PROP OF RIBBON HYPERGRAPHS AND STRONGLY HOMOTOPY INVOLUTIVE LIE BIALGEBRAS

SERGEI MERKULOV

Abstract. For any integer d we introduce a prop $\mathcal{RH}ra_d$ of d-oriented ribbon hypergraphs (in which "edges" can connect more than two vertices) and prove that there exists a canonical morphism $\mathcal{H}olieb_d^\diamond \longrightarrow \mathcal{RH}ra_d$ from the minimal resolution $\mathcal{H}olieb_d^\diamond$ of the (degree shifted) prop of involutive Lie bialgebras into the prop of ribbon hypergraphs which is non-trivial on each generator of $\mathcal{H}olieb_d^\diamond$. As an application we show that for any graded vector space W equipped with a family of cyclically (skew)symmetric higher products,

$$\Theta_n : (\otimes^n W[d])_{\mathbb{Z}_n} \longrightarrow \mathbb{K}[1+d], \qquad n \geq 1,$$

the associated vector space of cyclic words $Cyc(W) = \oplus_{n\geq 0}(\otimes^n W)_{\mathbb{Z}_n}$ has a combinatorial $\mathcal{H}olieb_d^\diamond$-structure. As an illustration we construct for each natural number $N \geq 1$ an explicit combinatorial strongly homotopy involutive Lie bialgebra structure on the vector space of cyclic words in N graded letters which extends the well-known Schedler's necklace Lie bialgebra structure from the formality theory of the Goldman-Turaev Lie bialgebra in genus zero.

1. Introduction

1.1. Involutive Lie bialgebras. Lie bialgebras were introduced by Drinfeld in [D1] in his studies of Yang-Baxter equations and the deformation theory of universal enveloping algebras. Nowadays (involutive) Lie bialgebras are used in many different areas of mathematics — in algebra, geometry, string topology, contact topology, theory of moduli spaces of algebraic curves, etc. (see, e.g., articles and books [AKKN1, AKKN2, Ba1, CFL, Ch, CS, D1, D2, DCTT, ES, Ma, MW1, Tu, S] as well as references cited there). The construction of a minimal resolution of the prop of involutive Lie bialgebras turned out to be a rather non-trivial problem which was solved rather recently in [CMW]. It is a remarkable fact that the deformation theory of that prop is controlled by the mysterious Grothendieck-Teichmüller group [MW2] which appears in many different areas of mathematics and explains, perhaps, the richness of important mathematical problems which involve involutive Lie bialgebras.

The main result of this paper is an explicit construction of a large family of highly non-trivial *strongly homotopy* involutive Lie bialgebras using a new prop of ribbon hypergraphs $\mathcal{HG}ra_d$, $\forall d \in \mathbb{Z}$. The structure of the paper is as follows: in §2 we remind basic facts about the prop of (degree $1-d$ shifted) involutive Lie bialgebras $\mathcal{L}ieb_d^\diamond$ and its minimal resolution $\mathcal{H}olieb_d^\diamond$, and their interrelations with $\mathcal{H}o\mathcal{BV}_d^{com}$-algebras (see [CMW] for more details and proofs). In §3 we describe in detail the prop of ribbon hypergraphs which is a rather obvious extension of the prop of ribbon graphs introduced in [MW1], and then construct its canonical representation in the space of cyclic words $Cyc(W)$ associated with any graded vector space W equipped with a family of cyclically (skew)symmetric higher products (which again is a rather straightforward generalization of a similar construction in [MW1]). In §4 we explain one of the main (and not that straightforward) results of this paper — a construction of an explicit morphism of props $\mathcal{H}olieb_d^\diamond \to \mathcal{RH}ra_d$ which is *non-trivial on every generator of* $\mathcal{H}olieb_d^\diamond$. This result gives us a large family of explicit strongly homotopy involutive Lie bialgebras; in particular, for any natural number $N \in \mathbb{N}_{\geq 1}$ we show in §5 an explicit strongly homotopy involutive Lie bialgebra structure on the vector space of cyclic words in $\mathbb{Z}$-graded formal letters which extends the well-known Schedler's necklace Lie bialgebra structure [S] from the formality theory of the Goldman-Turaev Lie bialgebra in genus zero.

1.2. More details on main results and motivation. Let us discuss in more detail one particular geometric example of an involutive Lie bialgebra which motivated much the present work. Let $\widehat{\mathbb{K}}\langle\pi_1(\Sigma_{0,N+1})\rangle$ stand for the completed group algebra of the fundamental group $\pi_1(\Sigma_{0,N+1}, \mathbb{K})$ of the genus zero Riemann surface $\Sigma_{0,N+1}$ with $N+1$ boundary components, $N\geq 2$, and $H_1(\Sigma_{0,N+1})$ for its first homology group over $\mathbb{K}$. Let

$$\widehat{\mathfrak{g}}[\Sigma_{0,N+1}] := \frac{\widehat{\mathbb{K}}\langle\pi_1(\Sigma_{0,N+1})\rangle}{[\widehat{\mathbb{K}}\langle\pi_1(\Sigma_{0,N+1}), \widehat{\mathbb{K}}\langle\pi_1(\Sigma_{0,N+1})\rangle]}$$

be the (completed) vector space spanned over a field $\mathbb{K}$ of characteristic zero by free homotopy classes of loops in $\Sigma_{0,N+1}$. Using intersections and self-intersection of loops Goldman and Turaev [G, Tu] made this vector space into a filtered involutive Lie bialgebra[1]. Let

$$\mathrm{gr}\widehat{\mathfrak{g}}[\Sigma_{0,N+1}] := \frac{\widehat{\otimes^\bullet} H_1(\Sigma_{0,N+1},\mathbb{K})}{[\widehat{\otimes^\bullet} H_1(\Sigma_{0,N+1},\mathbb{K}), \widehat{\otimes^\bullet} H_1(\Sigma_{0,N+1},\mathbb{K})]} \simeq Cyc(W_N) := \prod_{n\geq 0} (\otimes^n W_N)_{\mathbb{Z}_n}$$

be the *associated graded* involutive Lie bialgebra where

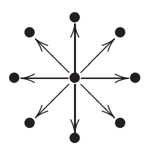
$$W_N = \mathrm{span}_{\mathbb{K}}\,\langle x_1, \dots x_N\rangle$$

stands for the vector spaces generated by N formal letters $x_1,\dots,x_N$ (corresponding to the standard generators of $H_1(\Sigma_{0,N+1},\mathbb{K})$). The *formality theorem* [AKKN1, AN, Ma] establishes a highly non-trivial isomorphism of Lie bialgebras

$$\widehat{\mathfrak{g}}[\Sigma_{0,N+1}] \longrightarrow \mathrm{gr}\widehat{\mathfrak{g}}[\Sigma_{0,N+1}]$$

which depends on the choice of a Drinfeld associator. Thus the Goldman-Turaev Lie bialgebra structure can be understood in terms of its much simpler graded associated version which admits a purely combinatorial description. In fact, it admits two purely combinatorial descriptions. The first one is due to the general construction by Schedler [S] which associates to any quiver a so called necklace Lie bialgebra; the particular involutive Lie bialgebra structure on $\mathrm{gr}\widehat{\mathfrak{g}}[\Sigma_{0,N+1}] \simeq Cyc(W_N)$ is the necklace one corresponding to the following quiver

(1)



with N legs. Put another way, for any natural number $N\geq 2$ Schedler's construction[2] gives us an involutive Lie bialgebra structure on $Cyc(W_N)$ which admits a nice geometric interpretation.

The second combinatorial description of the necklace Lie bialgebra structure on $Cyc(W_N)$ involves the $d=1$ case of a family of props of ribbon graphs $\mathcal{RG}ra_d$, $d\in\mathbb{Z}$, which come equipped with canonical morphisms [MW1]

$$\rho: \mathcal{L}ieb_d^\diamond \longrightarrow \mathcal{RG}ra_d$$

from the prop of involutive Lie bialgebras; the map ρ is non-trivial on both Lie and coLie generators of $\mathcal{L}ieb_d^\diamond$ (which are assigned homological degree $1-d$ so that the case $d=1$ corresponds to the ordinary Lie bialgebras). It was shown in [MW1] that for any $\mathbb{Z}$-graded vector space V equipped with a pairing (which for $d=1$ is nothing but a skew-symmetric scalar product in W)

$$\Theta: \quad \odot^2(W[d]) \quad\longrightarrow\quad \mathbb{K}[1+d]$$

there is an associated representation

$$\rho_\Theta : \mathcal{RG}ra_d \longrightarrow \mathcal{E}nd_{Cyc(W)}$$

of the prop of ribbon graphs in the vector space

$$Cyc(W) := \prod_{n\geq 0}(\otimes^n W)_{\mathbb{Z}_n}$$

[1]Strictly speaking this structure was originally defined modulo constant loops, i.e. on the quotient space $\widehat{\mathfrak{g}}[\Sigma_{0,N+1}]/\mathbb{K}$, but a choice of framing on $\Sigma_{0,N+1}$ permits us to extend that structure to the whole space. Put another way, the induced Lie bialgebra structure on $\widehat{\mathfrak{g}}[\Sigma_{0,N+1}]$ is canonical only on the quotient space $Cyc^\bullet(W_N)/\mathbb{K}1$, 1 being the empty cyclic word, and its extension to the whole space depends on some additional choices.

[2]It is worth emphasizing that Schedler's construction depends on the choice of a basis $(x_1,\dots,x_N)$ in W_N, i.e. it depends essentially only on the natural number N.

spanned by cyclic words in elements of W, and hence there is an induced via the composition $\rho_\Theta \circ \rho$ an involutive Lie bialgebra structure in $Cyc(W)$. The vector space W_N has no natural pairings so we can not apply this construction immediately to get the necklace Lie bialgebra structure on W_N. However a certain "doubling" trick explained in §**2.5.1** does the job and gives us a canonical representation

$$\rho_N : \mathcal{RG}ra_1 \longrightarrow \mathcal{E}nd_{Cyc(W_N)} \tag{2}$$

which induces via the composition $\rho_N \circ \rho$ the required necklace Lie bialgebra structure on $Cyc(W_N)$ (modulo terms depending on a particular choice of framing on $\Sigma_{0,N+1}$, i.e. both structures fully agree on the quotient space $Cyc(W_N)/\mathbb{K}\mathbb{1}$, where $\mathbb{1}$ stands for the empty cyclic word; in fact, the particular map ρ_N we construct in §**2.5.1** corresponds to the blackboard framing as was explained to the author by Yusuke Kuno [Ku].).

