On linear decompositions of $\mathcal{L}$-valued simple graphs

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June 1996

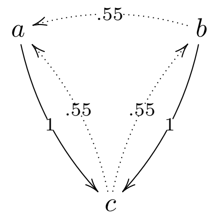
Abstract

In this report we will present a linear decomposition of a given $\mathcal{L}$-valued binary relation into a set of sub-relations of kernel-dimension one. We will apply this theoretical result to the design of a faster algorithm for computing $\mathcal{L}$-valued kernels on general $\mathcal{L}$-valued simple graphs.

1 Introductory example

Recent research work [3] [4] on defining and computing fuzzy kernels from $\mathcal{L}$-valued simple graphs [1] have shown that multiple kernel solutions correspond in fact to multiple possibilities for extracting independant alignements, that is fuzzy a-cyclic binary relations from given genral fuzzy simple graphs.

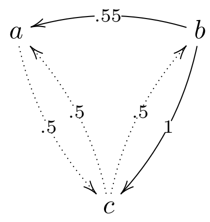
To give a first intuitive illustration of this idea consider the following numerical example: [2].

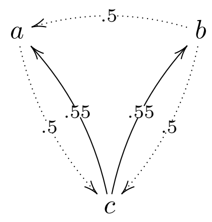
$G =$ 

This fuzzy outranking [9] graph G admits two fuzzy dominating kernel solutions $K_1 = [.45, 1, 0]$, $K_2 = [.45, .45, .55]$, [3], [4] to which coprrespond the following "independant" restrictions of of the original relation:

[1] R. Bisdorff, M. Roubens 1996a, 1996b

[2] This example was proposed by Bernard Roy at the 41th meeting of the EURO working group MCAD on "Multicriteria Aid for Decisions" in Lausanne, March 1995

$G_{/K_1} =$ 

$G_{/K_2} =$ 

with $K_1 = [.45, 1, 0]$ and $K_2 = [.45, .45, .55]$ as respective fuzzy kernel solution.

This example suggest that multiple kernel solutions are in fact hints towards a possible boolean "additive" decomposition of the original $\mathcal{L}$-valued relation.

2 On boolean "addition" of $\mathcal{L}$-valued binary relations

Let $G^{\mathcal{L}}(A, R, \mathcal{L})$ be a simple finite $\mathcal{L}$-valued graph, where $\mathcal{L} = (V, \leqslant, \max, \min,\text{-}\neg, \rightarrow, 0, 1)$ is a symmetric rational evaluation domain (cf. [3], where the set V is a completely ordered possibly infinite set of rational truth values with 0 as "certainly false" and1 as "certainly true" value, $\neg$ is a strong negation and $\rightarrow$ is a residual min-implication. [3] We shall denote in the sequel $\mathcal{B} = (\{0, 1\}, \leqslant, \max, \min, \neg, \Rightarrow, 0, 1)$ the standard boolean crisp logical denotation and $\mathcal{L}_3 = (\{0, \frac{1}{2}, 1\}, \leqslant, \max, \min, \neg, \rightarrow, 0, 1)$ the 'minimum trivalent symmetric evaluation domain.

Let $\mathcal{R}^{\mathcal{L}}_A$ be the set of all possible $\mathcal{L}$-valued relations on a given finite set A. We may define on $\mathcal{R}^{\mathcal{L}}_A$ a special addition-operator $\oplus$ in the following manner:

Definition 1 *Let* $R, S \in \mathcal{R}^{\mathcal{L}}_A : \forall (a, b) \in A \times A :$

$$(R \oplus S)(a, b) = \begin{cases} \max(R(a, b), S(a, b) & \Leftrightarrow \quad R(a, b) \geqslant \frac{1}{2} \wedge S(a, b) \geqslant \frac{1}{2}, \\ \min(R(a, b), S(a, b) & \Leftrightarrow \quad R(a, b) \leqslant \frac{1}{2}, \wedge S(a, b) \leqslant \frac{1}{2}, \\ \frac{1}{2} \quad \text{elsewhere.} \end{cases}$$

Proposition 1 *If* $M \in \mathcal{R}^{\mathcal{L}}_A$ *is the trivial all median-valued relation, the algebraic structure of* $(\mathcal{R}^{\mathcal{L}}_A, \oplus, M)$ *gives a commutative group with relation* M *as neutral element.*

Indeed, the trivial fuzziest $\mathcal{L}$-valued relation M is a neutral element for the $\oplus$ operator. Commutativity comes for the standard conjunction and disjunction operators min and max. Finally we may associate to every given relation $R \in \mathcal{R}^{\mathcal{L}}_A$ its "contradictory" relation $\neg R = 1 - R$ so that $R \oplus \neg R = M$. $\square$

[3] We will follow in this research report the terminology introduced in Bisdorff, Roubens 1996a and 1996b.

