Holomorphic Frobenius actions for DQ-modules

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Abstract

Given a complex manifold endowed with a $\mathbb{C}^\times$-action and a DQ-algebra equipped with a compatible holomorphic Frobenius action (F-action), we prove that if the $\mathbb{C}^\times$-action is free and proper, then the category of F-equivariant DQ-modules is equivalent to the category of modules over the sheaf of invariant sections of the DQ-algebra. As an application, we deduce the codimension three conjecture for formal microdifferential modules from the one for DQ-modules on a symplectic manifold.

1 Introduction

Relying on the notion of Frobenius action for Deformation quantization modules (DQ-modules) introduced in [KR08], we establish an equivalence between the category of coherent Frobenius equivariant DQ-modules and the category of modules over the sheaf of invariant sections of the DQ-algebra. This result applied to the special case of the canonical DQ-algebra $\widehat{\mathcal{W}}$ on the cotangent bundle provides an equivalence between coherent F-equivariant DQ-modules and coherent microdifferential modules on the projective cotangent bundle. This equivalence permits to deduce the codimension three conjecture for formal microdifferential modules [KV14] from the one for DQ-modules on a symplectic manifold [Pet17].

Deformation quantization algebras (DQ-algebras) are non-commutative formal deformations of the structure sheaf of a complex variety. They are used to quantize complex Poisson varieties. In the symplectic case, they are often presented as an extension of the ring of microdifferential operators to arbitrary symplectic manifolds. The ring of formal microdifferential operators $\widehat{\mathcal{E}}$, introduced in [SKK73], is a sheaf on the cotangent bundle of a complex manifold that quantizes it as a *homogeneous symplectic manifold*. DQ-algebras and in particular the canonical deformation quantization of the cotangent bundle $\widehat{\mathcal{W}}$ ignore the homogeneous structure and quantize this bundle as a symplectic manifold. This allows one to produce quantizations of arbitrary complex symplectic manifolds using $\widehat{\mathcal{W}}$ (see [PS04]) and in some sense extends formal microdifferential modules to arbitrary symplectic manifolds (Note that it is always possible to quantize complex Poisson varieties as proved in [CH11, Yek05] building upon ideas of Kontsevich [Kon01]).

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The ring $\widehat{\mathcal{W}}$ is an $\widehat{\mathcal{E}}$ algebra. Hence, it is natural to ask if it possible to identify those $\widehat{\mathcal{W}}$-modules which are extension of $\widehat{\mathcal{E}}$-modules. For that purpose, it is necessary to add an extra structure to encode the compatibility with the $\mathbb{C}^\times$-action on the fibers of the cotangent bundle. This can be achieved by using the notion of *holomorphic Frobenius action.* They were introduced by Masaki Kashiwara and Raphaël Rouquier in their seminal work [KR08] which introduced an analogue of Beilinson-Bernstein's localization for rational Cherednik algebras. Objects originating from deformation quantization are defined over the ring of formal power series $\mathbb{C}[[\hbar]]$ or its localization with respects to $\hbar$ that is the field of formal Laurent series $\mathbb{C}((\hbar))$. This makes these objects too large for many applications since what is often required is an object satisfying certain finiteness assumptions over $\mathbb{C}$. To overcome this difficulty, they introduced, in [KR08], the notion of $\widehat{\mathcal{W}}$-algebra with a holomorphic Frobenius action or F-action for short. Given a complex symplectic manifold X endowed with an action of $\mathbb{C}^\times$ and quantized by a DQ-algebra, a F-action is a compatible action of $\mathbb{C}^\times$ on the DQ-algebra, acting on the deformation parameter $\hbar$ with a weight. This allows one to rescale $\widehat{\mathcal{W}}$ and the $\widehat{\mathcal{W}}$-modules with respect to $\hbar$. These actions have been subsequently used by several authors in problems arising from the study of the representation theory of quantized conic symplectic singularities, and in particular rational Cherednik algebras (see for instance [BDMN17, BK12, BLPB12, McG12, Los12, Los15])

