CONFORMAL ACTIONS OF HIGHER RANK LATTICES ON COMPACT PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. We investigate conformal actions of cocompact lattices in higher-rank simple Lie groups on compact pseudo-Riemannian manifolds. Our main result gives a general bound on the real-rank of the lattice, which was already known for the action of the full Lie group ([33]). When the real-rank is maximal, we prove that the manifold is conformally flat. This indicates that a global conclusion similar to that of [1] and [17] in the case of a Lie group action might be obtained. We also give better estimates for actions of cocompact lattices in exceptional groups. Our work is strongly inspired by the recent breakthrough of Brown, Fisher and Hurtado on Zimmer's conjecture [7].

Contents

1. Introduction

Zimmer's conjectures concern actions of lattices in higher-rank semi-simple Lie groups on differentiable manifolds. It is expected that they share common features with standard algebraic actions. The most famous problem is when the dimension of the manifold is low compare to the lattice, and important breakthroughs ([7], [8]) have recently been made in this direction.

Originally, the conjectures of Zimmer were formulated for actions of lattices which preserve a geometric structure, such as a pseudo-Riemannian metric or a symplectic form. The general idea is that there should be a universal obstruction to the existence of a non-trivial action of a given lattice on a geometric structure of a given type, see [32], Conjecture I for a concrete formulation. In the differentiable case, the obstruction only depends on the dimension of the manifold, but for other geometric structures, more restrictive conclusions are naturally expected.

This program is motivated by earlier results of Zimmer, notably his cocycle super-rigidity theorem, that generalizes Margulis' super-rigidity phenomenon, and deals with *measure preserving* dynamics of semi-simple Lie groups and their lattices. Based on this result, he obtained very strong conclusions, for instance when a semi-simple Lie group G, or one of his lattices Γ, acts on a compact manifold M, preserving a *finite-type* H-structure and a volume form (see [31], Theorem A and F). Section 1.1 gives an illustration of these results.

When we consider dynamics on compact geometric structures that are not unimodular, there is a priori no natural, finite invariant measure, and the problem is notably different. Nonetheless, remarkable restrictions were observed in various non-measure preserving contexts such as [33], [1], [23], [2]. All these results concerned actions of connected semi-simple Lie groups, and to our knowledge, only few results about discrete group actions on non-unimodular geometric structures exist (see for instance Theorem 1.4 of [2] for "semi-discrete" group acting on parabolic geometries).

The contributions of the present article are results about actions of higher-rank lattices in the continuity of these works, in the framework of conformal pseudo-Riemannian geometry, which is a typical example of non-unimodular geometric structure. Our main result is Theorem 1 below. We obtain the same bound as in [33] on the real-rank of the lattice, and when the lattice has maximal real-rank we establish that the metric is conformally flat, indicating that a conclusion analogous to the main results of [1] and [17] is plausible.

Our approach is largely inspired by the proof of Brown-Fisher-Hurtado in [7]. However, significant simplifications appeared in this context, due to the fact that a *rigid geometric structure* is preserved (see [18], [21], Chapter I). The main point from their article that we use is the construction of invariant measures in some dynamical configurations, based on Ledrappier-Young's formula (see Section 4.2).

1.1. Actions by pseudo-Riemannian isometries.

A natural situation where the original results of Zimmer apply is when G or Γ acts by isometries on a compact pseudo-Riemannian manifold of signature (p, q), *i.e.* by automorphisms of an $O(p, q)$-structure.

We remind that a (smooth) pseudo-Riemannian metric g on a manifold M is a smooth distribution of non-degenerate quadratic forms of signature (p, q) on the tangent spaces of M. An isometry is a diffeomorphism that preserves this field of quadratic forms. Pseudo-Riemannian metrics are always rigid at order 1, implying that the group of isometries $\mathrm{Isom}(M, g)$ is a Lie transformation group.

For isometric actions of lattices, Zimmer's result reads:

Theorem (Consequence of Theorem F of [31])**.** *Let (M, g) be a closed pseudo-Riemannian manifold of signature (p, q) and Γ a lattice in a semi-simple Lie group G with finite center and all of whose simple factors have real-rank at least 2. Assume that Γ acts isometrically on (M, g).*

Then, either $\mathfrak{g}$ embeds into $\mathfrak{so}(p, q)$, or the action factorizes through a compact Lie group, i.e. the action is of the form $\Gamma \to K \hookrightarrow \mathrm{Isom}(M, g)$, where K is a compact Lie group.

Of course, the obstruction $\mathfrak{g} \hookrightarrow \mathfrak{so}(p, q)$ is stronger than a constraint formulated with the dimension of M. For instance, $\mathfrak{sl}(3, \mathbf{R})$ does not embed into $\mathfrak{so}(2, n)$ for all $n \geqslant 1$. Thus, if (M, g) is a closed pseudo-Riemannian manifold of signature $(2, n)$, then any isometric action of $\mathrm{SL}(3,\mathbf{Z})$ on (M,g) has finite image, even though $\dim M = n+2$ is large.

Example 1. If $\min(p,q) \geqslant 2$ and $(p,q) \neq (2,2)$, then $G = O(p,q)$ and $\Gamma = G_{\mathbf{Z}}$ satisfy the hypothesis of this theorem and Γ acts on the pseudo-Riemannian torus $\mathbf{T}^{p,q} = \mathbf{R}^{p,q}/\mathbf{Z}^{p+q}$, and its action is unbounded.

Remark 1.1. Even if $\mathrm{Isom}(M,g)$ is a Lie group, Margulis' super-rigidity does not imply that an isometric action $\Gamma \to \mathrm{Isom}(M,g)$ extends to an action of G. And this is wrong in general, as it can be observed in the example of $\mathbf{T}^{p,q}$.

Remark 1.2. Concerning pseudo-Riemannian isometric actions of simple Lie groups, the conclusion of Zimmer's embedding theorem - Theorem A of [31] - gives a complete obstruction: given a non-compact, simple Lie group G, the existence of a locally faithfull isometric action of G on a compact manifold of signature (p,q) is reduced to an algebraic question on representations of $\mathfrak{g}$.

1.2. **Conformal dynamics.** Let (M,g) be a pseudo-Riemannian manifold of signature (p,q).

1.2.1. *Definitions and standard examples.* The conformal class of g is defined as $[g] = \{\varphi g,\ \varphi \in \mathcal{C}^\infty(M),\ \varphi > 0\}$, and a diffeomorphism f of M is said to be conformal with respect to g if it preserves $[g]$ setwise. An important property is that when $\dim M \geqslant 3$, a conformal class $[g]$ defines a rigid geometric structure on M. This can be interpreted by the fact that the associated $(\mathbf{R}_{>0} \times O(p,q))$-structure is of finite type in the sense of Cartan ([21], Chapter I), see also [18]. As a consequence, the group of conformal diffeomorphisms $\mathrm{Conf}(M,g)$ has a natural Lie group structure.

An important example of compact pseudo-Riemannian manifold is the conformal compactification of the flat pseudo-Euclidean space $\mathbf{R}^{p,q}$, the (pseudo-Riemannian) *Einstein universe* $\mathbf{Ein}^{p,q}$. It is a parabolic space $\mathrm{PO}(p+1,q+1)/P$, where P is a maximal parabolic subgroup, isomorphic to the stabilizer of an isotropic line in $\mathbf{R}^{p+1,q+1}$. Otherwise stated, $\mathbf{Ein}^{p,q}$ is the projectivized nullcone of $\mathbf{R}^{p+1,q+1}$, and it inherits from it a natural conformal class of signature (p,q) such that $\mathrm{Conf}(\mathbf{Ein}^{p,q}) = \mathrm{PO}(p+1,q+1)$. When $p=0$, it is nothing else than the sphere with its standard conformal structure.

1.2.2. *Additional motivation: a generalization of Lichnerowicz conjecture.* The interest in conformal dynamics of semi-simple Lie groups and their lattices is moreover motivated by an older problem originally asked by Lichnerowicz.

It was settled in the case of Riemannian conformal geometry. Ferrand and Obata solved it ([12], [24]) and in the compact case, their result asserts that given a compact Riemannian manifold (M,g), its conformal group $\mathrm{Conf}(M,g)$ is non-compact if and only if (M^n,g) is conformally equivalent to the round sphere $\mathbf{S}^n$.

For other signatures, the situation is more complicated. The natural conjecture that arose from the theorem of Ferrand and Obata was that pseudo-Riemannian manifolds with an *essential conformal group* shall be classifiable, see [10] Section 7.6. We remind that a subgroup $H < \mathrm{Conf}(M,g)$ is said to be essential if $H \nsubseteq \mathrm{Isom}(M,g')$ for all metrics g' in the conformal class of g. It turned out that there are many essential pseudo-Riemannian manifolds, and that obtaining a classification seems not plausible, see [14], [15].

1.2.3. *Anterior results for connected semi-simple groups.* It is then natural to consider manifolds admitting an essential conformal group with a "rich" algebraic structure. The following result gives an interesting positive answer, when "rich" is interpreted as "containing a semi-simple Lie subgroup of maximal real-rank".

Theorem ([33],[1],[17])**.** *Let (M^n, g) be a closed pseudo-Riemannian manifold of signature (p, q), with $n \geqslant 3$ and $p \leqslant q$, and let G be a non-compact simple Lie group. Assume that we are given a locally faithful, conformal action $G \to \mathrm{Conf}(M, g)$. Then,*

- $\mathrm{Rk}_{\mathbf{R}}\, G \leqslant p + 1$ *(follows from Theorem 1 of* [33]*)*
- *and if* $\mathrm{Rk}_{\mathbf{R}}\, G = p + 1$*, then* $\mathfrak{g} = \mathfrak{so}(p + 1, k)$ *with* $p + 1 \leqslant k \leqslant q + 1$ *and* (M, g) *is conformally diffeomorphic to a quotient* $\Gamma \setminus \widetilde{\mathbf{Ein}}^{p,q}$*, where* Γ *is a discrete group acting freely, properly and conformally (*[1]*, Theorem 2 and* [17]*).*

We recently obtained results about conformal actions of semi-simple Lie groups whose real-rank is not maximal [25], [26].

1.3. **Main result: conformal actions of uniform lattices.** Our main result gives a similar statement for conformal actions of cocompact lattices of G.

Theorem 1. *Let (M^n, g) be a closed pseudo-Riemannian manifold of signature (p, q), with $n \geqslant 3$, and $\Gamma < G$ a uniform lattice in a non-compact simple Lie group of real-rank at least 2 and finite center. Assume that we are given a conformal action $\alpha : \Gamma \to \mathrm{Conf}(M, g)$ such that $\alpha(\Gamma)$ is unbounded in $\mathrm{Conf}(M, g)$. Then,*

- $\mathrm{Rk}_{\mathbf{R}}\, G \leqslant \min(p, q) + 1$*,*
- *and when* $\mathrm{Rk}_{\mathbf{R}}\, G = \min(p, q) + 1$*,* (M, g) *is conformally flat.*

We remind that a pseudo-Riemannian metric is said to be conformally flat if near every point, there are local coordinates in which the metric reads $\varphi(x)(-\mathrm{d}x_1^2 - \cdots - \mathrm{d}x_p^2 + \mathrm{d}x_{p+1}^2 + \cdots + \mathrm{d}x_n^2)$, where $\varphi > 0$.

Even though our conclusion is not as sharp as in the case of an action of a semi-simple Lie group, we suspect that nothing notably different may happen. The remaining problem is to consider the action of such a lattice on compact manifolds endowed with a $(\mathrm{Conf}(\mathbf{Ein}^{p,q}), \mathbf{Ein}^{p,q})$-structure, and we expect that these structures should be complete. We leave this problem for further investigations.