This $\oplus$ operator is linked to the sharpness ordering on $\mathcal{R}_A^{\mathcal{L}}$ in the following way:

Definition 2 *Let $R : A \times B \to V$ and $S : A \times B \to V$ be two $\mathcal{L}$-vbr's. We say that R is sharper than S, noted $R \succcurlyeq S$ iff $\forall (a,b) \in A \times B$: either $(R(a,b) \leqslant S(a,b) \leqslant \frac{1}{2})$ or $\frac{1}{2} \leqslant S(a,b) \leqslant R(a,b))$.*

The sharpness relation '$\succcurlyeq$' on the set $\mathcal{R}_{AB}^{\mathcal{L}}$ of all $\mathcal{L}$-vbr's defined between any finite sets A and B of respective dimensions n and p, gives a complete partial order $(R^{\mathcal{L}}, \succcurlyeq)$ with the constant median-valued relation M as unique minimum element and $R^{\mathcal{B}}$, the 2^{np} possible $\mathcal{B}$-valued crisp relations between sets A and B, as the set of maximal (sharpest) elements.

Proposition 2 *Let $R, S \in \mathcal{R}_A^{\mathcal{L}}$ be two $\mathcal{L}$-vbr's defined on the same set A.*

$$R \succcurlyeq S \quad \Leftrightarrow \quad (R \oplus S \succcurlyeq R) \wedge (R \oplus S \succcurlyeq S).$$

Indeed, the boolean addition of two $\succcurlyeq$-comparable relations gives an overall sharper relation as result. This is an immediate consequence of definition 2 and definition 1. □

3 Linear decompositions of $\mathcal{L}$-valued relations

Let R be an $\mathcal{L}$-vbr defined on a given finite set A and let $K(R) = \{K_1, K_2, \ldots, K_n\}$ be its n fuzzy $\mathcal{L}$-valued kernel solutions. To each kernel solution $K_i, i = 1, \ldots n$ corresponds a specific subgraph $G_{/K_i} = (A, R_{K_i})$ defined in the following way.

Definition 3 *Let K_i be a specific kernel solution on a given graph $G^{\mathcal{L}} = (A, R)$. We call the relation $R_{/K_i}$ defined as follows: $\forall (a,b) \in A \times A$:*

$$R_{/K_i}(a,b) = \begin{cases} R(a,b) & \Leftrightarrow \quad (K_i(a) > \frac{1}{2}) \quad \vee \quad (R(a,b) \leqslant \frac{1}{2}) \quad \vee \quad (a = b) \\ \frac{1}{2} & \text{elsewhere.} \end{cases}$$

a kernel- or K_i-restriction of the original relation R. We shall denote $R_{/K(R)} = \{R_{/K_i} / K_i \in K(R)\}$ the set of kernel restriction on R based on the set of kernel solutions from R.

Proposition 3 *To each specific kernel solution K_i corresponds a unique K_i-restriction R_{K_i}.*

Indeed, consider $R_{/K_i}$ and $R'_{/K_i}$ being two different kernel restrictions of R corresponding to a same given kernel solution K_i. Then there $\exists (a,b) \in A \times A$: $R_{/K_i}(a,b) \neq R'_{/K_i}(a,b)$. This contradicts definition 3. □

Proposition 4 *Let $G^{\mathcal{L}} = (A, R)$ be a finite $\mathcal{L}$-valued simple graph and $K(R) = \{K_1, K_2, \ldots, K_n\}$ the corresponding kernel solutions set. Let $R_{/K(R)}$ be the complete set of kernel restrictions corresponding to the set $K(R)$ of kernel solutions on R. The following relation are then verified:*

$$\hat{R} = \left[\bigoplus_{i=1}^{n} R_{/K_i}\right] \preccurlyeq R \quad (1)$$

$$K(R_{/K_i}) = \{K_i\}, i = 1, \ldots n \quad (2)$$

$$K(\hat{R}) = \bigcup_{i=1}^{n} K(R_{/K_i}) = K(R). \quad (3)$$

Every relation R may be decomposed into n relations $R_{/K_i}$ so that the $\oplus$-sum equals again a relation of same shape than the original relation R but possibly less sharper, and the kernel solution set for each relation R_i is exactly equal to the corresponding unique kernel solution K_i determined on the orginal relation R. This follows indeed from the K_i-decomposition construction principle of definition 3 and from the partial monotonicty of the kernel construction w.r.t. the sharpness ordering "$\succcurlyeq$" on $\mathcal{R}$. Finally, the set-union of all individual kernel solutions restitutes back again the complete set $K(R)$ of kernel solutions observed on R. □