In this paper, we study the notion of DQ-modules endowed with a F-actions. The definition of a F-action initially provided by Kashiwara and Rouqier is a punctual definition which makes it difficult to use for problems of global nature as questions of analytic extension (i.e. extending a F-action through an analytic subset). Hence, we provide a reformulation in the spirit of G-linearization of coherent sheaves (see [MFK94, Ch.1 §3]). Given a DQ-algebra $\mathcal{A}_X$, on a Poisson manifold X, endowed with a F-action, and assuming that this action is free and proper, we establish an equivalence between the category of coherent DQ-modules endowed with a F-action and the category of modules over the sheaf of invariant sections on the quotient space $Y = X/\mathbb{C}^\times$ (Theorem 6.11). Here we have to work on the quotient space since $\mathbb{C}^\times$ is not simply connected and F-equivariant DQ-modules are constant along the orbits. Our result generalizes the first example of [KR08, §2.3.3] (provided without a proof) which states an equivalence of categories between good $\widehat{\mathcal{W}}$-modules and good micro-differential modules. We extend this example to DQ-modules over arbitrary Poisson manifold and relax the finiteness conditions by only requiring the DQ-modules to be coherent. To obtain this equivalence of categories, we first prove that a locally finitely generated $\mathcal{A}_X$-module endowed with a F-action is locally finitely generated by locally invariant sections (Theorem 5.5). This implies that if $\mathcal{M}$ is coherent, it locally has an equivariant presentation of length one (Corollary 5.8). We prove that the invariant sections functor and the equivariant extension functor form an adjoint pair (Proposition 6.2) and establish the coherence of the sheaf of invariant sections (Theorem 6.9). Then we can prove the equivalence announced earlier (Theorem 6.11). As an example, we construct the weight one F-action on the canonical deformation quantization $\widehat{\mathcal{W}}$ of the cotangent bundle and obtain as a corollary of Theorem 6.11 an equivalence between coherent $\widehat{\mathcal{W}}$-modules and coherent formal microdifferential modules on the projective cotangent bundle (see Proposition 6.16 for a precise statement). Finally, we use this result to deduce the codimension three conjecture for formal microdifferential modules initially proved by Kashiwara and Vilonen (in the formal as well as in the analytic case) in [KV14] from its DQ-module version proved in [Pet17]. For that purpose, we have to extend F-action through analytic subsets, which is one of the reason, we defined F-actions in a non-punctual manner.

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2 Preliminaries on DQ-modules

We write $\mathbb{C}^\hbar$ for the ring of formal power series with complex coefficients in $\hbar$ and $\mathbb{C}^{\hbar,loc}$ for the field of formal Laurent series. Let $(X, \mathcal{O}_X)$ be a complex manifold. We define the sheaf of $\mathbb{C}^\hbar$-algebras

$$\mathcal{O}_X^\hbar := \varprojlim_{n\in\mathbb{N}} \mathcal{O}_X \underset{\mathbb{C}}{\otimes} (\mathbb{C}^\hbar/\hbar^n\mathbb{C}^\hbar).$$

Definition 2.1. A star-product denoted $\star$ on $\mathcal{O}_X^\hbar$ is a $\mathbb{C}^\hbar$-bilinear associative multiplication law satisfying

$$f \star g = \sum_{i\geq 0} P_i(f,g)\hbar^i \text{ for every } f,\ g \in \mathcal{O}_X,$$

where the P_i are holomorphic bi-differential operators such that for every $f, g \in \mathcal{O}_X, P_0(f,g) = fg$ and $P_i(1,f) = P_i(f,1) = 0$ for $i > 0$. The pair $(\mathcal{O}_X^\hbar, \star)$ is called a star-algebra.

Definition 2.2. A DQ-algebra $\mathcal{A}_X$ on X is a $\mathbb{C}_X^\hbar$-algebra locally isomorphic to a star-algebra as a $\mathbb{C}_X^\hbar$-algebra.

