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Alonso and the Scaling of Urban Profiles

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How urban characteristics change with total population, their scaling behavior, has become an important research field since one needs to better understand the challenges of urban densification. Yet urban scaling research is largely disconnected from intra-urban structure, and this seriously limits its operationalization. In contrast, the monocentric model of Alonso provides a residential choice-based theory to urban density profiles. However, dedicated comparative static analyses do not completely solve how the model scales with population. This article bridges this gap by simultaneously introducing power laws for land, income and transport cost in the Alonso model. We show that the equilibrium urban structure of this augmented model matches recent empirical findings about the scaling of European population density profiles and satisfactorily represents European cities. This result is however not compatible with the observed scaling power of housing land profiles, and challenges current empirical understanding of wage and transport cost elasticities with population. Our results call for revisiting theories about land development and housing processes as well as the empirics of agglomeration benefits and transport costs.

Introduction

It is theoretically elegant and empirically convenient to think of all the good and bad of cities simply in terms of their total population. We live in an increasingly urban World (UN-Habitat 2016), and liaising the social and environmental outcomes of cities to their size is an important question. Yet, we know that many outcomes of cities depend crucially on their internal structure, especially on how densely citizens occupy the land they have developed. The spatial distributions of developed land and densities within cities emerge from the location decisions of many interacting actors. A simple description of these distributions can be made in radial terms, that is, how far-reaching a city is (the urban fringe distance) and how flat/steep its density profile is. The radial profile of urban land and densities are a key interest of theoretical and empirical urban economics (see Anas, Arnott, and Small 1998), while land use zoning and setting density are the favorite playground of urban planners. The long dispute between compactness or sprawl

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(see Ewing et al. 2014) just shows how much this internal structure matters and is worth being studied. Therefore, before summing-up a city as the outcome of a single termed function of population, one first needs to make sure that the internal structure of cities is independent of their size (measured as total population), or is at least independent of a simple transformation of their total population. Second, if desirable actions need to be made with potential social impacts, one needs to know whether this internal structure results from similar mechanisms whatever the city size, that is, whether the same urban theory holds across the size distribution of cities.

Nordbeck (1971) provided an intuition to the first need by assuming that cities, similarly to biological objects, keep the same form across sizes. He opened up a vast literature strand in geographical research, based on allometric urban growth and urban scaling laws (Lee 1989; Batty 2008b, 2010, 2013; Bettencourt 2013; Arcaute et al. 2015; Jiang 2018). Recently, Lemoy and Caruso (2018) endorsed this idea with new data on European cities and empirically identified the homothetic transformations that affect density and developed land profiles across population sizes. A logical extra step is then to address the second of these needs: assessing whether observed scaling behaviors can be related to socioeconomic factors by means of an explicative model. This is the very objective of the present contribution.

Identifying a valid model that can be applied to any city after simple rescaling would definitely bear powerful implications for understanding cities and identifying generic planning recipes independent of size. The Alonso–Muth–Mills monocentric framework (Alonso 1964; Muth 1969; Mills 1972) is a perfect candidate because it issues micro-foundations to urban expansion limits and density gradients. It does so after fixing population in its closed equilibrium form, or after fixing its social outcome (utility) in its open form where equilibrium with other cities is then assumed and the population an output.

In this article, we first assess the theoretical ability and conditions for the Alonso model to replicate the scaling behavior of urban density and urban land profiles. For that purpose, we simultaneously introduce power laws for income, transport costs, and population density in its setting. This enables us to discuss the scaling of population density within residential areas. However, the Alonso model is silent about the share of urban land which is actually devoted to housing. Yet it is important in measuring population density profiles empirically. Thus, we also relax this assumption by introducing an exogeneous profile for urban land and its scaling behavior. In the second part of the article, we test how the standard form of the Alonso model and its relaxed land use form (named "Alonso-LU") empirically perform in Europe after a parsimonious calibration with only three parameters.

Background literature and research strategy

Urban scaling and allometry

The study of the relationship between the size and shape of an object is named "allometry" (Small 1996). Under the successive influences of general systems theory and complex systems, this concept has been progressively imported from biology to urban studies in order to describe how key attributes of cities change with their size, in particular their population (Woldenberg and Berry 1967; Nordbeck 1971; Naroll and von Bertalanffy 1973). Urban allometric equations are based on the elementary law of relative growth (Huxley and Teissier 1936) and specify that the relative growth of an urban attribute is a constant fraction of the relative growth of the urban population. Urban attributes and size are related by a scaling law of the form $x = aN^b$, where N denotes the population and x denotes any attribute of interest, such as the urban fringe, employment rate, mean wage, etc. The parameter a is a pre-factor and b is the scaling power. The value of b describes the allometric regime: sublinear ($b < 1$), superlinear ($b > 1$), or linear ($b = 1$), in which case the growth is said to be *isometric* (Huxley and Teissier 1936). Note that this scaling law has also been derived from other theoretical settings (Nordbeck 1971; Makse, Havlin, and Stanley 1995; Bettencourt 2013). Yet globally, a consistent literature strand of geographical research has emerged, which measures and explains scaling laws in diverse urban attributes (Lee 1989; Batty 2008b, 2010, 2013; Bettencourt 2013; Jiang and Okabe 2014; Arcaute et al. 2015; Jiang 2018).

Within the empirical literature committed to estimating the scaling power b, geographers and physicists focused mostly on attributes related to infrastructures, individual needs or social activities (Bettencourt et al. 2007; Batty 2008a; Bettencourt 2013; Arcaute et al. 2015; Bettencourt and Lobo 2016; Leitão et al. 2016; Cottineau et al. 2017). Besides, we should also remind that urban scaling laws are mostly used for the study of rank-size population distributions and empirical tests of Zipf's law (e.g., Gabaix 1999; Eeckhout 2004; Tabuchi, Thisse, and Zeng 2005; Cura et al. 2017). This is a related but distinct research topic where the focus is set on the self-organization of cities into urban systems. Still, elements of inter-city systems will be called here since our model must, given its scaling perspective, ensure the existence of cities of different sizes.

A major shortcoming of the research cited above is that, although they acknowledge that scaling laws result from internal urban processes and properties, they opt for an aggregate view that reduces cities to dimensionless points (Batty 2008b, Batty et al. 2008). Internal morphologies are not made explicit (except for instance the simulation-based approach of Pumain and Reuillon 2017), hence the link between the urban scaling regimes and the geography of underlying sociological, economic, or technical processes is simply broken.

Distance-based urban profiles

In geography, the study of population density profiles has been initiated by the empirical work of Clark (1951), who calibrated exponential curves on the population density profiles of 20 cities around the World. His work has been followed by numerous studies since then (Berry, Simmons, and Tennant 1963; Ishikawa 1980; Batty and Kim 1992). By placing numbers behind the notion of urban centrality (Baumont, Ertur, and Le Gallo 2004; Pereira et al. 2013), this literature has become crucial to measure urban sprawl and to underpin the debate of the sustainability of compact development (Wang and Zhou 1999; Boyko and Cooper 2011; Jiao 2015; OECD 2018). Relying on these studies, an original approach to *intra-urban* allometry has been proposed by Nordbeck (1971) who suggested that an urban population can be regarded as the volume obtained from integrating the population density function over a defined area. Given the decreasing population density as one moves away from central locations, measured by Clark (1951) and followers, this shape is suggested to look like a three-dimensional cone, that is, like a volcano in the words of Nordbeck (1971, p. 57). However, distance to the center is not explicit in Nordbeck's work. Lemoy and Caruso (2018) recently followed up on this idea and carried out a thorough empirical investigation of these cones for European cities. The distance to the main center of the city is explicit in their work and scaling laws of aggregate variables are therefore replaced by geometric transformations of distance-based profiles (i.e., of the slopes of the volcano). They compared how urban profiles change with respect to population size, hence proposing a new radial approach to allometric scaling.

In details, Lemoy and Caruso (2018) performed their radial analysis for over 300 European cities of more than 100 000 inhabitants. They analyzed population density profiles with distance to the city center and found that they superpose well after a rescaling in abscissae and ordinates. Hence, the three-dimensional city cones follow a homothetic scaling (or dilation) from their center (CBD).[1] Lemoy and Caruso (2018) similarly analyzed the radial profiles of the share of land devoted to housing and found that they superpose after their abscissae (*not* their ordinates) are rescaled with respect to population. Hence, a homothetic scaling in two-dimensions. The authors numerically obtain an optimal rescaling with the cube root of population for density profiles and the square root of population for land use profiles, yielding the average profiles shown on Fig. 1, where $H_N(r)$ is the share of housing land and $\rho_N(r)$ the population density as functions of distance r to the CBD for a city of arbitrary size N. While the allometric approach of Nordbeck (1971) and its recent empirical development by Lemoy and Caruso (2018) are key to understand the scaling of intra-urban forms, they still rely on physical analogies and are disconnected from any sociological or economic theories.