In this paper we introduce, for each integer $d \in \mathbb{Z}$, a prop of ribbon hypergraphs $\mathcal{HG}ra_d$ and show that there is a morphism of dg props

$$\rho^\diamond : \mathcal{H}olieb_d^\diamond \longrightarrow \mathcal{HG}ra_d$$

which is *non-trivial on every generator of* $\mathcal{H}olieb_d^\diamond$, the minimal resolution of the prop $\mathcal{L}ieb_d^\diamond$. There is a natural commutative diagram

$$\begin{tikzcd} \mathcal{H}olieb_d^\diamond \arrow[r, "\rho^\diamond"] \arrow[d, "p"'] & \mathcal{HG}ra_d \arrow[d, "q"] \\ \mathcal{L}ieb_d^\diamond \arrow[r, "\rho"] & \mathcal{RG}ra_d \end{tikzcd}$$

where the left vertical arrow p is a natural quasi-isomorphism and the right arrow q is a "forgetful" map sending to zero all ribbon hypergraphs with at least one non-bivalent hyperedge.

Given any graded vector space W equipped with cyclically (skew)invariant maps

$$\Theta_n : \quad \odot^n(W[d])_{\mathbb{Z}_n} \quad \longrightarrow \quad \mathbb{K}[1+d], \qquad n \in \mathbb{N}_{\geq 1}, \tag{3}$$

there is a canonical representation

$$\rho_{\Theta_\bullet} : \mathcal{HG}ra_d \longrightarrow \mathcal{E}nd_{Cyc(W)}$$

of the prop of ribbon hypergraphs, and hence a canonical strongly homotopy involutive Lie bialgebra structure

$$\rho_{\Theta_\bullet} \circ \rho^\diamond : \mathcal{H}olieb_d^\diamond \longrightarrow \mathcal{E}nd_{Cyc(W)}$$

on $Cyc(W)$. If all Θ_n vanish except for $n = 2$ we recover the previous result from [MW1]. Using multi-tuple generalization of the "doubling" trick used in the construction of representation (2) we construct an explicit highly non-trivial strongly homotopy involutive Lie bialgebra structure,

$$\hat{\rho}_N : \mathcal{H}olieb_1^\diamond \longrightarrow \mathcal{E}nd_{Cyc(\widehat{W}_N)}$$

in the vector space generated by cyclic words in $\mathbb{Z}$-graded letters $\{x_1[-p], \ldots, x_N[-p]\}_{p \in \mathbb{N}}$, where $x_i[-p]$ stands for the formal letter x_i to which we assigned homological degree p. When all letters are concentrated in degree zero, one recovers Schedler's necklace Lie bialgebra associated to the quiver (1). We conjecture that this family of $\mathcal{H}olieb_1^\diamond$-algebras is related to the equivariant string topology in manifolds of dimensions $D \geq 3$.

Some notation. The set $\{1, 2, \ldots, n\}$ is abbreviated to $[n]$; its group of automorphisms is denoted by $\mathbb{S}_n$. The trivial (resp., sign) one-dimensional representation of $\mathbb{S}_n$ is denoted by $\mathbb{1}_n$ (resp., sgn_n). The cardinality of a finite set A is denoted by $\#A$. If $V = \oplus_{i \in \mathbb{Z}} V^i$ is a graded vector space, then $V[n]$ stands for the graded vector space with $V[n]^i := V^{i+n}$; for $v \in V^i$ we set $|v| := i$.

For a prop(erad) $\mathcal{P}$ we denote by $\mathcal{P}\{n\}$ a prop(erad) which is uniquely defined by the following property: for any graded vector space W a representation of $\mathcal{P}\{n\}$ in W is identical to a representation of $\mathcal{P}$ in $W[n]$.

For a module V over a group G we denote by V_G the vector space of coinvariants: $V/\{g(v) - v \mid v \in V, g \in G\}$ and by V^G the subspace of invariants: $\{\forall g \in G \ : \ g(v) = v, \ v \in V\}$. We always work over a field $\mathbb{K}$ of characteristic zero so that if G is finite, then these spaces are canonically isomorphic, $V_G \cong V^G$.

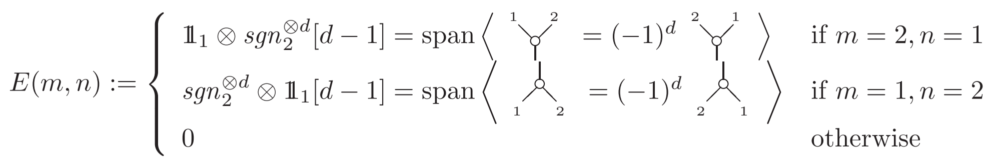
Acknowledgement. It is a pleasure to thank Anton Alekseev, Martin Kassabov, Nariya Kawazumi, Anton Khoroshkin, Yusuke Kuno, Florian Naef and Thomas Willwacher for valuable discussions. This text is based on the talk given by the author at the workshop "Poisson geometry of moduli spaces, associators and quantum field theory" organized at the Simons Center of Geometry and Physics in June 2018. Most of this text was also written there, and the author acknowledges with thanks excellent working conditions at the SCGP.

2. Strongly homotopy involutive Lie bialgebras as commutative BV algebras and vice versa

2.1. Reminder on the prop of strongly homotopy (involutive) Lie bialgebras [CMW]**.** Let

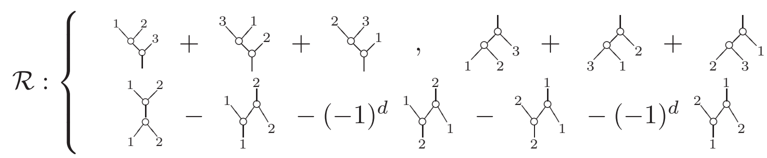
$$\mathcal{L}ieb_d := \mathcal{F}ree\langle E\rangle/\langle\mathcal{R}\rangle,$$

be the quotient of the free prop(erad) generated by an $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geq 1}$ with



by the ideal $\langle\mathcal{R}\rangle$ generated by the following relations

(4)

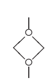


It is called the prop of (degree shifted) *Lie bialgebras*.

The prop of *involutive* Lie d-bialgebras is defined similarly,

$$\mathcal{L}ieb_d^\diamond := \mathcal{F}ree\langle E\rangle/\langle\mathcal{R}_\diamond\rangle$$

but with a larger set of relations,

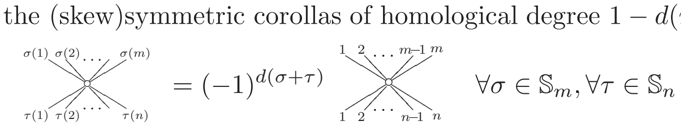
$\mathcal{R}_\diamond := \mathcal{R} \bigsqcup$ 

A representation $\rho : \mathcal{L}ieb_d \to \mathcal{E}nd_V$ (resp., $\rho : \mathcal{L}ieb_d^\diamond \to \mathcal{E}nd_V$) in a graded vector space V provides the latter with two operations

$[\ ,\ ] := \rho\Big($  $\Big) : \odot^2(V[d]) \to V[1+d], \qquad \triangle := \Big($  $\Big) : V[d] \longrightarrow \odot^2(V[d])[1-2d]$

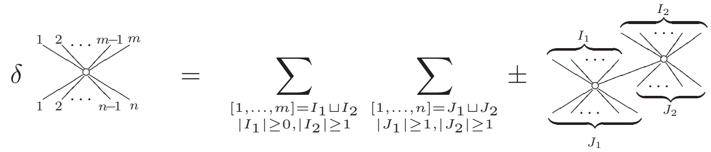
which satisfy the compatibility conditions controlled by the relations $\mathcal{R}$ (resp. $\mathcal{R}^\diamond$). If $d = 1$, it is precisely the prop of ordinary *involutive Lie bialgebras* and is often denoted by $\mathcal{L}ieb^\diamond$.

The properads behind the props $\mathcal{L}ieb_d$ and $\mathcal{L}ieb_d^\diamond$ are Koszul so that their minimal resolutions, $\mathcal{H}olieb_d$ and respectively $\mathcal{H}olieb_d^\diamond$, are relatively "small" (see [CMW, MaVo, V] and references cited there). The dg prop $\mathcal{H}olieb_d$ is generated by the (skew)symmetric corollas of homological degree $1 - d(m+n-2)$

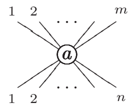
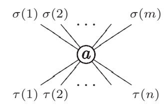


The differential is given on the generators by

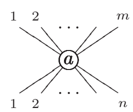
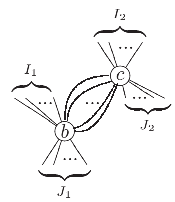
(5)



where the signs on the r.h.s are uniquely fixed by the fact that they all equal to $+1$ for d odd. On the other hand, the dg prop $\mathcal{H}olieb_d^\diamond$ is generated by the (skew)symmetric corollas of degree $1-d(m+n+2a-2)$,


$$= (-1)^{(d+1)(\sigma+\tau)}$$

$$\forall \sigma\in\mathbb{S}_m, \forall\tau\in\mathbb{S}_n,$$

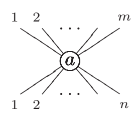
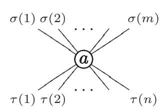
where $m+n+a\geq 3$, $m\geq 1$, $n\geq 1$, $a\geq 0$; the differential is given by

(6) δ 
$$=\sum_{l\geq 1}\ \sum_{a=b+c+l-1}\ \sum_{\substack{[m]=I_1\sqcup I_2\\ [n]=J_1\sqcup J_2}} \pm$$


where the summation parameter l counts the number of internal edges connecting the two vertices on the r.h.s., and the signs are fixed by the fact that they all equal to $+1$ for d even. If $d=1$, it is called the prop of *strongly homotopy involutive Lie bialgebras* and is denoted by $\mathcal{H}olieb^\diamond$.

There is a canonical injection of props $\mathcal{H}olieb_d\to\mathcal{H}olieb_d^\diamond$ sending a generator $\mathcal{H}olieb_d$ into the corresponding generator of $\mathcal{H}olieb_d^\diamond$ with $a=0$.