Example 1 *Let $G^{\mathcal{L}} = (A, R)$ with R being a $\mathcal{L}$-empty ($\leqslant \frac{1}{2}$) relation, that is every credibility value observed in R is either $\mathcal{L}$-untrue or $\mathcal{L}$-undetermined [3]. Such a relation has a unique fuzzy kernel solution K with every element strictly $\mathcal{L}$-true selected. As the K-restriction is neutral for such a relation, proposition 4 is trivially verified in this case.*

Example 2 *On the other hand let us take the following $\mathcal{L}$-clique of dimension* 3*, that is a relation with every term being $\mathcal{L}$-true.*

$$R = \begin{bmatrix} 1 & .75 & .75 \\ .70 & 1 & .70 \\ .60 & .60 & 1 \end{bmatrix}$$

Such a relation has 3 *fuzzy kernel solutions, $K(R) = \{K_1, K_2, K_3\}$ with*

$$K_1 = [\mathbf{.75}, .25, .25],$$
$$K_2 = [.30, \mathbf{.70}, .30],$$
$$K_3 = [.40, .40, \mathbf{.60}],$$

describing each a different fuzzy singleton. The corresponding K_i-restrictions give $\mathcal{L}$-valued relations with each time a different row of credibity values different from $\frac{1}{2}$.

$$R_{/K_1} = \begin{bmatrix} \mathbf{1} & \mathbf{.75} & \mathbf{.75} \\ .50 & 1 & .50 \\ .50 & .50 & 1 \end{bmatrix}, \quad R_{/K_2} = \begin{bmatrix} 1 & .50 & .50 \\ \mathbf{.70} & \mathbf{1} & \mathbf{.70} \\ .50 & .50 & 1 \end{bmatrix}, \quad R_{K_3} = \begin{bmatrix} 1 & .50 & .50 \\ .50 & 1 & .50 \\ \mathbf{.60} & \mathbf{.60} & \mathbf{1} \end{bmatrix}.$$

To each such K_i-restriction corresponds a unique individual kernel solution. In this example, the linear recomposition $\hat{R} = R_{/K_1} \oplus R_{/K_2} \oplus R_{K_3}$ is in fact identical to R so that properties (1) and (3) of proposition 4 are again trivially true.

4 A faster algorithm for computing $\mathcal{L}$-valued kernels

The above noted theoretical results suggest the idea for the design of a faster algorithm for $\mathcal{L}$-valed kernel computations as originally proposed by constrained finite domains enumeration [4]. Indeed, the linear decomposition of a given relation R into a set of mono-nuclear relations allows to use for each such computation the very fast dual fixpoint algorithm as originally proposed by Von Neumann (1944)[13], and applied to the fuzzy case by Kitaïnik (1993)[7].

Indeed, we may observe that the kernel shapes given by the kernel solution from median β-cut relation is sufficient to determine the K_i-decomposition for a given general $\mathcal{L}$-valued relation R.

Proposition 5 *Let R be an $\mathcal{L}$-valued relation and let $K(R)$ be its associated kernel solutions set of dimension n. Let $R_{\beta > \frac{1}{2}}$ be the associated β-cut relation. Let $K(R_{\beta > \frac{1}{2}})$ be the corresponding $\mathcal{L}_3$-valued kernel solutions sets. Then $R_{/K(R)} = R_{/K(R_{\beta > \frac{1}{2}}}$.*

Indeed, the median β-cut is a natural transformation (in the categorical sense) from general $\mathcal{L}$-valued relation to $\mathcal{L}_{\ni}$-valued relations (cf. Bisdorff & Roubens 1996a, 1996b), so that the median β-cut kernel solutions are $\succcurlyeq$-comparable limits. Or the definion 3 of the K_i decomposition does not rely on a precise value, but only on the $\succcurlyeq$-comparable shape of the kernel solution and K_i-restrictions of R from $K(R)$ and from $K(R_{\beta > \frac{1}{2}})$ are in fact identical.□

