Bi-pole ranking from pairwise fuzzy outrankings

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Abstract

In this paper we propose to apply the concept of $\mathcal{L}$-valued kernels (see Bisdorff & Roubens [1, 2]) to the problem of constructing a global ranking from a pairwise $\mathcal{L}$-valued outranking relation defined on a set of decision alternatives as encountered in the fuzzy preference modelling context (see Roy & Bouyssou [6] for instance). Our approach is based on a repetitive selection of best and worst candidates from sharpest $\mathcal{L}$-valued or most credible initial and terminal kernels (see Bisdorff [4]). A practical illustration will concern the global ranking of movies from individual evaluations of a given set of movie critics.

1 Introduction

In this paper we propose to apply the concept of $\mathcal{L}$-valued kernels (see Bisdorff & Roubens [1, 2]) to the problem of constructing a global ranking from a pairwise $\mathcal{L}$-valued binary outranking relation defined on a set of decision alternatives as encountered in the fuzzy preference modelling context (see Fodor & Roubens[5] or Roy & Bouyssou [6] for instance).

First we introduce the practical problem which concerns the construction of a global ranking of movies based on individual evaluations from a set of given movie critics. In a second part we briefly introduce the concepts of initial and terminal $\mathcal{L}$-valued kernels and show their eventual use in implementing an $\mathcal{L}$-valued ranking procedure. In a third section, we then illustrate and discuss our ranking approach the results obtained on the set of movies.

2 Ranking movies from the best to the worst

In this section we first present the practical ranking problem we propose for our investigation. In a second part, we introduce an Electre based construction of a global outranking relation between alternatives to consider(see [6]). Unfortunately our data contains necessarily a high rate of missing evaluations. Therefore we propose in a last subsection an innovative method for dealing with this problem in the scope of the Electre methods.

2.1 The movie critics in Luxembourg

In Luxembourg, the movie magazine "Graffiti" publishes monthly a list of appreciations some well known local journalists and cinema critics give periodically to currently shown movies in the Luxembourg movie theatres. The evaluation

Table 1: The Luxembourg Movie Critics in our data set

Identifier	Name	Press affiliation
jpt	JP Thilges	Revue & Graffiti
as	Alain Stevenart	La Meuse
mr	Martine Reuter	Tageblatt & RTL Radio Lëtzebuerg
dr	Duncan Roberts	Luxembourg News
pf	Peter Feist	Grengespoun
vt	Viviane Thill	Le Jeudi
jh	Joy Hoffmann	Zinemag
rei	Raoul Reis	Noticias & Radio Ara,
rr	Romain Roll	Zeitung
cs	Christian Spielman	Journal
h7	Rédaction Cinéma	Radio 100.7

data set we use in this paper is collected from the July/August 1998 issue of the Graffiti magazine (see the complete data set in Figure 2.1). Here in Table 2, we show an extract of the data. The critics express their opinions on the base of an ordinal preference scale ranging from four stars (****) (very much appreciated)

Movies	jpt	as	mr	dr	pf	vt	jh	rei	rr	cs	h7
Abre los Ojos (ao)	**	***	**	**	**	/	**	***	***	*	/
Amarilles (am)	**	**	*	/	/	o	**	**	/	**	/
American Werewolf in Paris (aw)	*	*	*	*	*	o	*	*	***	*	*
La Buona Estrella (bo)	**	/	**	/	***	**	***	***	**	/	/
La Buena Vida (bv)	**	/	**	/	***	**	***	**	/	*	/
Caricies (c)	**	/	/	/	/	/	*	*	/	**	/
Deep Rising (dr)	*	/	/	*	/	/	*	o	**	*	*
En la puta calle (epc)	**	**	/	/	/	**	**	*	**	**	/
Fairy Tale, a true Story (ft)	***	/	***	**	/	/	/	/	***	*	*
Flamenco (fl)	***	/	*	**	**	/	/	**	***	*	**
Gingerbread Man (gm)	**	o	**	*	*	*	**	**	**	**	*
Hola, estas Sola? (hes)	/	***	*	/	/	/	***	/	*	/	/
Kundun (k)	****	****	*	***	**	*	*	*	*	*	**
Liar (l)	**	*	**	**	/	**	***	**	***	*	****
Love!Valour!Compassion! (lvc)	**	/	/	*	***	*	o	/	/	**	**
The Magic Sword (ms)	**	/	/	/	/	/	*	/	**	*	/
La Mirada del otro (mo)	**	/	*	**	*	*	**	*	**	**	/
Paparazzi (pp)	*	*	**	/	*	*	**	**	*	**	*
La Pasion Turca (pt)	**	/	*	/	/	*	*	/	/	**	/
Perdita Durango (pd)	/	***	*	***	**	o	*	*	****	/	**
Primary Colours (pc)	***	**	**	**	/	**	**	**	*	*	/
Secretos del Corazon (scd)	***	**	**	/	***	***	***	***	***	***	/
Serial Lover (sl)	**	***	**	/	/	**	***	/	/	**	**
Swept from the Sea (ss)	*	/	/	**	**	/	o	o	/	*	*
TerritoComanche (tc)	o	/	/	**	*	/	**	**	*	**	/
A Thousand Acres (ta)	**	*	*	**	***	*	/	/	/	/	/
Vertigo 70mm (v)	****	****	***	****	****	****	***	****	****	****	****
The Wedding Singer (ws)	**	**	o	*	/	o	oo	**	**	/	*
Wings of the Dove (wd)	***	*	***	****	**	**	**	o	**	*	***

Figure 1: The complete evaluation data (Source: Graffiti July/August 1998)

to two zeros (oo) (very much disliked). A slash (/) indicates missing data, i.e. a critic missed to see a movie.

