Characterizations of idempotent n-ary uninorms

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Abstract. In this paper we provide a characterization of the class of idempotent n-ary uninorms on a given chain. When the chain is finite, we also provide an axiomatic characterization of the latter class by means of four conditions only: associativity, quasitriviality, symmetry, and nondecreasing monotonicity. In particular, we show that associativity can be replaced with bisymmetry in this axiomatization.

1 Introduction

Let X be a nonempty set and let $n \geq 2$ be an integer. Binary aggregation functions have been extensively investigated since the last decades due to their usefulness in merging data (see, e.g. [5] and the references therein). Among these functions, uninorms play an important role in fuzzy logic. Meanwhile, the study of n-ary uninorms also raised some interest (see, e.g. [6]).

In this paper, which is a shorter version of [4], we investigate the class of idempotent n-ary uninorms $F\colon X^n \to X$ on a chain $(X, \leq)$ (Definition 3). We provide in Section 2 a characterization of these operations and show that they are nothing other than idempotent binary uninorms (Proposition 1). We also provide a description of these operations as well as an alternative axiomatization when the chain is finite (Theorem 1). In Section 3 we investigate some subclasses of bisymmetric n-ary operations and derive an equivalence involving associativity and bisymmetry. More precisely, we show that if an n-ary operation is quasitrivial and symmetric, then it is associative if and only if it is bisymmetric (Proposition 3). This observation enables us to replace associativity with bisymmetry in our axiomatization (Corollary 1).

We use the following notation throughout. A chain $(X, \leq)$ will simply be denoted by X if no confusion may arise. For any chain X and any $x, y \in X$ we use the symbols $x \wedge y$ and $x \vee y$ to represent $\min\{x, y\}$ and $\max\{x, y\}$, respectively. For any integer $k \geq 0$, we set $[k] = \{1, \ldots, k\}$. Finally, for any integer $k \geq 0$ and any $x \in X$, we set $k \cdot x = x, \ldots, x$ (k times). For instance, we have $F(3 \cdot x, 2 \cdot y, 0 \cdot z) = F(x, x, x, y, y)$.

Definition 1. *An operation $F\colon X^n \to X$ is said to be*

- idempotent *if $F(n \cdot x) = x$ for all $x \in X$;*

– quasitrivial *(or* conservative*) if* $F(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\}$ *for all* $x_1,\ldots,x_n \in X$*;*
– symmetric *if* $F(x_1,\ldots,x_n)$ *is invariant under any permutation of* $x_1,\ldots,x_n$*;*
– associative *if*

$$F(x_1,\ldots,x_{i-1},F(x_i,\ldots,x_{i+n-1}),x_{i+n},\ldots,x_{2n-1})$$
$$= F(x_1,\ldots,x_i,F(x_{i+1},\ldots,x_{i+n}),x_{i+n+1},\ldots,x_{2n-1})$$

for all $x_1,\ldots,x_{2n-1} \in X$ *and all* $i \in [n-1]$*;*
– bisymmetric *if*

$$F(F(\mathbf{r}_1),\ldots,F(\mathbf{r}_n)) \ = \ F(F(\mathbf{c}_1),\ldots,F(\mathbf{c}_n))$$

for all $n \times n$ *matrices* $[\mathbf{c}_1 \ \cdots \ \mathbf{c}_n] = [\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T \in X^{n\times n}$.

If $(X,\leq)$ *is a chain, then* $F\colon X^n \to X$ *is said to be*

– nondecreasing *(for* $\leq$*) if* $F(x_1,\ldots,x_n) \leq F(x'_1,\ldots,x'_n)$ *whenever* $x_i \leq x'_i$ *for all* $i \in [n]$.

Definition 2. *Let* $F\colon X^n \to X$ *be an operation. An element* $e \in X$ *is said to be a* neutral element *of* F *if*

$$F((i-1)\cdot e, x, (n-i)\cdot e) \ = \ x$$

for all $x \in X$ *and all* $i \in [n]$.

2 First characterization

In this section we provide a characterization of the n-ary operations on the chain X that are associative, quasitrivial, symmetric, and nondecreasing. We will also show that in the case where the chain is finite these operations are nothing other than n-ary idempotent uninorms.

Recall that a *uninorm* on a chain X is a binary operation $U\colon X^2 \to X$ that is associative, symmetric, nondecreasing, and has a neutral element (see [3, 7]). It is not difficult to see that any idempotent uninorm is quasitrivial.