1.4. **Better bounds on the optimal index for exceptional groups.** Let Γ be a lattice in a higher rank semi-simple Lie group G with no compact factor. A famous question addressed in the Zimmer program is to determine the smallest integer n such that there exist a compact manifold M^n and an action $\Gamma \to \mathrm{Diff}(M)$ with infinite image. In the context of conformal actions of Γ, an analogous question would be to determine the "optimal signature(s)" for which there exists a non-trivial conformal action of Γ on a compact manifold. A natural quantity that we would like to optimize is the metric index $\min(p, q)$, which is the dimension of maximally isotropic subspaces of g.

Definition 1.3. We define the optimal index of Γ as the smallest integer k such that there exist a compact pseudo-Riemannian manifold (M, g) of metric index k and a conformal action $\alpha : \Gamma \to \mathrm{Conf}(M, g)$ such that $\alpha(\Gamma)$ is unbounded in $\mathrm{Conf}(M, g)$. We note k_Γ the optimal index of Γ.

The first point of Theorem 1 says that $k_\Gamma \geqslant \mathrm{Rk}_{\mathbf{R}}(G) - 1$ when Γ is cocompact. Even though this bound is sharp when Γ is a lattice in a group of the form $\mathrm{SO}(p,q)$, we expect that it will not be the case for other groups.

An analogy can be made with the article of Brown-Fisher-Hurtado. The proof of the upper bound on $\mathrm{Rk}_{\mathbf{R}}\, G$ in Theorem 1 is based on Proposition 4.7 (essentially contained in [7]), which is a property of differentiable actions. In the context of Zimmer's conjecture, this property is not strong enough to obtain the bounds proved in [7]. In fact, put together with the other techniques involved in their work, Proposition 4.7 would "only" imply that if a differentiable action of Γ on a compact manifold M has infinite image, then $\dim M \geqslant \mathrm{Rk}_{\mathbf{R}}\, G$ in the non-unimodular case (see [5], Theorem 11.1'). But this is the conjectured bound only when G is locally isomorphic to $\mathrm{SL}(n, \mathbf{R})$.

For split, simple Lie groups other than $\mathrm{SL}(n, \mathbf{R})$, Brown-Fisher-Hurtado obtained the expected bounds by applying another property involving the resonance of the Lyapunov spectrum with the restricted root-system of $\mathfrak{g}$, proved in [8].

We could expect that this more advanced methods would imply stronger bounds in the setting of conformal dynamics. Surprisingly, it is not what happens and we get the same bound as in Theorem 1, except when the restricted root system of G is exceptional. For these exceptional groups, we obtain the following result.

Theorem 2. *Let Γ be a uniform lattice in a non-compact simple Lie group G of real-rank at least 2, with finite center, and such that the restricted root-system Σ of G is exceptional. We have the following lower bounds for its optimal index, depending Σ:*

(1) *If $\Sigma = E_6$, then $k_\Gamma \geqslant 7$.*
(2) *If $\Sigma = E_7$, then $k_\Gamma \geqslant 13$.*
(3) *If $\Sigma = E_8$, then $k_\Gamma \geqslant 28$.*
(4) *If $\Sigma = F_4$, then $k_\Gamma \geqslant 7$.*
(5) *If $\Sigma = G_2$, then $k_\Gamma = 2$.*

See [20], Table 6.108 for the list of such exceptional simple Lie groups. The specific values of these lower bounds come from the *minimal resonance codimension* of $\mathfrak{g}$, see [7], Definition 2.1 and Example 2.3.

Even though we get a significant improvement compare to the bound $k_\Gamma \geqslant \mathrm{Rk}_{\mathbf{R}}(G) - 1$, we suspect that these lower bounds on the optimal index are still not sharp.

1.5. **Organization of the article and ideas of proofs.** As said above, our approach is inspired by that of Brown-Fisher-Hurtado in the differentiable case. Let $\Gamma < G$ be a cocompact lattice in a higher-rank simple Lie group G with finite center, and let $\alpha : \Gamma \to \mathrm{Conf}(M, g)$ be a conformal action such that $\alpha(\Gamma)$ is unbounded.

Given an action $\alpha : \Gamma \to \mathrm{Diff}(M)$, a classic construction gives an auxiliary space M^α on which G acts naturally, and in which the action of Γ is encoded. Given a Cartan subspace $A < G$ and an A-invariant, A-ergodic measure μ on M^α, the higher-rank Oseledec's theorem gives a simultaneous Oseledec's splitting of the vertical tangent bundle F^α of M^α for all elements in A, and the Lyapunov exponents are linear functionals $\chi_1, \ldots, \chi_r \in \mathfrak{a}^*$.

The point that we use from [7] is that in general, if r is small compare to a data extracted from the restricted root-system of G, and if the measure is well chosen, then

μ must be G-invariant. See Section 4.2 for a concrete formulation. In the general differentiable case, the only possible control on the maximal number of Lyapunov functionals is given by the dimension.

The starting point of our work is that when a geometric structure is preserved, we have a better control of r. In our case of a conformal action in signature (p, q), the number of Lyapunov functionals is bounded by $2\min(p, q) + 1$ and they moreover satisfy linear relations (Proposition 3.5). We explain this in Section 3, after having detailed how the conformal structure of M can be recovered in the vertical tangent bundle of M^α in Section 2.

In Section 4, we prove Proposition 4.1, which gives an important simplification compare to the differentiable case. It asserts that if G is not locally isomorphic to a subgroup of $\mathrm{SO}(p, q)$, then Γ does not preserve any finite measure on M. This proposition is almost stated in anterior works of Zimmer and relies essentially on cocycle super-rigidity and the fact that the conformal structure is rigid, which implies that Γ acts freely and properly on a principal bundle over M, the Cartan bundle.

The bound on the real-rank follows easily in Section 5. Essentially, if the rank of G was larger than $\min(p, q) + 1$, then the number of Lyapunov functionals would be too small compare to $\mathrm{Rk}_{\mathbf{R}} G$ and we would obtain a G-invariant measure by the above-mentioned argument of differentiable dynamics. This would contradict the fact that Γ does not preserve any finite measure. We also prove Theorem 2 by applying a more advanced argument given in [8] (Proposition 4.10).

Section 6 is devoted to the proof of the geometric part of Theorem 1. The idea is that when $\mathrm{Rk}_{\mathbf{R}} G$ is maximal, then there still cannot exist a finite Γ-invariant measure on M. This forces the Lyapunov functionals of a well chosen A-invariant, A-ergodic measure on M^α to be in a special configuration, always because of the the results cited in Section 4.2 and because of the linear relations they satisfy.

In particular, this configuration singles out a direction in $\mathfrak{a}$ having a uniform vertical Lyapunov spectrum. Using local stable manifolds of the corresponding flow in M^α, we interpret this fact in terms of the dynamics in M of some diverging sequence (γ_k) in Γ. This is Proposition 6.1. We then use a property of stability of sequences of conformal maps of Frances ([16]) to prove that the sequence (γ_k) has a uniform contracting behavior on an open set, and we finally derive conformal flatness of this open set by using standard arguments of conformal geometry. We conclude that the whole manifold is conformally flat by observing that any compact Γ-invariant subset of M will intersect such an open set.

1.6. **Conventions and notations.** Throughout this article, unless otherwise specified, $(M^n, \overline{g})$ will always denote a smooth compact pseudo-Riemannian manifold of signature (p, q), with $n = p + q \geqslant 3$, G a non-compact, simple Lie group of real-rank at least 2 and with finite center, and $\Gamma < G$ a uniform lattice. We will consider a conformal action of Γ on M, noted $\alpha : \Gamma \to \mathrm{Conf}(M, \overline{g})$. We note $[\overline{g}] = \{\varphi\overline{g},\ \varphi \in \mathcal{C}^\infty(M, \mathbf{R}_{>0})\}$ the conformal class of $\overline{g}$.

We will note $A < G$ a Cartan subspace of G, *i.e.* a closed connected abelian subgroup of G such that $\mathrm{Ad}_{\mathfrak{g}}(A)$ is $\mathbf{R}$-split. The set of restricted roots of $\mathrm{ad}(\mathfrak{a})$ is noted Σ and for $\lambda \in \Sigma$, we let $\mathfrak{g}_\lambda$ denote the corresponding restricted root-space, and G_λ the closed connected subgroup to which it is tangent.

Given a differentiable action of G on a manifold N, we will identify an element $X \in \mathfrak{g}$ with the vector field on N defined by $X(x) = \frac{\mathrm{d}}{\mathrm{d}t}_{t=0} e^{tX}.x$ for all $x \in N$. For convenience, we also note $V(x) = \{X(x),\ X \in V\} \subset T_xN$ for any vector subspace $V \subset \mathfrak{g}$.

The (linear) frame bundle of a vector bundle $E \to N$ of rank n is the $\mathrm{GL}(n, \mathbf{R})$-principal bundle $L(E) = \{u : \mathbf{R}^n \to E_x,\ x \in N,\ u \text{ linear isomorphism}\}$. A frame field (with a given regularity) is a section $\sigma : N \to L(E)$. Given a morphism F of E over $f : N \to N$, we note $\mathrm{Jac}_x^\sigma(F) := \sigma(f(x))^{-1} F \sigma(x) \in \mathrm{GL}(n, \mathbf{R})$ its Jacobian matrix at x with respect to a frame field σ.

2. Vertical conformal structure on the suspension space

Let G be a simple Lie group, with $\mathrm{Rk}_{\mathbf{R}}\, G \geqslant 2$, $\Gamma < G$ be a lattice and M^n be a compact manifold. A differentiable action $\alpha : \Gamma \to \mathrm{Diff}(M)$ gives rise to an action of G on a fibered manifold.

2.1. Suspensions space.

Definition 2.1. The suspension space of α is the fibration $\pi : M^\alpha \to G/\Gamma$ given by $M^\alpha = (G \times M)/\Gamma$, where Γ acts on the product via $\gamma.(g, x) = (g\gamma, \gamma^{-1}.x)$, and where π is the natural projection.

We note $[(g, x)]$ the equivalence class of $(g, x) \in G \times M$. The fibers of π are diffeomorphic to M, and the total space M^α is compact when Γ is uniform. The full Lie group G acts locally freely on M^α via $g.[(g_0, x)] = [(gg_0, x)]$. The G-orbits are transverse to the fibers, and define in this way a natural horizontal distribution, and the action is fiber-preserving. Moreover, the original action of Γ is encoded in this continuous action via return maps in the fibers.

We note $F^\alpha \subset TM^\alpha$ the sub-bundle tangent to the fibers. It is a vector bundle over M^α, of rank $n = \dim M$ and on which G acts linearly. The tangent distribution to the G-orbits - namely $\{\mathfrak{g}(x^\alpha),\ x^\alpha \in M^\alpha\}$ - is a natural G-invariant distribution in direct sum with the vertical bundle.

If $A < G$ is a Cartan subspace and $\Sigma \subset \mathfrak{a}^*$ are its restricted-roots, then the splitting:

$$\mathfrak{g}(x^\alpha) = \mathfrak{g}_0(x^\alpha) \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda(x^\alpha)$$

diagonalizes the action of A on the horizontal distribution since any $g \in A$ commutes with vector fields generated by elements of $\mathfrak{g}_0$ and if $X \in \mathfrak{g}_\lambda$ and if $g = e^{X_0}$, then $D_{x^\alpha} g.X_{x^\alpha} = e^{\lambda(X_0)} X_{gx^\alpha}$. In particular, for any A-invariant probability measure μ, the Lyapunov spectrum of any $g \in A$ with respect to μ is completely known in the horizontal direction and does not depend on μ. A central question is to undestand its vertical part.

2.2. The conformal class induced on the vertical distribution of M^α.

Let p, q be two non-negative integers with $n = p + q \geqslant 3$. We assume that the action of Γ on M is conformal with respect to a pseudo-Riemannian metric g of signature (p, q). This geometric data on M gives rise to a conformal structure on the vertical bundle of M^α.

2.2.1. *Classic definitions.* Let us first remind:

Definition 2.2. Let $p : E \to N$ be a vector bundle of rank n over a differentiable manifold N. A pseudo-Riemannian metric of signature (p,q) on E is a smooth assignment of quadratic forms $(g_x)_{x \in N}$ of signature (p,q) on the fibers of E.