Unfortunately, missing data is rather natural and we will propose below an original method for dealing with this uncertainty. In order to clearly separate the positive stars from the negative zeros, we furthermore introduce a neutral null point as separator between positive stars and negative o's, i.e. we will extend the original scale to a set of seven ordinal grades $\{-2, -1, 0, 1, 2, 3, 4\}$.

For an individual critic, this preference scale gives a complete ordering $\geq$ from the best (**** = 4) to the worst (00 = -2) evaluation. For instance, critic *jpt* certainly accepts the movie *Kundum* as being at least as good as the movie *Liar*, but not the reverse. On the contrary, critic *mr* just expresses the opposite opinion.

Table 2: The Movie Critics' opinions in Luxembourg

Movies	*jpt*	*mr*	*vt*	*jh*	...
Kundum	****	*	*	*	...
Liar	**	**	**	***	...
The Wedding Singer	**	o	o	oo	...
The Magic Sword	**	/	/	*	...
...	...	...	...	...	...

2.2 Constructing a global outranking index

Following the general Electre methodology (see Roy & Bouyssou [6])[1], we may additively aggregate the individual outranking relations we observe from the evaluation table by considering each of the eleven critics as an independent criteria associated with a weight of 1/11.

In general, let M denote the set of considered movies and for each $m \in M$, let C_m be the subset of critics who have expressed their opinions about the movie m. For each movie $m_i \in M$ and critic $c \in C_{m_i}$, let $m_i(c)$ denote the evaluation the critic has expressed. A natural outranking index s_{ij} logically evaluating the proposition "*movie m_i is evaluated at least as good as movie m_j*" may be computed in the following way:

$$s_{ij} = \frac{\mid \{c \in C_{m_i} \cap C_{m_j} \quad : \quad m_i(c) \ \geq \ m_j(c)\} \mid}{\mid C_{m_i} \cap C_{m_j} \mid} \tag{1}$$

We may see in s_{ij} the result of a voting in favour of the proposition "*movie m_i is evaluated at least as good as movie m_j*" and we could take such a proposition as logically verified if it is supported by at least a majority of critics. In Table 3 we may see the resulting global outranking index on the illustrative sample given in Table 2 above.

Table 3: The global outranking index s_ij

Movies	k	l	ws	ms	...
Kundum (k)	–	**.40**	.78	.75	...
Liar (l)	.70	–	.90	.100	...
The Wedding Singer (ws)	.22	.40	–	.75	...
The Magic Sword (ms)	**.75**	.60	.100	–	...
...	...	...	...	...	...

Unfortunately, the given evaluation data frequently contains missing values, namely in case a critic has not had the opportunity to see and/or to express his opinion about a movie on the given evaluation list (see Figure 2.1 above).

[1] In fact, we only take into account the concordance part of the Electre method. The discordance part being irrelevant in our problem

2.3 Taking into account missing evaluations

Our idea is that two movies who have not been both seen by a critic may not be ranked, i.e. the credibility of the proposition that "*the first movie is considered by the critic at least as good as the second movie*" must admit the $\mathcal{L}$-undetermined value, i.e. the negational fix-point $\frac{1}{2}$(see [2]).

Now the more a movie is missing comparisons from the critics, the more its global outranking relation wrt to all the other's, is tending to the $\mathcal{L}$-undetermined value $\frac{1}{2}$.

Formally, we adjust the above outranking index (see equation 1) in the following way. Let s_{ij} be the original outranking index computed from the evaluations of movies m_i and m_j and let m_{ij} be the ratio of common evaluations wrt to the number of possible critics. Then the proposed adjusted outranking index s^m_{ij} is defined in the following way:

$$s^m_{ij} \;=\; m_{ij} \cdot s_{ij} \;+\; (1 - m_{ij}) \cdot \tfrac{1}{2} \qquad (2)$$

A graphical representation of the transformation may be seen in Figure 2.3.

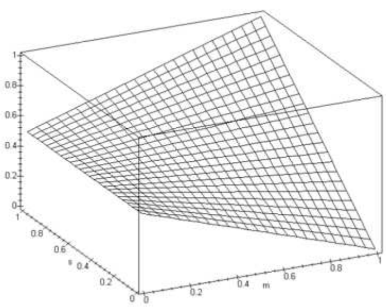

Figure 2: Taking into account missing evaluations

The resulting adjusted outranking index is shown in Table 4.

