On a natural fuzzification of Boolean logic

Raymond Bisdorff

Faculty of Law, Economics and Finance

University of Luxembourg

Content

- introducing logical fuzziness
  - expressions, contradiction positive truth projection, positive and negative assertions
- a first example of logical fuzzification
  - operator triple $< -, \min, \max >$, $\mathcal{L}_o$-tautologies and antilogies, $\mathcal{L}_o$-valued modus ponens
- fuzzification/polarization: an adjoint pair
  - median cut operator, natural fuzzification, examples
- a Bochvar-like fuzzification
  - conjunction and disjunction, De Morgan duality,
- Moving On
  - t-norms are unnatural, semiotical foundation

Introducing logical fuzziness

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The set E of all *well formulated finite expressions* will be generated inductively from the following grammar:

$$\begin{aligned} \forall p \in P &: \quad p \in E, \\ \forall x, y \in E &: \quad \neg x \mid (x) \mid x \vee y \mid x \wedge y \in E. \end{aligned}$$

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Such a credibility domain is called $\mathcal{L}$, and we denote $E^{\mathcal{L}} = \{(x, r_x) \mid x \in E, r_x \in [-1, 1]\}$ a given set of such more or less credible propositional expressions, also called for short $\mathcal{L}$-expressions.

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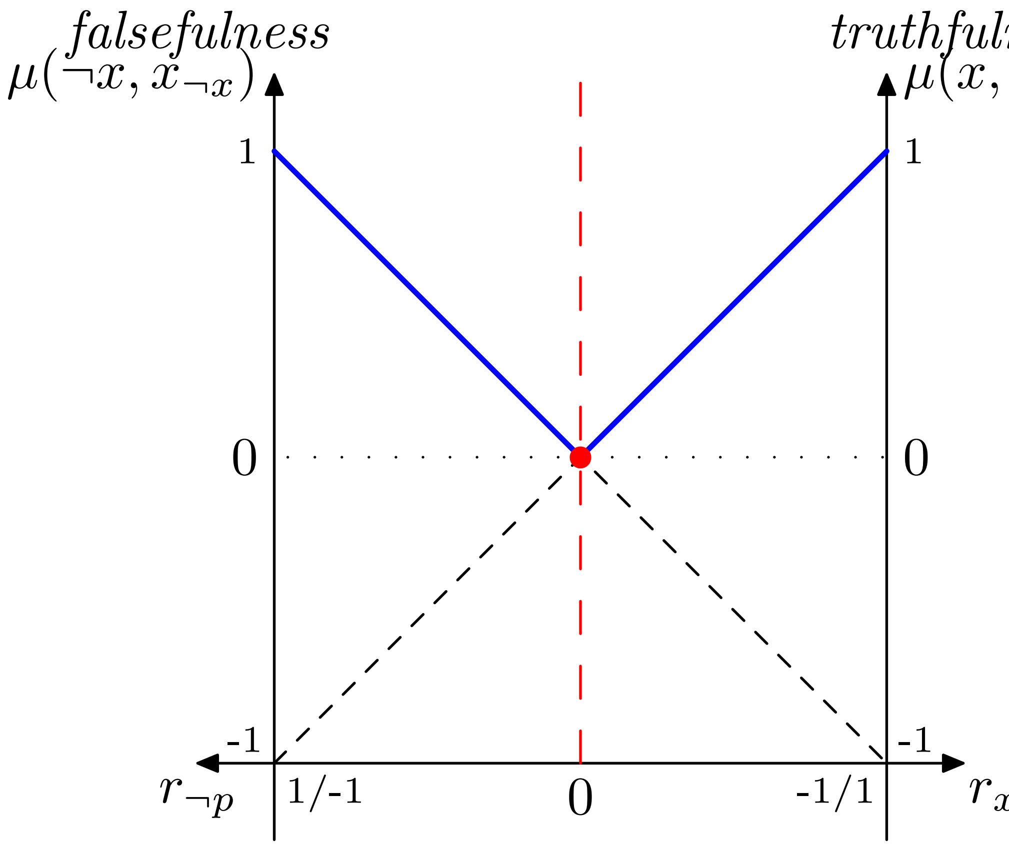
The sign exchange thus implements an antitone bijection on the rational interval $[-1, 1]$ where the *zero* value appears as contradiction fix-point.