Furthermore, from proposition 4 we know that any K_i-restriction has a unique kernel solution as it is an $\mathcal{L}$-alignement [3]. Indeed this follows from the resultbelow:

Proposition 6 *Let $G^{\mathcal{L}} = (A, R)$ be an $\mathcal{L}$-valued simple graph with $K(R)$ its corresponding $\mathcal{L}$-valued kernel solutions set. Every corresponding K_i-restriction $R_{/K_i}$ gives an $\mathcal{L}$-alignement.*

Indeed, from definition 3 it follows that the only $\mathcal{L}$-true elements of $R_{/K_i}$ are those coming from kernel members. All other elements are either $\mathcal{L}$-untrue or $\mathcal{L}$-undetermined. □

Finally, application of the dual fixpoint algorithm to this $\mathcal{L}$-alignements R_i gives following result:

Proposition 7 *Let R be a given $\mathcal{L}$-alignement defined on a finite set A of dimension m and $K(R)$ its $\mathcal{L}$-valued kernel solution. Let Y be a possible $\mathcal{L}$-valued kernel-membership function and $R^{-1} \circ Y = \overline{Y}$ and $\overline{R^{-1} \circ Y'} = Y'$ be the* *dual anti-eigenvalue fixpoint equations corresponding to the stability conditions of the kernel construction.* [4] *Let the inital solution for Y be the all 0-valued vector and for Y' be the all 1-valued vector. The dual iteration will exchange at each step the corresponding values of Y and Y' in the two equations in order to reach rapidly in a finite number of steps (bound by m) two fixpoints $\hat{Y}$ and $\hat{Y}'$ such that $K(R) = \hat{Y} \oplus \hat{Y}'$*

Demonstration of this rather technical property of the dual fixpoint algorithm is given by Kitaïnik [7].

The general algorithm we propose is the following:

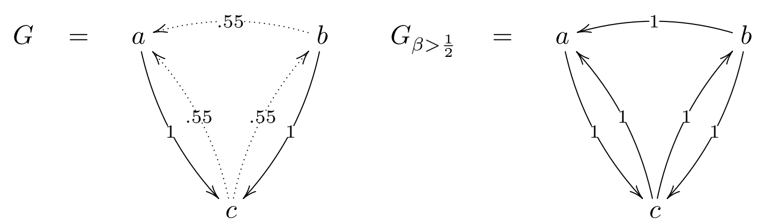
Algorithm 1 ***input****: a given $\mathcal{L}$-valued relation R:*
1) determination of the median β-cut kernel solutions $K(R_{\beta > \frac{1}{2}})$,
2) computing the K_i-decompositions R^K from $K(R_{\beta > \frac{1}{2}})$,
3) for every $R_i \in R^K$
determine K_i by the dual fixpoint algorithm
output*: $K(R) = \{K_1, \ldots, K_n\}$.*

Theoretical computing complexity of the first step is in $\mathcal{O}(3^m)$ if m is the dimension of the underlying finite set A. This exponential complexity must be reconsidered in the light of the efficiency of the dynamic propagation of the min and max operators by the finite domain solver we intend to use. For a rather large class of practical examples this step is rather quick, but a detailed exploration of the necessary limit in dimension for the set A has still to be done. Step 2 has a polynomial computing complexity $\mathcal{O}(nm^3)$, where n is the kernel dimension of R. For $\mathcal{L}$-connected graphs, this dimension is bound by the dimension of the set A, so that it may be approximated in the worst case by $\mathcal{O}(m^4)$. Finally, step 3 is again, for the worst case, in polynomial complexity $\mathcal{O}(nm^4)$.

Example 3 *We may reconsider the introductory example and illustrate the output of the different steps of our algorithm.*

step 1

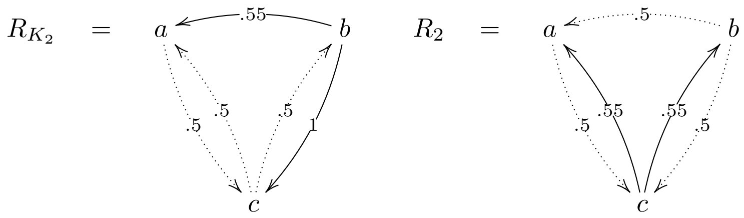
The median β-cut [3] will gives following $\mathcal{B}$-valued graph $G_{\beta > \frac{1}{2}}$:



The kernel solutions set for this crisp relation is the following:
$K(R_{\beta > \frac{1}{2}}) = \{[0,1,0],[0,0,1]\}$.