Sometimes it is more suitable to work with the degree shifted version $\mathcal{H}olieb_d^\diamond\{d\}$ of the prop $\mathcal{H}olieb_d^\diamond$ which is defined uniquely by the following property: a representation of $\mathcal{H}olieb_d^\diamond\{d\}$ in a vector space V is identical to the representation of $\mathcal{H}olieb_d^\diamond$ in $V[d]$. The prop $\mathcal{H}olieb_d^\diamond\{d\}$ is generated by the symmetric corollas of degree $1-2d(n+a-1)$,


$$=$$

$$\forall \sigma\in\mathbb{S}_m, \forall\tau\in\mathbb{S}_n,$$

The differential is given by the above formula with the slightly ambiguous symbol $\pm$ replaced by $+1$ (i.e. omitted); hence the sign rules become especially simple in this case.

2.2. $\mathcal{H}olieb_d^\diamond$-algebras as Maurer-Cartan elements [CMW]**.** According to the general theory [MV], the set of $\mathcal{H}olieb_d^\diamond\{d\}$-algebra structures in a dg vector space (V,δ) can be identified with the set of Maurer-Cartan elements of a graded Lie algebra,

$$\mathfrak{g}_V^\diamond := \mathrm{Hom}(V,V)[1]\ \oplus\ \mathsf{Def}\left(\mathcal{H}olieb_d^\diamond\{d\}\stackrel{0}{\to}\mathcal{E}nd_V\right), \tag{7}$$

controlling deformations of the trivial morphism which sends all the generators of $\mathcal{H}olieb_d^\diamond\{d\}$ to zero in $\mathcal{E}nd_V$. The summand $\mathrm{Hom}(V,V)[1]$ takes care about deformations of the given differential δ in V. Using the explicit description of the dg prop $\mathcal{H}olieb_d^\diamond\{d\}$ given at the end of the previous subsection, one can identify $\mathfrak{g}_V^\diamond$ as a $\mathbb{Z}$-graded vector space with

$$\begin{aligned}
\mathfrak{g}_V^\diamond &= \prod_{a\geq 0,m,n\geq 1}\mathrm{Hom}_{\mathbb{S}_m\times\mathbb{S}_n}\left(\mathrm{Id}_m\otimes\mathrm{Id}_n[2d(n+a-1)-1],\mathrm{Hom}\left(V^{\otimes n},V^{\otimes m}\right)\right)[-1]\\
&= \prod_{a\geq 0,m,n\geq 1}\mathrm{Hom}\left((V[2d])^{\odot n},V^{\odot m}\right)[-2da+2d]
\end{aligned}$$

Assume V has a countable basis $(x_1,x_2,\ldots)$, and let $(p^1,p^2,...)$ stand for the associated set of dual generators of $\mathrm{Hom}(V[2d],\mathbb{K})$ (with $|p^i|+|x_i|=2d$), then the degree shifted vector space $\mathfrak{g}_V^\diamond[-2d]$ can be identified with the subspace of a graded commutative ring

$$\mathfrak{g}_V^\diamond[-2d]\subset\mathbb{K}[[x^i,p_i,\hbar]]$$

spanned by those formal power series $f(x,p,\hbar)$ which satisfy the conditions

$$f(x,p,\hbar)|_{x_i=0}=0,\qquad f(x,p,\hbar)|_{p^i=0}=0$$

i.e. which belong to the maximal ideal generated by the products $x_i p^j$. Here $\hbar$ is a formal parameter[3] of degree $2d$. The algebra $\mathbb{K}[[x^i, p_i, \hbar]]$ has a classical associative star product given explicitly (up to standard Koszul signs) as follows

$$f *_\hbar g := \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \sum_{i_1,\dots,i_k} \pm \frac{\partial^k f}{\partial p^{i_1} \cdots \partial p^{i_k}} \frac{\partial^k g}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

The Lie brackets in the degree shifted deformation complex $\mathfrak{g}_V^\diamond[-2d]$ are then given by [DCTT]

$$[f, g] = \frac{f *_\hbar g - (-1)^{|f||g|} g *_\hbar f}{\hbar}$$

Hence $\mathcal{H}olieb_d^\diamond\{d\}$-algebra structures in a graded vector space V (with a countable basis) are in 1-1 correspondence with homogeneous formal power series $\Gamma \in \mathfrak{g}_V^\diamond[-2d]$ of degree $1 + 2d$ satisfying the equation

$$\Gamma *_\hbar \Gamma = \sum_{k=1}^{\infty} \frac{\hbar^{k-1}}{k!} \sum_{i_1,\dots,i_k} \pm \frac{\partial^k \Gamma}{\partial p^{i_1} \cdots p^{i_k}} \frac{\partial^k \Gamma}{\partial x_{i_1} \cdots \partial x_{i_k}} = 0, \tag{8}$$

This compact description of all higher homotopy involutive Lie bialgebra operations in V is quite useful in making an explicit link between $\mathcal{H}olieb_d^\diamond$ algebras and *commutative BV-algebras* which is outlined next.

2.3. Commutative Batalin-Vilkovisky d-algebras. A *commutative Batalin-Vilkovisky d-algebra* or, shortly, a $\mathcal{H}o\mathcal{BV}_d^{com}$*-algebra* is, by definition [Kr], a differential graded commutative algebra (V, δ) equipped with a countable collections of homogeneous linear maps, $\{\Delta_a : V \to V,\ |\Delta_a| = 1 - 2da\}_{a\geq 1}$, such that each operator Δ_a is of order $\leq a + 1$ (with respect to the given multiplication) and the equations,

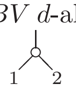
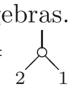
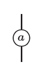
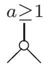
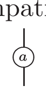
$$\sum_{a=0}^{n} \Delta_a \circ \Delta_{n-a} = 0, \quad \text{with } \Delta_0 := -\delta \tag{9}$$

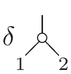
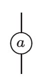
hold for any $n \in \mathbb{N}$. These equations are equivalent to one equation,

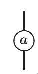
$$\Delta_\hbar^2 = 0$$

for the formal power series of operators

$$\Delta_\hbar := \sum_{a=0}^{\infty} \hbar^a \Delta_a,$$

where the formal power variable $\hbar$ is assigned degree $2d$. Let us denote by $\mathcal{H}o\mathcal{BV}_d^{com}$ the dg operad governing commutative BV d-algebras. This operad is the quotient of the free operad generated by one binary operation of degree zero  =  (standing for graded commutative multiplication) and a countable family of unary operations $\Big\{$  $\Big\}_{a\geq 1}$ of homological degree $1 - 2da$ modulo the ideal I generated by the standard associativity relation for  and the compatibility relations involving the latter and the unary operations which assure that each unary operation  is of order $\leq a + 1$ with respect to the multiplication. The differential δ in $\mathcal{BV}_\infty^{com}$ is given by

(10) δ  $= 0, \quad \delta$  $:= \sum_{\substack{a=b+c \\ b,c\geq 1}}$ 

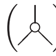
Let J be the differential closure of an ideal in $\mathcal{H}o\mathcal{BV}_d^{com}$ generated by operations  with $a \geq 2$. The quotient $\mathcal{H}o\mathcal{BV}_d^{com}/J$ is precisely of the operad of (degree shifted) Batalin-Vilkovisky algebras $\mathcal{BV}_d$. It was proven in [CMW] that the canonical projection $\mathcal{H}o\mathcal{BV}_d^{com} \longrightarrow \mathcal{BV}_d$ is quasi-isomorphism of operads.

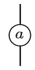
[3]For a vector space W we denote by $W[[\hbar]]$ the vector space of formal power series in $\hbar$ with coefficients in W. For later use we denote by $\hbar^m W[[\hbar]]$ the subspace of $W[[\hbar]]$ spanned by series of the form $\hbar^m f$ for some $f \in W[[\hbar]]$.

2.4. From $\mathcal{H}olieb_d^{\diamond}$-algebras to $\mathcal{H}o\mathcal{BV}_d^{com}$-algebras and back. Recall that a $\mathcal{H}olieb_d^{\diamond}\{d\}$-algebra structure in a graded vector space V (i.e. a $\mathcal{H}olieb_d^{\diamond}$ structure in $V[d]$) can be identified with a degree $1+2d$ element Γ in $\mathfrak{g}_V^{\diamond}[-2d] \subset \mathbb{K}[[p^i, x_i, \hbar]]$ satisfying equation (8). Out of this datum one creates a $\mathcal{H}o\mathcal{BV}_d^{com}$-algebra structure on $\widehat{\odot^{\bullet}} V$ (the completed symmetric tensor algebra on V), i.e. a representation

$$\rho : \mathcal{H}o\mathcal{BV}_d^{com} \longrightarrow \mathcal{E}nd_{\widehat{\odot^{\bullet}}(V)},$$

which is given explicitly as follows [CMW],

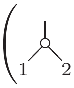
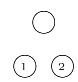
$\rho\Big($  $\Big)$:= the standard multiplication in $\widehat{\odot^{\bullet}}(V)$

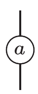
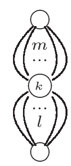
$\Delta_a := \rho\Big($  $\Big) := \displaystyle\sum_{a+1=k+l} \frac{1}{k!l!} \sum_{i_1,\dots,i_l} \frac{\partial^{a+1}\Gamma}{\partial^k \hbar \partial p^{i_1}\cdots\partial p^{i_l}}\Big|_{\hbar=p^i=0} \frac{\partial^l}{\partial x_{i_1}\cdots\partial x_{i_l}}$

This explicit correspondence can be equivalently understood as a morphism of dg operads

$$F : \mathcal{H}o\mathcal{BV}_d^{com} \longrightarrow \mathcal{O}(\mathcal{H}olieb_d^{\diamond}\{d\})$$

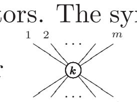
given explicitly on generators as follows,