Split Truth/Falseness Semantics



positive (*truth oriented*) view point

positive truth projection

We denote the truthfulness possibly induced from the underlying credibility calculus through a truth projection operator μ, acting as a *positive* domain and range restriction on the credibility operator r. Let $(x, r_x) \in \mathcal{E}^{\mathcal{L}}$ be an $\mathcal{L}$-expression:

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$$\mu(x, r_x) = \begin{cases} (x, r_x) \text{ if } r_x \geq r_{\neg x}, \\ (\neg x, r_{\neg x}) \text{ otherwise.} \end{cases}$$

Truthfulness of a given expression x is thus only defined in case the expression's credibility r_x exceeds the credibility $r_{\neg x}$ of its contradiction $\neg x$, otherwise the logical point of view is switched to $\neg x$, i.e the contradicted version of the expression.

positive and negative assertions

As $r_x \geq r_{\neg x} \Leftrightarrow r_x \geq 0$ it follows that the sign ($+$ or $-$) of r_x immediately carries the truth functional semantics of $\mathcal{L}$-expressions,

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Only 0-valued expressions appear to be both $\mathcal{L}$-true and $\mathcal{L}$-false, therefore they are called $\mathcal{L}$-undetermined.

back to content

A first example of natural logical fuzzification

The operator triple $< -, \min, \max >$

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$\forall (x, r_x), (y, r_y) \in \mathcal{E}^{\mathcal{L}}$:

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\begin{aligned}
(x, r_x) \vee (y, r_y) &= (x \vee y, \max(r_x, r_y)) \\
(x, r_x) \wedge (y, r_y) &= (x \wedge y, \min(r_x, r_y))
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The operator triple $< -, \min, \max >$ implements on the rational interval $[-1, 1]$ an ordinal credibility calculus, denoted for short $\mathcal{L}_o$ that gives a first example of what we shall call a *natural fuzzification* of propositional calculus.

$\mathcal{L}_o$-tautologies and $\mathcal{L}_o$-antilogies

truthfulness of the tautology $(x \vee \neg x)$ is always given, as $\max(\, r_x, -r_x)\,) \geq 0$ in any case.

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More or less "untruthfulness" of such an expression is however always given as $\max(\, -r_x, -(-r_x)\,) \geq 0$ in any case and we may call such propositions *$\mathcal{L}_o$-antilogies*.

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the $\mathcal{L}$-valued modus ponens is an $\mathcal{L}_o$-tautology.

back to content

Fuzzification/Polarization: an adjoint pair

a logical polarization operator

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π is also called a *median* cut operator.

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a credibility calculus $\mathcal{L}$ with operator triple $< \neg^{\mathcal{L}}, \wedge^{\mathcal{L}}, \vee^{\mathcal{L}} >$ verifying the categorical equation above is called *natural*.

examples of natural fuzzifications

example 1 :
the ordinal $\mathcal{L}_o$ credibility calculus with the operator triple $< -, \min, \max >$ defined on $[-1, 1]$ implements a natural fuzzification on the category of propositional expressions.

back to content

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example 2 :
the classic operator triple $< 1 - r_x, \min, \max >$ defined on $[0, 1]$ implements a natural fuzzification on the category of propositional expressions, where $\frac{1}{2}$ captures the $\mathcal{L}$-undeterminedness.