Two metrics g_1 and g_2 on E are said to be conformal if there exists a smooth function $\phi : N \to \mathbf{R}_{>0}$ such that $g_2 = \phi g_1$. A conformal structure of signature (p,q) on E is an equivalence class of conformal pseudo-Riemannian metrics of signature (p,q).

Remark 2.3. Equivalently, a conformal structure on E is the data of a covering U_i of N, together with a smooth metric g_i of signature (p,q) on $p^{-1}(U_i)$ such that for all i,j, there exists $f_{i,j} : U_i \cap U_j \to \mathbf{R}_{>0}$ such that $g_j = f_{i,j} g_i$ on $U_i \cap U_j$.

Let $p : E \to N$ be a vector bundle, and let g be a pseudo-Riemannian metric on E. A bundle morphism $F : E \to E$ over a map $f : N \to N$ is said to be conformal with respect to g if $F^* g$ is conformal to g. The function $\phi : N \to \mathbf{R}_{>0}$ such that $F^* g = \phi g$ is called the *conformal distortion* of F with respect to g.

If a group H acts by conformal bundle automorphisms of E, and if for all $h \in H$ we note $\lambda(h,.)$ the conformal distortion of h with respect to g, then $\lambda : H \times N \to \mathbf{R}_{>0}$ is a cocycle over the action of H on N.

2.2.2. *Vertical conformal class on the suspension.* Let now $\alpha : \Gamma \to \mathrm{Conf}(M, \overline{g})$ be a conformal action of a cocompact higher rank lattice on a compact pseudo-Riemannian manifold $(M, \overline{g})$. Let $\pi : M^\alpha \to G/\Gamma$ be the associated suspension. As said previously, it is a bundle with fiber M. In fact, we have a family of natural parametrizations of the fibers. For any $g \in G$, let $\psi_g : M \to M^\alpha$ be the map $\psi_g(x) = [(g,x)]$. Of course, ψ_g is a proper injective immersion of M into M^α whose image is the fiber over $g\Gamma$. Because any two such parametrizations of a given fiber differ by an element of $\alpha(\Gamma) < \mathrm{Conf}(M, \overline{g})$, we can push-forward the conformal class $[\overline{g}]$ of M onto conformal classes on every fiber of M^α. The following proposition asserts that the result is a G-invariant smooth object.

Proposition 2.4. *To the conformal class* $[\overline{g}]$ *on* M *corresponds a conformal class* $[\overline{g}^\alpha]$ *on the vertical distribution* $F^\alpha \subset TM^\alpha$ *such that all the maps* φ_g *are conformal diffeomorphisms between* M *and the fibers of* M^α*. The vertical differential action of* G *preserves this conformal class.*

Proof. Let $p : G \to G/\Gamma$ be the natural projection. Let $\{D_i\}$ be a collection of trivializing open sets of G, such that $\{p(D_i)\}$ is a covering of G/Γ and $p(D_i) \cap p(D_j)$ is connected for all i,j. Let $\sigma_i : p(D_i) \to D_i$ be the associated section. Then, for all i,j, the map $\sigma_i^{-1}\sigma_j$ defined on $p(D_i) \cap p(D_j)$ takes values in Γ. By continuity, it is constant equal to some γ_{ij}.

We fix $\overline{g}$ a metric in the conformal class of M and we note $\lambda : \Gamma \times M \to \mathbf{R}_{>0}$ the conformal distortion of Γ with respect to $\overline{g}$. Let $U_i = \pi^{-1}(p(D_i))$, $\psi_i : D_i \times M \to U_i$ the trivialization $(g,x) \mapsto [(g,x)]$. Then, we define a metric $\overline{g}_i$ on the vertical tangent bundle of U_i by sending the obvious one on the vertical tangent bundle of $D_i \times M$ via ψ_i. If $f_{ij} : U_i \cap U_j \to \mathbf{R}_{>0}$ is defined for all $(g,x) \in D_i \cap D_j \gamma_{ij}^{-1}$ by $f_{ij}(\psi_i(g,x)) = \lambda(\gamma_{ij}, x)$, then $\overline{g}_i = f_{ij} \overline{g}_j$ over $U_i \cap U_j$.

This conformal structure induces on each fiber of M^α the natural conformal class given by the parametrizations ψ_g. The G-invariance is immediate since $g_0.\psi_g = \psi_{g_0 g}$ for all $g, g_0 \in G$. □

This geometric data on the vertical bundle F^α restricts the vertical Lyapunov spectrum of any A-invariant measure, where $A < G$ is any Cartan subspace.

3. Linear relations between Lyapunov functionals

We assume in this section that $E \to N$ is a vector bundle of rank $n = p + q$, over a compact manifold N, and that E is endowed with a conformal class $[g]$ of pseudo-Riemannian metrics of signature (p, q). Assume that we are given a conformal action of $A = \mathbf{R}^k$ on E. Given an A-invariant, A-ergodic measure on N, we establish general linear relations among the associated Lyapunov functionals given by Oseledec's theorem, that we recall below.

3.1. **Higher rank Oseledec's theorem.** As a consequence of the higher rank version of Oseledec's theorem ([6], Theorem 2.4), we obtain here:

Theorem. *Assume that a connected abelian group $A \simeq \mathbf{R}^k$ acts differentiably on N and that its action lifts to an action by bundle automorphisms of E. Let μ be an A-invariant, A-ergodic probability measure on N. Then, there exist:*

1. *a measurable set $\Lambda \subset N$ of μ-measure 1,*
2. *a finite set of linear forms $\chi_1, \ldots, \chi_r \in \mathfrak{a}^*$,*
3. *and a measurable, A-invariant splitting $E = E_1 \oplus \cdots \oplus E_r$ defined over Λ,*

such that for any Riemannian norm $\|.\|$ on E and for every $x \in \Lambda$ and every $v \in E_i(x) \setminus \{0\}$,

$$\frac{1}{|X|}(\log \|e^X.v\| - \chi_i(X)) \xrightarrow[\substack{|X|\to\infty \\ X\in\mathfrak{a}}]{} 0,$$

and

$$\frac{1}{|X|}(\log |\det \mathrm{Jac}_x(e^X)| - \sum_{1\leqslant i\leqslant r} \chi_i(X) \dim E_i(x)) \xrightarrow[\substack{|X|\to\infty \\ X\in\mathfrak{a}}]{} 0,$$

where $\mathrm{Jac}_x(e^X)$ denotes the matrix of $e^X : E(x) \to E(e^X.x)$ with respect to some bounded measurable frame field of E.

Remark 3.1. By compactness of N, the classic integrability condition of the cocycle of the action is immediate since we assume the action of A smooth. We also skip the conclusion on the angles which we will not use.

3.2. **Asymptotic conformal distortion and orthogonality relations.** Let $\Lambda \subset N$ be the set of full measure where the conclusions of Oseledec's theorem are valid. Let $E(x) = E_1(x) \oplus \cdots \oplus E_r(x)$ be the corresponding A-invariant decomposition given for all $x \in \Lambda$, and let $\chi_1, \ldots, \chi_r \in \mathfrak{a}^*$ be the Lyapunov functionals.

We fix a metric g on E in the conformal class. We note $\lambda : A \times N \to \mathbf{R}_{>0}$ the conformal distortion of A with respect to g. It is a cocycle over the action of A on N with values in $\mathbf{R}_{>0}$. Restricting Λ if necessary, there is another linear form $\chi : \mathfrak{a} \to \mathbf{R}$ such that $\frac{1}{|X|}(\log |\lambda(e^X, x)| - \chi(X)) \to 0$ as $|X| \to \infty$ in $\mathfrak{a}$.

Remark 3.2. By compactness of N, any other metric in the conformal class $[g]$ is of the form φg, where $\varphi : N \to \mathbf{R}_{>0}$ takes values in a bounded interval. Thus, the linear form χ does not depend on the choice of g in the conformal class.

Remark 3.3. In fact, $n\chi/2$ coincides with $\sum_{1\leqslant i\leqslant r} \dim E_i \chi_i$. This can be seen by considering a linear cocycle of the A-action, lying in $\mathbf{R}_{>0} \times O(p,q)$. The Jacobian determinant will then be the conformal distortion to the power $n/2 = \dim E/2$.

There are more linear relations between the χ_i's which are coming from orthogonality relations between the Oseledec's spaces. They will be obtained by using the following observation.

Lemma 3.4. *For any i,j and $x \in \Lambda$, if $\chi_i + \chi_j \neq \chi$, then $E_i(x) \perp E_j(x)$.*

Proof. Let us choose an element $X \in \mathfrak{a}$ such that $\chi_i(X) + \chi_j(X) < \chi(X)$. If $\|.\|$ denotes an arbitrary Riemannian metric on E, then by compactness of N, there is $C > 0$ such that for all $x \in N$ and $u, v \in E$, we have $|g_x(u,v)| \leqslant C\|u\|\|v\|$. Thus, if $x \in \Lambda$ and u, v are in $E_i(x)$ and $E_j(x)$ respectively, then from

$$\lambda(e^X, x)|g_x(u,v)| = |g_{e^X.x}(e^X u, e^X v)| \leqslant C\|e^X u\|\|e^X v\|,$$

we get

$$\chi(X) \leqslant \chi_i(X) + \chi_j(X),$$

unless $g_x(u,v) = 0$. By the choice of X, we obtain $E_i(x) \perp E_j(x)$ for all $x \in \Lambda$. □

3.3. **General linear relations.** We still consider a vector bundle $E \to N$ over a compact manifold N endowed with a conformal structure of signature (p,q) with $p \leqslant q$, preserved by an action of an abelian Lie group $A = \mathbf{R}^k$. Let μ be a finite A-invariant, A-ergodic measure on N, and let $\Lambda \subset N$ such that $\mu(\Lambda) = 1$ and $E|_\Lambda = \bigoplus_{1\leqslant i\leqslant r} E_i|_\Lambda$ be the associated decomposition given by Oseledec's theorem. Since A acts ergodically on (M,μ) and conformally on E, we can assume that for all i, the signature of E_i is constant over Λ, as well as the orthogonality relations among the E_i's.

Proposition 3.5. *Let $\chi_1, \dots, \chi_r$ be the Lyapunov functionals of μ, and let $\chi \in \mathfrak{a}^*$ be the Lyapunov functional of the distortion cocyle. Then, $r \leqslant 2p+1$. Moreover, we can reorder the χ_i's such that μ-almost everywhere:*

1. *If $i + j \neq r+1$, then $E_i \perp E_j$.*
2. *If $i \leqslant r/2$, the subspace $E_i \oplus E_{r+1-i}$ is non-degenerate, and E_i and E_{r+1-i} are maximally isotropic in it. Thus, they have the same dimension.*
3. *If r is even, then $p = q$ and all E_i's are totally isotropic.*
4. *If r is odd, then $E_{(r+1)/2}$ is non-degenerate.*

Consequently, when $r = 2s$ is even, the Lyapunov functionals satisfy the relations:

$$\chi_1 + \chi_r = \cdots = \chi_s + \chi_{s+1} = \chi.$$

And when $r = 2s+1$ is odd, they satisfy the relations:

$$\chi_1 + \chi_r = \cdots = \chi_s + \chi_{s+2} = 2\chi_{s+1} = \chi.$$

Remark 3.6. It has to be noted that these linear forms generate a linear subspace of $\mathfrak{a}^*$ of dimension at most $p+1$.

Proof. We permute the indices such that there is $X \in \mathfrak{a}$ such that $\chi_1(X) < \cdots < \chi_r(X)$.

Case 1: There exists i such that E_i is not totally isotropic.