$F\Big($  $\Big) :=$ 

(11) $F\Big($  $\Big) = \displaystyle\sum_{m\geq 1} \sum_{a+1=p+q}$ 

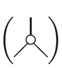
where

$$\begin{array}{rccc} \mathcal{O}: & \text{Category of props} & \longrightarrow & \text{Category of operads} \\ & \mathcal{P} & \longrightarrow & \mathcal{O}(\mathcal{P}) \end{array}$$

is the *polydifferential functor* introduced in [MW1] (we refer to §5.1 of [MW1] or §2.2 of [MW2] for full details explaining, in particular, the symbols on the r.h.s. of the above formula; these sections can be read independently of the rest of both papers). Its main defining property is that, given any representation $\rho : \mathcal{P} \to \mathcal{E}nd_V$ of a prop $\mathcal{P}$ in a graded vector space X, there is an associated representation $\mathcal{O}(\rho) : \mathcal{O}(\mathcal{P}) \to \mathcal{E}nd_{\widehat{\odot^{\bullet}} X}$ of the operad $\mathcal{O}(\mathcal{P})$ in the completed free graded commutative algebra $\widehat{\odot^{\bullet}} X$ such that elements of $\mathcal{P}$ acts on $\widehat{\odot^{\bullet}}(X)$ as polydifferential operators. The symbol on the r.h.s. of (11) is precisely the polydifferential operator corresponding to the generator  of $\mathcal{H}olieb_d^{\diamond}\{d\}$. Reversely, given a representation

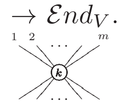
$$\rho : \mathcal{H}o\mathcal{BV}_d^{com} \longrightarrow \mathcal{E}nd_{\widehat{\odot^{\bullet}}(V)},$$

such that

$\rho\Big($  $\Big)$:= the standard multiplication in $\widehat{\odot^{\bullet}}(V)$

it was proven in [DCTT] that it factors through the composition

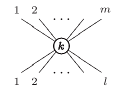
$$\rho : \mathcal{H}o\mathcal{BV}_d^{com} \xrightarrow{F} \mathcal{O}(\mathcal{H}olieb_d^{\diamond}\{d\}) \xrightarrow{\mathcal{O}(\rho')} \mathcal{O}(\mathcal{E}nd_V) = \mathcal{E}nd_{\widehat{\odot^{\bullet}} V}$$

for some representation $\rho' : \mathcal{H}olieb^{\diamond}\{d\} \to \mathcal{E}nd_V$. Put another way, one can read all the higher homotopy involutive Lie bialgebra operations $\rho'\Big($  $\Big)$ in V from the explicit representation of

$\rho\Big($  $\Big) = \displaystyle\sum_{m\geq 1} \frac{1}{m!} \sum_{a+1=k+l} \sum_{i_\bullet, j_\bullet} C^{(k)}{}_{i_1\dots i_l}^{j_1\dots j_m} x_{j_1}\dots x_{j_m} \frac{\partial^l}{\partial x_{i_1}\cdots\partial x_{i_l}}$

of the generator  as a differential operator of order $\leq a+1$ on $\widehat{\odot^\bullet V} \simeq \mathbb{K}[[x_i]]$: the linear map

$$\mu^{(k)m}_{\ \ l} := \rho' \left(\right) : \otimes^m V \to \otimes^n V$$



is given in the basis $\{x_i\}$ by (modulo the standard Koszul signs) by

$$\mu^{(k)m}_{\ \ l}(x_{i_1} \otimes \ldots \otimes x_{i_l}) = \sum_{j_\bullet} C^{(k)}{}^{j_1 \ldots j_m}_{i_1 \ldots i_l} x_{j_1} \otimes \ldots \otimes x_{j_m}$$

We shall use this one-to-one correspondence heavily in the proof of the Main Theorem **4.3** below.

2.5. A basic example of an involutive Lie bialgebra. Let W be a graded vector space equipped with a linear map $\Theta : \odot^2(W[d]) \to \mathbb{K}[1+d]$ for some $d \in \mathbb{Z}$. This map is the same as a degree $1-d$ pairing $\Theta : W \otimes W \to \mathbb{K}[1-d]$ satisfying the (skew) symmetry condition,

$$\Theta(w_1, w_2) = (-1)^{d+|w_1||w_2|}\Theta(w_2, w_1), \quad \forall\ w_1, w_2 \in W.$$

A symplectic structure on W corresponds to the case $d = 1$ and Θ non-degenerate.

The associated vector space of "cyclic words in W",

$$Cyc^\bullet(W) := \sum_{n\geq 0} (W^{\otimes n})^{\mathbb{Z}_n},$$

admits a $\mathcal{L}ieb^\diamond_d$-structure given by the following well-known formulae for the Lie bracket and cobracket,

$$[(w_1 \otimes ... \otimes w_n)_{\mathbb{Z}_n}, (w'_1 \otimes ... \otimes w'_m)^{\mathbb{Z}_n}] :=$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \pm\Theta(w_i, w'_j)(w_1 \otimes ... \otimes w_{i-1} \otimes w'_{j+1} \otimes ... \otimes w'_m \otimes w'_1 \otimes ... \otimes w'_{j-1} \otimes w_{i+1} \otimes \ldots \otimes w_n)^{\mathbb{Z}_{n+m-2}}$$

$$\vartriangle (w_1 \otimes \ldots \otimes w_n)_{\mathbb{Z}_n} := \sum_{i\neq j} \pm\Theta(w_i, w_j)(w_{i+1} \otimes ... \otimes w_{j-1})^{\mathbb{Z}_{j-i-1}} \bigotimes (w_{j+1} \otimes ... \otimes w_{i-1})^{\mathbb{Z}_{n-j+i-1}}$$

where $\pm$ stands for the standard Koszul sign. A very short (and pictorial) proof of this claim can be found in [MW1]. Note that the vector space $Cyc^\bullet(W)$ is naturally *weight*-graded

$$Cyc^\bullet(W) = \bigoplus_{n\geq 0} Cyc^n(W), \qquad Cyc^n(W) := (\otimes^n W)^{\mathbb{Z}_n}$$

by the length of cyclic words, and both operations Δ and $[\ ,\ ]$ have weight-degree -2 with respect to this weight-grading (which should not be confused with the *homological* grading).

2.5.1. A special case: Schedler's necklace Lie bialgebra. A special case of the above construction for $d = 1$ gives us Schedler's necklace Lie bialgebra structure [S] associated with the quiver (1). Consider a set of N formal letters

$$\{x_1, \ldots, x_n\}$$

and denote by W_N their linear span over a field $\mathbb{K}$. We shall make $Cyc(W)$ into a weight-degree -1 (not -2 as in the above example!) involutive Lie bialgebra using the following "doubling" trick.

Consider two copies $W^{(1)}_N, W^{(2)}_N$ of W_N and equip their direct sum

$$\hat{W}_N := W^{(1)}_N \oplus W^{(2)}_N$$

with the unique symplectic structure $\theta : \wedge^2\hat{W} \to \mathbb{K}$ making the basis $\{x^{(1)}_\alpha, x^{(2)}_\beta\}_{1\leq\alpha,\beta\leq N}$ a Darboux one,

$$\Theta(x^{(1)}_\alpha, x^{(2)}_\beta) = -\Theta(x^{(2)}_\beta, x^{(1)}_\alpha) = \delta^\alpha_\beta, \quad \Theta(x^{(1)}_\alpha, x^{(1)}_\beta) = 0, \qquad \Theta(x^{(2)}_\alpha, x^{(2)}_\beta) = 0.$$

Then the above formulae for $[\ ,\ ]$ and Δ make the space $Cyc(\hat{W}_N)$ into an involutive Lie bialgebra. It is easy to see that the subspace

$$Cyc(\hat{W}_N)^{(12)} \subset Cyc(\hat{W}_N)$$

spanned by cyclic words of the form

$$\left(x^{(1)}_{\alpha_1} \otimes x^{(2)}_{\alpha_1} \otimes x^{(1)}_{\alpha_2} \otimes x^{(2)}_{\alpha_2} \otimes \ldots \otimes x^{(1)}_{\alpha_n} \otimes x^{(2)}_{\alpha_n}\right)^{\mathbb{Z}_{2n}}$$

is closed with respect to the above Lie bracket and co-bracket. The canonical isomorphism

$$\begin{array}{ccc} Cyc(W_N) & \longrightarrow & Cyc(\hat{W}_N)^{(12)} \\ (x_{\alpha_1} \otimes x_{\alpha_2} \otimes \ldots \otimes x_{\alpha_n})^{\mathbb{Z}_n} & \longrightarrow & \left(x^{(1)}_{\alpha_1} \otimes x^{(2)}_{\alpha_1} \otimes x^{(1)}_{\alpha_2} \otimes x^{(2)}_{\alpha_2} \otimes \ldots \otimes x^{(1)}_{\alpha_n} \otimes x^{(2)}_{\alpha_n}\right)^{\mathbb{Z}_{2n}} \end{array}$$

makes $Cyc(W_N)$ into an involutive Lie bialgebra with the Lie bracket identical to the one introduced earlier by Schedler in [S], $[\ ,\ ] = [\ ,\ ]^S$, but with the Lie cobracket slightly different,

$$\Delta\left(e_{\alpha_1} \otimes \ldots \otimes x_{\alpha_n}\right)^{\mathbb{Z}_n} = \Delta^S\left(x_{\alpha_1} \otimes \ldots \otimes x_{\alpha_n}\right)^{\mathbb{Z}_n} + \sum_{i=1}^{n} 1 \wedge \left(x_{\alpha_1} \otimes ... \otimes x_{\alpha_{i-1}} \otimes x_{\alpha_{i+1}} \otimes \ldots \otimes x_{\alpha_n}\right)^{\mathbb{Z}_{n-1}}.$$

This purely combinatorial Lie bialgebra structure on $Cyc(W_N)$ has a beautiful geometric interpretation — it is *isomorphic* [AKKN1, AN, Ma] to the Goldman-Turaev Lie bialgebra structure on the space of free loops in $\Sigma_{0,N+1}$, the two dimensional sphere with $n+1$ non-intersecting open disks removed.