Table 4: The adjusted global outranking index s^m_{ij}

Movies	k	l	ws	ms	...
Kundum (k)	−	**.41**	.73	.59	...
Liar (l)	.68	−	.86	.73	...
The Wedding Singer (ws)	.27	.41	−	.59	...
The Magic Sword (ms)	**.59**	.55	.68	−	...
...	...	...	...	...	...

Semantically spoken, we adjust the outranking index by adding halve of the relatively missing evaluations as outranking and the other halve as not outranking propositions. In the limit, if m_{ij} approaches 1 (both movies have been seen by nearly all critics), s_{ij} remains rather unchanged. This is observed for the movies *Kundum* and *Liar* where $m_{k,l} = 10/11$, $s_{k,l} = .40$ and $s^m_{k,l} = .41$.

On the other hand, if m_{ij} approaches the value 0, (no common evaluations), s_{ij} is more and more restricted to close values around $\frac{1}{2}$. In case a small number of critics largely prefers a movie to another one, this local preference is always transformed into an $\mathcal{L}$-true global outranking but, the more tending to $\mathcal{L}$-undeterminedness the less the actual voting critics are.[2] This case is observed for instance with the comparison of the movies *Kundum* and *The Magic Sword* where $m_{k,ms} = 4/11$, $s_{k,ms} = .75$, and $s^m_{k,ms} = .59$.

Movies	Abre los Ojos	Amantes	American Werewolf in Paris	La Buena Estrella	La Buena Vida	Caricies	Deep Rising	En la puta calle	Fairy Tale, a true Story	Flamenco	Gingerbread Man	Hola, estas Sola?	Kundun	Liar	Love!Valour!Compassion	The Magic Sword	La Mirada del otro	Paparazzi	La Pasion Turca	Perdita Durango	Primary Colours	Secretos del Corazon	Serial Lover	Swept from the Sea	TerritioComanche	A Thousand Acres	Vertigo 70mm	The Wedding Singer	Wings of the Dove
Abre los Ojos	-	68	91	59	59	59	77	68	55	73	82	59	64	77	55	68	77	77	59	64	77	50	55	77	73	64	9	82	64
Amantes	50	-	82	36	50	68	68	68	45	59	64	45	55	55	59	64	68	64	64	64	55	27	41	68	68	59	18	77	55
American Werewolf in P	27	36	-	27	27	50	82	36	50	41	45	50	45	32	36	59	45	50	45	45	27	18	18	64	45	41	0	64	36
La Buena Estrella	68	73	73	-	77	64	68	73	36	55	82	64	73	68	68	64	82	82	68	68	68	55	68	68	73	68	27	77	64
La Buena Vida	68	68	82	68	-	59	68	64	45	64	73	59	73	77	64	64	73	73	64	73	68	45	64	73	64	68	27	73	64
Caricies	50	50	68	45	50	-	68	59	50	45	50	45	59	50	64	64	59	50	64	59	41	32	55	68	50	55	32	55	50
Deep Rising	32	32	64	41	41	41	-	36	45	32	45	50	45	27	45	59	32	50	45	36	41	27	32	68	41	41	18	59	45
En la puta calle	50	68	73	55	55	68	73	-	45	41	73	45	64	55	68	68	77	73	68	55	64	27	55	68	64	64	18	68	732
Fairy Tale, a true Story	73	55	77	64	64	50	73	55	-	68	68	59	50	68	50	64	64	64	55	41	73	59	50	68	59	64	32	73	59
Flamenco	64	59	86	45	55	55	77	59	68	-	68	59	68	64	55	64	73	64	55	59	68	41	50	77	68	59	14	77	59
Gingerbread Man	45	73	82	45	55	68	82	64	41	41	-	50	55	41	64	68	82	86	73	45	55	18	45	64	73	50	0	82	36
Hola, estas Sola?	50	64	59	45	50	55	50	55	41	50	50	-	59	50	55	50	55	59	59	59	59	50	55	55	59	59	41	59	50
Kundun	55	55	91	27	36	59	73	45	59	68	55	59	-	41	64	59	64	59	64	82	55	27	45	82	55	68	18	73	45
Liar	68	64	95	68	77	59	82	64	59	73	86	59	68	-	68	68	77	82	64	59	73	41	64	77	68	73	23	82	68
Love!Valour!Compassion	55	59	73	50	55	55	64	50	50	55	73	45	55	41	-	55	59	68	59	55	36	36	55	68	55	59	18	73	36
The Magic Sword	50	45	59	55	55	55	68	50	45	45	50	50	59	50	55	-	50	50	55	50	50	32	45	64	50	55	32	64	50
La Mirada del otro	50	68	82	36	36	68	77	68	45	45	73	55	64	41	68	68	-	68	73	55	50	14	45	68	73	64	9	73	45
Paparazzi	41	64	86	27	45	59	68	45	45	45	77	50	59	45	50	50	68	-	64	50	59	18	36	68	77	55	5	59	41
La Pasion Turca	50	64	73	41	45	64	64	50	45	55	55	50	64	45	68	64	64	55	-	64	36	27	45	64	55	64	27	68	36
Perdita Durango	55	55	91	32	27	59	73	55	59	68	55	59	73	41	55	59	64	50	55	-	45	36	45	73	55	55	18	77	45
Primary Colours	59	73	82	50	68	59	68	64	55	68	73	50	64	73	64	59	68	77	64	55	-	41	50	73	68	73	9	77	64
Secretos del Corazon	77	82	91	82	82	68	73	82	59	77	91	59	73	86	73	68	86	91	73	64	86	-	68	73	77	73	18	82	82
Serial Lover	73	77	82	68	73	64	68	73	50	59	82	64	64	73	73	64	73	82	73	73	68	50	-	68	64	68	27	77	55
Swept from the Sea	50	32	64	32	36	32	68	32	59	50	45	45	36	41	41	45	41	50	36	36	45	27	32	-	50	45	18	55	45
TerritioComanche	45	59	64	27	45	59	59	55	50	50	64	50	55	50	55	50	64	68	55	45	68	23	45	59	-	45	18	55	45
A Thousand Acres	55	59	77	50	50	55	59	45	45	59	68	50	50	55	68	55	73	64	64	55	36	36	41	64	64	-	23	64	41
Vertigo 70mm	91	82	100	82	82	68	82	82	77	86	100	68	100	95	82	68	91	95	73	91	91	91	82	82	82	77	-	91	100
The Wedding Singer	27	59	64	41	45	55	68	59	36	32	64	41	27	36	45	55	45	59	41	59	41	27	32	55	55	45	9	-	36
Wings of the Dove	64	55	82	55	55	50	82	55	68	68	82	50	73	59	64	68	73	77	64	64	73	27	55	82	64	68	18	73	-