Lemma 3.7. *The space E_i is non-degenerate and orthogonal to $\bigoplus_{j \neq i} E_j$, which has signature (p', p') for some $p' \leqslant p$, and $\bigoplus_{j<i} E_j$ and $\bigoplus_{j>i} E_j$ are totally isotropic.*

Proof. By Lemma 3.4, we get $\chi = 2\chi_i$. Thus, if $j \leqslant i$ and $k < i$, then we have $\chi_j(X) + \chi_k(X) < \chi(X)$, and the same lemma implies that $\bigoplus_{1 \leqslant j < i} E_j$ is totally isotropic and orthogonal to E_i. Similar arguments work of course for indices greater than i and we obtain that $E_i^\perp$ contains $\bigoplus_{1 \leqslant j < i} E_j \oplus \bigoplus_{i < j \leqslant r} E_j$. The dimensions imply equality, and finally $E_i \cap E_i^\perp = 0$. The other claim is immediate because $E_i^\perp$ is non-degenerate and if $\mathbf{R}^{p',q'} = V_1 \oplus V_2$ with V_1, V_2 totally isotropic, then $p' = q'$ and $\dim V_1 = \dim V_2 = p'$. □

Inside $\bigoplus_{j \neq i} E_j$, the subspaces $\bigoplus_{1 \leqslant j < i} E_j$ and $\bigoplus_{r \geqslant j > i} E_j$ are maximally isotropic. Thus, for all $j < i$, there exists $f(j) > i$ such that E_j and $E_{f(j)}$ are not orthogonal, because if not $\bigoplus_{r \geqslant j > i} E_j$ would not be maximally isotropic. Moreover, the integer $f(j)$ is uniquely determined by $\chi_j(X) + \chi_{f(j)}(X) = \chi(X)$. The same relation also implies that $\{j \mapsto f(j)\}$ is strictly decreasing because $\chi_1(X) < \cdots < \chi_r(X)$.

By symmetry, $f : \{1, \ldots, i-1\} \to \{i+1, \ldots, r\}$ must be a bijection, and r is odd, equal to $2i - 1$, and $f(j) = r + 1 - j$. We obtain that E_j and E_{r+1-j} are not orthogonal for $j < i$, implying $\chi_j + \chi_{r+1-j} = \chi$. Consequently, always by Lemma 3.4, all other couples $E_j, E_{j'}$ are orthogonal. Indeed, if for instance $j < i$ and $i < k < r + 1 - j$, then $\chi_j(X) + \chi_k(X) < \chi_j(X) + \chi_{r+1-j}(X) = \chi(X)$. Thus, $\chi_j + \chi_k \neq \chi$ and we can apply Lemma 3.4.

For all $j < i$, $E_j \oplus E_{r+1-j}$ is not totally isotropic and orthogonal to the sum of all other spaces. By the same argument as in the proof of Lemma 3.7, it must be non-degenerate. Consequently, E_j and E_{r+1-j} are maximally isotropic in it, and thus have the same dimension.

Case 2: For all i, E_i is totally isotropic.

Since the metric is non-degenerate, for all i, there exists $f(i)$ such that E_i and $E_{f(i)}$ are not orthogonal. Thus $f(i)$ is uniquely determined by $\chi_i + \chi_{f(i)} = \chi$, proving that f is strictly decreasing. Necessarily, $f(i) = r+1-i$ and r must be even (if not, $E_{(r+1)/2}$ would not be totally isotropic). Consequently, if $i + j \neq r + 1$ then E_i and E_j are orthogonal. Therefore, $\bigoplus_{i \leqslant r/2} E_i$ is totally isotropic, and so is $\bigoplus_{i > r/2} E_i$. These subspaces being in direct sum, the full space must have split signature (p, p).

Similarly to the end of Case 1, we conclude that $E_i \oplus E_{r+1-i}$ is non-degenerate and E_i and E_{r+1-i} are maximally isotropic in it. □

4. Invariant measures and cocyle super-rigidity

From now on, we consider the main objects of this article, that are the data of a conformal action $\alpha : \Gamma \to \mathrm{Conf}(M, \overline{g})$, where Γ is a cocompact lattice in a non-compact simple Lie group G with finite center and of real-rank at least 2, and $(M, \overline{g})$ a closed pseudo-Riemannian manifold of signature (p, q), with $p + q \geqslant 3$ and $p \leqslant q$. The global assumption that we make is that the image of α in $\mathrm{Conf}(M, \overline{g})$ is unbounded. We still note $\pi : M^\alpha \to G/\Gamma$ the suspension of this action.

4.1. **Finite Γ-invariant measures.** The aim of this section is to establish the proposition below, valid also when Γ is non-uniform, and saying that when G is large enough, there are no finite, Γ-invariant measures on M. For instance, Γ will have no finite orbit on M and the action will be essential.

Proposition 4.1. *Let G be as above and assume moreover that $\mathfrak{g}$ cannot be embedded into $\mathfrak{so}(p,q)$. Let $\Gamma < G$ be a lattice that acts conformally on a compact pseudo-Riemannian manifold $(M,\overline{g})$ of signature (p,q), and such that the image of Γ in $\mathrm{Conf}(M,\overline{g})$ is unbounded. Then, Γ does not preserve any finite measure on M.*

Remark 4.2. We emphasize that the ideas we use in its proof are not new, and largely inspired from former works of Zimmer. To our knowledge, even though several similar results were already established, such a statement is not explicitly written or proved in the literature. For the sake of self-completeness, we have chosen to give a complete exposition of the arguments.

We first observe that under our assumptions, the image of Γ in $\mathrm{Conf}(M,\overline{g})$ is closed. It is a consequence of the general following result.

Lemma 4.3. *Let G' be a Lie group and $\rho : \Gamma \to G'$ a morphism such that $\rho(\Gamma)$ is not relatively compact in G'. Then, $\rho(\Gamma)$ is closed in G'.*

Proof. Let H be the closure of $\rho(\Gamma)$ in G'. Let $\Gamma_0 = \rho^{-1}(H_0)$ be the preimage of the identity component of H. Then, Γ_0 is normal in Γ and as such, must be either finite or has finite index in Γ. We claim that Γ_0 is finite. To see it, we assume to the contrary that it has finite index in Γ. Since Γ_0 has property (T), we deduce that H_0 also has property (T) according to Theorem 1.3.4 of [3]. Let $R \lhd H_0$ be its solvable radical.

- Case 1: H_0/R is non-compact. Composing ρ with the projection, we obtain a morphism $\Gamma_0 \to H_0/R$ with dense image. As it follows from Margulis' super-rigidity theorem, there does not exist morphism $f : \Gamma_0 \to S$ into a connected, non-compact, semi-simple Lie group S such that $f(\Gamma_0)$ is dense in S for the strong topology, and we obtain a contradiction. We omit details here, the idea is that the projection of $f(\Gamma_0)$ on a non-compact simple factor would still have to be strongly dense, but at the same time a lattice by super-rigidity.
- Case 2: H_0/R is compact. In this case, H_0 is amenable. Since it also has (T), H_0 itself is compact. This contradicts the fact that $\rho(\Gamma)$ is unbounded in G'.

Finally, we get that Γ_0 is finite, and since it is dense in H_0, we conclude that $\Gamma_0 = \mathrm{Ker}\,\rho$ and $H_0 = \{e\}$, *i.e.* $\rho(\Gamma) = H$. □

Thus Γ is closed in $\mathrm{Conf}(M,\overline{g})$ - without assuming that $\mathfrak{g}$ does not embed into $\mathfrak{so}(p,q)$. We remind that the Lie group structure of $\mathrm{Conf}(M,\overline{g})$ is defined by making it act on B, the second prolongation of the $(\mathbf{R}_{>0} \times O(p,q))$-structure associated to the conformal class $[\overline{g}]$ (see [21], Ch. I, Theorem 5.1). All we need to know here is that B is a principal bundle over M, with structure group $P := (\mathbf{R}_{>0} \times O(p,q)) \ltimes \mathbf{R}^n$, and that the action of $\mathrm{Conf}(M,\overline{g})$ on M lifts to an action by automorphisms of B, which is *free and proper*. The differential structure on $\mathrm{Conf}(M,\overline{g})$ is then obtained by identifying it with any of its orbits in B.

Assume now that a closed subgroup $H < \mathrm{Conf}(M, \overline{g})$ acts on B, preserving a finite measure μ, which we can assume to be H-ergodic. Then, H has to be compact. To see it, consider the natural projection $p : B \to H\backslash B$. Since H is closed, its action on B is proper, and the target space is Hausdorff. Therefore, p must be μ-essentially constant by ergodicity, meaning that H has an orbit of full measure in B (this argument is a basic case of Proposition 2.1.10 of [34]). Since the action is free, this implies that H has finite Haar measure, so H is compact.

Thus, the proof of Proposition 4.1 will be completed with the following lemma based on cocycle super-rigidity, and inspired from the arguments of [30], page 23.

Lemma 4.4. *If $\mathfrak{g}$ does not embed in $\mathfrak{so}(p,q)$ and if there exists a finite Γ-invariant measure μ on M, then there exists a finite Γ-invariant measure μ_B on the prolongation bundle B.*

Proof. Considering an ergodic component, we may assume that μ is Γ-ergodic. Let us note $p : B \to M$ the projection of the bundle, whose fibers are given by the free and proper right action of P on B. The key point is that the action of Γ on the bundle B has to preserve a measurable sub-bundle with compact fiber, and this comes from Zimmer's cocycle super-rigidity. To be precise, the claim is the following.

Sub-lemma 4.5. *The exist a compact subgroup $K \subset P$, a measurable section $\sigma_K : M \to B$, and a cocycle $c_K : \Gamma \times M \to K$ such that*

$$\gamma.\sigma_K(x) = \sigma_K(\gamma.x).c_K(\gamma, x)$$

for all $\gamma \in \Gamma$ and for μ-almost every $x \in M$.

Proof (Sub-lemma 4.5). Let us fix a bounded measurable section $\sigma : M \to B$, and let $c : \Gamma \times M \to P$ be the associated cocycle. We use Fisher-Margulis' extension of Zimmer's cocycle super-rigidity, formulated in Theorem 1.5 of [13]. Up to passing to a finite cover of G and lifting Γ to it, they satisfy the hypothesis of this theorem. We note $P' = (\mathbf{R}^* \times O(p,q)) \ltimes \mathbf{R}^n$ the Zariski closure of P in $O(p+1, q+1)$. We note $\varepsilon = (-1, -\mathrm{id}) \in \mathbf{R}^* \times O(p,q)$ the central element of P' such that $P' = P \sqcup \varepsilon P$.

By assumption, any morphism $\mathfrak{g} \to \mathfrak{so}(p,q)$ is trivial. By simplicity, the first cohomology group of $\mathfrak{g}$ is trivial (this is Whitehead's Lemma), and then any morphism from $\mathfrak{g}$ to $(\mathbf{R} \oplus \mathfrak{so}(p,q)) \ltimes \mathbf{R}^n$ is also trivial. By connectedess of G, every morphism from G to P' is also trivial. Consequently, Theorem 1.5 of [13] gives a compact subgroup $K' < P'$ such that c is cohomologous to a K'-valued cocycle. It means that there exists a measurable $f' : M \to P'$ such that $f'(\gamma.x)^{-1} c(\gamma, x) f'(x) \in K'$ for all γ and for almost every x.

We define $f : M \to P$ by $f(x) = f'(x)$ if $f(x) \in P$ and $f(x) = \varepsilon f'(x)$ if not. Then, for all γ and for μ-almost every x, $f(\gamma.x)^{-1} c(\gamma, x) f(x) \in P \cap (K' \cup \varepsilon K') =: K$. The latter is a compact subgroup of P since $K' \cup \varepsilon K'$ is a compact subgroup of P'. The section $\sigma_K(x) = \sigma(x).f(x)$ is the announced one. □

The set $\Lambda \subset M$ of points of M at which the conclusion of Sub-lemma 4.5 is valid for any $\gamma \in \Gamma$ has full measure and is Γ-invariant. The section σ_K provides a measurable trivialization $\varphi : B \to M \times P$ through which the action of an element γ on $p^{-1}(\Lambda)$ reads $(x, p) \mapsto (\gamma.x, c_K(\gamma, x).p)$ for all $x \in \Lambda$ and $p \in P$. Thus, Γ preserves the Borel set

$\varphi^{-1}(\Lambda \times K)$, and preserves the measure $(\varphi^{-1})_*(\mu \otimes m_K)$ on it, where m_K denotes the Haar measure of K. □

4.2. **Arguments from differentiable dynamics.** We cite in this section general results about differentiable actions of Γ on compact manifolds which give sufficient conditions for the existence of invariant measures. They are proved and used in [8] and [7], but do not require the manifold to be low-dimensional.