Population effects in urban economics

Although they usually do not refer to urban scaling research, economists draw links with concepts and theories of agglomeration economies and costs. They discuss and measure population-elasticities of wages or other economic outputs (Glaeser and Maré 2001; Rosenthal and Strange 2004; Combes, Duranton, and Gobillon 2012), which actually correspond to the scaling power b discussed by geographers.[2] Urban economics provides the most predominant socioeconomic theory of the internal structure of cities. Its backbone is the monocentric model of Alonso (1964), which spread out of urban economics, became a standard component of many urban models (Batty 2008a, b; Wegener 2014; Acheampong and Silva 2015), and is still a landmark for discussing urban development issues (Brueckner and Fansler 1983; McGrath 2005; Oueslati, Alvanides, and Garrod 2015).

This model explains how population density and land/housing market variables change as functions of distance to an exogenous CBD (Fujita 1989). It is key in representing the trade-offs that agents make between land/housing costs and transport costs, and has been added a series of theoretical extensions (Fujita and Thisse 2013). Comparative static analyses are performed to

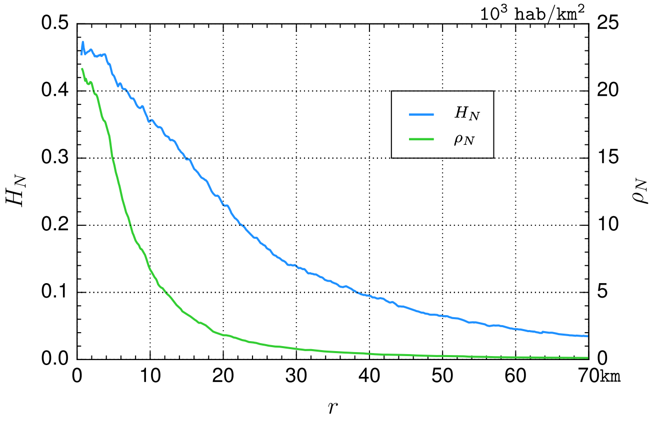

Figure 1. Average share of housing land and population density profiles. These profiles have been rescaled without loss of generality to London's population (the largest European Larger Urban Zone in 2006), taken as $N = 1.21 \times 10^7$ (see Lemoy and Caruso 2018).

understand the separate effects of key parameters within the model. Among these comparative statics analyses, some provide insights on the scaling behavior of urban attributes after "closing" the city, that is, exogenously fixing its population. Mills (1972), Wheaton (1974), Pines and Sadka (1986) and Brueckner (1987) show that a greater population expands the urban fringe and raises population density everywhere. If we think that population increases wages and lowers transport costs, it is interesting to note that comparative statics shows that a greater wage level and lower transport costs, under mild assumptions, expand the urban fringe and flatten the population density profile (Fujita 1989; Papageorgiou and Pines 1999). Conversely, however, using an alternative monocentric model with Cobb-Douglas utility, Anas, Arnott, and Small (2000) found a steeper population density profile for increased populations.

However, those results were, to the best of our knowledge, never used in urban scaling research. Their main drawback in this perspective is that comparative static analyses are carried out *ceteris paribus*, that is, parameters are changed one at a time (Fig. 2). For example, we may end up comparing large and small cities while assuming similar wage levels and transport costs, which is hardly admissible from an empirical perspective. More complex theoretical models and empirical evidence actually suggest that wages rise with total population (Glaeser and Maré 2001; D'Costa and Overman 2014) whilst congestion affects transport costs in larger cities (see for example Small and Verhoef 2007). Comparative statics thus strongly opposes the urban allometry approaches described above, which consider the simultaneous change of all urban attributes with population, but lack micro-economic foundations. A reconciling approach could be achieved by endogenising wages and transport costs so that they will vary with exogenous shocks in population. Yet the complexity of these processes will necessarily introduce a lot of new variables and parameters to the model. This is then detrimental to parsimony, which is necessary to having a practical understanding of the model, as well as to its empirical testing, which is already a very demanding task (Brueckner and Fansler 1983; McGrath 2005; Ahlfeldt 2008; Oueslati, Alvanides, and Garrod 2015).

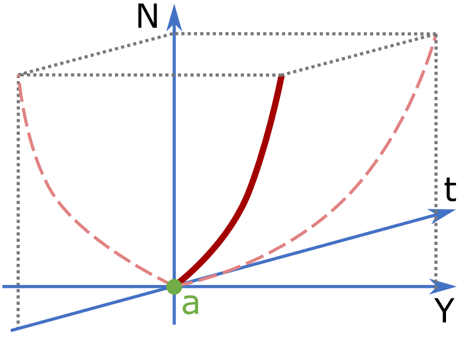

Figure 2. Sketch of scaling laws in the parameters space $\{N \times t \times Y\}$. Starting from a given model's parametrization, comparative statics consists in comparing the model results with the results from another parametrization, for which the value of a single parameter has been changed. That is, starting from a in the parameter space (green dot), the second parametrization is taken by moving along only one of the three axes directions (in blue). On the other hand, in this article we assume that scaling laws hold between parameters. Thus we explore another part of the parameters space, for example along the red curve.

Research strategy

In this article, we follow a hybrid strategy that falls in between a comparative static analysis of population effects in the Alonso model and a full endogenization of population effects on every parameter of the model. We exogenously assume a form of scaling for income and transport costs – which is in line with the literature on agglomeration economies (e.g., Rosenthal and Strange 2004) – and deduce their scaling power from the allometric relationships of their radial profiles as of Lemoy and Caruso (2018). The relationships are first deduced analytically and then checked empirically. By these means, we distinguish from a traditional comparative static analysis by comparing outcomes of models whose parametrizations differ from more than a single parameter value (Fig. 2).

From a theoretical perspective, the advantage of our approach compared to comparative static analyses is to consider the simultaneous change of population, income and transport costs. The advantage compared to a model with more endogeneous effects of population is its parsimony, allowing analytical tractability and empirical testability. In addition, it provides insights on the scaling properties and functional forms that would be requested for a potential future model that would contain all endogeneous scaling effects and would want to realistically fit with data.

From an empirical perspective, our approach also constitutes an unprecedented confrontation of the Alonso model to radial density and housing land profiles for a large range of city sizes. More precisely, we raise the question of whether the population density profile resulting from the monocentric city model of Alonso (1964) with log-linear utility can provide a good description of the European average profile of Fig. 1 and, by extension, whether it can represent any European city, after being rescaled.

However, empirical density profiles do not only depend on the housing intensity within residential areas, which is at the heart of the Alonso model, but also on the share of urban land devoted to housing, which is usually kept constant in this model. Lemoy and Caruso (2018) provide empirical scaling results which we use to exogenously introduce a housing land profile and its scaling in our model, in line with our treatment of wages and transport costs. As in the standard Alonso model, the consumption of land and total city size will be endogenously determined by households' choices, but we add more realistic radial description of urban land use as an additional constraint. We are aware of models that permit non-urbanized land to be interspersed within the urban footprint (Cavailhès et al. 2004; Caruso et al. 2007, 2009) but at this stage of the research, and given our primary focus on scaling, we keep the treatment of the housing land development process as exogeneous.

We organize the remainder of the article into a theoretical section and an empirical one. In the next section, we introduce power laws for income, transport costs, and housing land profiles in a relaxed version of the Alonso model where housing does not necessarily fully occupy land around the CBD. We then derive conditions at which the equilibrium profiles scale homothetically. In another, empirical section, we use the European data of Lemoy and Caruso (2018) to calibrate the model, respectively, its standard form with constant occupation of land (Alonso) and the relaxed version with exogenously given land profile function (Alonso-LU), thus leaving the model to produce densities within these constraints. We conclude in the last section.

Theory

First, we define the setting and introduce homothetic scaling in density and housing land profiles. Second, we define the decision-making of households and introduce scaling for parameters of this choice (income and transport cost). Third, we take an intra-urban perspective, and solve the equilibrium for the closed form (given population, endogenous utility) of the Alonso model with log-linear utility. Total land, housing land and transport cost functions are kept general and conditions for the homothetic scaling of the population density profile are derived. Fourth, we analyze whether the homothetic scaling is compatible with a system of cities where cities of different populations coexist at equilibrium with the same utility level. Finally, we operationalize the model with functional forms to prepare the empirical validation.

Alonso-LU and homothetic scaling profiles

The setting is a featureless plain except for a unique Central Business District (CBD), which concentrates all jobs on a point and is accessed by a radial transport system without congestion. Let r be the Euclidean distance to the CBD and $L(r)$ the exogenous distribution of developable land around the CBD. In reality, $L(r)$ is not necessarily a circle of radius r typically because of water bodies (port cities). In our model, whatever the form of $L(r)$, we depart from Alonso by introducing $H(r)$, the share of $L(r)$ that is actually used for housing, hence we have an urban land use augmented model, which we name "Alonso-LU". The function $H(r)$ enables to distinguish the urban land that is actually used for housing from the urban land which is used for complementary land uses such as agriculture, transport infrastructures or economic activity. Note that $H(r)$ is only used for housing, while its complement $L(r)[1-H(r)]$ cannot be used for housing. In the Alonso standard model $H(r) = 1$ (or any other constant), which obviously contrasts with the blue curve in Fig. 1. In Alonso-LU, we impose $H(r)$ as a portion of $L(r)$ and provide its form exogenously. Residential densities emerge endogenously but are constrained by the available space $H(r)$ which we know is decreasing with r (Fig. 1).