The concept of uninorm can be easily extended to n-ary operations as follows.

Definition 3 (see [6]). *An* n*-ary* uninorm *is an operation* $F\colon X^n \to X$ *that is associative, symmetric, nondecreasing, and has a neutral element.*

The next proposition provides a characterization of idempotent n-ary uninorms.

Proposition 1. *Let* X *be a chain and let* $F\colon X^n \to X$ *be an operation. Then* F *is an idempotent* n*-ary uninorm if and only if there exists an idempotent uninorm* $U\colon X^2 \to X$ *such that*

$$F(x_1,\ldots,x_n) \ = \ U(\textstyle\bigwedge_{i=1}^n x_i \, , \bigvee_{i=1}^n x_i), \qquad x_1,\ldots,x_n \in X.$$

In this case, the uninorm U *is uniquely defined as* $U(x,y) = F((n-1)\cdot x, y)$.

We now introduce the concept of single-peaked linear ordering which first appeared for finite chains in social choice theory (see Black [1, 2]).

Definition 4. *Let $(X, \leq)$ and $(X, \preceq)$ be chains. We say that the linear ordering $\preceq$ is* single-peaked for $\leq$ *if for any $a, b, c \in X$ such that $a < b < c$ we have $b \prec a$ or $b \prec c$.*

The following theorem provides a characterization of the class of associative, quasitrivial, symmetric, and nondecreasing operations $F\colon X^n \to X$. We observe that it generalizes Proposition 1 since the latter class does not require the existence of a neutral element. In particular, it provides a new axiomatization as well as a description of idempotent n-ary uninorms when the chain X is finite.

Theorem 1. *Let $F\colon X^n \to X$ be an operation. The following assertions are equivalent.*

(i) *F is associative, quasitrivial, symmetric, and nondecreasing.*
(ii) *There exists a quasitrivial, symmetric, and nondecreasing operation $G\colon X^2 \to X$ such that*

$$F(x_1, \ldots, x_n) \ = \ G(\textstyle\bigwedge_{i=1}^n x_i \, , \bigvee_{i=1}^n x_i), \qquad x_1, \ldots, x_n \in X.$$

(iii) *There exists a linear ordering $\preceq$ on X that is single-peaked for $\leq$ and such that F is the maximum operation on $(X, \preceq)$, i.e.,*

$$F(x_1, \ldots, x_n) \ = \ x_1 \curlyvee \cdots \curlyvee x_n \, , \qquad x_1, \ldots, x_n \in X.$$

If $X = [k]$ for some integer $k \geq 1$, then any of the assertions (i)–(iii) above is equivalent to the following one.

(iv) *F is an idempotent n-ary uninorm.*

3 Second characterization

In this section we investigate bisymmetric n-ary operations and derive an equivalence involving associativity and bisymmetry. More precisely, if an n-ary operation is quasitrivial and symmetric, then it is associative if and only if it is bisymmetric. In particular this latter observation enables us to replace associativity with bisymmetry in Theorem 1.

Definition 5. *We say that a function $F\colon X^n \to X$ is* ultrabisymmetric *if*

$$F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_n)) \ = \ F(F(\mathbf{r}'_1), \ldots, F(\mathbf{r}'_n))$$

for all $n \times n$ matrices $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T, [\mathbf{r}'_1 \ \cdots \ \mathbf{r}'_n]^T \in X^{n \times n}$, where $[\mathbf{r}'_1 \ \cdots \ \mathbf{r}'_n]^T$ is obtained from $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T$ by exchanging two entries.

Proposition 2. *Let $F\colon X^n \to X$ be an operation. If F is ultrabisymmetric, then it is bisymmetric. The converse holds whenever F is symmetric.*

Proposition 3. *Let $F\colon X^n \to X$ be an operation. Then the following assertions hold.*

(a) If F is quasitrivial and ultrabisymmetric, then it is associative and symmetric.
(b) If F is associative and symmetric, then it is ultrabisymmetric.
(c) If F is quasitrivial and symmetric, then it is associative if and only if it is bisymmetric.

From Proposition 3(c) we immediately derive the following corollary, which is an important but surprising result.

Corollary 1. *In Theorem 1 we can replace associativity with bisymmetry.*

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