back to content

A Bochvar-like fuzzification

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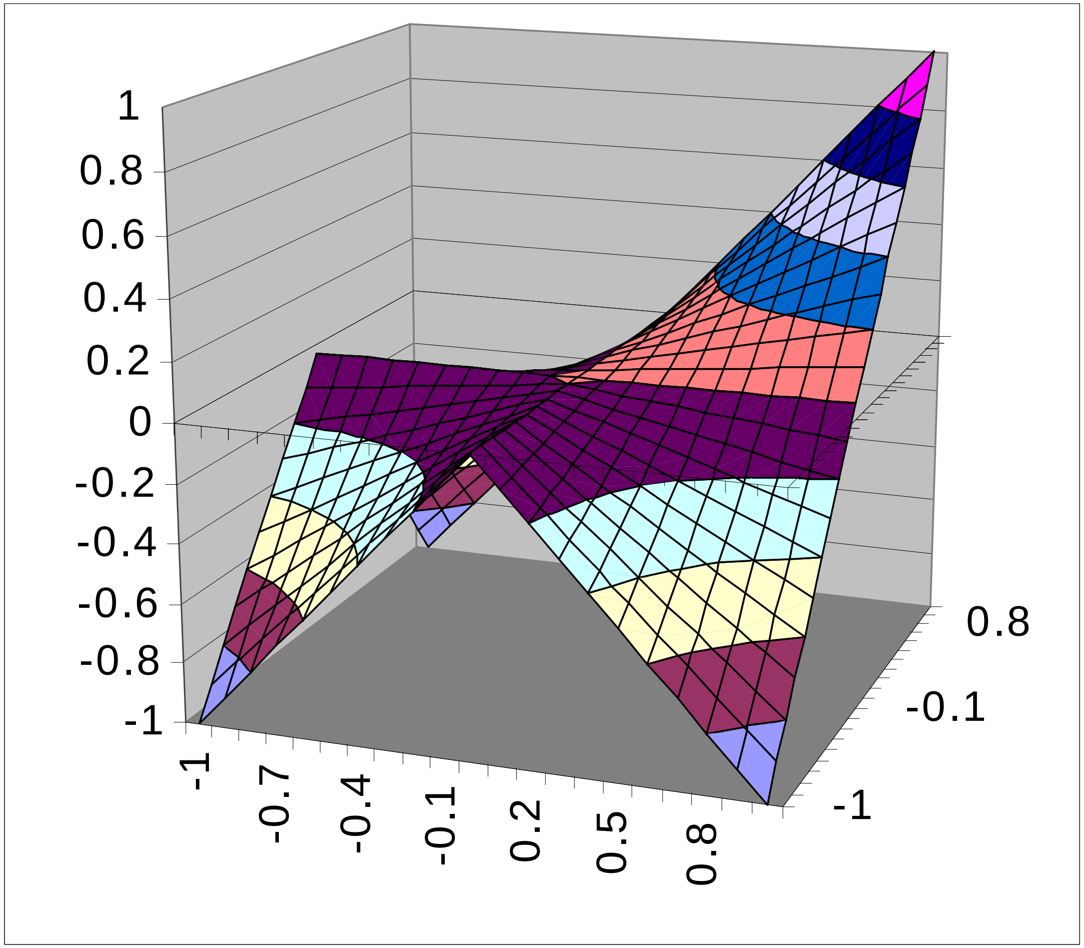