We now introduce the scaling of population and housing profiles. Let us denote by $\rho_N(r)$ the population density profile and by $H_N(r)$ the profile of the share of urban land used for housing for a city of total population N. Inspired by Lemoy and Caruso (2018), we assume there exists α and γ such that population density profiles $\rho_N(r)$ scale homothetically in three dimensions with the power α of population N, and that housing land radial profiles $H_N(r)$ scale homothetically in the two horizontal dimensions with the power γ of population.[3] These homothetic scalings are formalized with the following functional sequences:

$$\rho_N(r) = N^\alpha \rho_1\left(\frac{r}{N^\alpha}\right), \tag{1}$$

$$H_N(r) = H_1\left(\frac{r}{N^\gamma}\right), \tag{2}$$

where $\rho_1(r)$ and $H_1(r)$ are the population and land use radial profiles of an abstract unitary city of population $N = 1$, which are the first elements of both sequences. Note that according to Lemoy Caruso (2018), European urban areas obey equations 1 and 2 (up to some fluctuations which are illustrated later in this work) with the exponents $\alpha \simeq 1/3$ and $\gamma \simeq 1/2$. Using these values, one can rescale any European profile to any arbitrary population size. In doing so, one gets the European average profiles of population density and housing land, which have been presented on Fig. 1.

Residential choice and scaling parameters

Each household in the model requires space s for housing, works in the CBD and consumes a composite commodity z that is produced out of the region and imported at constant price. In that context, residential choice depends only on the distance r to the CBD. Households are rational and their utility function U is

$$U(z, s) = (1-\beta)\ln\big(z(r)\big) + \beta \ln\big(s(r)\big), \tag{3}$$

where $z(r)$ is the amount of composite good (including all consumption goods except housing surface) consumed at distance r from the CBD, $s(r)$ is the housing surface[4] at the same distance r and $\beta \in\ ]0, 1[$ is a parameter representing the share of income (net of transport expenses) devoted to housing, or the relative expenditure in housing. Note that β is assumed to remain constant across cities of different sizes, which is empirically supported (Davis and Ortalo-Magné 2011).

Equation (3) is a log-linear utility function, that is, the logarithmic transformation of the traditional Cobb–Douglas utility function (from Cobb Douglas 1928), and gives the same results in the present case since we work with an ordinal utility. We choose it here for several reasons, which also explain why in urban economic literature it is the form of utility function which is the most used. First, it matches the assumption of a *well-behaved utility function*[5] (Fujita 1989, p. 12), which is central in the basic monocentric model and ensures that $U(z, s)$ is defined only for positive values of z and s. Second, it contains only a single parameter, β, which can be discussed empirically. Third, β is independent of prices, as found in the empirical literature (Davis and Ortalo-Magné 2011). Generalization to more general representations of preferences, such as utility functions with constant elasticity of substitution (CES), is left for further studies.[6]

We choose the composite commodity (z) as the numeraire (unit price) and the budget constraint of each household is binding since the households' utility function is monotonic and does not include any incentive to spend money otherwise. The budget constraint at distance r from the CBD is

$$z + R(r)s(r) = Y_N - T_N(r), \tag{4}$$

where $R(r)$ is the housing rent at distance r, Y_N is the wage of households, and $T_N(r)$ is the commuting cost at r.

As announced earlier, we now introduce important new scaling assumptions: wages and transport costs are assumed to depend on the total population N of the city. Their variations with city size will strive to reproduce the empirical radial profiles of small and large cities. The measure of agglomeration economies and costs through elasticities of wages and transport costs is well-established in the empirical economic literature (Rosenthal and Strange 2004; Combes et al. 2010; Combes, Duranton, and Gobillon, 2011, 2012). This implies power law functions, which are also most often used in urban scaling laws literature (Bettencourt et al. 2007; Shalizi 2011; Bettencourt 2013; Leitão et al. 2016). Thus, following both strands, we introduce power laws, such that $Y_N = N^{\phi} Y_1$, where Y_1 is the wage in a unitary city, and ϕ is the elasticity of wage with respect to urban population. Similarly, we assume that the transport cost function $T_N(r)$ is a scaling transformation (not necessarily homothetic) of $T_1(r)$, the transport cost function in a unitary city (assumed to be continuously increasing and differentiable in r). The exact form of this transformation will be derived from the conditions for a homothetic scaling, as will be clarified in the next subsection (equation 8).

The households' problem consists in maximizing their utility (3) such that the budget constraint (4) holds.

Intra-urban equilibrium

Consider a closed urban region with population N. Solving the maximization problem of households yields the bid rent function $\Psi(r,u)$ (see appendix A.1), which is the maximal rent per unit of housing surface they are willing to pay for enjoying a utility level u (exogenous) while residing at distance r.

Closing the model by linking the utility level u to the population size N yields two additional conditions. First, the quantity of land $L(r)$ at each commuting distance r is finite. Then, summing the population density over the whole (finite) extent of the city, up to the fringe f_N, must yield the total population N. Second, this fringe is determined by a competition between urban (i.e., housing) and agricultural (default) land uses. We suppose that rents are caught by absentee landowners and that agricultural rent is null for mathematical convenience (see appendix A.2). This assumption is common in urban economic theory (Fujita 1989) and is empirically supported by the low values of agricultural rents relative to housing rents (Chicoine 1981). Consequently, the urban fringe f_N is the distance at which households spend their entire wage in commuting (and pay a null rent):

$$f_N = T_N^{-1}(Y_N) \Leftrightarrow Y_N = T_N(f_N). \tag{5}$$

Finding the unique equilibrium utility and urban fringe satisfying the equilibrium conditions[7] yields the equilibrium population density function $\rho_N(r)$ (appendix A.2). Its homotheticity relies on the three following conditions (appendix A.3).

$$\forall \lambda \in \mathbb{R}: \quad L(\lambda r) = \lambda L(r), \tag{6}$$

$$\gamma = \alpha, \tag{7}$$

$$\exists \theta \in \mathbb{R}: \quad T_N(r) = N^{\theta} T_1\left(\frac{r}{N^{\alpha}}\right). \tag{8}$$

Condition (6) is simply the linearity of L, which is clearly satisfied in a two-dimensional circular framework (where $L(r) = 2\pi r$). Condition (7) states that the horizontal scaling exponents of the share of housing land and population density profiles of equations (1) and (2) must be the same. Finally, condition (8) actually refines the power-law form of the transport cost function by specifying that the transport cost function is at least (since θ can be zero) horizontally scaling with power α. If these three assumptions hold, then the equilibrium population density function writes

$$\rho_N(r) = N^{1-2\alpha} H_1(r_1)\left[T_1(f_1) - T_1(r_1)\right]^{1/\beta-1}\left[\int_0^{f_1} L(r_1)H_1(r_1)\left[T_1(f_1) - T_1(r_1)\right]^{1/\beta-1}\mathrm{d}r_1\right]^{-1}, \tag{9}$$

where $r_1 = r/N^{\alpha}$ and $f_1 = f_N/N^{\alpha}$. We find that this population density profile follows the three-dimensional homothetic scaling (1) if and only if $\alpha = 1/3$, which coincidentally matches the empirical evidence of Lemoy and Caruso (2018). This means that Alonso's fundamental trade-off between transport and housing is able to explain the observation that cities are similar objects across sizes, provided land profiles, wages and transport costs scale with total population. In other words, a single density profile can be defined from Alonso-LU to match any European city.

Among the assumptions underlying this result, equation (7) deserves some discussion since the empirical work of Lemoy and Caruso (2018) has shown that $1/3 = \alpha \neq \gamma = 1/2$. Imposing $\gamma = 1/3$ reduces the scaling, thus yielding an underestimation of the share of urbanized land devoted to housing in large cities, and to an overestimation in small cities. We note that if we had used a scaling assumption similar to the one used for the transport cost function, that is, relaxing assumption (2) by $H_N(r) = N^{\eta} H_1(r/N_{\alpha})$, we would have obtained $\alpha = 1/3$ as well, which indicates that we deal with a process linked to three-dimensional geometry.

Inter-urban analysis

Up to now, a closed city of size *N* has been considered. Yet cities belong to an urban system where households may move from one city to another. This perspective holds two implications. First, since cities of different population sizes coexist in real urban systems, the equilibrium of the model should be able to reproduce this fact. As a consequence, the benefits and costs of urban agglomeration should vary together when population size changes, to compensate each other whatever the size of the city. If one force would dominate the other, the urban system would either collapse to a single giant city or be peppered into countless unitary cities. Second, since by definition households' location decisions are mutually consistent at equilibrium, the equilibrium utility level has to be the same whatever the city population *N*. Otherwise, households would have an incentive to move to larger or smaller cities.

To find out whether the inter-urban equilibrium holds, we substitute the power-law expressions of the wage and transport cost function into the boundary rent and total population conditions. Accounting for the equality of equilibrium utilities across cities of different sizes then yields the following two equalities (see appendix A.4)

$$\phi = \theta = \frac{\beta}{3(1-\beta)}. \tag{10}$$

The left-hand side equality in equation (10) implies that the elasticity of wages with respect to urban population (ϕ) equals the elasticity of the transport cost function once it has been horizontally rescaled (θ). Thus, following the approach of Dixit (1973), ϕ is representative of urban agglomeration economies whilst θ results from agglomeration costs.[8] Hence, the condition for several cities of different populations to coexist at equilibrium is met. We note that this equality is supported by recent developments in the limited empirical literature on agglomeration costs (Combes, Duranton, and Gobillon 2012).