Figure 3: Global outranking index (s^m_{ij})

The result of our construction finally gives an aggregate $\mathcal{L}$-valued pairwise outranking relation on the set of all 29 movies we consider (see Figure 2.3). On the basis of this fuzzy outranking relation, we would like now to construct a global ranking of the movies from the best to the worst evaluated ones.

3 Ranking by repetitive best and worst choices

In this section, we first show how the concepts of initial and terminal $\mathcal{L}$-valued kernels (see Bisdorff[4]) allow to implement a best and/or worst choice procedure from a pairwise $\mathcal{L}$-valued based outranking index. In a second part we then show how a recursive use of this approach allows to generate a global ranking.

3.1 Initial and terminal $\mathcal{L}$-valued kernels

Let $G(A, R)$ be a simple graph with R being a crisp binary relation on a finite set A of dimension n. A subset Y of A is a dominant (initial) or absorbent (terminal) kernel of the graph G, if it verifies conjointly the following right and left interior stability and corresponding exterior stability conditions:
right and left interior stability:

$$\forall a, b \in A (a \neq b) : (aRb)(\text{respectively } (bRa)) \wedge (b \in Y) \Rightarrow (a \in Y) \qquad (3)$$

initial (respectively terminal) exterior stability:

$$\forall a \in A : (a \notin Y) \Rightarrow (\exists b \in A : (b \in Y)) \wedge (bRa)(\text{respectivlely } (aRb)) \qquad (4)$$

[2]In fact, the simple majority principle for asserting an outranking situation is not restricted by any required minimal quorum of effectively given evaluations

Terminal kernels on simple graphs were originally introduced by J. Von Neumann and O. Morgenstern ([?]) under the name 'game solution' in the context of game theory. J. Riguet ([?]) introduced the name 'noyau (kernel)' for the Von Neumann 'game solution' and B. Roy ([?] introduced the reversed terminal or initial kernel construction as possible dominant choice procedure in the context of the multicriteria Electre decision methods. Terminal kernels were studied by C. Berge ([?, ?]) in the context of the Nim game modelling. Let $G_{\mathcal{L}} = (A, R)$ be a simple $\mathcal{L}$-valued graph with R being a binary relation on a set A of decision alternatives. The relation R is logically evaluated in a symmetric credibility domain $\mathcal{L} = \{V, \leq, \min, \max, \neg, \rightarrow, 0, \frac{1}{2}, 1\}$ (see [1]), where V is a finite set of $2m + 1$ rational values between 0 and 1 with min and max as t-norm and co-t-norm, '$\neg$' in V being a strictly anti-tonic bijection with $\frac{1}{2}$as negational fix-point and the implication operator '$\rightarrow$' verifying the following condition: $\forall u, v \in V : (u \leq v) \Leftrightarrow (u \rightarrow v) = 1$. All degrees of credibility $v \in V$ such $v > \frac{1}{2}$, are denoted as being $\mathcal{L}$-*true*, that is more supporting the truthfulness than the falseness of a relational proposition and all degrees $v < \frac{1}{2}$ are denoted as being $\mathcal{L}$-*false*, that is more supporting the falseness than the truthfulness of a given relational proposition. The median truth value $\frac{1}{2}$ appears as logically undetermined and therefore expresses most uncertainty towards truthfulness or falseness of a given relational proposition. Let $\{k_R\}$ be a singleton set. We assume Y to be an $\mathcal{L}$-valued binary relation defined on $A \times \{k_R\}$, that is a function $Y : A \times \{k_R\} \rightarrow V$, where each $Y(a, k_R), \forall a \in A$, is supposed to indicate the degree of credibility of the proposition that the 'element a is included in the kernel k_R'. As k_R is a constant, we will simplify our notation by dropping the second argument and in the sequel $Y(a), \forall a \in A$, is to be seen as an $\mathcal{L}$-valued characteristic vector for the kernel membership function defined on a given R. As degrees of credibility of the propositions that 'a is a right (respectively left) interior stable element of A' we choose a value $Y(a)$ verifying the following conditions:

$$\max_{b \in A, (a \neq b)} [\min(aRb), Y(b))] \quad \rightarrow \quad \neg Y(a) = 1 \tag{5}$$

$$\max_{b \in A, (a \neq b)} [\min((aR^{-1}b), Y(b))] \quad \rightarrow \quad \neg Y(a) = 1 \tag{6}$$

where $\neg Y$ represents the $\mathcal{L}$-negation of Y. And similarly, as degrees of credibility $Y(a)$ of the propositions that 'a is an initial (respectively terminal) stable element of A' we choose a value $Y(a)$ verifying the following respective condition:

$$\max_{b \in A, (a \neq b)} [\min(aRb), Y(b))] \quad \leftarrow \quad \neg Y(a) = 1 \tag{7}$$

$$\max_{b \in A, (a \neq b)} [\min((aR^{-1}b), Y(b))] \quad \leftarrow \quad \neg Y(a) = 1 \tag{8}$$

It is worthwhile noticing that these conditions may be naturally expressed in a synthetical way with the help of relational $\mathcal{L}$-valued products and inequations.

$$Y \text{is right interior stable} \quad \Leftrightarrow \quad R \circ Y \leq Y \tag{9}$$

$$Y \text{is left interior stable} \quad \Leftrightarrow \quad R^{-1} \circ Y \leq Y \tag{10}$$

$$Y \text{is absorbent stable} \quad \Leftrightarrow \quad R \circ Y \geq Y \tag{11}$$

$$Y \text{is dominant stable} \quad \Leftrightarrow \quad R^{-1} \circ Y \geq Y \tag{12}$$

On the basis of theses above stability inequations, we may now generalize the concept of dominant or absorbent kernel as follows:
Y^{rt} is a right absorbent (terminal) $\mathcal{L}$-valued kernel if

$$Y^{rt} = \max_{Y}\{Y : (R \circ Y \leq Y) \wedge (R \circ Y \geq Y)\} \tag{13}$$

Y^{ri} is a right dominant (initial) L-valued kernel if

$$Y^{ri} = \max_{Y}\{Y : (R \circ Y \leq Y) \wedge (R \circ Y \geq Y)\} \tag{14}$$

Y^{la} is a left absorbent (terminal)L-valued kernel if

$$Y^{rt} = \max_{Y}\{Y : (R \circ Y \leq Y) \wedge (R \circ Y \geq Y)\} \tag{15}$$

Y^{ld} is a left dominant (initial) L-valued kernel if

$$Y^{rt} = \max_{Y}\{Y : (R \circ Y \leq Y) \wedge (R \circ Y \geq Y)\} \tag{16}$$

We denote K^k with $k = \{rt, ri, lt, li\}$ the different solution sets for the corresponding $\mathcal{L}$-valued relational inequality systems. We shall call the set $K^i = \{Y : Y = \max(K^{ri} \cup K^{li})\}$ its dominant kernels and the set $K^t = \{Y : Y = \max(K^{rt} \cup K^{lt})\}$ its absorbent kernels. One may see our kernel definitions as residual constructions, in the sense that we consider as dominant or absorbent kernel candidates, only the maximal sharpest admissible kernel solutions.

For $\mathcal{L}$-un-cyclic graphs, i.e. $\mathcal{L}$-valued graphs not containing any $\mathcal{L}$-true supported circuit, $\mathcal{L}$-valued initial and terminal kernel solutions are unique and recursive elagation of best and worst choices makes apparent the underlying transitive $\mathcal{L}$-valued ordering of the alternatives.

In general, we may observe several admissible initial as well as terminal $\mathcal{L}$-valued kernel solutions. Therefore we introduce a special ordering on $\mathcal{L}$-valued kernel solutions which is inspired by the concept of distributional dominance as used in the context of stochastic dominance.

Let $K = \{K_1, K_2, \ldots, K_k\}$ be a set of kernel solutions defined on a given $\mathcal{L}$-valued graph $G^{\mathcal{L}} = (A, R)$ where the set A contains a finite number n of alternatives. We say that a kernel solution K_i *is at least as credible as* a kernel solution K_j, denoted as $K_i \succeq K_j$ iff the cumulative frequencies of $\mathcal{L}$-true values of K_i are all shifted towards truth value 1 (certainly true) if compared to the cumulative frequencies of $\mathcal{L}$-true values of K_j and vice versa the cumulative frequencies of $\mathcal{L}$-false values of K_i are all shifted towards the truth value 0 (certainly false) if compared to the cumulative frequencies of $\mathcal{L}$-false values of K_j. Now, from the resulting most credible initial kernel solutions we extract all maximal dominating alternatives as best choices and similarly, from the most credible terminal kernel solutions, we extract the maximal dominated alternatives as worst choices.