Notations 2.3. (i) If $\mathcal{A}_X$ is a DQ-algebra, we set $\mathcal{A}_X^{loc} := \mathbb{C}^{\hbar,loc} \underset{\mathbb{C}^\hbar}{\otimes} \mathcal{A}_X$,

(ii) if X and Y are two complex manifolds endowed with DQ-algebras $\mathcal{A}_X$ and $\mathcal{A}_Y$ then $X \times Y$ is canonically equiped with a DQ-algebra $\mathcal{A}_{X\times Y} := \mathcal{A}_X \boxtimes \mathcal{A}_Y$ (see [KS12, §2.3]). There is a canonical morphism of $\mathbb{C}^\hbar$-algebras

$$p_2^\sharp \colon p_2^{-1}\mathcal{A}_X \to \mathcal{A}_X \boxtimes \mathcal{A}_Y \to \mathcal{A}_{X\times Y}$$

and this morphism is flat ([KS12, lemma 2.3.2]).

(iii) We denote by $\mathrm{Mod}(\mathcal{A}_X)$ the Grothendieck category of left $\mathcal{A}_X$-modules, by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{A}_X)$ its full abelian subcategory whose objects consist of coherent $\mathcal{A}_X$-modules. We use similar notation for the left $\mathcal{A}_X^{loc}$-modules.

There is a unique isomorphism $\mathcal{A}_X/\hbar\mathcal{A}_X \xrightarrow{\sim} \mathcal{O}_X$ of $\mathbb{C}_X$-algebra. We denote by $\sigma_0 : \mathcal{A}_X \twoheadrightarrow \mathcal{O}_X$ the epimorphism of $\mathbb{C}_X$-algebras defined as the following composition

$$\mathcal{A}_X \to \mathcal{A}_X/\hbar\mathcal{A}_X \xrightarrow{\sim} \mathcal{O}_X.$$

These data induce a Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{O}_X$ defined by:

$$\text{for every } a,\ b \in \mathcal{A}_X,\ \{\sigma_0(a), \sigma_0(b)\} = \sigma_0(\hbar^{-1}(ab - ba)).$$

Lemma 2.4. *Let $(\mathcal{O}_X^\hbar, \star)$ be a star algebra and $v : \mathcal{O}_X^\hbar \to \mathcal{O}_X^\hbar$ be a $\mathbb{C}$-linear derivation of $(\mathcal{O}_X^\hbar, \star)$ such that there exists $v_0 \in Der(\mathcal{O}_X)$ such that for every $u \in \mathcal{O}_X^\hbar$, $\sigma_0 \circ v(u) = v_0 \circ \sigma_0(u)$ and $v(\hbar) = m\hbar$. Then, there exists a unique sequence $(v_k)_{k\geq 0}$ of differential operators such that for any $f \in \mathcal{O}_X$,*

$$v(f) = \sum_{i\geq 0} \hbar^i v_i(f).$$

In particular, for every $u = \sum_i \hbar^i u_i \in \mathcal{O}_X^\hbar$,

$$v(u) = \sum_i \left(\sum_k \hbar^{i+k} v_k(u_i) + m\, i\, \hbar^i u_i \right) \tag{2.1}$$

$$= \sum_n \hbar^n \left(\sum_{i+k=n} v_k(u_i) + m\, n\, u_n \right). \tag{2.2}$$

Proof. This proof is an adaptation of the proof of [KS12, Lemma 2.2.3]. It is clear that there exists a sequence $(v_k)_{k\geq 0}$ of endomorphism of $\mathcal{O}_X$ such that, for every $f \in \mathcal{O}_X$

$$v(f) = \sum_i \hbar^i v_i(f).$$

By assumption v_0 is a differential operator. We will prove by induction that the v_k are differential operators. Assume that this is true for $k < l$ with $l \in \mathbb{N}$. Let $(P_n)_{n\in\mathbb{N}}$ be the sequence of bidifferential operators associated with the star products $\star$. By assumption v is continuous for the $\hbar$-adic topology, thus for every $f,\ g \in \mathcal{O}_X$,

$$v(f \star g) = \sum_{j\geq 0} v(\hbar^j P_j(f,g)) = \sum_{n\geq 0} \hbar^n \left(\sum_{i+j=n} v_i(P_j(f,g)) + mn\, P_n(f,g) \right)$$

and

$$f \star v(g) + v(f) \star g = \sum_{n\geq 0} \hbar^n \sum_{j+k=n} \left(P_k(f, v_j(g)) + P_k(v_j(f), g) \right).$$