The right-hand side equality in equation (10) provides a relationship between the vertical scaling exponent of the value of transport cost θ (or the population-elasticity of wages ϕ) and households' relative expenditure in housing β. This relation is increasing and suggests that a relative expenditure $\beta = 1/3$, which is in the range of empirically supported values (Accardo and Bugeja 2009; Davis and Ortalo-Magné 2011), would be associated to exponents $\phi = \theta = 1/6$ (Fig. 3). This value is the same as the superlinearity of socio-economic outputs discussed in Bettencourt (2013) and Bettencourt and Lobo (2016). Consequently, the inter-urban perspective inferred by Alonso-LU is definitely compatible with some former theoretical and empirical researches. However, it diverges from other authors who consider this elasticity to range from 2% to 5% (Combes et al. 2010; Combes, Duranton, and Gobillon 2012). In addition, following other measures of agglomeration economies that are not only based on wages, the elasticity of productivity with respect to city population is considered to be of maximum 3% to 8% (Rosenthal and Strange 2004). Alonso-LU does not solve these empirical incompatibilities. Yet it provides some hints for further research. In particular, the functional form of the transport cost function may reweight equation (10) so as to refine the analysis of scale economies. The issue of functional forms is discussed in the next subsection.

Functional form

We now propose an operational version of the previous model by selecting appropriate functional forms for the land distribution $L(r)$, the housing profile $H_N(r)$ and the transport cost function

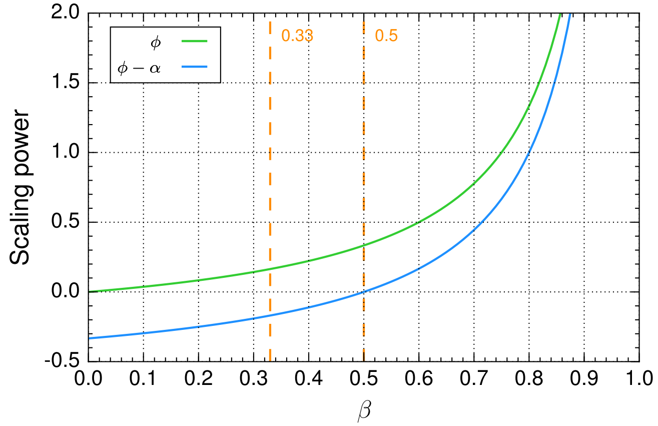

Figure 3. Population-elasticities of wage and transport cost with respect to housing relative expenditure. Dashed orange lines highlight values of reference discussed in the text. Recall from equation 10 that $\phi = \theta$.

$T_N(r)$. The theoretical implications of those forms are discussed as well as their empirical supports. In brief, the functional model is specified by

$$L(r) = 2\pi r, \tag{11}$$

$$H_N(r) = b \exp\left(\frac{-r}{dN^{1/3}}\right), \tag{12}$$

$$T_N(r) = cN^{\theta-\alpha} r, \tag{13}$$

where $\theta = \beta/(1-\beta)/3$ (equation 10), $\alpha = 1/3$, b is the share of housing land at the CBD, d is the characteristic distance of the housing land profile in a unitary city and c is the transport cost per unit distance in a unitary city. One can easily check that the functional forms (11)–(13) follow the conditions for homotheticity (6)–(8), as well as for consistency with the inter-urban approach (10). The land distribution (11) is simply the usual two-dimensional circular framework and the exponential form (12) of the housing land profile has been chosen for its simplicity and goodness of fit, which is discussed in the next section.

We choose a linear form (13) for the transport cost function, because it is largely practised by urban economists. The elasticity of unitary transport cost with respect to urban population $\theta - \alpha = (2\beta - 1)/(1-\beta)/3$ is then endogenously determined by the conditions of homothetic scaling (8) and homogeneous utility across cities (10). It suggests that for $\beta < 1/2$ – which is empirically supported – the unitary transport cost should decrease with urban population (Fig. 3). This strives against Dixit (1973) and the expectation that unitary transport cost is increasing with urban population due to congestion. Hence, the linear transport is not consistent with the scaling of urban profiles. We leave the complete study of a nonlinear transport cost to further research but show in appendix that using a concave transport cost function, which is intuitively more realistic, a positive elasticity of unitary transport cost appears for realistic values of the housing expenditure, for example, $\beta = 1/3$ (appendix A.5). In particular, changing (13) to $T_N(r) = c\sqrt{r}$ (i.e., no scaling with population size N) would respect condition (8).

Finally, with the functional form (11)–(13), the equilibrium population density function (9) becomes (appendix A.6)

$$\rho_N(r) = \frac{N^{1/3}}{2\pi} e^{-r_1/d} \left(f_1 - r_1\right)^{\frac{1}{\beta}-1} \left[\beta f_1^{1/\beta+1} - \left(\beta f_1 + d\right) e^{-f_1/d} d^{1/\beta} \int_0^{f_1/d} e^x x^{1/\beta} \mathrm{d}x\right]^{-1}, \tag{14}$$

where $r_1 = r/N^{1/3}$. This expression depends on three parameters: the unitary urban fringe $f_1 = Y_1/c$, the housing expenditure β and the characteristic distance d of the housing land profile in a unitary city. This density profile model is suitable for empirical calibration. Note that this is a daring exercise since all cities in Europe are calibrated at once using only those three parameters. Its success will expose the descriptive power of the homothetic scaling.

Empirics

In this section, the functional model (14) is calibrated to the average European population density profiles of Fig. 1 using nonlinear least squares. The calibration procedure is performed in two steps. First, the optimal value of d is calibrated by comparing the share of housing land (12) to the average profile for a reference city of population $\bar{N}$. Second, the optimal value of d is substituted into the population density function (14), which in turn is calibrated to the average population density profile once by optimizing the values of f_1 and β, and once by optimizing only the value of f_1 with a fixed β = 1/3. Results are visualized for four individual cities.

Housing land profile

We calibrate the share of housing land (12) to the average profile (Fig. 1) for a population of reference $\bar{N}$. This population can be chosen arbitrarily, yet the condition for homothetic scaling imposes a scaling power of γ = 1/3 which is different from the empirical one (γ = 1/2). As a result, the model is optimal for the population of reference, but rescaling to other population sizes generates an error. Using the error function detailed in appendix A.7, we choose a population of reference $\bar{N} = 7.04 \times 10^5$ For a city with this population, the best fit suggests that the characteristic distance is $d = 65.2m$ (Table 1). Besides, we see that 52.3% of land is dedicated to housing at the CBD, which slightly offsets the average empirical value (Fig. 1). In the Alonso model, the best constant value of housing share is around 17% (Table 1), which is a poor description of data.

Four cities of different sizes are chosen as illustrations, namely London (Ldn), the largest urban area of the data set with a population of $N = 1.21 \times 10^7$ in 2006, Brussels (Bxl), the capital of Belgium with $N = 1.83 \times 10^6$, Luxembourg (Lux), capital of the country of the same name, with $N = 4.52 \times 10^5$ and Namur (Nam), the capital of Wallonia in Belgium, with $N = 1.39 \times 10^5$. The population of reference $\bar{N}$, for which the error is minimized, is between those of Luxembourg and Brussels. We see that because of the wrong scaling exponent, the larger the difference between the population N of the considered city and the reference population $\bar{N}$, the larger is the error on housing land share (Fig. 4). For $N > \bar{N}$, the housing share is underestimated, and overestimated for $N < \bar{N}$. In the case of the four considered cities, the absolute error does not exceed 12 points (35% in relative terms, see Fig. 4).

Population density profile

We turn now to the calibration of the population density function (14) with the optimal value d (Table 1) obtained in the previous subsection to the average population density profile (Fig. 1).

Table 1. Nonlinear least square results

A. Housing usage			B. Population density				
	Alonso	Alonso-LU		Alonso		Alonso-LU	
b	0.167 (0.005)	0.523 (0.001)	β	0.02	0.34	0.02	0.34
d		0.065 (0.001)	$f_{\bar{N}}$	172.9 (0.5)	12.94 (0.07)	409 (2)	23.1 (0.2)
$C\ (b, d)$		−0.74					
BIC	−2 808	−6 332	BIC	7 613	8 744	7 544	7 893

Calibration are performed on European average profiles made up with 694 points, for a population of reference $\bar{N} = 7.04 \times 10^5$. Distances d and $f_{\bar{N}}$ are expressed in kilometers. C(b, d) is correlation between parameters. BIC is the Bayesian information criterion.