3. A prop of ribbon hypergraphs

3.1. Ribbon hypergraphs. A *ribbon hypergraph* Γ is, by definition, a triple $(E(\Gamma), \sigma_1, \sigma_0)$ consisting of a finite set $E(\Gamma)$ *of edges* and two arbitrary bijections ("permutations") $\sigma_0, \sigma_1 : E(\Gamma) \to E(\Gamma)$. The orbits

$$V(\Gamma) := E(\Gamma)/\sigma_0$$

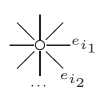
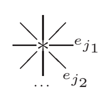
or, equivalently, the cycles of the permutation σ_0 are called the *vertices* of Γ while the orbits

$$H(\Gamma) := E(\Gamma)/\sigma_1$$

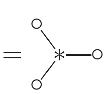
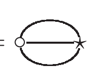
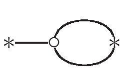
of the permutation σ_1 are called *hyperedges* of Γ (cf. [LZ]). Let $p_\circ : E(\Gamma) \to V(\Gamma)$ and $p_* : E(\Gamma) \to H(\Gamma)$ be canonical projections. For any vertex $v \in V(\Gamma)$ and any hyperedge $h \in H(\Gamma)$ the associated sets of edges

$$p_\circ^{-1}(v) = \{e_{i_1}, \ldots, e_{i_k}\}, \quad p_*^{-1}(h) = \{e_{j_1}, \ldots, e_{j_l}\}, \quad k, l \in \mathbb{N}$$

come equipped with induced cyclic orderings and hence can be represented pictorially as planar corollas,

$v \Leftrightarrow$  , $h \Leftrightarrow$ 

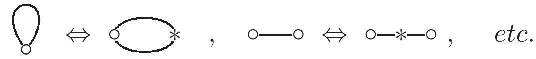
Each edge $e \in E(\Gamma)$ belongs to precisely one vertex, $e \in p_0^{-1}(v)$ for some $v \in V(\Gamma)$, and precisely one hyperedge, $e \in p_*^{-1}(h)$ for some $h \in H(\Gamma)$. Hence we can glue the vertex v to the corresponding hyperedge h along the the common edge(s) to get a pictorial (CW complex like) representation of a ribbon hypergraph Γ as an ordinary ribbon graph whose vertices are bicoloured, e.g.

$\Gamma_1 =$  , $\Gamma_2 =$  , $\Gamma_3 =$ 

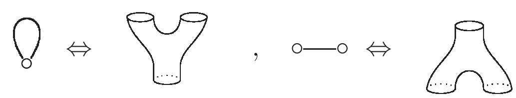
The vertices of Γ get represented pictorially as white vertices while hyperedges as asterisk vertices; sometimes we call a hyperedge an *asterisk vertex* when commenting some pictures. Note that each edge $e \in E(\Gamma)$ connects precisely one white vertex to precisely one asterisk vertex; such bicoloured ribbon graphs are called *hypermaps* in [LZ].

The orbits of the permutation $\sigma_\infty := \sigma_0^{-1} \circ \sigma_1$ are called *boundaries* of the ribbon hypergraph Γ; the set of boundaries is denoted by $B(\Gamma)$. For example, in the case of the above ribbon hypergraphs we have $\#V(\Gamma_1) = 3$, $\#H(\Gamma_1) = 1$, $\#B(\Gamma_1) = 1$, $\#E(\Gamma_1) = 3$, $\#V(\Gamma_2) = 1$, $\#H(\Gamma_2) = 1$, $\#B(\Gamma_2) = 3$, $\#E(\Gamma_2) = 3$ and $\#V(\Gamma_3) = 1$, $\#H(\Gamma_3) = 2$, $\#B(\Gamma_3) = 2$, $\#E(\Gamma_3) = 3$. For a vertex $v \in V(\Gamma)$ (resp, a hyperedge $h \in H(\Gamma)$) we denote its *valency* by $|v| := \#p_\circ^{-1}(v)$ (resp., $|h| := \#p_*^{-1}(h)$).

A ribbon hypergraph with each hyperedge having valency 2 is called a *ribbon graph*. Ribbon graphs are depicted pictorially with asterisk vertices omitted as they contain no extra information,



3.2. On geometric interpretation of ribbon hypergraphs. Every connected ribbon graph Γ can be interpreted geometrically as a topological 2-dimensional surface with $\#B(\Gamma)$ boundary circles and $\#V(\Gamma)$ punctures which is obtained from its CW-complex realization by thickening its every vertex into a closed disk punctured in the center and then thickening its every edge $e \in E(\Gamma)$ into a 2-dimensional strip. For example (cf. [MW1]),

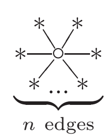


with punctures represented as bottom "in-circles", and boundaries as top "out-circles".

A ribbon hypergraph Γ is often used [LZ] to encode combinatorially a Belyi map, that is, a ramified covering $f : X \to \mathbb{CP}^1$ of the sphere whose ramification locus is contained in the set $\{0, 1, \infty\}$. A famous Belyi theorem says that every smooth projective algebraic curve X defined over the algebraic closure $\overline{\mathbb{Q}}$ (in $\mathbb{C}$) of rational numbers can be realized as such a ramified covering of $\mathbb{CP}^1$. Moreover, the universal Galois group $\mathsf{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$ acts faithfully on (equivalence classes of) Belyi maps. Given a Belyi map $f : X \to \mathbb{CP}^1$, the associated ribbon hypergraph (*dessin d'enfants*) Γ is embedded into the Riemann surface X as the pre-image

$$f^{-1}(\bullet\!\!-\!\!\!-\!\!\circ)$$

of the unit interval $[0, 1] \subset \mathbb{CP}^1$ with the point 0 presented as the white vertex vertex and the point 1 as the asterisk. For example, the hypergraph



corresponds to the Belyi map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ given by $f(z) = z^n$, while the hypergraph

corresponds to the Belyi map $f(z) = (1 - z)^n$. However, ribbon hypergraphs used in this paper are $\mathbb{Z}$-graded and *oriented* (see the next subsection) and it is not clear how this extra structure may fit this particular geometric interpretation of hypergraphs. *Oriented* ribbon graphs have a useful interpretation as combinatorial objects parameterizing cells of a certain cell decomposition of the moduli spaces $\mathcal{M}_{g,n}$ of algebraic curves of genus g with n punctures; they also play an important role in the deformation theory of the Goldman-Turaev Lie bialgebra. We show in this paper that *oriented* ribbon hypergraphs have much to do with strongly homotopy involutive Lie bialgebras and hence might be useful in the string topology.

3.3. Orientation and $\mathbb{Z}$ grading. Let d be an arbitrary integer. To a ribbon hypergraph Γ we assign its *homological degree*

$$|\Gamma| = (d+1)\#H(\Gamma) - d\#E(\Gamma),$$

i.e. each hyperedge has degree $d+1$ and every edge has degree $-d$.

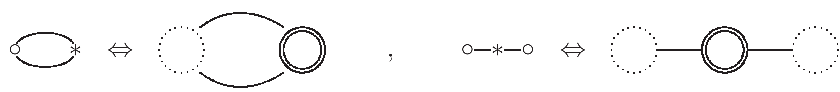
An *orientation* on a ribbon hypergraph Γ is, by definition, a

- choice of the total ordering (up to an even permutation) on the set of hyperedges $H(\Gamma)$ *for d even*,
- choice of the total ordering (up to an even permutation) on the set of edges $E(\Gamma)$ *for d odd*.

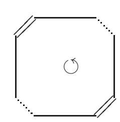
As $E(\Gamma) = \sqcup_{h\in H(\Gamma)} p_*^{-1}(h)$ and each set $p_*^{-1}(h)$ is cyclically ordered, a choice of the total ordering in $E(\Gamma)$ *for d odd* is equivalent to a choice of a total ordering of $H_{odd}(\Gamma) := \{h \in H(\Gamma) | \ |h| \in 2\mathbb{Z}+1\}$ (up to an even permutation), and a choice of a total ordering of each set $p_*^{-1}(h)$, $h \in H_{even}(\Gamma) := \{h \in H(\Gamma) | \ |h| \in 2\mathbb{Z}\}$ which is compatible with the given cyclic ordering (again up to an even permutation).
Note that every ribbon hypergraph has precisely two possible orientations. If Γ is an oriented ribbon hypergraph, then the same hypergraph equipped with an opposite orientation is denoted by Γ^{opp}.

Also note that if Γ has all hyperedges bivalent, then the above definition agrees with the notion of orientation in the prop of ribbon graphs $\mathcal{RG}ra_d$ introduced in [MW].

3.4. Boundaries and corners of a hypergraph. Let us represent pictorially a ribbon hypergraph Γ with vertices and hyperedges blown up into dashed and, respectively, double solid circles, for example



Edges attached to dashed (resp. double solid) circles divide the latters into the disjoint union of chords which can be called *vertex (resp. hyperedge) corners*; thus with any vertex $v \in V(\Gamma)$ we associate a cyclically ordered set $C(v)$ of its corners (which is, of course, isomorphic to its set $p_\circ^{-1}(v)$ of attached edges but has a different geometric incarnation). The motivation for this terminology is that any boundary of a hypergraph can be understood as a polytope glued from edges at that corners. For example, the unique boundary b of the right graph just above is given the following polytope

$b =$ 

where small dashed (resp. double solid) intervals stand for the vertex (resp. hyperedge) corners. Thus with any boundary b of a hypergraph one can associate a cyclically ordered set $C(b)$ of its vertex corners. We shall use these cyclically ordered sets $C(v)$ and $C(b)$ in the definition of the prop composition of hypergraphs below, and in the construction of canonical representations of that prop in spaces of cyclic words.