[4]The dominated or absorbent kernel is obtained by replacing the reversed relation R^{-1} by the original relation R [**?**], [4].

step 2
From this kernel solutions we may construct the same $R_{/K_i}$-restrictions as allready mentioned in the introduction above:



step 3
Finally the dual fixpoint algorithm, computing in an integer percents domain, will give as required, for the K_1-restriction, following results:

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Computing the dominating L-valued kernel :
 R-Supporting set:  [a, b, c]
 Evaluation domain L:  [0, 1, ..., 100]
 Relation matrix R_K1 : [[ 0,  55, 50],
                         [ 0,   0, 50],
                         [50, 100,  0]]
 Iteration : 1
Y   : [0, 0, 0]
Y'  : [100, 100, 100]
 Iteration : 2
Y   : [45, 50, 0]
Y'  : [100, 100, 100]
 Iteration : 3
Y   : [45, 50, 0]
Y'  : [50, 100, 50]
```

with $Y \oplus Y' = [.45, 1, 0]$, and for this kernel restriction and

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 Relation matrix R_K2 : [[ 0, 50, 55],
                         [ 0,  0, 55],
                         [50, 50,  0]]
 Iteration : 1
Y   : [0, 0, 0]
Y'  : [100, 100, 100]
 Iteration : 2
Y   : [45, 45, 50]
Y'  : [100, 100, 100]
 Iteration : 3
Y   : [45, 45, 50]
Y'  : [50, 50, 55]
```

with $Y \oplus Y' = [.45, .45, .55]$ for the second kernel restriction, so that we recover the original set $K(R)$ of $\mathcal{L}$-valued kernel solutions as announced.

5 Conclusion

We have defined in this report a linear decomposition of a given$\mathcal{L}$-valued simple graph into a set of independant $\mathcal{L}$-alignements. The original relationmay be naturally recomposed in a global relation of same shape and of same kernel solutions as the original graph. This interesting linear de- and recomposition may be used to implement on the basis of the median β-cut kernels, a fast algorithm for computing the corresponding $\mathcal{L}$-valued kernel solutions. Practical experiments have shown a very significant amelioration (1 to 50) in time for solving even small-sized examples as the well known car selection data of the Electre IS method [9].

References

[1] Berge, Cl., Théorie des graphes et ses applications, Dunod, Paris, 1958.

[2] Berge, Cl., Graphes and Hypergraphes, Dunod, Paris, 1970.

[3] Bisdorff, R. and Roubens, M., On defining fuzzy kernels from L-valued simple graphs, in proceedings of IPMU'96 (Information Processing and Management of Uncertainty in Knowledge-Based Systems), Granada, July, 1996a

[4] Bisdorff, R,and Roubens, M., On defining and computing fuzzy kernels from L-valued simple graphs, in proceedings of FLINS'96 workshop on Intelligent Systems and Soft Computing for Nuclear Science and Industry, Mol, Belgium, September 1996b

[5] Bolc, L and Borowik, P, Many-valued Logics: Theoretical foundations, Springer-Verlag, Berlin, 1992

[6] Fodor, J. and Roubens, M, Fuzzy preference modelling and multicriteria decision support, Kluwer Academic Publishers, 1994

[7] Kitaïnik, L, Fuzzy decision procedures with binary relations: towards a unified theory, Kluwer Academic Publ., Boston, 1993

[8] Roy, B., Algèbre moderne et théorie des graphes, Dunod, Paris, 1969

[9] Roy, B. and Bouyssou, D., Aide multicritère à la dècision: Mèthodes et cas, Economica, Paris, 1993, chap. 5

[10] Riguet, J., Relations binaires, fermetures, correspondances de Galois, Bull. Soc. Math. France, 76 (1948), pp. 114-155

[11] Schmidt, G. and Ströhlein, Th., Relationen und Graphen, Springer-Verlag, Berlin, 1989

[12] Schmidt, G. and Ströhlein, Th., On kernels of graphs and solutions of games: A synopsis based on relations and fixpoints, SIAM, J. Algebraic Discrete Methods, 6 (1985), pp. 54-65

[13] Von Neumann, J. and Morgenstern, O, Theory of games and economic behaviour, Princeton Univ. Press, Princeton, N.J., 1944