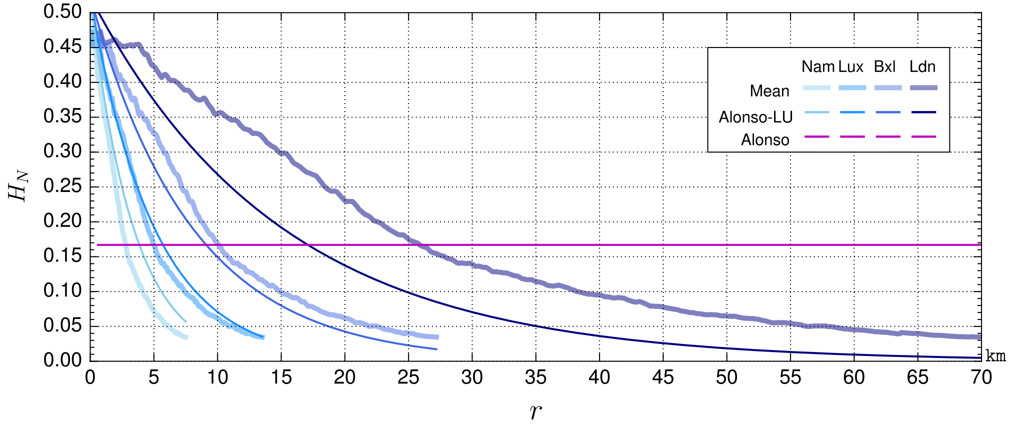

Figure 4. Calibration of the average housing usage profile. Mean and best-fit, rescaled to the four example cities.

We focus again on a city of size $\bar{N} = 7.0410^5$, this time without loss of generality since the scaling of population density in the model is in agreement with empirical results. The optimal values of the urban fringe f_N and of the relative expenditure in housing β turn out to be negatively correlated. The best fit is therefore a corner solution with arbitrarily small values of β and arbitrarily high values of f_N (Fig. 5). In the following we consider the optimal model with $\beta = 0.02$. However, this value is unrealistic (Davis and Ortalo-Magné 2011) and thus questions the ability of monocentric models to describe real cities. As mentioned in the previous section, this is a problem of the model which results from the linear form of the transport cost function (see Appendix A.5). Yet nonlinear forms challenge the mathematical tractability of the model. At this stage, our solution is to also consider a constrained model with $\beta = 0.34 \simeq 1/3$ as a reference case (Fig. 5).

We look at the best-fit population density profile and focus on the case of London on Fig. 6, knowing that smaller cities are obtained by homothetic rescaling. Note that the relative errors are exaggerated because of the semi-logarithmic plot. We observe that the Alonso-LU model outperforms the standard Alonso model, especially for realistic values of β. Both models display densities whose logarithms are concave because density is going to zero at $r = f_N$. Conversely, the empirical population density profile appears convex. As a result, the best fit model is almost linear in the semi-logarithmic plot (hence almost exponential with linear axes). This form has been long studied empirically in urban economics since Clark (1951). Theoretical justifications for this exponential form have been provided by Mills (1972), Brueckner (1982) after adding building construction in the Alonso model, or by Anas, Arnott, and Small (2000) who used exponential unitary commuting costs. We contribute a different explanation that is parsimonious and works across city sizes.

Using the four cities of reference, Fig. 7 shows that the Alonso-LU model gives a good description of population density profiles for European cities, whatever their size. Four additional cities are provided in appendix B, Fig. 8. Visual inspection reveals that the error is mostly due to deviations of individual data from the average profile, and less to deviations of the model from the average profile (Figs. 7 and 8). Those deviations may be due, for example, to multicentricity at finer scales. Correcting these would however require to endogenize complex agglomeration

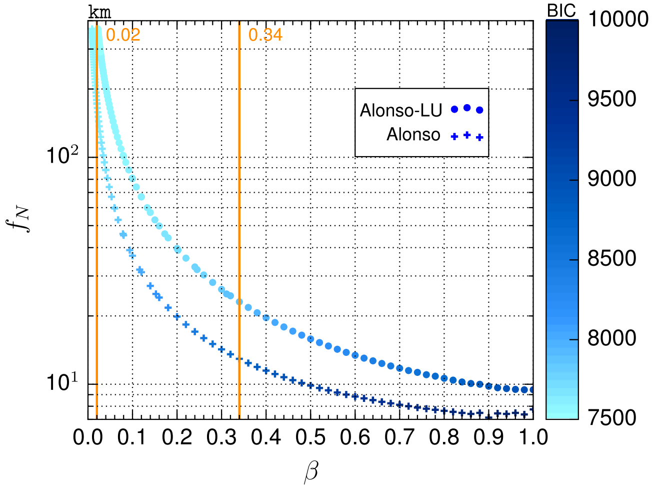

Figure 5. Best fit parameters for the average population density profile. The average profile has been rescaled without loss of generality to a reference city of size $\bar{N} = 7.04 \times 10^5$. Colors represent the Bayesian information criterion (BIC). Orange lines show parameter values of the optimal ($\beta = 0.02$) and constrained ($\beta = 0.34$) models.

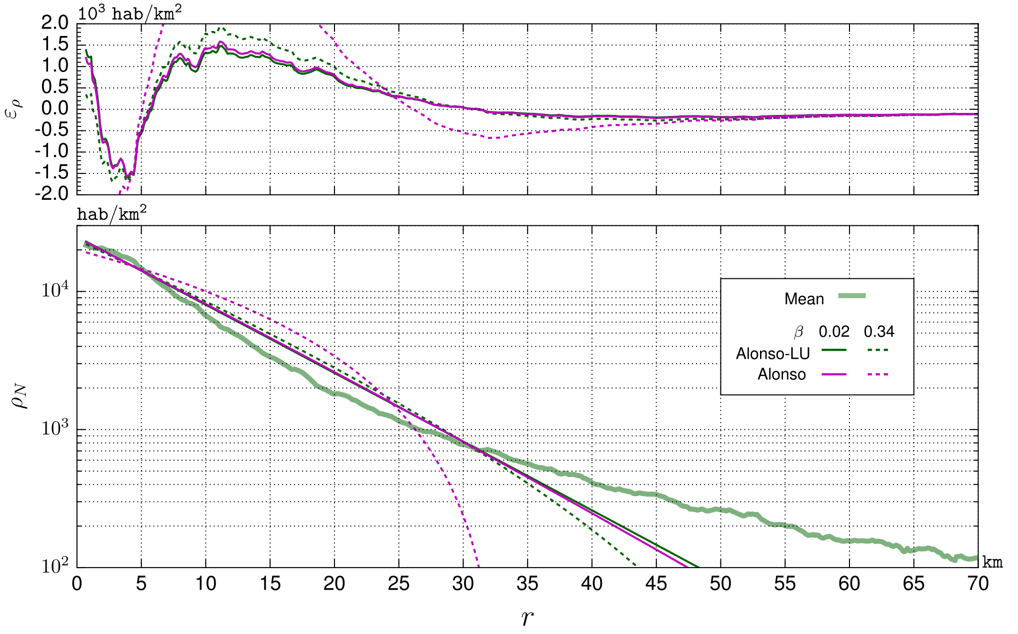

Figure 6. Calibration of the average population density profile Note that the semi-logarithmic scale of the bottom subplot visually exacerbates the error at larger distance from the CBD. On the top subplot, ε_ρ denotes residuals.