We remind the general fact:

Lemma 4.6 ([22], Lem. 6.1)**.** *If G preserves a finite measure on M^α, then Γ preserves a finite measure on M.*

Thus, in our situation, the previous section implies that when $\mathfrak{g}$ does not embed into $\mathfrak{so}(p,q)$, it is not possible to construct any G-invariant finite measure on M^α.

Let $A < G$ be a Cartan subspace. The heuristic of an important step in the proof of [7] is that if the restricted root-system of G is "large" compare to the number of vertical Lyapunov functionals of a well-chosen A-invariant, A-ergodic measure μ on M^α, then μ is invariant under a lot of restricted root-spaces G_λ, and necessarily G-invariant.

Consequently, Proposition 4.4 forbids such a configuration and implies interesting restrictions on the Lyapunov functionals.

4.2.1. *Non-zero vertical Lyapunov exponent.* The proof of Theorem 1 uses the following general property of differentiable actions. It does not appear explicitly in [7], and is used in a simpler approach of Zimmer's conjecture for cocompact lattices of $\mathrm{SL}(n,\mathbf{R})$. An exposition of this simpler proof can be found in [5] and [9].

Proposition 4.7 ([7])**.** *Let $\pi : M^\alpha \to G/\Gamma$ be the suspension of an action $\alpha : \Gamma \to$ $\mathrm{Diff}(M)$, and let $A < G$ be a Cartan subspace. Let μ be an A-invariant, A-ergodic measure on M^α such that $\pi_*\mu$ is the Haar measure of G/Γ. If there exists a non-trivial element $g \in A$ all of whose vertical Lyapunov exponents are zero, then μ is G-invariant.*

Proof. See [5], proof of Theorem 11.1 and 11.1' in Section 11, or [9] Proposition 8.7. The assumption $\dim M < \mathrm{Rk}_{\mathbf{R}}\, G$ is only used to exhibit an element $g \in A$ whose vertical Lyapunov spectrum is reduced to $\{0\}$ (claim (11.1) in the proof of Theorem 11.1 of [5], p. 46). The above statement follows from the arguments presented after this claim. □

In our situation of an unbounded conformal action $\alpha : \Gamma \to \mathrm{Conf}(M,\overline{g})$, the combination of Proposition 4.7 and Proposition 4.1 immediately gives:

Corollary 4.8. *Let $\alpha : \Gamma \to \mathrm{Conf}(M,\overline{g})$ be an unbounded conformal action in signature (p,q). Let μ be an A-invariant, A-ergodic finite measure on M^α which projects to the Haar measure of G/Γ, and let $\chi_1,\dots,\chi_r \in \mathfrak{a}^*$ be the vertical Lyapunov exponents of μ. If $\mathfrak{g}$ does not embed into $\mathfrak{so}(p,q)$, then $\chi_1,\dots,\chi_r$ linearly span $\mathfrak{a}^*$.*

4.2.2. *Resonance.* A more advanced property, proved in [8], is used in [7] to obtain G-invariant measures on M^α. Let $A < G$ be a Cartan subspace.

Definition 4.9. Let μ be an A-invariant, A-ergodic measure on M^α, with vertical Lyapunov functionals $\chi_1,\dots,\chi_r$. A restricted root $\lambda \in \Sigma$ is said to be μ-resonant if there exist a vertical Lyapunov exponent χ_i and $c > 0$ such that $\lambda = c\chi_i$.

Proposition 4.10 ([8], Prop. 5.1)**.** *Let μ be a probability measure on M^α which is A-invariant and A-ergodic and projects to the Haar measure of G/Γ. If $\lambda \in \Sigma$ is not μ-resonant, then μ is G_λ-invariant.*

Following [7], we note $r(\mathfrak{g}) = \min\{\dim(\mathfrak{g}'/\mathfrak{p}'),\ \mathfrak{p}' \text{ proper parabolic subalgebra of } \mathfrak{g}'\}$, where $\mathfrak{g}'$ denotes the real split simple algebra of type $\hat{\Sigma}$, where $\hat{\Sigma} = \Sigma$ when Σ is reduced, and $\hat{\Sigma} = B_\ell$ when $\Sigma = (BC)_\ell$. This integer is called the *resonant codimension* of $\mathfrak{g}$.

Corollary 4.11 ([7])**.** *Assume that any finite A-invariant, A-ergodic measure μ on M^α has at most $r(\mathfrak{g})-1$ vertical Lyapunov functionals. Then, there exists a finite G-invariant measure on M^α.*

Proof. This is proved in Section 5.5 of [7]. Let us give an idea of the proof in the split case. We pick μ any A-invariant, A-ergodic measure on M^α, projecting to the Haar measure of G/Γ, and with vertical Lyapunov exponents $\chi_1, \ldots, \chi_r$ and consider $H < G$ the stabilizer of μ. Since $\mathfrak{a} \subset \mathfrak{h}$, $\mathfrak{h} = \mathfrak{a} \oplus \bigoplus_{\lambda \in S} \mathfrak{g}_\lambda$ for some subset S of Σ. Any root $\lambda \in \Sigma$ which is not positively related to some χ_i must belong to S by Proposition 4.10. This implies that $|S| \geqslant |\Sigma| - r$, and then that $\mathfrak{h}$ has codimension at most $r < r(\mathfrak{g})$. This implies that $\mathfrak{h} = \mathfrak{g}$ (Lemma 2.5 of [7]). ☐

5. Bound on the real-rank and further restrictions

In this section, Γ still denote a cocompact lattice in a non-compact simple Lie group G of real-rank at least 2, and Γ is still assumed to have an unbounded conformal action $\alpha : \Gamma \to \mathrm{Conf}(M, \overline{g})$ on a compact pseudo-Riemannian manifold $(M, \overline{g})$ of signature (p, q), with $p \leqslant q$.

5.1. **Bound on the real-rank.** We have all the ingredients to obtain the announced bound on the real-rank, that is $\mathrm{Rk}_{\mathbf{R}}\, G \leqslant p + 1$.

Let $\pi : M^\alpha \to G/\Gamma$ be the suspension of the action. Let $A < G$ be a Cartan subspace, $B < G$ be a Borel subgroup containing A and let ν be a B-invariant measure on M^α, which exists by amenability of B. Then, $\pi_* \nu$ is a B-invariant measure on G/Γ, thus it must be G-invariant (this follows from Ratner's theory, see for instance Theorem 5.1(d) of [7]), *i.e.* proportional to the Haar measure. Let now μ be any A-ergodic component of ν. Since the action of A on G/Γ is ergodic with respect to the Haar measure, it follows that $\pi_* \mu$ is also proportional to the Haar measure.

Let $\chi_1, \ldots, \chi_r \in \mathfrak{a}^*$ be the vertical Lyapunov exponents of A with respect to μ. By Proposition 3.5, we know that they span a subspace of $\mathfrak{a}^*$ of dimension at most $p + 1$. Thus, if $\mathrm{Rk}_{\mathbf{R}}\, G$ was greater than $p + 1$, then Corollary 4.8 would imply that $\mathfrak{g}$ embeds into $\mathfrak{so}(p, q)$, which is obviously false since $\mathrm{Rk}_{\mathbf{R}}\, \mathfrak{g} > \mathrm{Rk}_{\mathbf{R}}\, \mathfrak{so}(p, q)$.

5.2. **Optimal index for exceptional Lie groups.** Let us observe what could be derived from Proposition 4.10 and Corollary 4.11 in our situation. By Proposition 3.5, for any A-invariant, A-ergodic measure μ on M^α, there are at most $2p+1$ vertical Lyapunov functionals when $p < q$, and at most $2p$ when $p = q$. Thus, from Corollary 4.11 and Proposition 4.1, we deduce that if $\mathfrak{g}$ does not embed into $\mathfrak{so}(p, q)$, then $r(\mathfrak{g}) \leqslant 2p + 1$. Let $\ell \geqslant 2$.

- If $\Sigma = A_\ell$, then $r(\mathfrak{g}) = \ell$ and we obtain $\ell \leqslant 2p + 1$.

- If $\Sigma = B_\ell, C_\ell, (BC)_\ell$, then $r(\mathfrak{g}) = 2\ell - 1$, and we get $\ell \leqslant p+1$.
- If $\Sigma = D_\ell$, then $r(\mathfrak{g}) = 2\ell - 2$, and we get $\ell \leqslant p+1$.
- If $\Sigma = E_6$, then $r(\mathfrak{g}) = 16$, and we get $p \geqslant 7$.
- If $\Sigma = E_7$, then $r(\mathfrak{g}) = 27$, and we get $p \geqslant 13$.
- If $\Sigma = E_8$, then $r(\mathfrak{g}) = 57$, and we get $p \geqslant 28$.
- If $\Sigma = F_4$, then $r(\mathfrak{g}) = 15$, and we get $p \geqslant 7$.
- If $\Sigma = G_2$, then $r(\mathfrak{g}) = 5$, and we get $p \geqslant 2$.

Therefore, in all non-exceptional cases, we obtain either the same inequality as in Section 5.1, or a less good one in the case of A_ℓ.

In the case where Γ is a cocompact lattice in a simple Lie algebra with restricted root system $\mathfrak{g}_2$, *i.e.* when $\Gamma < G_2^{(2)}$, we obtain that the metric index of $(M, \overline{g})$ is at least 2, and this is optimal because if V is the 7-dimensional representation of $\mathfrak{g}_2$, then V admits a $\mathfrak{g}_2$-invariant quadratic form of signature $(3,4)$. Consequently, the full Lie group $G_2^{(2)}$ acts locally faithfully and conformally on $\mathbf{Ein}^{2,3}$.

This completes the proof of Theorem 2.

6. Conformal flatness in maximal real-rank

In this section, we prove the geometric part of our main theorem. We fix a signature (p,q), with $p+q \geqslant 3$ and $p \leqslant q$, a non-compact simple Lie group G of real-rank $p+1$ and a cocompact lattice $\Gamma < G$. We assume that we are given a conformal action $\alpha : \Gamma \to \mathrm{Conf}(M, \overline{g})$ on a compact pseudo-Riemannian manifold $(M, \overline{g})$ of signature (p,q) such that $\alpha(\Gamma)$ is unbounded, and we will prove that $(M, \overline{g})$ is conformally flat.

6.1. **Organization of the proof.** The starting point is that $\mathfrak{g}$ does not embed in $\mathfrak{so}(p,q)$ because of the real-ranks. Thus, Corollary 4.8 implies that for any Cartan subspace $A < G$ and any finite A-invariant, A-ergodic measure μ on the suspension M^α, which projects to the Haar measure of G/Γ, the vertical Lyapunov exponents $\chi_1, \ldots, \chi_r$ span linearly $\mathfrak{a}^*$. In Section 6.2, we deduce from the linear relations satisfied by the χ_i's that there exists a unique $X \in \mathfrak{a}$ such that $\chi_1(X) = \cdots = \chi_r(X) = -1$.

A guiding principle in conformal geometry is that when there exists a sequence of conformal maps (f_k) collapsing an open set to a singular set, say a point or a segment, then we can derive interesting conclusions on the conformal curvature by using conformally invariant tensors. For instance, in Lorentzian signature, if a sequence of conformal maps contracts topologically an open set to a point, then this open set is conformally flat, see [16], Théorème 1.3. However, in general signature this is not true and we have to ask some notion of "uniformity of contraction" to derive conformal flatness.