In the case where no non trivial, i.e. not $\mathcal{L}$-undetermined kernel solutions exist, we stop the procedure and exhibit an unrankable residue as middle ranking class. The earlier an alternative is selected as best or worst, the more reliable this choice is. So that the interior unrankable residual class appears as the less credible result of all.

We have defined now all formal ingredients to implement our bipolar ranking procedure.

3.2 The bi-pole ranking algorithm

The general algorithm we propose is the following:

Algorithm : Bipolar ranking procedure

initialisation step:
$I \leftarrow 1$
$A_I \leftarrow A$
$R_I \leftarrow R$

main step
bipoleranking $(G_I = (A_I, R_I))$
  if $\mid A_I \mid < 1$ **then**
    do
      output *unrankable residue* : $\emptyset$
      stop
    enddo
  else
  do
    $K_I^i \leftarrow$ *initial kernels on* G_I
    $\check{K}_I \leftarrow \max(\succeq)\{K \in K_I^i\}$
    $\check{A}_I \leftarrow \{a \in A_I \mid \exists K \in \check{K}_I : K(a) > \frac{1}{2}\}$

    $K_I^t \leftarrow$ *terminal kernels on* G_I
    $\hat{K}_I \leftarrow \max(\succeq)\{K \in K_I^t\}$
    $\hat{A}_I \leftarrow \{a \in A_I \mid \exists K \in \hat{K}_I : K(a) > \frac{1}{2}\}$

    if $\check{K}_I$ $\mathcal{L}$-undetermined **and** $\hat{K}_I$ $\mathcal{L}$-undetermined **then**
      do
        output *unrankable residue* : A_I
        stop
      enddo

    $J \leftarrow I + 1$
    $A_J \leftarrow A_I - (\check{A}_I \cup \hat{A}_I)$
    $R_J \leftarrow$ *restriction of* R_I *to* A_J

    output *Ith best choices* : $\check{A}_I$
    output bipoleranking $(G_J = (A_J, R_J))$
    output *Ith worst choice* : $\hat{A}_I$
  enddo
endbipoleranking

The main step of the procedure consists in a recursive computing of initial and terminal $\mathcal{L}$-valued kernels solutions on successive restrictions of the original graph by elagating the alternatives corresponding to $\mathcal{L}$-valued disjonction of the Ith most credible kernels in the sense of the above introduced first order credibility dominance. The complexity of the kernel computation is theoretically in $\mathcal{O}(3^n)$ with n the dimension of set A, but efficient concurrent finite domains enumeration techniques in a constraint logic programming environment allow to solve problems up to 50 or even 60 alternatives (see[3]).

On the small sample of four movies of Table 2 above, we obtain the following results:

```
Bi-pole ranking of relation : Table 2
action set A :[k, l, ws, ms]

choices :

1rst step:
Ki =[32, 68, 32, 32]
best choice  : [l](68)

    2d step
    Ki =[73, 27]
    best choice  : [k](73)

        residual class : [](50)

    2d step
    Kt =[27, 73]
    worst choice  : [ms](73)

1rst step:
Kt =[38, 32, 68, 32]
worst choice  : [ws](68)
```

Among the four movies, *Liar* (**l**) appears as first best choice with credibility 68% and *The Wedding Singer* (**ws**) as first worst choice with same credibility. Second best (resp. worst) choice gives *Kundum* (**k**) (resp. *The Magic Sword* (**ms**)). The eventual unrankable middle class is empty in this example.

To illustrate our approach we will solve now the complete movie ranking problem.

4 Global ranking of all movies

In a first part, we show the outcome of our algorithm on the complete data and in a second part we discuss some methodological considerations with respect to our bipolar ranking approach and our treatment of missing values.

4.1 Bipolar ranking results

The outcome of our bipolar ranking procedure is the following:

```
Bi-pole ranking of relation : Complete data set

1rst best  : Vertigo 70mm (v)(68)
2nd best   : Secretos del Corazon (csd) (59)
3rd best   : Liar (l) (59)
```

```
4th best   : Abre los Ojos (ao),  (55)
residual class : Amantes (am), La Buena Estrella (be), La Buena Vida (bv),
                 Caricies (c), Deep Rising (dr), En la puta calle (epc),
                 Fairy Tale, a true story (ft), Flamenco (fl),
                 Gingerbread Man (gm), Hola, esta sola? (hes),
                 Kundum (k), Love!Valour!Compassion! (lvc),
                 La Mirada del otro (mo), The Magic Sword (ms),
                 Paparazzi (pp), Perdita Durango (pd), La Pasion Turca (pt),
                 Primary Colours (pc), Serial Lover (sl),
                 A Thousand Acres (ta), TeritioCommanche (tc),
                 Wings of the Dove (wd)(50)
2nd worst  : Swept from the Sea (ss), The Wedding Singer (ws)(55)
1rst worst : American Werewolf in Paris (aw)(59)
```

The ranking result may be graphically represented as in Figure 4.1.