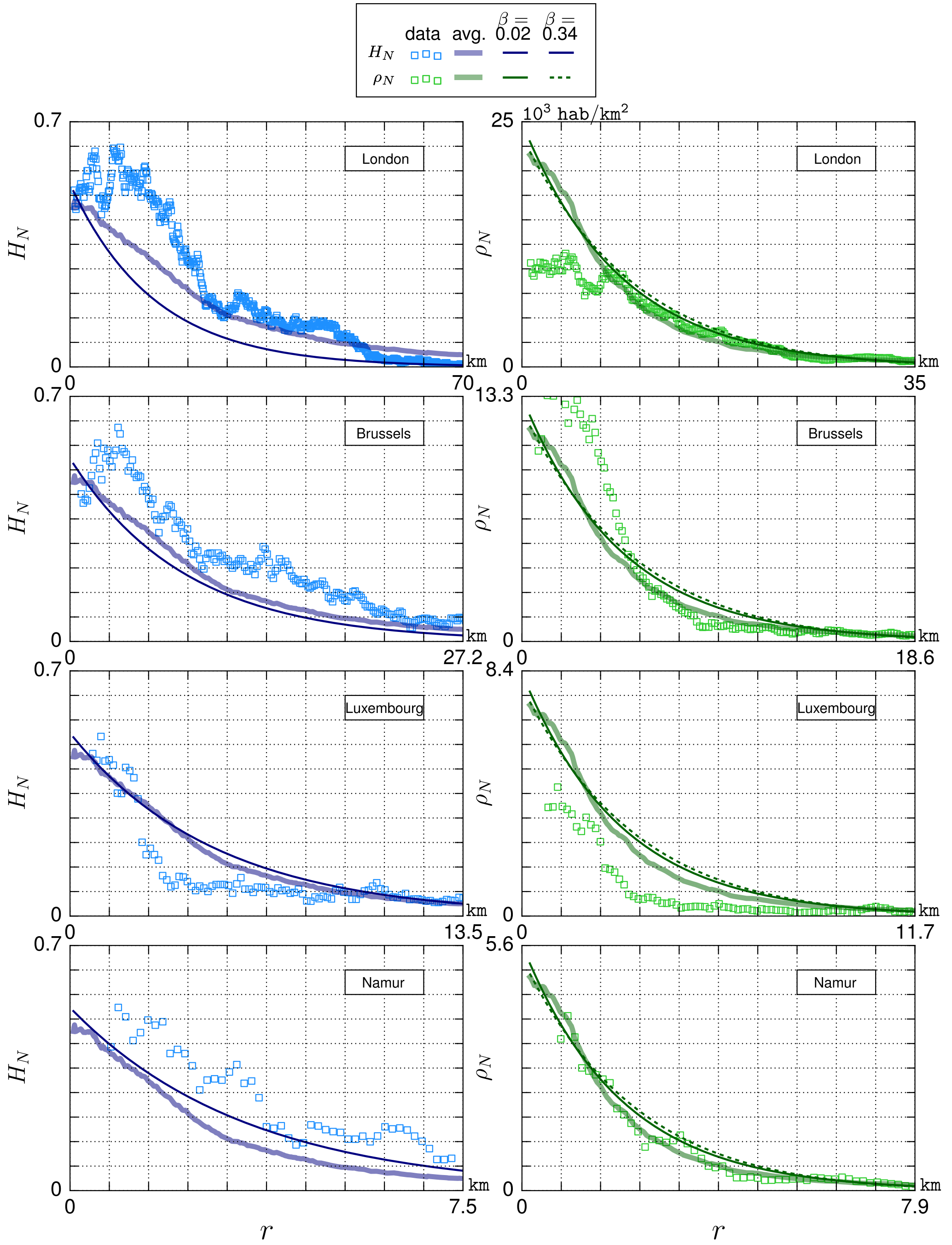

Figure 7. Summary plot of the results. Fitted average profiles compared to individual profiles. Left panel: housing share profile. Right panel: population density profile. Axes have been rescaled to maintain the average curves at the same position across subplots.

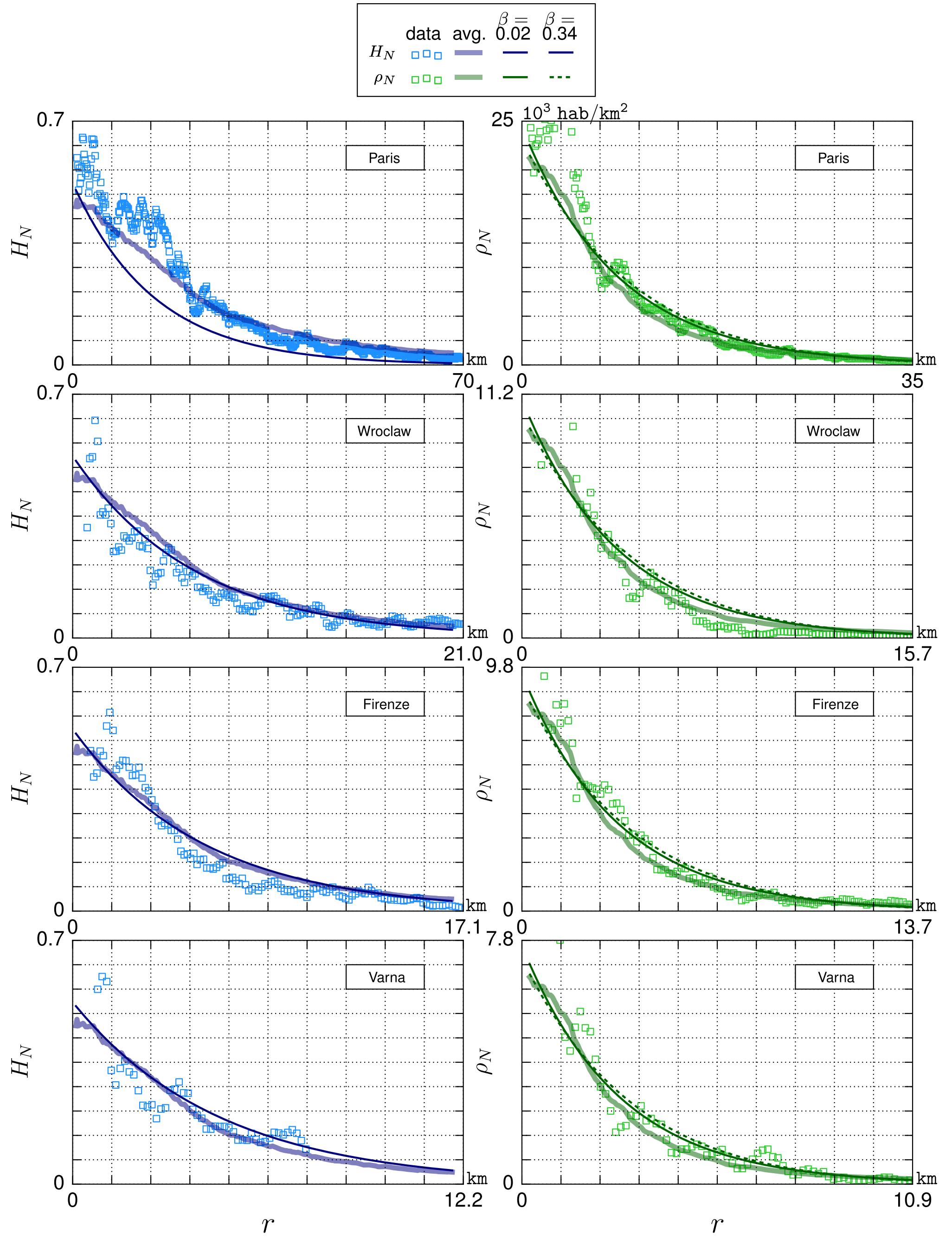

Figure 8. Summary plot of the results. Fitted average profiles compared to individual profiles. Left panel: housing share profile. Right panel: population density profile. Axes have been rescaled to maintain the average curves at the same position across subplots.

processes, which is left for further studies. We can therefore consider our Alonso-LU model to be a successful representation of European urban shapes at this geographical scale.

Let us note that we do not fix the values of the income Y_1 and the unit distance transport cost c in a unitary city since they do not appear in the expression of the population density (14). Our calibration is only performed on land use and population density. We leave to further work a more comprehensive calibration including land prices or rents. Alonso-LU model outputs rent profiles that scale non-homothetically with power 1/3 in the horizontal dimensions and with power $(1/3 + \theta)$ vertically (see Appendix A.3, also comparison with Duranton and Puga 2015). It is flatter than the density profile because the (exponential) $H_N(r)$ factor present in the density disappears in the equation of rents (A.18). This flatter profile seems realistic. However, we do not have radial data for rents across European cities to go further.

Conclusion

The internal structure of cities obeys a homothetic scaling relationship with total population, which is important to model and explain in order to bridge intra-urban and inter-urban research, and eventually provide new normative hints for urban planning. In this article, we showed that the homothetic scaling of European population density profiles uncovered by Lemoy and Caruso (2018) can be understood in light of the fundamental trade-off between transport and housing costs initially exposed by Alonso (1964).

To show this, we have proposed an original, augmented version of Alonso's monocentric model (Alonso-LU) that exogenously introduces urban land profile and the scaling of this profile, of wages and transport costs. The model succeeds in reproducing the three-dimensional homothetic scaling of the European population density profiles. Moreover, the model infers the scaling power of 1/3 empirically found by Lemoy and Caruso (2018), and is consistent with an inter-urban perspective, that is, the coexistence of cities of different sizes. The operationalized version of the Alonso-LU model performs better than the original Alonso model in reproducing the two empirical average profiles. Not only is the fit good, but it is also very parsimonious in parameters (the urban fringe, the housing expenditure, and the decay of the exponential housing land profile). Moreover, comparison with data from individual cities turns out to be surprisingly good in light of the fact that a single parameter (population) is used to adapt the model to different cities.

Our analysis brings those significant new findings but also comes up with three new challenges. First, the inferred scaling power of the land use profile is significantly smaller than the empirical value of Lemoy and Caruso (2018). Second, we still miss an explanation of this land use profile, which is exogenous here. Third, the proposed model challenges current empirical understanding of wage and transport costs elasticities with population.

Further research should address those points. In particular, we have shown that an endogenous model of housing land development is crucial to explain the presence and increase of non-housing land with distance, as well as the scaling of the housing land profile. Potential candidates to this explanation are models of leapfrog urban land development, such as Cavailhès et al. (2004), Turner (2005), Caruso et al. (2007), Peeters et al. (2014) which invoke interaction with agricultural land, or dynamic models with uncertainty like Capozza and Helsley (1990) and Irwin and Bockstael (2002). In the spirit of Muth (1969), the intensity of housing development (including vertical development) within this urban land should also be addressed in order to better describe cities in their vertical dimension. Finally, we have also shown that the implications of using a nonlinear transport cost need to be taken up in order to shed light on urban agglomeration economies and costs across sizes.

Notes

[1]An homothetic scaling is an affine transformation defined by a projection center and a real ratio (see for example, Meserve 2010).

[2]Indeed, if $x = aN^b$, then $\frac{\partial x}{x} / \frac{\partial N}{N} = b$.