Here, the existence of an $\mathbf{R}$-split element $X \in \mathfrak{g}$ with a uniform Lyapunov spectrum on M^α indicates that uniform contractions might be observed in the dynamics of Γ on M. Using local stable manifolds of the flow of X in M^α, we will obtain the following in Section 6.3. A Riemannian norm on M is fixed $\|.\|$, and balls refer to its length distance.

Proposition 6.1. *Let $X \in \mathfrak{a}$ and μ be a finite ϕ_X^t-invariant, ϕ_X^t-ergodic measure on M^α admitting exactly one vertical Lyapunov exponent, which is non-zero, and let $(\lambda_k) \to 0$ be a decreasing sequence. Then, there exist $x \in M$ and $g \in G$ such that $[(g,x)] \in \operatorname{Supp} \mu$,* *a sequence* (γ_k) *in* Γ*, an increasing sequence of positive numbers* $(T_k) \to \infty$ *and* $r > 0$ *such that:*

(1) $\gamma_k B(x,r) \subset B(x,r)$ *for all* k*,*
(2) $\gamma_k : B(x,r) \to B(x,r)$ *is* λ_k*-Lipschitz for all* k*,*
(3) $\gamma_k.x \to x$*,*
(4) *For all* $v \in T_xM \setminus \{0\}$*,* $\frac{1}{T_k} \log \|D_x\gamma_k.v\| \to -1$*.*
(5) $\frac{1}{T_k} \log |\det \mathrm{Jac}_x \gamma_k| \to -n$*.*

For the last point, $\mathrm{Jac}_x \gamma_k \in \mathrm{GL}(n, \mathbf{R})$ is the Jacobian matrix of $D_x\gamma_k$ with respect to a given measurable bounded frame field on $B(x,r)$, *i.e.* a measurable section of the frame bundle of $B(x,r)$ whose image is contained in a compact subset of the bundle. Any change of this bounded frame field will not modify point (5) in the Proposition.

The next step makes a crucial use of the rigidity of the conformal structure of $(M, \overline{g})$. Using results of Frances ([16]) on degeneracy of conformal maps, we will prove that the derivatives of the sequence (γ_k) obtained in Proposition 6.1 have the same exponential growth *at any point* in $B(x,r)$. It will directly follow from the proposition below, proved in Section 6.4.

Proposition 6.2. *Let* $x \in M$*,* U *be a connected neighborhood of* x *and* $(f_k) \in \mathrm{Conf}(M, \overline{g})$ *be a sequence such that:*

(1) *any* $y \in U$ *admits a neighborhood* V *such that* $f_k(\overline{V}) \to \{x\}$ *for the Hausdorff topology,*
(2) *there exists* $x_0 \in U$ *such that for all* $v \in T_{x_0}M \setminus \{0\}$*,* $\frac{1}{T_k} \log \|D_{x_0} f_k v\| \to -1$ *and* $\frac{1}{T_k} \log |\det \mathrm{Jac}_{x_0} f_k| \to -n$*.*

Then, for all $y \in U$ *and* $v \in T_yM \setminus \{0\}$*,* $\frac{1}{T_k} \log \|D_y f_k v\| \to -1$*.*

Using these uniform contractions, we will deduce the following corollary in Section 6.5.

Corollary 6.3. *Let* $X \in \mathfrak{a}$ *and* μ *be a* ϕ_X^t*-invariant,* ϕ_X^t*-ergodic measure on* M^α *admitting exactly one vertical Lyapunov exponent, which is non-zero. Then, there exist* $x \in M$ *and* $g \in G$ *such that* $[(g,x)] \in \mathrm{Supp}\,\mu$ *and such that* x *admits a conformally flat neighborhood.*

Finally, we will conclude in Section 6.6 that any compact, Γ-invariant subset of M intersects a conformally flat open set, and conformal flatness of all of M will easily follow.

We remind that throughout this section, G is assumed to have real-rank $p+1$, that $\Gamma < G$ is a cocompact lattice, $\alpha : \Gamma \to \mathrm{Conf}(M, \overline{g})$ is an unbounded conformal action on a compact pseudo-Riemannian manifold of signature (p,q), with $p \leqslant q$. We fix $A < G$ a Cartan subspace.

6.2. **A direction with uniform vertical Lyapunov spectrum.** Let μ be a finite A-invariant, A-ergodic measure on M^α which projects to the Haar measure of G/Γ. Let $\chi_1, \ldots, \chi_r \in \mathfrak{a}^*$ be the vertical Lyapunov functionals of A with respect to μ and let $\chi \in \mathfrak{a}^*$ be the linear form associated to the conformal distortion (see Section 3.2). The aim of this section is to prove the following.

Lemma 6.4. *If $p = q$, then $r = 2p$ and if $p < q$ then $r = 2p + 1$. Moreover, there exists a unique $X \in \mathfrak{a}$ such that:*

$$\chi_1(X) = \cdots = \chi_r(X) = -1.$$

Proof. Since $\mathrm{Rk}_{\mathbf{R}}\,\mathfrak{g} > \mathrm{Rk}_{\mathbf{R}}\,\mathfrak{so}(p,q)$, $\mathfrak{g}$ does not embed into $\mathfrak{so}(p,q)$ and Corollary 4.8 implies that $\chi_1, \ldots, \chi_r$ linearly span $\mathfrak{a}^*$.

Assume that the Lyapunov functionals have been indexed such that they satisfy the relations given by Proposition 3.5. The latter ensures that $r \leqslant 2p+1$. For all $1 \leqslant i \leqslant r/2$, we note P_i the space spanned by χ_i and χ_{r+1-i}. We note that $\chi \in \bigcap_{1\leqslant i\leqslant r/2} P_i$.

Sub-lemma 6.5. *$\lfloor \frac{r}{2} \rfloor = p$ and $P_1, \ldots, P_p$ are planes such that for all i, $P_i \nsubseteq \sum_{j\neq i} P_j$.*

Proof. For all $1 \leqslant i, j \leqslant r/2$, we have $P_j = P_i + \mathbf{R}.\chi_j$ because $\chi_{r+1-j} = \chi - \chi_j$. Thus, for all $i \leqslant r/2$, we get $p + 1 = \dim \mathfrak{a}^* = \dim \sum_{j\leqslant r/2} P_j \leqslant \dim P_i + (\lfloor \frac{r}{2} \rfloor - 1) \leqslant p + 1$. Therefore, $\dim P_i = 2$, $\lfloor \frac{r}{2} \rfloor = p$, and $\chi_i \notin \sum_{j\neq i} P_j$. □

Thus, $r \geqslant 2p$. So, if $p = q$, then $r = 2p$ since $r \leqslant \dim M$. And if $p < q$, then r is odd by Proposition 3.5, and then $r = 2p + 1$. We note that for all i, χ_i and χ are linearly independent.

We identify $\mathfrak{a}^*$ with $\mathfrak{a}$ with some Euclidean norm so that we are now dealing with a problem in a Euclidean space. For all $i \leqslant p$, we note that $\chi_i - \chi/2$ and $\chi_{r+1-i} - \chi/2$ are proportional and non-zero. Let ξ_i be a unit vector giving this direction and $H_i = \xi_i^{\perp}$. Because χ and ξ_i span P_i, the ξ_i's are linearly independent.

Then, $L = \bigcap_{1\leqslant i\leqslant p} H_i = \mathrm{Span}(\xi_1, \ldots, \xi_p)^{\perp}$ is a line, and $\chi \notin L^{\perp}$ because if not L would be orthogonal to $\chi, \xi_1, \ldots, \xi_p$ which span $\mathfrak{a}^*$. Thus, the unique $X \in L$ such that $\chi(X) = 2$ is the announced one. □

6.3. **From dynamics in M^{α} to dynamics in M.** In this section, we assume that there exists $X \in \mathfrak{a}$ and a ϕ_X^t-invariant, ϕ_X^t-ergodic measure μ on M^{α} whose vertical Lyapunov spectrum is $\{-1\}$, with multiplicity $n = \dim M$. We let $\Lambda \subset \mathrm{Supp}(\mu)$ denote the set of full measure where the conclusion of Oseledec's theorem are valid.

6.3.1. *Local stable manifolds.* Horizontally, the Lyapunov spectrum of $e^X \in A$ is simply given by the restricted root-spaces. Let us note $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots of $\mathfrak{a}$, and $\Sigma_X^- = \{\xi \in \Sigma \ : \ \xi(X) < 0\}$. We identify any element $X_0 \in \mathfrak{g}$ with the vector field on M^{α} whose flow is $x^{\alpha} \mapsto e^{tX_0}.x^{\alpha}$. Note that its projection on G/Γ is the right-invariant vector field X_0^R and for any $g \in G$, $g_* X_0(x^{\alpha}) = (\mathrm{Ad}(g)X_0)(x^{\alpha})$. In particular, $(e^{tX})_* X_{\xi} = e^{t\xi(X)} X_{\xi}$ for any $X_{\xi} \in \mathfrak{g}_{\xi}$. Thus, the action on the horizontal distribution being completely known, we get that the full Lyapunov spectrum of ϕ_X^t is $\{-1\} \cup \{\xi(X), \ \xi \in \Sigma\}$ and the stable distribution in Λ is $F_{x^{\alpha}} \oplus \sum_{\xi\in\Sigma_X^-} \mathfrak{g}_{\xi}(x^{\alpha})$.

We fix $0 < \lambda < 1$ such that $\xi(X) < -\lambda$ for all $\xi \in \Sigma_X^-$. Then, Pesin theory gives us a ϕ_X^t-invariant set of full μ measure $\Lambda' \subset \Lambda$ such that for all $x^{\alpha} \in \Lambda'$, there exists a local stable manifold $W_s^{\mathrm{loc}}(x^{\alpha})$ near x^{α}. This $W_s^{\mathrm{loc}}(x^{\alpha})$ is an embedded ball containing x^{α} and whose tangent space at x^{α} is the stable distribution, and for all x^{α}, there exists $C(x^{\alpha}) > 0$ such that for any $y^{\alpha}, z^{\alpha} \in W_s^{\mathrm{loc}}(x^{\alpha})$, $d(\phi_X^t(y^{\alpha}), \phi_X^t(z^{\alpha})) \leqslant C(x^{\alpha}) d(y^{\alpha}, z^{\alpha}) e^{-\lambda t}$.

Lemma 6.6. *For all $x^{\alpha} \in \Lambda'$, projecting on $g\Gamma$, the local stable manifold $W_s^{loc}(x^{\alpha})$ contains an open neighborhood of x^{α} in the fiber $\pi^{-1}(g\Gamma)$.*

Proof. Consider the projection $\pi(W_s^{\mathrm{loc}}(x^\alpha))$ in G/Γ. It is contained in the (future) maximal stable manifold of $g\Gamma$ for the action of e^{tX} on G/Γ

$$W_s(g\Gamma) = \{g'\Gamma \ : \ \overline{\lim}_{t\to+\infty} \frac{1}{t} \log d(e^{tX}g'\Gamma, e^{tX}g\Gamma) < 0\},$$

because the projection $\pi : M^\alpha \to G/\Gamma$ is a Lipschitz map. If $G_X^- < G$ is the analytic Lie subgroup associated to $\sum_{\xi\in\Sigma_X^-} \mathfrak{g}_\xi$, then $W_s(g\Gamma) = G_X^-.g\Gamma$.

Shrinking $W_s^{\mathrm{loc}}(x^\alpha)$ if necessary, there exists a neighborhood V of id in G such that $\pi(W_s^{\mathrm{loc}}(x^\alpha)) \subset Vg\Gamma$ and such that in $Vg\Gamma$, every leaf of the foliation defined by the local action of G_X^- is a closed, connected submanifold of $Vg\Gamma$ of dimension $\dim G_X^-$. By connectedness, $\pi(W_s^{\mathrm{loc}}(x^\alpha))$ is contained in a single such leaf, say $\mathcal{L}$. Shrinking V and $W_s^{\mathrm{loc}}(x^\alpha)$ once more if necessary, $\pi^{-1}(\mathcal{L}) \subset M^\alpha$ is a closed submanifold, of dimension $\dim M + \dim G_X^-$, which contains $W_s^{\mathrm{loc}}(x^\alpha)$. The latter having the same dimension, the result follows. □

6.3.2. *Proof of Proposition 6.1.* Let (λ_k) be a decreasing sequence of positive numbers. We exhibit here a sequence (γ_k) in Γ as claimed in Proposition 6.1. We still note $\Lambda' \subset \mathrm{Supp}(\mu)$ the set of full measure where the local stable manifolds are defined. Since μ-almost every point is recurrent for the flow ϕ_X^t, we choose a recurrent point $x^\alpha \in \Lambda'$. The idea is to consider suitable "pseudo-return maps" in a trivialization tube near x^α.