Since $v(f \star g) = f \star v(g) + v(f) \star g$, we obtain

$$\sum_{i+j=n} v_i(P_j(f,g)) + mn\, P_n(f,g) = \sum_{j+k=n} \left(P_k(f, v_j(g)) + P_k(v_i(f), g) \right).$$

Using the induction hypothesis, we deduce from the above expressions that

$$v_l(fg) + Q_l(f, g) = f v_l(g) + v_l(f)g + R_l(f, g)$$

where Q_l and R_l are bidifferential operators. This implies that

$$[v_l, g](f) = v_l(fg) - g\, v_l(f) = f\, v_l(g) - Q_l(f, g) + R_l(f, g).$$

Since $Q_l(\cdot, g)$ and $R_l(\cdot, g)$ are differential operators, it follows from [KS12, Lemma 2.2.4] that v_l is a differential operator. □

2.1 The canonical deformation quantization of the cotangent bundle

Let M be a complex manifold. The cotangent bundle of M, $X := T^*M$ is equipped with the sheaf $\widehat{\mathcal{E}}_X$ of formal microdifferential operators. This is a filtered, conic sheaf of $\mathbb{C}$-algebras. We denote by $\widehat{\mathcal{E}}_X(0)$ the subsheaf of $\widehat{\mathcal{E}}_X$ formed by the operators of order $m \leq 0$. These sheaves were introduced in [SKK73]. The reader can consult [Sch85] for an introduction to the theory of microdifferential modules.

On X, there is DQ-algebra $\widehat{\mathcal{W}}_X(0)$ which was constructed in [PS04]. Here, we review their construction.

Let $\mathbb{C}$ be the complex line endowed with the coordinate t and denote by $(t; \tau)$ the associated symplectic coordinate on $T^*\mathbb{C}$. We set

$$\widehat{\mathcal{E}}_{T^*(M\times\mathbb{C}),\hat{t}}(0) = \{P \in \widehat{\mathcal{E}}_{T^*M}; [P, \partial_t] = 0\}.$$

We consider the following open subset of $T^*(M \times \mathbb{C})$

$$T^*_{\tau\neq 0}(M \times \mathbb{C}) = \{(x, t; \xi, \tau) \in T^*(M \times \mathbb{C}) | \tau \neq 0\}$$

and the morphism

$$\rho\colon T^*_{\tau\neq 0}(M \times \mathbb{C}) \to T^*M,\ (x, t; \xi, \tau) \mapsto (x; \xi/\tau).$$

We obtain the $\mathbb{C}^\hbar_X$-algebra

$$\widehat{\mathcal{W}}_X(0)\colon\ = \rho_*(\widehat{\mathcal{E}}_{T^*(M\times\mathbb{C}),\hat{t}}(0)|_{T^*_{\tau\neq 0}(M\times\mathbb{C})}) \tag{2.3}$$

where $\hbar$ acts as τ^{-1}. A section P of $\widehat{\mathcal{W}}_X(0)$ can be written in a local symplectic coordinate system $(x_1, \dots, x_n, u_1, \dots, u_n)$ as

$$P = \sum_{j\leq 0} f_j(x, u_i)\tau^j,\ f_j \in \mathcal{O}_X,\ j \in \mathbb{Z}.$$

Setting $\hbar = \tau^{-1}$, we obtain

$$P = \sum_{k\geq 0} f_k(x, u_i)\hbar^k,\ f_k \in \mathcal{O}_X,\ k \in \mathbb{N}.$$