[3]Throughout this article, scaling properties will implicitly refer to scaling with respect to urban population N. Thus, indices "N" are used to indicate exogenous variables or functions that are assumed to vary with N. Accordingly, indices "1" are used to indicate the value of those variables for an abstract unitary city of population 1.

[4]In the Alonso model, there is no development of land into housing commodities (land development was introduced into the monocentric theory by Muth 1969). Hence the housing market is not distinguished from the land market. Throughout this article, it is referred as the housing market in order to emphasize Alonso's focus on households' choice. Note also that the term "housing" is used in a broad sense without distinguishing, for example, gardens from built space.

[5]Formally, U must be twice continuously differentiable, strictly quasi-concave with decreasing marginal rates of substitution, positive marginal utilities and all goods must be essentials. See Fujita (1989, p. 311).

[6]Actually, the log-linear utility is a homothetic function as well since it is the logarithmic transformation of the Cobb–Douglas utility, which is itself homogeneous. This corresponds to a representation of homothetic preferences (see for example Varian 2011).

[7]To get more information on equilibrium conditions, existence and uniqueness, see Fujita (1989).

[8]According to Dixit (1973), urban size is mainly determined by the balance between economies of scale in production and diseconomies in transport. Yet in a competitive labor market, labor is paid to its marginal productivity, so that wage-elasticity is representative of labor productivity, which capitalizes itself different effects of urban economies of agglomeration. Similarly, the elasticity of the horizontally-rescaled transport cost function catches agglomeration diseconomies.

Appendix

A. Mathematical appendices

A.1 Households consumption problem

Taking all the assumptions and notations from the main text as given, households' consumption problem in a city of population N is

$$\max \left\{ U(z, s) = (1-\beta)\ln\big(z(r)\big) + \beta \ln\big(s(r)\big) \right\} \tag{A1}$$

$$\text{s.t.} \;\; z + R(r)s(r) = Y_N - T_N(r). \tag{A2}$$

From the utility function, one computes the marginal rate of substitution

$$\frac{\partial U(z, s)}{\partial z}\left(\frac{\partial U(z, s)}{\partial s}\right)^{-1} = \frac{s(1-\beta)}{z\beta}, \tag{A3}$$

which can be equalized to the ratio of prices in order to have the optimal choice equation, that is,

$$\frac{s(1-\beta)}{z\beta} = \frac{1}{R(r)}. \tag{A4}$$

Simultaneously solving the optimal choice equation (A.4) and the budget constraint (A.2) by appropriate substitutions yields the solution of the households consumption problem,

$$z(r) = (1-\beta)\left[Y_N - T_N(r)\right], \tag{A5}$$

$$s(r) = \frac{\beta\left[Y_N - T_N(r)\right]}{R(r)}. \tag{A6}$$

Substituting back the optimal consumptions (A.5) and (A.6) into the utility function (A.1) yields the indirect utility, which can be set to an arbitrary level u in order to express the bid rent function

$$\Psi(r,u) = e^{-u/\beta}\beta(1-\beta)^{1/\beta-1}\left[Y_N - T_N(r)\right]^{1/\beta}, \tag{A7}$$

Finally substituting the bid rent (A.7) into the optimal housing consumption (A.6) yields the bid-max lot size

$$s(r,u) = e^{u/\beta}\left[(1-\beta)\left[Y_N - T_N(r)\right]\right]^{1-1/\beta}. \tag{A8}$$

A.2 Equilibrium problem

Turning now to the urban equilibrium, let u denote the equilibrium utility level and let $n(r)$ be the population distribution (the population living between r and $r + \mathrm{d}r$) at distance r from the CBD, which is a continuous and continuously differentiable function. Then the following equilibrium relationship states that land available for housing at a given commuting distance r within the city is finite and entirely occupied by households:

$$L(r)H_N(r) = n(r)s(r, u), \tag{A9}$$

where $H_N(r)$ follows the horizontal scaling (2). From this follows the definition of the population density $\rho_N(r)$ in this model:

$$\rho_N(r) = n(r)/L(r) = H_N(r)/s(r, u). \tag{A10}$$

We express now the two equilibrium conditions. The first one is the boundary rent condition

$$\Psi(f_N, u) = a, \tag{A11}$$

where a is the exogenous agricultural land rent. As traditionally in urban economic theory, the agricultural land use is no more than a default land use, that is why the agricultural sector is reduced to its most simple form, represented by a constant rent, although it is not really the case empirically (Chicoine 1981; Colwell and Dilmore 1999; Cavailhès et al. 2003). The second equilibrium condition is the population condition

$$\int_0^{f_N} n(r)\mathrm{d}r = N. \tag{A12}$$

On the one hand, substituting the bid rent function (A.7) into the boundary rent condition (A.11) yields the equilibrium urban fringe

$$f_N = T_N^{-1}\left(Y_N - (1-\beta)^{\beta-1}\beta^{-\beta}a^{\beta}e^{u}\right). \tag{A13}$$

On the other hand, consecutively substituting the optimal housing consumption (A.8) into equation (A.9), and the resulting value of population distribution into the population condition (A.12) yields

$$e^{-u/\beta}(1-\beta)^{1/\beta-1}\int_0^{f_N} L(r)H_N(r)\left[Y_N - T_N(r)\right]^{1/\beta-1}\mathrm{d}r = N. \tag{A14}$$

In general, an analytical solution for the equilibrium utility u cannot be obtained by substituting the equilibrium urban fringe (A.13) into the expression of total population (A.14). However, with the assumption that the agricultural land rent is null ($a = 0$), the equilibrium urban fringe becomes

$$f_N = T_N^{-1}(Y_N) \Leftrightarrow Y_N = T_N(f_N), \tag{A15}$$

which means that the urban fringe is the distance at which households spend their entire wage in commuting. Equation (A.15) is very powerful since it enables us to express the results with respect either to the urban fringe f_N or to the wage Y. It is also linking the two quantities in terms of scaling properties. Now, substituting the right-hand-side equation of (A.15) into the population constraint yields the equilibrium utility

$$e^{u/\beta} = N^{-1}(1-\beta)^{1/\beta-1}\int_0^{f_N} L(r)H_N(r)\left[T_N(f_N) - T_N(r)\right]^{1/\beta-1}\mathrm{d}r, \tag{A16}$$

which can be consecutively substituted into the optimal housing consumption (A.8) and into equation (A.10) in order to express the population density function

$$\rho_N(r) = NH_N(r)\left[T_N(f_N) - T_N(r)\right]^{1/\beta-1}\left[\int_0^{f_N} L(r)H_N(r)\left[T_N(f_N) - T_N(r)\right]^{1/\beta-1}\mathrm{d}r\right]^{-1}. \tag{A17}$$

We note also that the bid rent $\psi_N(r)$ is given by $\psi_N(r) = \beta(T_N(f_N) - T_N(r))\rho_N(r)/H_N(r)$, that is,

$$\psi_N(r) = N\beta\left[T_N(f_N) - T_N(r)\right]^{1/\beta}\left[\int_0^{f_N} L(r)H_N(r)\left[T_N(f_N) - T_N(r)\right]^{1/\beta-1}\mathrm{d}r\right]^{-1}. \tag{A18}$$

A.3 Conditions of homothetic scaling

In order to derive conditions under which the population density function (A.17) respects the homothetic scaling (1), one first rescales distances accordingly. Formally, under the following change of variable

$$r_1 = \frac{r}{N^\alpha}, \tag{A19}$$

the population density function (A.17) rewrites

$$\rho_N(r) = \frac{N^{1-\alpha}H_N(r_1N^\alpha)\left[T_N(f_1N^\alpha) - T_N(r_1N^\alpha)\right]^{1/\beta-1}}{\int_0^{f_1} L(r_1N^\alpha)H_N(r_1N^\alpha)\left[T_N(f_1N^\alpha) - T_N(r_1N^\alpha)\right]^{1/\beta-1}\mathrm{d}r_1}, \tag{A20}$$

where we note that the urban fringe f_N has to be rescaled as well, following

$$f_1 = \frac{f_N}{N^\alpha}. \tag{A21}$$

This has, due to equation (A.15), important consequences on the scaling properties of Y_N and T_N.