Shrinking $W_s^{\mathrm{loc}}(x^\alpha)$ if necessary, we can assume that there is an open ball $D \subset G$ such that the $D\gamma$, $\gamma \in \Gamma$ are pairwise disjoint, and letting $\overline{D}$ denote its projection in G/Γ, such that $W_s^{\mathrm{loc}}(x^\alpha) \subset \pi^{-1}(\overline{D}) =: T_D$. We note $\psi_D : (g,x) \in D \times M \mapsto [(g,x)] \in T_D$, and we let π_M denote the map $\mathrm{pr}_2 \circ \psi_D^{-1} : T_D \to M$ and $\pi_D = \mathrm{pr}_1 \circ \psi_D^{-1} : T_D \to D$.

We fix $\|.\|_M$ and $\|.\|_G$ Riemannian metrics on M and G. Reducing D if necessary, there exists a Riemannian metric $\|.\|$ on M^α whose restriction to T_D is the push-forward by ψ_D of the orthogonal sum of $\|.\|_M$ and $\|.\|_G$. All the distances and balls in M^α, M or $M \times G$ will implicitly refer to these metrics. These choices have of course no importance, the aim is to simplify the notations.

Let $x = \pi_M(x^\alpha)$ and $g = \pi_D(x^\alpha) \in D$. By a general fact of Riemannian geometry, there exists $r_0 > 0$ such that for all $r < r_0$, the ball $B(x^\alpha, r)$ is strongly geodesically convex, in the sense that for any two points of this ball, there exists a unique minimizing geodesic joining them and contained in the ball. Thus, if r is small enough, $B(x^\alpha, r) \subset T_D$ and ψ_D^{-1} embeds isometrically $B(x^\alpha, r)$ into $D \times M$, as metric spaces. This proves that the restrictions of π_M and π_D to $B(x^\alpha, r)$ are 1-Lipschitz. Reducing r once more if necessary, by Lemma 6.6, we may assume that we also have $B(x^\alpha, r) \cap \pi^{-1}(g\Gamma) \subset W_s^{\mathrm{loc}}(x^\alpha)$.

We fix once and for all such an $r > 0$. By construction, $\psi_D(\{g\} \times B(x,r)) \subset B(x^\alpha, r) \cap \pi^{-1}(g\Gamma)$, and the map $y \in B(x,r) \mapsto \psi_D(g,y) \in B(x^\alpha, r)$ is isometric. Now we choose an increasing sequence $(T_k) \to \infty$ such that:

- $d(\phi_X^{T_k}(x^\alpha), x^\alpha) < \frac{r}{k+1}$ for all $k \geqslant 1$;
- $2rC(x^\alpha)e^{-\lambda T_k} < \min(\lambda_k, r/2)$.

We have the following observation.

Fact 1. *Let $T>0$ be such that $e^{TX}g\Gamma \in \overline{D} \subset G/\Gamma$, and let $\gamma \in \Gamma$ be the unique element such that $e^{TX}g\gamma^{-1} \in D$. Then, for all $y^\alpha \in \pi^{-1}(g\Gamma)$, we have*

$$\pi_M(e^{TX}y^\alpha) = \gamma.\pi_M(y^\alpha). \tag{1}$$

Proof. If $y = \pi_M(y^\alpha)$, then $y^\alpha = \psi_D(g,y) = [(g,y)]$, and $e^{TX}y^\alpha = [(e^{TX}g,y)] = [(e^{TX}g\gamma^{-1},\gamma.y)]$, proving the claim. □

For any $k \geqslant 1$, $\phi_X^{T_k}(x^\alpha) \in B(x^\alpha,r) \subset T_D$, thus $e^{T_kX}g\Gamma \in \overline{D}$. Let γ_k be the unique element of Γ such that $e^{T_kX}g\gamma_k^{-1} \in D$. Let us see that these choices are convenient.

1. Let $y \in B(x,r)$. Then, $y^\alpha := \psi_D(g,y) \in W_s^{\mathrm{loc}}(x^\alpha) \cap B(x^\alpha,r)$. By the choice of T_k, we get that $d(\phi_X^{T_k}(x^\alpha),\phi_X^{T_k}(y^\alpha)) < r/2$. By the previous observation, $\pi_M(\phi_X^{T_k}(x^\alpha)) = \gamma_k.x$ and $\pi_M(\phi_X^{T_k}(y^\alpha)) = \gamma_k.y$. Thus, since π_M is 1-Lipschitz in restriction to $B(x^\alpha,r)$, we obtain $d(x,\gamma_k.x) < r/(k+1)$ and $d(\gamma_k.x,\gamma_k.y) < r/2$, proving the first and the third point of Proposition 6.1.
2. For any $y,z \in B(x,r)$, since $\psi_D(g,y),\psi_D(g,z) \in W_s^{\mathrm{loc}}(x^\alpha) \cap B(x^\alpha,r)$, the same considerations as above give $d(\gamma_k.y,\gamma_k.z) \leqslant d(\phi_X^{T_k}\psi_D(g,y),\phi_X^{T_k}\psi_D(g,z)) < \lambda_k d(\psi_D(g,y),\psi_D(g,z))$. Thus, γ_k is λ_k-Lipschitz since $d(\psi_D(g,y),\psi_D(g,z)) = d(y,z)$.
3. The third point has already been observed.
4. If we differentiate the relation (1), we obtain for any vector v^α tangent to $\pi^{-1}(g\Gamma)$ and for all $k \geqslant 1$, $D\pi_M \circ D\phi_X^{T_k}(v^\alpha) = D\gamma_k \circ D\pi_M(v^\alpha)$. Since $D\pi_M$ preserves the Riemannian norm in the vertical direction, this proves the fourth point because $\frac{1}{T}\log\|D_{x^\alpha}\phi_X^T v^\alpha\| \to -1$ by assumption on X.
5. We choose a bounded measurable frame field on TM, that we pullback on the vertical tangent bundle of T_D via π_M. We can then arbitrarily complete it into a bounded measurable frame field on F^α. With respect to it, we have by construction $\mathrm{Jac}_{x^\alpha}\phi_X^{T_k} = \mathrm{Jac}_x\gamma_k$ for all $k \geqslant 1$, and the result follows.

6.4. **Strong stability of sequences of conformal maps.** In this section, we establish Proposition 6.2. We start by introducing some tools of conformal geometry.

6.4.1. *Definitions and general results.* We note $P = (\mathbf{R}_{>0} \times O(p,q)) \ltimes \mathbf{R}^n$ and $\pi : B \to M$ the P-principal bundle obtained by the prolongation procedure (we no longer work with the suspension of the action, and forget that π used to denote its projection). We call $\pi : B \to M$ the Cartan bundle associated to $(M,[\overline{g}])$. We remind that any conformal map f of M lifts to a bundle automorphism of B, and that the action of $\mathrm{Conf}(M,\overline{g})$ on B is free and proper. Consequently, if a sequence $(f_k) \to \infty$ is such that $f_k(x) \to x_\infty$ for $x, x_\infty \in M$, then $f_k(b) \to \infty$ for any b in the fiber of x. The notion of holonomy sequence quantifies the divergence of $(f_k(b))$ in the fiber direction.

Definition 6.7. Let $(f_k) \in \mathrm{Conf}(M,\overline{g})$ be a sequence of conformal maps and $x,y \in M$ such that $f_k(x) \to y$. A sequence (p_k) in P is said to be a *holonomy sequence of* (f_k) *at* x if there exists a bounded sequence (b_k) in $\pi^{-1}(x)$ such that $f_k(b_k).p_k^{-1}$ also stays in a bounded domain of B.

The holonomy sequence of (f_k) at x is uniquely defined up to compact perturbations, *i.e.* for any two holonomy sequences (p_k) and (p'_k), there exist $(l_k^1),(l_k^2)$ bounded sequences in P such that $p'_k = l^1_k p_k l^2_k$ for all k (we remind that the action of P on B is free and proper).

We will use the following important observation of Frances.

Lemma 6.8 ([16], Lem. 6.1)**.** *Assume that a sequence of conformal maps (f_k) of $(M, \overline{g})$ converges for the $\mathcal{C}^0$-topology to a continuous map $f : M \to M$. Then, there exists a sequence (p_k) which is a holonomy sequence of (f_k) at any point of M.*

6.4.2. *Lyapunov regularity and holonomy sequences.* Let $(T_k) \to \infty$ be a sequence of positive numbers. We remind the following definition (see for instance [19], Definition in Section 4).

Definition 6.9. Let (g_k) be a sequence of matrices in $\mathrm{GL}(n, \mathbf{R})$. We say that (g_k) is (T_k)-Lyapunov regular if there exist a flag $\mathbf{R}^n = E_r \supsetneq E_{r-1} \supsetneq \cdots \supsetneq E_1 \supsetneq E_0 = \{0\}$ and numbers $\chi_1 < \cdots < \chi_r$ such that $\frac{1}{T_k} \log |g_k v| \to \chi_i$ for all $1 \leqslant i \leqslant r$ and $v \in E_i \setminus E_{i-1}$, and $\frac{1}{T_k} \log |\det g_k| \to \sum_i \chi_i (\dim E_i - \dim E_{i-1})$. We will say that (g_k) is uniformly (T_k)-Lyapunov regular when $r = 1$.

A sequence (f_k) of diffeomorphisms of M is said to be (resp. uniformly) (T_k)-Lyapunov regular at x if in some (equivalently any) bounded measurable frame field, $\mathrm{Jac}_x f_k$ is (resp. uniformly) (T_k)-Lyapunov regular.

The following fact follows easily from Theorem 4.1 of [19] when $T_k = k$. The proof is easily adaptable, but we give elementary arguments for this basic situation.

Lemma 6.10. *A sequence (g_k) in $\mathrm{GL}(n, \mathbf{R})$ is uniformly (T_k)-Lyapunov regular if and only if the limit $\chi_{\det} := \lim \frac{1}{T_k} \log |\det g_k|$ exists and $\frac{1}{T_k} \log \|g_k\| \to \frac{\chi_{\det}}{n}$.*

Proof. Replacing g_k by $|\det g_k|^{-1/n} g_k$, we may assume that $|\det g_k| = 1$ and the statement is $\frac{1}{T_k} \log |g_k v| \to 0$ for any $v \neq 0$ if and only if $\frac{1}{T_k} \log \|g_k\| \to 0$.

$\boxed{\Rightarrow}$: If $|v| = 1$, then $|g_k v| \leqslant \|g_k\|$ implies $0 \leqslant \underline{\lim} \frac{1}{T_k} \log \|g_k\|$. If $(v_1, \ldots, v_n)$ is an orthonomal basis, then $\|g_k\| \leqslant n \max |g_k v_i|$, implying $\overline{\lim} \frac{1}{T_k} \log \|g_k\| \leqslant 0$.

$\boxed{\Leftarrow}$: For any $v \neq 0$, $\overline{\lim} \frac{1}{T_k} \log |g_k v| \leqslant 0$. For any basis $(v_1, \ldots, v_n)$, there is $C > 0$ such that for all k, $1 = |\det g_k| \leqslant C |g_k v_1| \ldots |g_k v_n|$. Therefore, $\underline{\lim}(\frac{1}{T_k} \sum_i \log |g_k v_i|) \geqslant 0$.