We write $\widehat{\mathcal{W}}_X$ for the localization of $\widehat{\mathcal{W}}_X(0)$ with respect to the parameter $\hbar$. There is the following commutative diagram of morphisms of algebras.

$$\begin{array}{ccc} \widehat{\mathcal{E}}_X & \overset{\iota}{\hookrightarrow} & \widehat{\mathcal{W}}_X \\ \uparrow & & \uparrow \\ \widehat{\mathcal{E}}_X(0) & \hookrightarrow & \widehat{\mathcal{W}}_X(0) \end{array}$$

where the algebra map $\iota : \widehat{\mathcal{E}}_X \to \widehat{\mathcal{W}}_X$ is given in a local symplectic coordinate system $(x_1, \dots, x_n, u_1, \dots, u_n)$ by $x_i \mapsto x_i$, $\partial_{x_i} \mapsto \hbar^{-1} u_i$.

3 Section depending on a complex parameter

Let X be a complex manifold endowed with a DQ-algebra $\mathcal{A}_X$. We consider the DQ-algebras $\mathcal{A}_{\mathbb{C}\times X} = \mathcal{O}_{\mathbb{C}}^{\hbar} \boxtimes \mathcal{A}_X$ on $\mathbb{C} \times X$. We denote by t the coordinate on $\mathbb{C}$, by $p_2 \colon \mathbb{C} \times X \to X$ the projection on X and by p_1 the projection on $\mathbb{C}$. Note that $\mathcal{A}_{\mathbb{C}\times X}$ is a left $\mathcal{D}_{\mathbb{C}}$-modules and in particular a left $\mathcal{O}_{\mathbb{C}}$-module. Let $t \in \mathbb{C}$, denote by $\mathfrak{m}_t$ the maximal ideal of $\mathcal{O}_{\mathbb{C},t}$ and consider the morphism

$$i_t \colon X \to \mathbb{C} \times X,\ x \mapsto (t, x).$$

Then, we have an evaluation morphism

$$\begin{aligned} ev_t \colon i_t^{-1}\mathcal{A}_{\mathbb{C}\times X} &\to i_t^{-1}\mathcal{A}_{\mathbb{C}\times X}/\mathfrak{m}_t(i_t^{-1}\mathcal{A}_{\mathbb{C}\times X}) \simeq \mathcal{A}_X \\ u &\mapsto u(t). \end{aligned}$$

and

$$\begin{aligned} ev_t \colon i_t^{-1}\mathcal{A}^{loc}_{\mathbb{C}\times X} &\to i_t^{-1}\mathcal{A}^{loc}_{\mathbb{C}\times X})/\mathfrak{m}_t(i_t^{-1}\mathcal{A}^{loc}_{\mathbb{C}\times X}) \simeq \mathcal{A}^{loc}_X \\ u &\mapsto u(t). \end{aligned}$$

Notations 3.1. (i) Let $(f, f^\sharp) \colon (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a morphism of ringed spaces. As usual, we denote by f^* the functor

$$f^* \colon \mathrm{Mod}(\mathcal{R}_Y) \to \mathrm{Mod}(\mathcal{R}_X),\ \mathcal{M} \mapsto f^*\mathcal{M} := \mathcal{R}_X \underset{f^{-1}\mathcal{R}_Y}{\otimes} f^{-1}\mathcal{M}.$$

(ii) In order to keep the number of notations to a bearable level, we will write indistinctly $p_2^*\mathcal{M}$ for $\mathcal{A}_{\mathbb{C}\times X} \underset{p_2^{-1}\mathcal{A}_X}{\otimes} p_2^{-1}\mathcal{M}$ and for $\mathcal{A}^{loc}_{\mathbb{C}\times X} \underset{p_2^{-1}\mathcal{A}^{loc}_X}{\otimes} p_2^{-1}\mathcal{M}$ depending of whether $\mathcal{M}$ is considered as an $\mathcal{A}_X$-module or an $\mathcal{A}^{loc}_X$-module.