3.5. Prop of ribbon hypergraphs. Let $\mathcal{RH}_{m,n}^{k,l}$ be the set of (isomorphism classes of) oriented ribbon hypergraphs Γ with n vertices labelled by elements of $[n]$, k edges, l unlabelled hyperedges and m boundaries labelled by elements of $[m]$. Consider a collection of quotient $\mathbb{S}$-bimodules,

$$\mathcal{RH}ra_d := \left\{ \mathcal{RH}ra_d(m,n) := \bigoplus_{k,l\geq 1} \frac{\mathbb{K}\langle \mathcal{RH}_{m,n}^{k,l}\rangle}{\{\Gamma = -\Gamma^{opp}, \Gamma \in \mathcal{H}_{m,n}^{k,l}\}}[dk-(d+1)l] \right\}_{m,n\geq 1}$$

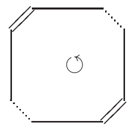
Thus elements of $\mathcal{RH}ra$ are isomorphisms classes of $\mathbb{Z}$-graded oriented ribbon hypergraphs Γ whose vertices and boundaries are enumerated. Ribbon hypergraphs admitting automorphisms which reverse their orientations are equal to zero in $\mathcal{RH}ra_d$. The $\mathbb{S}$-module $\mathcal{RH}ra_d$ contains a submodule $\mathcal{RG}ra_d$ generated by hypergraphs with all hyperedges bivalent. This submodule has a prop structure [MW1] which can be easily extended to $\mathcal{RH}ra_d$. Indeed, the horizontal composition

$$\begin{array}{rccc} \circ: & \mathcal{RH}ra_d(m_1,n_1) \otimes_{\mathbb{K}} \mathcal{RH}ra_d(m_2,n_2) & \longrightarrow & \mathcal{RH}ra_d(m_1+m_2, n_1+n_2) \\ & \Gamma_2 \otimes \Gamma_1 & \longrightarrow & \Gamma_2 \sqcup \Gamma_1 \end{array}$$

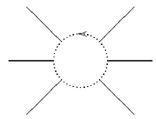
is defined as the disjoint union of ribbon hypergraphs, and the vertical composition,

$$\begin{array}{rccc} \circ: & \mathcal{RH}ra_d(p,m) \otimes_{\mathbb{K}} \mathcal{RH}ra_d(m,n) & \longrightarrow & \mathcal{RH}ra_d(p,n) \\ & \Gamma_2 \otimes \Gamma_1 & \longrightarrow & \Gamma_2 \circ \Gamma_1 \end{array}$$

is defined by gluing, for every $i \in [m]$, the i-th oriented boundary b of Γ_1,

$b \sim$ 

with the i-th vertex v of Γ_2,

$v \sim$ 

by erasing the vertex v from Γ_2 and taking the sum over all possible ways of attaching "hanging in the air" (half)edges from the set $p_\circ^{-1}(v)$ to the set of dashed corners from $C(b)$ while respecting the cyclic structures of both sets; put another way we take a sum over all morphisms $p_\circ^{-1}(v) \to C(b)$ of cyclically ordered sets. Every ribbon graph in this linear combination comes equipped naturally with an induced orientation, and belongs to $\mathcal{RH}ra_d(p,n)$. The graph $\circ$ consisting of a single white vertex acts as the unit in $\mathcal{RH}ra_d$. The subspace of $\mathcal{RH}ra_d$ spanned by *connected* ribbon graphs forms a *properad* which we denote by the same symbol $\mathcal{RH}ra_d$.

3.5.1. Proposition. *Let W be an arbitrary graded vector space and $Cyc(W) = \oplus_{n\geq 0}(W^{\otimes n})^{\mathbb{Z}_n}$ the associated space of cyclic words. Then any collection*

$$\Theta_n : (W[d])^{\otimes n}_{\mathbb{Z}_n} \longrightarrow \mathbb{K}[1+d], \qquad n \geq 1,$$

of cyclically (skew)invariant maps gives canonically rise to a representation

$$\rho_{\Theta_\bullet} : \mathcal{RH}ra_d \longrightarrow \mathcal{E}nd_{Cyc(W)}.$$

of the prop of hypergraphs in $Cyc(W)$.

Proof. If only Θ_2 is non-zero, the associated representation

$$\rho_{\Theta_2} : \mathcal{RG}ra_d \longrightarrow \mathcal{E}nd_{Cyc(W)}$$

was constructed in Theorem 4.2.2 of [MW1]. The general case is a straightforward hypergraph extension of that construction. Let us sketch this extension for d even (the case d odd is completely analogous). Consider a hypergraph $\Gamma \in \mathcal{RH}ra_d(m,n)$ with n vertices $(v_1,\dots,v_i,\dots,v_n)$ and m boundaries $(b_1,\dots,b_j,\dots,b_m)$. Using $\Theta_\bullet$ we will construct a linear map

$$\begin{array}{rccc} \rho^\Gamma_{\Theta_\bullet}: & \otimes^n Cyc(W) & \longrightarrow & \otimes^m Cyc(W) \\ & \mathcal{W}_1 \otimes \ldots \otimes \mathcal{W}_n & \longrightarrow & \rho^\Gamma_{\Theta_\bullet}(\mathcal{W}_1,\dots,\mathcal{W}_n) \end{array}$$

where

$$\left\{\mathcal{W}_i := (w_{i_1} \otimes \ldots \otimes w_{p_i})_{\mathbb{Z}_{p_i}}\right\}_{1\leq i\leq n,\ \ p_i\in\mathbb{N}} \tag{12}$$

is an arbitrary collection of n cyclic words from $Cyc(W)$. If $\#p_\circ^{-1}(v_i) > p_i$ for at least one $i \in [n]$, i.e. if number of edges attached to v_i greater than the length of the word $\mathcal{W}_i$, we set $\rho^\Gamma_{\Theta_\bullet}(\mathcal{W}_1,\dots,\mathcal{W}_n) = 0$. Otherwise it makes sense to consider a *state* s which is by definition a collection of fixed injective morphisms of cyclically ordered sets

$$s_i : p_\circ^{-1}(v_i) \longrightarrow \{w_{i_1},\dots,w_{p_i}\}, \quad \forall i \in [n],$$

that is, an assignment of some letter $w_{i_\bullet}$ from the word $\mathcal{W}_i$ to each edge $e_{i_\bullet} \in p_\circ^{-1}(v_i)$ of each vertex v_i in a way which respects cyclic orderings of both sets. Note that for each state the complement, $(w_{i_1},\dots,w_{i_{l_p}}) \setminus \mathsf{Im}\, s_i$, splits into a disjoint (cyclically ordered) union of *totally ordered* subsets, $\coprod_{c\in C(v)} I_c$, parameterized by the set of corners of the vertex v. Note also that to each boundary $b_j \in B(\Gamma)$ we can associate a cyclic word

$$\mathcal{W}'_j := \left(\bigotimes_{c\in C(b_j)} I_c\right)_{\mathbb{Z}_{\sum_{c\in C(b_j)}\# I_c}}$$

where the tensor product is taken along the given cyclic ordering in the set $C(b_j)$,

Recall that the set of hyperedges $H(\Gamma)$ is defined as the set of orbits $E(\Gamma)/\sigma_1$ of the permutation σ_1. To any hyperedge $h \in H(\Gamma)$ of valency $|h| \in \mathbb{N}$ there corresponds therefore a cyclically ordered set of $|h|$ edges $p_*^{-1}(h) \subset E(\Gamma)$. Let us choose for a moment a compatible total order on this set, i.e. write it as

$$p_*^{-1}(h) = \{e_1^h, e_2^h = \sigma_1(e_1^h), \ldots, \ldots e_{|h|}^h = \sigma_1^{|h|-1}(e_1^h)\}$$

for some chosen edge $e_1^h \in p_*^{-1}(h)$. As the set of edges decomposes into the disjoint union

$$E(\Gamma) = \coprod_{h \in H(\Gamma)} p_*^{-1}(h),$$

we can use the given maps $\Theta_k : (\otimes^k W)_{\mathbb{Z}_k} \to \mathbb{K}[1+d-dk]$ to define the *weight* of any given state s on Γ as the following number,

$$\lambda_s := \prod_{h \in H(\Gamma)} \Theta_{|h|}\left(s(e_1^h), \ldots s(e_{|h|}^h)\right).$$

Thus to each state $s = \{s_i\}_{i \in [n]}$ we associate an element

$$\rho^s_{\Theta_\bullet}(\mathcal{W}_1, \ldots, \mathcal{W}_n) := (-1)^\sigma \lambda_s \mathcal{W}'_{b_1} \otimes \ldots \otimes \mathcal{W}'_{b_m} \in \otimes^m Cyc(W)$$

where $(-1)^\sigma$ is the standard Koszul sign of the regrouping permutation,

$$\sigma : \mathcal{W}_1 \otimes \ldots \otimes \mathcal{W}_n \longrightarrow \prod_{h \in (\Gamma)} \left(s(e_1^h), \ldots s(e_{|h|}^h)\right) \otimes \mathcal{W}'_{b_1} \otimes \ldots \otimes \mathcal{W}'_{b_m}.$$

Note that $\rho^s_{\Theta_\bullet}(\mathcal{W}_1, \ldots, \mathcal{W}_n)$ does not depend on the choices of compatible total orderings in the sets $p_*^{-1}(h)$ made above.

Finally we define a linear map,

$$\begin{array}{rccc} \rho_{\Theta_\bullet}: & \mathcal{RG}ra_d(m,n) & \longrightarrow & \mathrm{Hom}(\otimes^n Cyc(W), \otimes^m Cyc(W)) \\ & \Gamma & \longrightarrow & \rho^\Gamma_{\Theta_\bullet} \end{array}$$

by setting the value of $\rho^\Gamma_{\Theta_\bullet}$ on cyclic words (12) to be equal to

$$\rho^\Gamma_{\Theta_\bullet}(\mathcal{W}_1, \ldots, \mathcal{W}_n) := \left\{ \begin{array}{cl} 0 & \text{if } \#p_\circ^{-1}(v_i) > p_i \text{ for some } i \in [n] \\ \sum\limits_{\substack{\text{all possiblle} \\ \text{states } s}} \rho^s_{\Theta_\bullet}(\mathcal{W}_1, \ldots, \mathcal{W}_n) & \text{otherwise} \end{array}\right.$$

It is now straightforward to check that the map $\rho_{\Theta_\bullet}$ respects prop compositions in $\mathcal{RH}ra_d$ and $\mathcal{E}nd_{Cyc(W)}$ because the prop structure in former has been just read off from the compositions of operators $\rho^\Gamma_{\Theta_\bullet}$ in the latter. $\square$

4. Strongly homotopy involuyive Lie bialgebras and ribbon hypergraphs

4.1. Reminder from [MW1]**.** *There is a morphism of props,*

$$\rho : \mathcal{L}ieb_d^\diamond \longrightarrow \mathcal{RH}ra_d$$

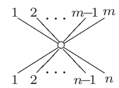
given on generators as follows,

The main new result of this note is an observation that this maps lifts to a morphism of dg props $\mathcal{H}olieb_d^\diamond \to \mathcal{RH}ra_d$ which is non-trivial on all generators (see below).

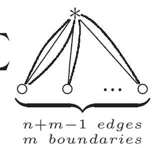
4.2. Proposition. *There is a morphism of dg props,*

$$\rho : \mathcal{H}olieb_d \longrightarrow \mathcal{RH}ra_d$$

given on generators as follows,



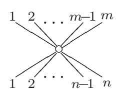
$$\xrightarrow{\rho} \sum$$



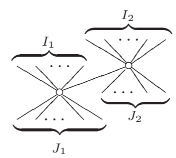
where the sum on the right hand side is over all possible ways of attaching $n+m-1$ edges beginning at the asterisk vertex to n white vertices (whose numerical labels are (skew)symmetrized) in such a way that every white vertex is hit and the total number of boundaries of the resulting hypergraph equals precisely m (and their numerical labels are also (skew)symmetrized).

