}$ and $\mathcal{A}_{\Gamma_2}$). Note that the space $\mathcal{P}'$ (resp. $\mathcal{A}'$) also has two connected components corresponding to Γ_1 and Γ_2 (see e.g. [DMS14, Section 7.3]). Applying the same argument as the case $N \geq 5$ to the angle-assignation map $\Psi_{\Gamma_i} : \mathcal{P}_{\Gamma_i} \to \mathcal{A}_{\Gamma_i}$ for $i = 1, 2$, we have that Ψ_{Γ_i} is a homeomorphism, which implies that Ψ is a homeomorphism.

6.4. **The proof of Theorem 1.12.** We recall another known result of [DMS14] for the ideal case, which provides the parameterization of the space $\bar{\mathcal{P}}'$ of all marked degenerate and non-degenerate convex ideal polyhedra in $\mathbb{AdS}^3$ with N vertices (up to isometries) in terms of induced metric on the boundary of polyhedra. Let $\mathcal{T}'_{0,N}$ denote the space of complete hyperbolic metrics on $\Sigma_{0,N}$ with finite area, considered up to isotopy fixing each marked point. It was shown (see [DMS14, Theorem 1.5]) that the map $\Phi' : \bar{\mathcal{P}}' \to \mathcal{T}'_{0,N}$ which takes a polyhedron $P' \in \bar{\mathcal{P}}'$ to the induced metric on $\partial P'$ is also a homeomorphism.

In this proof we discuss directly the case of $N \geq 3$, since the space $\mathcal{T}_{0,N}$ is connected for $N \geq 3$. The argument is almost the same as above for the map Ψ in the case $N \geq 5$. We include the proof for completeness. Indeed, the map $\Phi : \bar{\mathcal{P}} \to \mathcal{T}_{0,N}$ is a local immersion by Lemma 1.15. Note that the spaces $\bar{\mathcal{P}}$ and $\mathcal{T}_{0,N}$ have the same dimension $3N - 6$ for all $N \geq 3$ (see Proposition 1.13), so Φ is a local homeomorphism. Combined with the properness of Φ (see Lemma 1.14) and the fact that $\mathcal{T}_{0,N}$ is connected, Φ is an m-sheeted covering for some positive integer m. Using the known result (see [DMS14, Theorem 1.5]) that the map $\Phi' : \bar{\mathcal{P}}' \to \mathcal{T}'_{0,N}$, which is exactly the restriction of Φ to $\bar{\mathcal{P}}'$, is a homeomorphism and the fact that the complete metric induced near a strictly hyperideal vertex on the boundary of a hyperideal polyhedron has infinite area, we can show that each metric $h \in \mathcal{T}'_{0,N}$ has a unique preimage in $\bar{\mathcal{P}}$, which implies that $m = 1$. This concludes that Φ is a homeomorphism.

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