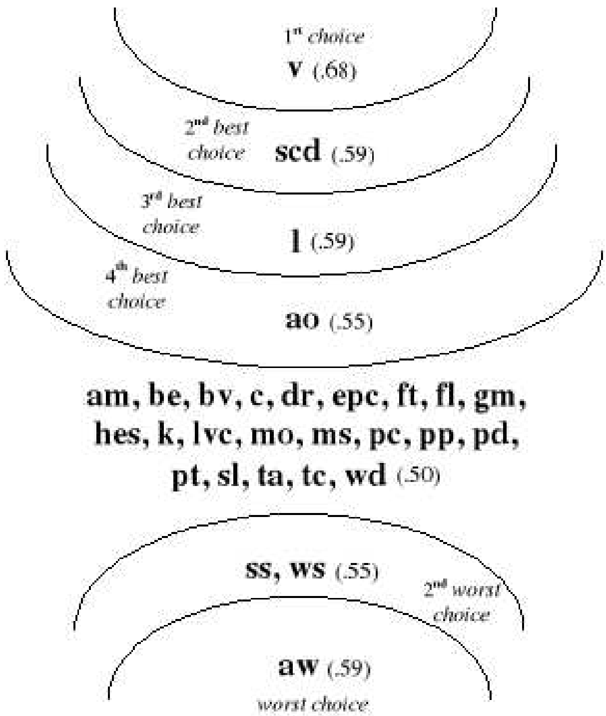

Figure 4: Best against worst choices

The movie *Vertigo 70 mm* (**v**), a recent 70mm restauration of a classic Hitchcock appears as global winner with a credibility of 68%. This result is not surprising as its evaluations are unanymously very high with 9×'****' and 2×'***' evaluations. Second selected is *Secretos del Corazon* (**scd**) with 7×'***' and 2×'**' evaluations (see Figure 2.1).

On the contrary, one movie is immediately designated as worst evaluated: *Americam Werewolf in Paris* (**aw**) with 1×'***', 9×'*' and 1×'o' evaluations.

If we sort the rows of our complete data set on the rank obtained through our bipolar ranking procedure, we obtain an interesting image of the distribution of stars and zeros (see Figure 4.1).

Movies	jpt	as	mr	dr	pf	vt	jh	rei	rr	cs	h7	rank
Vertigo 70mm	****	****	***	****	****	****	***	****	****	****	****	1
Secretos del Corazon	***	**	**	/	***	***	***	***	***	***	/	2
Liar	**	*	**	**	/	**	***	**	***	*	****	3
Abre los Ojos	**	***	**	**	**	/	**	***	***	*	/	4
Amantes	**	**	*	/	/	o	**	**	/	**	/	5
La Buena Estrella	**	/	**	/	***	**	***	***	**	/	/	5
La Buena Vida	**	/	**	/	***	**	***	**	/	*	/	5
Caricies	**	/	/	/	/	/	*	*	/	**	/	5
Deep Rising	*	/	/	*	/	/	*	o	**	*	*	5
En la puta calle	**	**	/	/	/	**	**	*	**	**	/	5
Flamenco	***	/	*	**	**	/	/	**	***	*	**	5
Fairy Tale, a true Story	***	/	***	**	/	/	/	/	***	*	*	5
Gingerbread Man	**	o	**	*	*	*	**	**	**	**	*	5
Hola, estas Sola?	/	***	*	/	/	/	***	/	*	/	/	5
Kundun	****	****	*	***	**	*	*	*	*	*	**	5
Love!Valour!Compassion!	**	/	/	*	***	*	o	/	/	**	**	5
La Mirada del otro	**	/	*	**	*	*	**	*	**	**	/	5
The Magic Sword	**	/	/	/	/	/	*	/	**	*	/	5
Primary Colours	***	**	**	**	/	**	**	**	*	*	/	5
Perdita Durango	/	***	*	***	**	o	*	*	****	/	**	5
Paparazzi	*	*	**	/	*	*	**	**	*	**	*	5
La Pasion Turca	**	/	*	/	/	*	*	/	/	**	/	5
Serial Lover	**	***	**	/	/	**	***	/	/	**	**	5
A Thousand Acres	**	*	*	**	***	*	/	/	/	/	/	5
TerritoComanche	o	/	/	**	*	/	**	**	*	**	/	5
Wings of the Dove	***	*	***	****	**	**	**	o	**	*	***	5
Swept from the Sea	*	/	/	**	**	/	o	o	/	*	*	6
The Wedding Singer	**	**	o	*	/	o	oo	**	**	/	*	6
American Werewolf in Paris	*	*	*	*	*	o	*	*	***	*	*	7

Figure 5: Final ranking of the movies

4.2 Methodological discussion

4.2.1 non-independance with respect to the relevant set of alternatives

Reconsidering our illustrative sample, we may notice that *Liar* (**l**) is indeed ranked before *Kundum* (**k**) and *The Magic Sword* (**ms**) unranked in the residual class, whereas *The Wedding Singer* (**ws**) is designated as worst choice against all three. This fact reminds us that we must consider our bipolar ranking result as immediately related to the actually considered set of alternatives. A same couple of alternatives, especially appearing near the unrankable middle class may very well undergo profound and contradictory ranking variations if considered with different reference alternatives, especially if missing evaluations are involved. This problem may become critic with certain applications, but in our case, as the considered reference set is independantly defined by the editor of the 'Graffiti' magazine, we are not really concerned.