Proof. As the prop $\mathcal{RH}ra_d$ has vanishing differential, the Propsoition holds true if and only of

$$\rho\Big(\delta$$



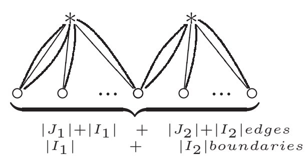
$$\Big) = \rho\Bigg(\sum_{\substack{[1,\ldots,m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \sum_{\substack{[1,\ldots,n]=J_1\sqcup J_2\\ |J_1|\geq 1, |J_2|\geq 1}} \pm$$



$$\Bigg) = 0.$$

This is almsot obvious as the r.h.s. is given by the sum

$$\sum_{\substack{[1,\ldots,m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \sum_{\substack{[1,\ldots,n]=J_1\sqcup J_2\\ |J_1|\geq 1, |J_2|\geq 1}} \sum \pm$$



which vanishes in $\mathcal{RH}ra_d$ for symmetry reasons (it is most easy to check this claim in the case d is even when the asterisk vertices are odd, and the symbol $\pm$ becomes $+$). $\square$

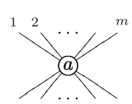
There is a canonical morphism of props $\mathcal{H}olieb_d \to \mathcal{H}olieb_d^\diamond$. The above morphism factors through the composition $\mathcal{H}olieb_d \to \mathcal{H}olieb_d^\diamond \xrightarrow{\rho^\diamond} \mathcal{RH}ra_d$.

4.3. Theorem. *There is a morphism of dg props,*

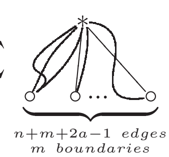
$$\rho^\diamond : \mathcal{H}olieb_d^\diamond \longrightarrow \mathcal{RH}ra_d$$

given on generators as follows,

(13)



$$\xrightarrow{\rho^\diamond} \sum$$



where the sum on the right hand side is over all possible ways to attach $n+m+2a-1$ edges beginning at the asterisk vertex to n white vertices (whose numerical labels are (skew)symmetrized) in such a way that every white vertex is hit and the total number of boundaries of the resulting hypergraph equals precisely m (and their numerical labels are (skew)symmetrized).

The orientations of the hypergraphs shown in the above formula are determined uniquely by a simple $\mathcal{H}o\mathcal{BV}_d^{com}$ operator which is constructed in the proof.

Proof. The composition of the above map $\rho^\diamond$ with the canonical representation $\rho_{\Theta_\bullet}$ from Proposition **3.5.1** implies that for any collection of cyclically (skew)symmetric maps 3 on a graded vector space W the associated vector space $Cyc(W)$ is canonically a $\mathcal{H}olieb_d^\diamond$ algebra. In fact one can read the Theorem from this conclusion provided the latter is independent of choices of W and the higher products $\Theta_\bullet$.

Assume d even. To make the construction of the representation $\rho_{\Theta_\bullet}\circ\rho^\diamond$ as simple and transparent as possible we shall employ, following Barannikov [Ba2], the invariant theory and identify the space $Cyc(W)$ with the space of $GL(\mathbb{K}^N)$-invariants,

$$Cyc(W) = \lim_{N\to\infty} \oplus_{n\geq 0} \left(\otimes^n (W\otimes \mathsf{End}(\mathbb{K}^n)\right)^{GL(\mathbb{K}^N)},$$

that is, we interpret a cyclic word, $\mathcal{W}=(w_{a_1}\otimes\ldots\otimes w_{a_n})^{\mathbb{Z}_n}$, $\{w_i\}_{i\in I}$ a basis in W, with the trace of the product of $N\times N$ matrices,

$$\mathcal{W} = tr\left(A_{a_1}A_{a_2}\cdots A_{a_n}\right) = \sum_{\alpha_\bullet} A^{\alpha_0}_{a_1\,\alpha_1} A^{\alpha_1}_{a_2\,\alpha_2}\cdots A^{\alpha_n}_{a_n\,\alpha_0}, \qquad A_{i_a} := \left(A^{\alpha}_{i_a\,\beta}\right)\in \mathsf{End}(\mathbb{K}^N), \quad a\in[n],\ \alpha,\beta\in[N],$$

for sufficiently large $N\in\mathbb{N}$. The graded cyclically symmetric maps

$$\begin{array}{rccc} \Theta_n: & \otimes^n W & \longrightarrow & \mathbb{K}[1+d-nd] \\ & w_{a_1}\otimes\ldots\otimes w_{a_n} & \longrightarrow & \Theta_{a_1\ldots a_n} \end{array}$$

define a degree 1 operator

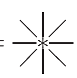
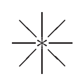
$$\Delta = \sum_{n\geq 1} \hbar^{n-1}\Theta_{a_1\ldots a_n} \frac{\partial^n}{\partial A^{\alpha_0}_{a_1\,\alpha_1}\partial A^{\alpha_1}_{a_2\,\alpha_2}\cdots\partial A^{\alpha_n}_{a_n\,\alpha_0}}$$

on $\odot^\bullet(Cyc(W)[-d])[[\hbar]]$ whose square is obviously zero; here the formal parameter $\hbar$ has degree $2d$. The latter defines a $\mathcal{H}o\mathcal{BV}^{com}_d$-structure in $\odot^\bullet(Cyc(W)[-d])[\hbar]]$ and, as explained in §**2.4**, an associated $\mathcal{H}olieb^\diamond_d\{d\}$ algebra structure in $Cyc(W)[-d]$ which in turn defines a $\mathcal{H}olieb^\diamond_d$-structure in $Cyc(W)$.

If d is odd, one can again use a trick from [Ba2] which replaces the ordinary trace of the standard matrix superalgebra with the *odd trace* of the Bernstein-Leites matrix sub-superalgebra. In fact the construction in [Ba2] explains the construction of the $\mathcal{H}o\mathcal{BV}^{com}_d$ operator in the case when only Θ_2 is non-zero, and its extension to the general case (3) is completely analogous to the d even case discussed above. □

4.4. Corollary. *Given any graded vector space V equipped with a collection of linear maps (3). There is an associated explicit $\mathcal{H}olieb^\diamond_d$-structure in $Cyc(W)$ given by the formulae (13).*

4.5. Rescaling freedom. Note that each map Θ_k from the family (3) can be independently rescaled, $\Theta_k\to\lambda_k\Theta_k$, $\forall\lambda_n\in\mathbb{K}$, so that the morphism $\rho^\diamond$ in (13) can also be rescaled by infinitely many independent parameters — just rescale in that formula each hyperedge h of valency k by

$h =$  $\longrightarrow \lambda_k\cdot$  , $\forall\lambda_k\in\mathbb{K}$,

and get a new morphism $\rho^\diamond_{\lambda_\bullet}$ from $\mathcal{H}olieb^\diamond_d$ to $\mathcal{RH}ra_d$. Such a phenomenon occurs in the string topology — see Theorem 6.2 and Corollary 6.3 in [CS] — and its main technical origin in our case is that the prop $\mathcal{RH}ra_d$ has vanishing differential. We shall discuss a *differential* version of $\mathcal{RH}ra_d$ elsewhere.

5. An algebraic application: a new family of combinatorial $\mathcal{H}olieb^\diamond$-algebras

Given any collection of formal letters $e_1,\ldots,e_N$, i.e. given any natural number $N\geq 1$, there is an associated involutive Lie bialgebra structure on the vector space $Cyc(W_N)$ of cyclic words,

$$W_N := \mathrm{span}_{\mathbb{K}}\langle e_1,e_2,\ldots,e_N\rangle$$

which belongs to the family of combinatorial $\mathcal{L}ieb^\diamond$-algebras constructed by Schedler in [S] out of any quiver. This particular $\mathcal{L}ieb^\diamond$-algebra has an important geometric meaning (discussed in §1) and corresponds to the quiver (1).

In this section we use Theorem **4.3** and Corollary **4.4** in the case $d=1$ to extend that particular Schedler's construction to a highly non-trivial (i.e. with *all* higher homotopy operations non-zero) $\mathcal{H}olieb^\diamond$-algebra structure on the vector space $Cyc(\widehat{W}_N)$ of cyclic words generated by $\mathbb{Z}$-*graded* formal letters,

$$\widehat{W}_N := \mathrm{span}_{\mathbb{K}}\langle e_1[-p],\ldots,e_N[-p]\rangle_{p\in\mathbb{N}} = \bigoplus_{p\geq 0} W_N[-p]$$

where $e_\alpha[-p]$, $\alpha \in [N]$, stands for the copy of the formal letter e_α to which we assign the homological degree p. Note that $\widehat{W}_N$ has no natural higher products (3) so we can not apply Corollary **4.4** immediately. The idea of our construction is to inject first

$$u: \quad \widehat{W}_N \quad \longrightarrow \quad \widehat{\mathbf{W}}_N$$

$\widehat{W}_N$ into a larger space $\widehat{\mathbf{W}}_N$ which does have a natural family of cyclically (skew)symmetric higher products

$$\left\{\Theta_{k+2} : \left(\otimes^{k+2}(\widehat{\mathbf{W}}_N[1])\right)_{\mathbb{Z}_{k+2}} \longrightarrow \mathbb{K}[2]\right\}_{k\geq 0}$$

so that the associated vector space of cyclic words $Cyc(\widehat{\mathbf{W}}_N)$ comes equipped with a canonical $\mathcal{H}olieb^\diamond$-algebra structure by Corollary **4.4**. The second step will be to check that the image of u is closed with respect to all strongly homotopy involutive Lie bialgebra operations. To realize this programme, consider, for any integer $p \geq 0$, a set of $p+2$ copies of the vector space W_N

$$W_N^{0_p} := W_N[-p]\ ,\ W_N^{1_p} := W_N\ , \ldots,\ W_N^{p+1_p} := W_N\ ,$$

one of them (say, labelled by zero) assigned a shifted homological degree, and define

$$\widehat{\mathbf{W}}_N := \bigoplus_{p\geq 0} W_{N,p}, \qquad W_{N,p} := W_N^{0_p} \oplus W_N^{1_p} \oplus \ldots \oplus \ldots W_N^{p+1_p}$$

The vector space $\widehat{\mathbf{W}}_N$ is countably dimensional, and is equipped by construction with a distinguished basis

$$\left\{e_\alpha^{l_p}\right\}_{p\geq 0, 0\leq l\leq p+1, 1\leq \alpha\leq N}$$

where $\{e_\alpha^{l_p}\}_{\alpha\in[N]}$ stands for the standard basis of the copy $W_N^{l_p}$. Note that for any $\alpha \in N$ the basis vector $e_\alpha^{l_p}$ has homological degree 0 if $l \geq 1$, and $-p$ of $l = 0$. Note also that summands $W_N^{l_{p'}}$ and $W_N^{l_{p''}}$ in $\widehat{\mathbf{W}}_N$ are viewed as different copies of W_N for $p' \neq p''$ (as they belong to different vector spaces $W_{N,p'}$ and $W_{N,p''}$).