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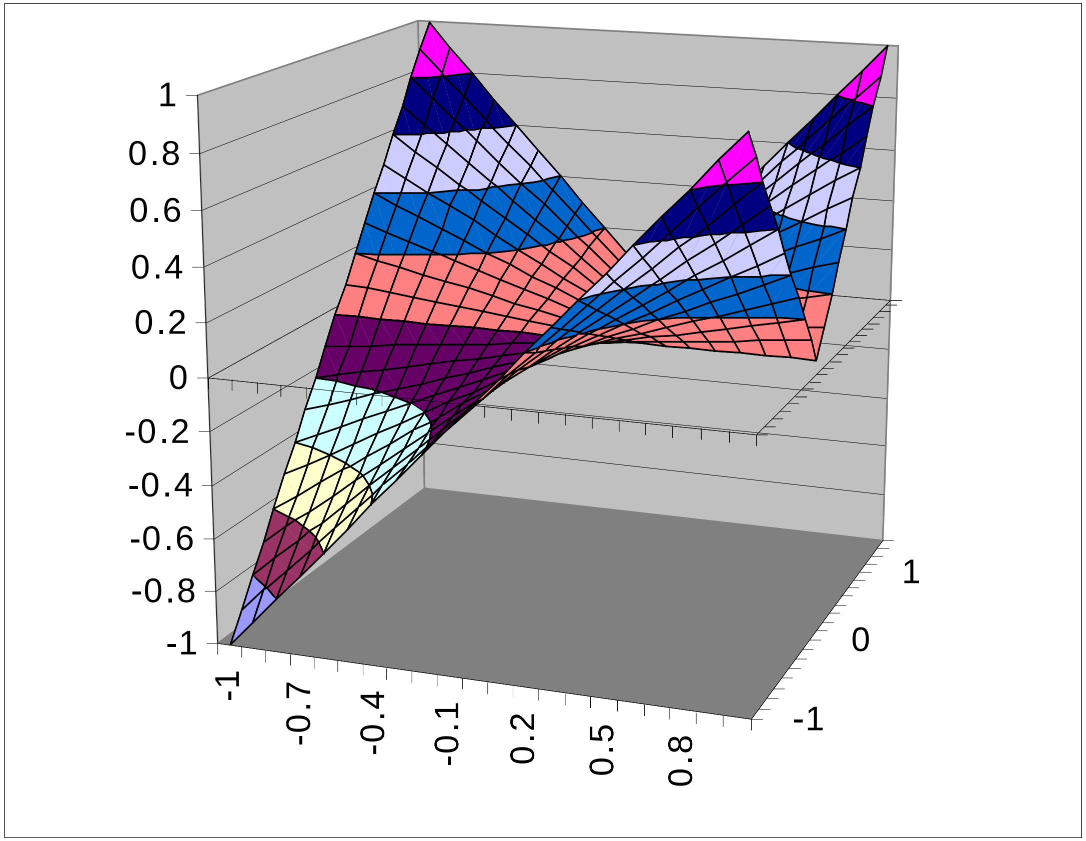

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$\mathcal{L}_b$-valued De Morgan duality

First, we may verify that the De Morgan duality properties are verified in $\mathcal{L}_b$.

$$\forall(x, r_x), (y, r_y) \in E^{\mathcal{L}_b} : r_{x \wedge y} = r_{(\neg(\neg x \vee \neg y))}.$$

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On the contrary, if $r_x, r_y < 0, r_x \curlywedge r_y = -(r_x \times r_y)$, then $r_{\neg x} \curlyvee r_{\neg y}) = (r_{\neg x} \times r_{\neg y}) = (-r_x \times -r(_y) = r_x \times r_y$.

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If either $r_x > 0$ and $r_y < 0$ or vice versa, the duality relation is equally verified.

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$\forall (x, r_x), (y, r_y) \in E^{\mathcal{L}_b}$ such that $r_x \neq 0$ we have:

$$\begin{aligned} \mid r_x \mid &> \mid r_x \curlywedge r_y \mid, \\ \mid r_x \mid &> \mid r_x \curlyvee r_y \mid. \end{aligned}$$

$\mathcal{L}_b$ is a natural fuzzification

we must show that the curly operators $\curlyvee$ and $\curlywedge$ verify $\mu \circ \pi \; = \; \pi \circ \mu$:

$$\mu(\pi(x \vee y, r_x \curlyvee r_y)) = \pi(\mu(x \vee y, r_x \curlyvee r_y)),$$
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if $r_x > 0$ or $r_y > 0$, $\mu(\pi(x \vee y, r_x \curlyvee r_y))) = \mu(x \vee y, 1) = (x \vee y, 1) = \pi(x \vee y, r_x \curlyvee r_y) = \pi(\mu(x \vee y, r_x \curlyvee r_y)$;

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Moving On

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1) consider the t-norm concept as potential generalization
2) follow the semiotical intuitions of C.S. Peirce

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the multiplicative conjunctive operator $\curlywedge$ verifies three of these axioms, i.e. all except the fourth one, so $\curlywedge$ is not a t-norm.

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in some sense we would recover the triangular axiom in "*absolute*" terms, i.e. T non-decreasing in both arguments, either in the positive or in the negative point of view.

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- this leaves open the case where each ground expression $p \in P$ is completely supported by different and disjoint semiotical references (aggregational logic, multiple logical criteria approach)