4.2.2 Partial versus complete ranking

A second practical problem may give the rather large unrankable (somehow equivalent) middle class we obtain. This result depends to some degree on the high rate of missing evaluations which introduce a considerable part of $\mathcal{L}$-undeterminedness into our adjusted global outranking index. But it also depends on the existance or not of contradictory evaluations as observed for instance about *The Wings of Dove* with 1×'****', 3×'***', 4×'**', 1×'****', 2×'*' and even 1×'0'. Such evaluations make a refined global ranking little credible. Indeed, the critics express in this case very diverging opinions which make it difficult to situate this movie against all the others. The size of the residual middle class gives therefore a hint towards the extistance of either missing values or the presence of contradictory evaluations. In our opinion, this prudent ranking approach, keeping in the final result traces of contradictory as well as missing evaluations constitutes precisely the strength of our use of recursive initial and terminal kernels elagation technique.

4.2.3 Other methods for treating missing evaluations

A third practical problem concerns naturally the treatment of missing evaluations. Another idea could consist in replacing missing evaluations by the neutral point on the preference scale, i.e. the separator between stars and zeros. In our case, this approach indeed largely reduces the size of the unrankable middle class and selects with certainty *Vertigo 70mm* as first best choice, but the worst choices are somehow changed and the result is less convincing. Indeed, in view of our data set, one star '*' evaluations appear as already very weak evaluations and adding artificially a lot of even lower evaluations in replacement with the missing ones, modifies quite a lot the original bottom ranking results (compare Figures 4.2.3 and 4.1).

Yet another and classic idea consists therefore in replacing missing evaluations with a mean evaluation from all observed evaluations in the row. In our case, the resulting complete bipolar ranking appears more or less compatible with the original one, except that higher credibilities are generally associated with the results and the residual unrankable middle class is reduced to only three items. Unfortunately, this greater precision is artificially introduced and is not originally supported by the observed data. To appreciate the difference in results, we may notice that the evident best choice, i.e. *Vertigo 70mm* (**v**) is selected in this case with certainty (100%), whereas it is only supported by a credibility of 68% in our approach. This increase in uncertainty is induced by our explicit consideration of the rather large part of missing evaluations.

5 Conclusion

In this paper, we introduce an innovative bipolar ranking approach based on the concepts of initial and terminal kernel solutions from a pairwise $\mathcal{L}$-valued comparison index. We illustrate our approach with the help of a real-size ranking problem of movies on the basis of a set of evaluations from known movie critics. An original method for dealing with numerous missing evaluations is introduced and discussed.

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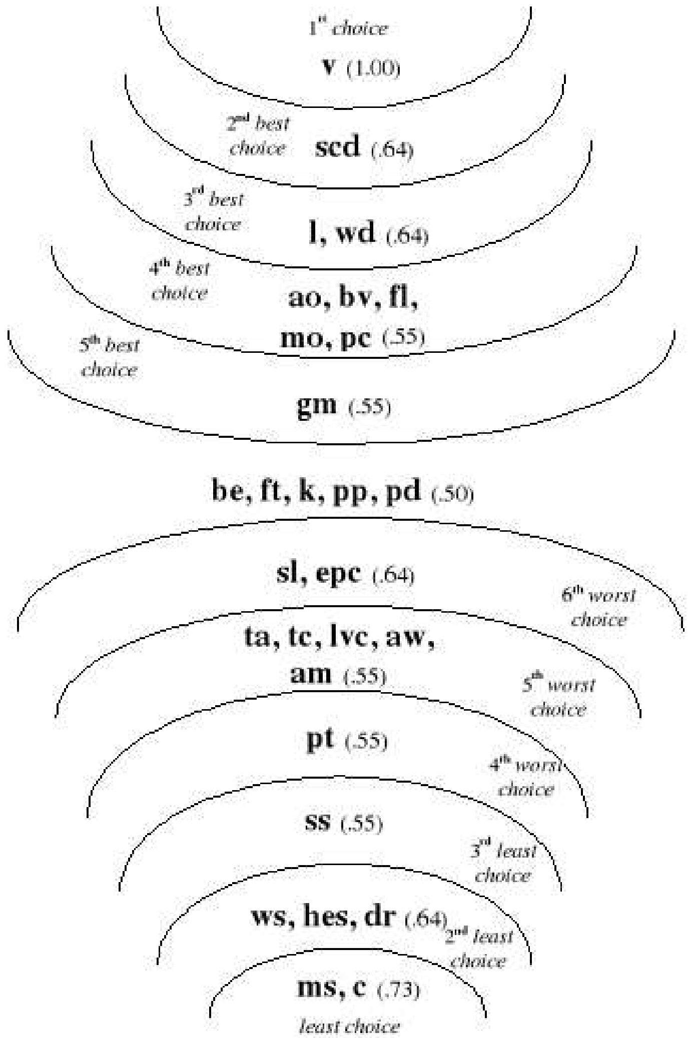

Figure 6: Replacing missing values with a neutral evaluation