Let us introduce next an infinite family of cyclically (skew)symmetric higher products (the case $d = 1$ in the notation (3))

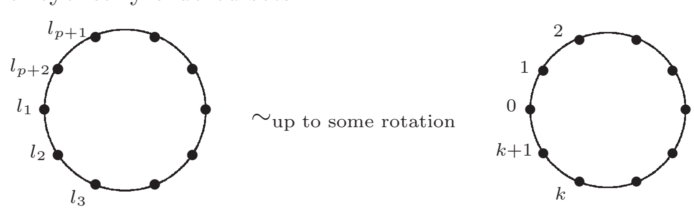
$$\Theta_{k+2} : \otimes^{k+2}\hat{\mathbf{W}}_N \longrightarrow \mathbb{K}[-k], \qquad \forall\ k \geq 0,$$

by setting

$$\Theta_{k+2}\left(e_{\alpha_1}^{{l^1}_{p_1}} \otimes \ldots \otimes e_{\alpha_{k+2}}^{{l^{k+2}}_{p_{k+2}}}\right) := 0$$

unless

(i) $\alpha_1 = \alpha_2 = \ldots = \alpha_{k+1}$;
(ii) $p_1 = p_2 = \ldots = p_{k+2} = k$;
(iii) $(l^1, \ldots, l^{k+2}) = (j, j-1, \ldots, 2, 1, 0, k+1, k, \ldots, j+1)$ for some $j \in \{0, 1, 2, \ldots, k+1\}$, i.e. we have an isomorphism of cyclically ordered sets



If all the above conditions are satisfied, we set

$$\Theta_{k+2}\left(e_\alpha^{j_k} \otimes e_\alpha^{j-1_k} \otimes \ldots e_\alpha^{1_k} \otimes e_\alpha^{0_p} \otimes e_\alpha^{k+1_k} \otimes e_\alpha^{k_k} \otimes \ldots \otimes e_\alpha^{j+1_k}\right) := (-1)^{j(k+1)}$$

By Corollary **4.4**, the graded vector space $Cyc^\bullet(\widehat{\mathbf{W}}_N)$ is a $\mathcal{H}olieb^\diamond$-algebra equipped with quite explicit strongly homotopy operations. There is a canonical (homogeneous of homological degree zero) injection

$$\begin{array}{cccc} u: & Cyc^\bullet(\mathbf{W}_N) & \longrightarrow & Cyc^\bullet(\widehat{\mathbf{W}}_N) \\ & (e_{\alpha_1}[-p_1] \otimes \ldots \otimes e_{\alpha_n}[-p_n])^{\mathbb{Z}_n} & \longrightarrow & u\left(e_{\alpha_1}[-p_1] \otimes \ldots \otimes e_{\alpha_n}[-p_n]\right)^{\mathbb{Z}_n} \end{array}$$

identifying each letter $e_\alpha[-p]$ in a cyclic word from $Cyc(\mathbf{W}_N)$ with a (totally ordered) word in $p+2$ letters

$$e_\alpha[-p] \longrightarrow e_\alpha^{(0_p)} \otimes e_\alpha^{(1_p)} \otimes \ldots \otimes e_\alpha^{(p+1_p)}$$

i.e.

$$u\left(e_{\alpha_1}[-p_1] \otimes ... \otimes e_{\alpha_n}[-p_n]\right)^{\mathbb{Z}_n} := \left(e_{\alpha_1}^{(0)_{p_1}} \otimes e_{\alpha_1}^{(1)_{p_1}} \otimes ... \otimes e_{\alpha_1}^{(p_1+1)_{p_1}} \otimes \ldots \otimes e_{\alpha_n}^{(0)_{p_n}} \otimes e_{\alpha_n}^{(1)_{p_n}} \otimes ... \otimes e_{\alpha_n}^{(p_n+1)_{p_n}}\right)^{\mathbb{Z}_m}$$

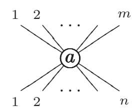
where

$$m = n(p_1 + \ldots + p_n + 2n)$$

A remarkable an almost obvious fact is that the linear subspace

$$u(Cyc(\widehat{W}_N)) \subset Cyc(\widehat{\mathbf{W}}_N)$$

is closed with respect to all strong homotopy involutive Lie bialgebra operations and hence is itself a $\mathcal{H}olieb^\diamond$-algebra. In this way we induce a $\mathcal{H}olieb^\diamond$-algebra structure

$\rho\Bigg($  $\Bigg) : \otimes^n(Cyc(\widehat{W}_N)) \longrightarrow \otimes^n(Cyc(\widehat{W}_N))[3-(m+n+2a)]$

on $Cyc(\widehat{W}_N)$. It is immediate to see that all these operations have degree -1 with respect to the weight-grading by the lengths of cyclic words. On the linear subspace

$$Cyc(W_N) \subset Cyc(\widehat{W}_N)$$

this $\mathcal{H}olieb^\diamond$-algebra structure reduces precisely to Schedler's necklace Lie bialgebra structure corresponding to the quiver (1).

References

[AKKN1] A. Alekseev, N. Kawazumi, Y. Kuno and F. Naef, *The Goldman-Turaev Lie bialgebra in genus zero and the Kashiwara-Vergne problem*, arxiv arXiv:1703.0581.

[AKKN2] A. Alekseev, N. Kawazumi, Y. Kuno and F. Naef, *The Goldman-Turaev Lie bialgebra and the Kashiwara-Vergne problem in higher genera*, arxiv arXiv:1703.0581.

[AN] A. Alekseev and F. Naef, *Goldman-Turaev formality from the Knizhnik-Zamolodchikov connection*, arXiv:1708.03119.

[Ba1] S. Barannikov, *Noncommutative Batalin-Vilkovisky geometry and matrix integrals.* C. R. Math. Acad. Sci. Paris **348** (2010), no. 7-8, 359-362

[Ba2] S. Barannikov, *Matrix De Rham complex and quantum A-infinity algebras*, preprint arXiv:1001.5264. Lett. Math. Phys. **104** (2014) 373-395.

[CS] M. Chas and D. Sullivan, *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, arXiv:math/0212358.

[CMW] R. Campos, S. Merkulov and T. Willwacher *The Frobenius properad is Koszul*, Duke Math. J. **165**, No.1 (2016), 2921-2989.

[Ch] M. Chas, *Combinatorial Lie bialgebras of curves on surfaces*, Topology **43** (2004), no. 3, 543–568.

[CFL] K. Cieliebak, K. Fukaya and J. Latschev, *Homological algebra related to surfaces with boundary*, preprint arXiv:1508.02741, 2015.

[CS] M. Chas and D. Sullivan, *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, in: The legacy of Niels Henrik Abel, pp. 771–784, Springer, Berlin, 2004.

[D1] V. Drinfeld, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations*, Soviet Math. Dokl. **27** (1983) 68–71.

[D2] V. Drinfeld, *On quasitriangular quasi-Hopf algebras and a group closely connected with* $Gal(\bar{Q}/Q)$, Leningrad Math. J. **2**, No. 4 (1991), 829–860.

[DCTT] G. C. Drummond-Cole, J. Terilla and T. Tradler, *Algebras over Cobar(coFrob)*, J. Homotopy Relat. Struct. **5** (2010), no.1, 15–36.

[ES] P. Etingof and O. Schiffmann, Lectures on Quantum Groups, International Press, 2002.

[G] W. Goldman, *Invariant functions on Lie groups and Hamil- tonian flows of surface group representations*, Invent. Math. **85** (1986), no. 2, 263-302.

[Kr] O. Kravchenko, *Deformations of Batalin-Vilkovisky algebras*, In: Poisson geometry (Warsaw, 1998), Banach Center Publ., vol. 51, Polish Acad. Sci., Warsaw, 2000, pp. 131-139.

[Ku] Yusuke Kuno, private communication, December 2016.

[LZ] S.K. Lando and A.K. Zvonkin, Graphs on Surfaces and their Applications, Springer, 2004

[MaVo] M. Markl and A.A. Voronov, *PROPped up graph cohomology*, in: Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, Progr. Math., 270, Birkhäuser Boston, Inc., Boston, MA (2009) pp. 249–281.

[Ma] G. Massuyeau, *Formal descriptions of Turaev's loop operations*, Quantum Topol. **9** (2018) 39-117.

[MV] S. Merkulov and B. Vallette, *Deformation theory of representations of prop(erad)s I & II*, Journal für die reine und angewandte Mathematik (Qrelle) **634**, 51-106, & **636**, 123-174 (2009)

[MW1] S. Merkulov and T. Willwacher, *Props of ribbon graphs, involutive Lie bialgebras and moduli spaces of curves*, preprint arXiv:1511.07808 (2015) 51pp.

[MW2] S.A. Merkulov and T. Willwacher, *Classification of universal formality maps for quantizations of Lie bialgebras*, preprint arXiv:1605.01282 (2016)

[P] R.C. Penner, *Perturbative series and the moduli space of Riemann surfaces*, J. Differential Geometry **27** (1988), no. 1, 35-53

[S] T. Schedler, *A Hopf algebra quantizing a necklace Lie algebra canonically associated to a quiver*. Intern. Math. Res. Notices (2005), 725-760.

[Tu] V.G. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. Ecole Norm. Sup. (4) 24, no. 6, (1991) 635–704.

[TZ] T. Tradler and M. Zeinalian, *Algebraic string operations*, K-Theory **38** (2007), no. 1, 59 - 82.

[V] B. Vallette, *A Koszul duality for props*, Trans. Amer. Math. Soc., **359** (2007), 4865–4943.

Sergei Merkulov: Mathematics Research Unit, Luxembourg University, Grand Duchy of Luxembourg
E-mail address: sergei.merkulov@uni.lu