Finally assume that $L(r)$ is linearly homogeneous, that γ (equation 2), the scaling power of H_N, is the same as α (equation 1), the scaling power of ρ_N, and that $T_N(r)$ is at least horizontally scaling. Formally,

$$\forall \lambda \in \mathbb{R}: \quad L(\lambda r) = \lambda L(r), \tag{A22}$$

$$\gamma = \alpha, \tag{A23}$$

$$\exists \theta \in \mathbb{R}: \quad T_N(r) = N^{\theta} T_1\left(\frac{r}{N^{\alpha}}\right). \tag{A24}$$

The first assumption will add a "−α" term to the power of N in the population density function (A.20). The second assumption implies that the horizontal scaling of the housing usage function (2) balances the effect of total population. The third assumption, equivalent to $T_N(rN^{\alpha}) = N^{\theta}T_1(r)$, will enable us to factorize $N^{(1/\beta-1)\theta}$ both in the numerator and the denominator, so that they cancel out. Altogether, this yields

$$\rho_N(r) = N^{1-2\alpha} H_1(r_1)\left[T_1(f_1) - T_1(r_1)\right]^{1/\beta-1} \left[\int_0^{f_1} L(r_1) H_1(r_1)\left[T_1(f_1) - T_1(r_1)\right]^{1/\beta-1} \mathrm{d}r_1\right]^{-1}, \tag{A25}$$

which is simply a power function of N. In order to finally get the homothetic scaling (1) of the population density function, one has to assume that $1-2\alpha = \alpha$ holds, resulting in

$$\alpha = \frac{1}{3}. \tag{A26}$$

The bid rent can be expressed accordingly as

$$\psi_N(r) = N^{1/3+\theta} \beta\left[T_1(f_1) - T_1(r_1)\right]^{1/\beta} \left[\int_0^{f_1} L(r_1) H_1(r_1)\left[T_1(f_1) - T_1(r_1)\right]^{1/\beta-1} \mathrm{d}r_1\right]^{-1}. \tag{A27}$$

A.4 Consistency with an inter-urban approach

Substituting the scaling of wages into the right-hand-side equation of relationship (A.15) yields

$$Y_1 N^{\phi} = T_N(f_N) = T_N(f_1 N^{\alpha}) = N^{\theta} T_1(f_1) = N^{\theta} Y_1, \tag{A28}$$

where we used also the scalings of the urban fringe (A.21) and of the transport cost function (A.24). This implies

$$\phi = \theta. \tag{A29}$$

Second, successively applying the two changes of variable (A.19) and (A.21) to the equilibrium utility (A.16), and substituting the conditions of homothetic scaling (A.22) and (A.23) yields

$$e^{u/\beta} = N^{(1/\beta-1)\theta-(1-2\alpha)}(1-\beta)^{1/\beta-1} \int_0^{f_1} L(r_1) H_1(r_1)\left[T_1(f_1) - T_1(r_1)\right]^{1/\beta-1} \mathrm{d}r_1 . \tag{A30}$$

Since at equilibrium households have no incentive to move to another city, equilibrium utility (A.30) should not change with N. Thus, equalizing the power of N in equation (A.30) to zero (the rest is independent of N) and substituting the value of $\alpha = 1/3$ (equation A.26) gives

$$\theta = \frac{\beta}{3(1-\beta)}. \tag{A31}$$

Finally, simultaneously solving equations (A.29) and (A.31) yields

$$\phi = \theta = \frac{\beta}{3(1-\beta)} \ . \tag{A32}$$

A.5 Functional transport cost function

Consider the following form of the transport cost function

$$T_N(r) = cN^{\mu} r^{\sigma}, \tag{A33}$$

where $\mu, \sigma \in \mathbb{R}^+$. Then the scaling condition (A.24) requires

$$\theta = \alpha\sigma + \mu, \tag{A34}$$

where the elasticity θ of the transport cost function has been broken into two parts. On the one hand, the nonlinear effect of distance contributes by ασ to the elasticity θ because of the horizontal scaling. On the other hand, the contribution of μ stands for the urban population effects. Further substituting (10) and α = 1/3 into (A.34) yields

$$\mu = \frac{\beta - \sigma(1-\beta)}{3(1-\beta)} \ . \tag{A35}$$

On the one hand, assuming σ = 1 yields the linear case presented in the functional form (13). On the other hand, assuming β = 1/3 yields μ = 1/6–σ/3, which is zero for σ = 1/2.

A.6. Functional monocentric model

Substituting the functional form equations (11)–(13) into the equilibrium population density function (9) with α = 1/3 yields

$$\rho_N(r) = \frac{N^{1/3}}{2\pi} e^{-r_1/d} \left(f_1 - r_1\right)^{\frac{1}{\beta}-1} \left[\int_0^{f_1} r_1 e^{-r_1/d} \left(f_1 - r_1\right)^{1/\beta - 1} \mathrm{d}r_1\right]^{-1}, \tag{A36}$$

with $r_1 = r/N^{1/3}$ Now, under the change of variable $y = f_1 - r_1$ the integral in equation (A.36) becomes

$$f_1 e^{-f_1/d} \int_0^{f_1} e^{y/d} y^{1/\beta - 1} \mathrm{d}y - e^{-f_1/d} \int_0^{f_1} e^{y/d} y^{1/\beta} \mathrm{d}y, \tag{A37}$$

that the second change of variable $x = y/d$ turns to

$$f_1 e^{-f_1/d} d^{1/\beta} \int_0^{f_1/d} e^{x} x^{1/\beta - 1} \mathrm{d}x - e^{-f_1/d} d^{1/\beta + 1} \int_0^{f_1/d} e^{x} x^{1/\beta} \mathrm{d}x. \tag{A38}$$

The first integral in (A.38) can be integrated by parts using

$$x^{1/\beta - 1} = \frac{\partial(\beta x^{1/\beta})}{\partial x}. \tag{A39}$$

After algebraic simplifications, this yields

$$\beta f_1^{1/\beta + 1} - \left(\beta f_1 + d\right) e^{-f_1/d} d^{1/\beta} \int_0^{f_1/d} e^{x} x^{1/\beta} \mathrm{d}x, \tag{A40}$$

which can be finally substituted to the integral into equation (A.36) to give

$$\rho_N(r) = \frac{N^{1/3}}{2\pi} e^{-r_1/d} \left(f_1 - r_1\right)^{\frac{1}{\beta}-1} \left[\beta f_1^{1/\beta+1} - \left(\beta f_1 + d\right) e^{-f_1/d} d^{1/\beta} \int_0^{f_1/d} e^x x^{1/\beta} \mathrm{d}x\right]^{-1}, \quad \text{(A41)}$$

and for the bid rent

$$\psi_N(r) = \frac{N^{1/3+\theta}}{2\pi} \beta \left(f_1 - r_1\right)^{\frac{1}{\beta}} \left[\beta f_1^{1/\beta+1} - \left(\beta f_1 + d\right) e^{-f_1/d} d^{1/\beta} \int_0^{f_1/d} e^x x^{1/\beta} \mathrm{d}x\right]^{-1}. \quad \text{(A42)}$$

A.7. Population of a reference city

From equations (2) and (12), the model of housing usage considered here is a negative exponential with a scaling characteristic distance. Considering the empirical exponents of Lemoy and Caruso (2018), the best model of housing usage is

$$H_N(r) = b \exp\left(\frac{-r}{gN^{1/2}}\right), \quad \text{(A43)}$$

whereas the approximate model is

$$H_N(r) = b \exp\left(\frac{-r}{dN^{1/3}}\right). \quad \text{(A44)}$$

The absolute error between the best model (A.43) and the approximate model (A.44) is given by

$$b \exp\left(\frac{-r}{dN^{1/3}}\right) - b \exp\left(\frac{-r}{gN^{1/2}}\right) = b \exp\left(-\frac{r}{gN^{1/2}}\right) \left[\exp\left(\frac{-r}{dN^{1/3}} - \frac{-r}{gN^{1/2}}\right) - 1\right], \quad \text{(A45)}$$

where the relative error is the term between braces. By definition, $\bar{N}$ is a population size chosen arbitrarily, for which the two characteristic distances are equal, thus annihilating the relative error. That is,

$$d = g\bar{N}^{1/6}, \quad \text{(A46)}$$

such that the relative error rewrites

$$\exp\left(\left[\left(\frac{N}{\bar{N}}\right)^{1/6} - 1\right] \frac{-r}{gN^{1/2}}\right) - 1. \quad \text{(A47)}$$

It appears from (A.47) that for any European city with $N > \bar{N}$, the housing share is underestimated and *vice versa* (Fig. 4). The relative error is bigger, the bigger the difference between N and $\bar{N}$. Hence a first desirable property is that the relative error for the smallest city is the same as for the largest one. This is equivalent to minimizing the maximal relative error. However, this cannot be true for any value of r since the relative error is increasing in r. On the opposite, the absolute error (A.45) has a maximum value at

$$\bar{r} = -\frac{gN^{1/2}}{6} \ln\left(\frac{\bar{N}}{N}\right) \left[\left(\frac{N}{\bar{N}}\right)^{1/6} - 1\right]^{-1}, \quad \text{(A48)}$$

and at this distance the relative error is simply

$$\left(\frac{\bar{N}}{N}\right)^{\frac{1}{6}} - 1. \quad \text{(A49)}$$

Finally, the critical population $\bar{N}$ is chosen as the value for which the absolute value of the relative error at the critical distance $\bar{r}$ is the same for the smallest city in the database, Derry (UK, 1.03×10^5hab), and for the largest, London. This yields

$$\bar{N} = \left(\frac{2}{(1.03 \times 10^5)^{-1/6} + (1.21 \times 10^7)^{-1/6}} \right)^6 \simeq 7.04 \times 10^5. \quad \text{(A50)}$$

B. Further example cities

We illustrate on Fig. 8 the results of the Alonso-LU model on four additional European cities, in order to complement Fig. 7: Paris ($N = 1.14 \times 10^7$), the second biggest city of the database, Wroclaw (Poland, $N = 1.03 \times 10^6$), Florence (Firenze, in Italy, $N = 6.81 \times 10^5$) and Varna (Bulgaria, $N = 3.48 \times 10^5$).

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