Sub-lemma 6.11. *Let $u^{(1)}_k, \ldots, u^{(n)}_k$ be n sequences of real numbers such that*

1. $\underline{\lim}(\sum_i u^{(i)}_k) \geqslant 0$
2. *for all i, $\overline{\lim}\, u^{(i)}_k \leqslant 0$.*

Then, for all i, $u^{(i)}_k \to 0$.

Proof. Assume for instance that $\underline{\lim}\, u^{(1)}_k < 0$. Then, there exist $\varepsilon > 0$ and an extraction ϕ such that $u^{(1)}_{\phi(k)} < -\varepsilon$. For all $i \geqslant 2$, and for k large enough, $u^{(i)}_{\phi(k)} \leqslant \frac{\varepsilon}{n}$. We get $\sum_i u^{(i)}_{\phi(k)} \leqslant -\frac{\varepsilon}{n}$ for k large enough, implying $\underline{\lim}(\sum_i u^{(i)}_k) < 0$, a contradiction. □

Therefore $\frac{1}{T_k} \log |A_k v_i| \to 0$ for all i, and this for any basis $(v_1, \ldots, v_n)$. □

Consequently, if $(C_k), (C'_k)$ are bounded sequences, then (A_k) is uniformly (T_k)-Lyapunov regular if and only if $(C_k A_k C'_k)$-is uniformly (T_k)-Lyapunov regular, and they have the same Lyapunov exponent. Indeed, let $B_k = C_k A_k C'_k$. Immediately, $\frac{1}{T_k}\log|\det B_k| \to \chi_{\det}$ and from $\|B_k\| \leqslant \|C_k\|\|A_k\|\|C'_k\|$ and $\|A_k\| \leqslant \|C_k^{-1}\|\|B_k\|\|C_k'^{-1}\|$ we get

$$\overline{\lim}\frac{1}{T_k}\log\|B_k\| \leqslant \frac{\chi_{\det}}{n} \leqslant \underline{\lim}\frac{1}{T_k}\log\|B_k\|.$$

We note $\rho : P \to \mathrm{GL}(\mathbf{R}^n)$ the linear representation given by the projection on the first factor of $P = (\mathbf{R}_{>0} \times O(p,q)) \ltimes \mathbf{R}^n$. We prove now:

Lemma 6.12. *Let (f_k) be a sequence of conformal maps of $(M,\overline{g})$ and $x \in M$ such that $(f_k(x)) \to x_\infty$. The following are equivalent.*

1. *(f_k) is Lyapunov regular at x, with Lyapunov exponents χ_i of multiplicity d_i.*
2. *For any b in the fiber of x and any sequence (p_k) in P such that $f_k(b).p_k^{-1} \to b_\infty$, for some b_∞ in the fiber of x_∞, the sequence $\rho(p_k)$ is Lyapunov regular with Lyapunov exponents χ_i and multiplicity d_i.*

Proof. We note $G' = \mathrm{PO}(p+1,q+1)$ and identify P with the parabolic subgroup of G' fixing a given isotropic line in $\mathbf{R}^{p+1,q+1}$. Then, the representation ρ is the representation $P \to \mathrm{GL}(\mathfrak{g}'/\mathfrak{p})$ induced by the adjoint representation of G'.

We make use of the Cartan connection $\omega \in \Omega^1(B,\mathfrak{g}')$ defined by the conformal structure of M. For any $b \in B$, following [28], Chap. 5, Theorem 3.15., we let $\psi_b : T_x M \to \mathfrak{g}'/\mathfrak{p}$ denote the linear isomorphism defined by $\psi_b(v) = \omega_b(\hat{v})$ mod.$\mathfrak{p}$, for any $\hat{v} \in T_b B$ projecting to v. It satisfies the equivariance property $\psi_{b.p} = \rho(p^{-1})\psi_b$ for any $b \in B$ and $p \in P$. Moreover, for any conformal map f, we have $\psi_{f(b)} \circ D_x f = \psi_b$.

Therefore, if (p_k), b, b_∞ are as in (2), and if $x_k := f_k(x)$, $b_k := f_k(b)p_k^{-1}$, $x_0 = x$ and $b_0 = b$, then we have for $k \geqslant 1$

$$\psi_{b_k} \circ D_x f_k = \rho(p_k)\psi_b.$$

The map $b \mapsto \psi_b$ from B to the frame bundle of M being continuous, the sequence ψ_{b_k} is a bounded sequence of linear frames. Thus, if $\sigma : M \to \mathcal{F}(M)$ is a bounded measurable frame field, then there is a bounded sequence (l_k) in $\mathrm{GL}(\mathfrak{g}'/\mathfrak{p})$ such that $\sigma(x_k) = l_k.\psi_{b_k}$, and we get that $\mathrm{Jac}_x^\sigma(f_k) = l_k.\rho(p_k).l_0^{-1}$. Thus, $\mathrm{Jac}_x^\sigma(f_k)$ is Lyapunov regular if and only if $\rho(p_k)$ is Lyapunov regular, with the same exponents and multiplicities. □

6.4.3. *Proof of Proposition 6.2.* Let (f_k) be a sequence in $\mathrm{Conf}(M,\overline{g})$, $x \in M$, a neighborhood U of x such that any $y \in U$ admits a neighborhood V such that $f_k(\overline{V}) \to \{x\}$, and $x_0 \in U$ such that $\frac{1}{T_k}\log\|D_{x_0}f_k v\| \to -1$ for all $v \in T_{x_0}M\setminus\{0\}$ and $\frac{1}{T_k}\log|\det \mathrm{Jac}_{x_0} f_k| \to -n$, *i.e.* (f_k) is uniformly (T_k)-Lyapunov regular at x_0 with exponent -1.

Since $(f_k|_U)$ converges for the $\mathcal{C}^0$-topology to the constant map equal to x, Lemma 6.8 implies that there exists a sequence (p_k) in P which is a holonomy sequence for any point in U. If b is fixed in the fiber of x, then for any point $y \in U$, there exist bounded sequences $(l_k), (l'_k)$ in P and a point b' in the fiber of y such that $f_k(b').(l_k p_k l'_k)^{-1} \to b$.

Applying Lemma 6.12 at x_0 we obtain that there exist l_k, l'_k such that $\rho(l_k p_k l'_k)$ is uniformly (T_k)-Lyapunov regular with exponent -1. Thus, Lemma 6.10 implies that $\rho(p_k)$ has the same property. Now, if $y \in U$, and if m_k, m'_k are relatively compact and such that $f_k(b').(m_k p_k m'_k)^{-1} \to b$, then $\rho(m_k p_k m'_k)$ is also uniformly (T_k)-Lyapunov regular with exponent -1 and Lemma 6.12 implies that (f_k) is uniformly (T_k)-Lyapunov regular at y, with exponent -1, completing the proof of Proposition 6.2.

6.5. **Local vanishing of the conformal curvature: proof of Corollary 6.3.**

6.5.1. *Weyl and Cotton tensors.* A standard way of proving conformal flatness is to prove that a specific conformally-invariant component of the curvature tensor vanishes identically.

Let W be (3,1)-Weyl tensor of $(M, \overline{g})$. It is conformally invariant, and when $\dim M \geqslant 4$, an open subset $U \subset M$ is conformally flat if and only if $W|_U = 0$ (see [4] Th. 1.159, 1.165). When $\dim M = 3$, the Weyl tensor is always zero. However, there is $(3,0)$-tensor T, called the Cotton tensor, which is also conformally invariant and such that an open set U is conformally flat if and only if $T|_U = 0$.

6.5.2. *Proof of Corollary 6.3.* Let $\|.\|$ denote a Riemannian metric on M. Let $X \in \mathfrak{a}$ and assume that there exists a ϕ_X^t-invariant, ϕ_X^t-ergodic measure μ which admits a non-zero uniform vertical Lyapunov spectrum. Then, let $x \in M$, $g \in G$, $r > 0$, (γ_k) and (T_k) be as in the conclusions of Proposition 6.1. Then, applying Proposition 6.2 to $U = B(x,r)$ and $(f_k) = (\gamma_k)$, we obtain that (γ_k) is uniformly (T_k)-Lyapunov regular with exponent -1 at any point in $B(x,r)$, in particular $\frac{1}{T_k} \log \|D_y \gamma_k v\| \to -1$ for any $y \in B(x,r)$ and $v \in T_y M \setminus \{0\}$.

Let us assume first $\dim M \geqslant 4$. By compactness of M, there is $C > 0$ such that for all $y \in M$ and $u, v, w \in T_y M$, $\|W_y(u,v,w)\| \leqslant C\|u\|\|v\|\|w\|$. Let now $y \in B(x,r)$ and $u, v, w \in T_y M$. The γ_k-invariance of W means

$$(\gamma_k)_* W_y(u,v,w) = W_{\gamma_k . y}((\gamma_k)_* u, (\gamma_k)_* v, (\gamma_k)_* w)$$

If $W_y(u,v,w)$ was non-zero, then we would have $\frac{1}{T_k} \log \|(\gamma_k)_* W_y(u,v,w)\| \to -1$. But the conformal invariance of W implies

$$\|(\gamma_k)_* W_y(u,v,w)\| \leqslant C \|(\gamma_k)_* u\| \|(\gamma_k)_* v\| \|(\gamma_k)_* w\|.$$

Thus, we would obtain $\overline{\lim} \frac{1}{T_k} \log \|(\gamma_k)_* W_y(u,v,w)\| \leqslant -3$, contradicting $W_y(u,v,w) \neq 0$. Thus, W vanishes identically on $B(x,r)$. Since $[(g,x)] \in \operatorname{Supp} \mu$ by construction, this finishes the proof in the case $\dim M \geqslant 4$.

Assume now that $\dim M = 3$, *i.e.* $(M, \overline{g})$ is a closed Lorentzian 3-manifold. The argument is essentially the same: the invariance of the Cotton tensor implies that for any $y \in B(x,r)$ and $u, v, w \in T_y M$,

$$|T_y(u,v,w)| \leqslant C \|(\gamma_k)_* u\| \|(\gamma_k)_* v\| \|(\gamma_k)_* w\|,$$

for some $C > 0$. Thus, we get $T_y(u,v,w) = 0$ since the three factors in the right hand side converge to 0, completing the proof of Corollary 6.3.

6.6. **Conclusion.**

We can now conclude that under the assumption $\mathrm{Rk}_{\mathbf{R}} G = p + 1$, the whole manifold is conformally flat. It will follow from the

Claim. *Any compact, Γ-invariant subset of M intersects a conformally flat open set.*

Proof. Let K be a compact Γ-invariant subset of M. Then, G preserves the compact subset $K^\alpha := (G \times K)/\Gamma \subset M^\alpha$. Fix $A < G$ a Cartan subspace and let B a Borel sugroup of G containing A. By B-invariance of K^α and amenability of B, the same argument as in the second paragraph of Section 5 gives the existence of a finite A-invariant, A-ergodic measure μ supported in K^α, and whose projection on G/Γ is the Haar measure.

By Lemma 6.4, there exists $X \in \mathfrak{a}$ whose vertical Lyapunov spectrum is reduced to $\{-1\}$. If $\Lambda \subset K^\alpha$ denotes the set of full measure μ where the conclusions of the higher-rank Oseledets theorem are valid, we choose μ' a probability measure ϕ_X^t-invariant, ϕ_X^t-ergodic such that $\mu'(\Lambda) = 1$, so that the vertical Lyapunov spectrum of X with respect to μ' is the same. By Corollary 6.3, there exists $x \in K$ admitting a conformally flat neighborhood, because $\operatorname{Supp} \mu' \subset K^\alpha$ and $[(g, x)] \in K^\alpha$ implies $x \in K$. $\square$

Let now $x \in M$, and consider the compact Γ-invariant subset $K = \overline{\Gamma.x}$. If U is a conformally flat open set which meets K, then there is $\gamma \in \Gamma$ such that $\gamma.x \in U$, proving that $\gamma^{-1}U$ is a conformally flat neighborhood of x. Finally, any point of M admits a conformally flat neighborhood, and Theorem 1 is established.

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