Definition 3.2. Let $\mathcal{M}$ be an $\mathcal{A}_X$-module (resp. a $\mathcal{A}^{loc}_X$-module) and set $\mathcal{N} = p_2^*\mathcal{M}$ and consider $s \in \mathcal{N}$. The module $\mathcal{N}$ is a $\mathcal{D}_{\mathbb{C}}$-module. The derivative with respect to t of a section s in $\mathcal{N}$ is the section $\partial_t s$. It is denoted s' and called the derivative of s.

Definition 3.3. Let U be an open subset of $\mathbb{C}$ and let $\mathcal{M}$ be a coherent $\mathcal{A}_X$-module (resp. $\mathcal{A}_X^{loc}$-module). Let $(s(t))_{t\in U}$ be a family of section of $\mathcal{M}$. We say that $(s(t))_{t\in U}$ depends holomorphically on t, if locally there exists a section $s \in p_2^*\mathcal{M}$ such that $ev_t(s) = s(t)$.

Proposition 3.4. *Let X be a complex manifold and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module on X and U an open subset of $\mathbb{C} \times X$, $u \in p_2^*\mathcal{F}(U)$ such that for every $t \in p_2(U)$, $u(t) = 0$. Then $u = 0$.*

Proof. This question is local. So, we can assume that we are working in the vicinity of a point $(t_0, x) \in \mathbb{C} \times X$. We identify the local ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ with a subring of the local ring $(\mathcal{O}_{\mathbb{C}\times X,(t_0,x)}, \mathfrak{m}_{(t_0,x)})$ via the morphism of locally ringed spaces induced by the projection $p_2 \colon \mathbb{C} \times X \to X$. We denote by $\mathfrak{r}_{(t_0,x)}$ the ideal of $\mathcal{O}_{\mathbb{C}\times X,(t_0,x)}$ generated by $\mathfrak{m}_x$. For every $q \in \mathbb{N}$, we have

$$(p_2^*\mathcal{F})_{(t_0,x)}/\mathfrak{r}^q_{(t_0,x)}(p_2^*\mathcal{F})_{(t_0,x)} \simeq \mathcal{O}_{\mathbb{C},t_0} \otimes \mathcal{F}_x/\mathfrak{m}_x^q\mathcal{F}_x.$$

Writing $u_{t_0}(x)$ for the image of u in $(p_2^*\mathcal{F})_{(t_0,x)}/\mathfrak{r}^q_{(t_0,x)}(p_2^*\mathcal{F})_{(t_0,x)}$ and choosing an isomorphism $\mathcal{F}_x/\mathfrak{m}_x^q\mathcal{F}_x \simeq \mathbb{C}^r$, we can identify $u_{t_0}(x)$ with a vector $(f_1, \dots, f_r)$ where the $f_i \in \mathcal{O}_{\mathbb{C},t_0}$. It follows from the assumption that there exists a neighbourhood V of t_0 such that for every $t \in V$, $f_i(t) = 0$. This implies that $u_{t_0}(x) = 0$ that is $u_{(t_0,x)} \in \mathfrak{r}^q_{(x,t_0)}(p_2^*\mathcal{F})_{(x,t_0)}$. As $\mathfrak{r}_{(x,t_0)} \subset \mathfrak{m}_{(x,t_0)}$ and $(p_2^*\mathcal{F})_{(t_0,x)}$ is a finitely generated $\mathcal{O}_{\mathbb{C}\times X,(t_0,x)}$-module, it follows from the Krull intersection lemma that $u_{(t_0,x)} = 0$. ∎

Proposition 3.5. *Let $\mathcal{M}$ be a coherent $\mathcal{A}_X$-module, set $\mathcal{N} = p_2^*\mathcal{M}$, U an open subset of $\mathbb{C} \times X$ and let $u \in \mathcal{N}(U)$ such that for every $t \in p_1(U)$, $u(t) = 0$. Then $u = 0$.*

Proof. The $\mathcal{O}_{\mathbb{C}\times X}$-modules $\hbar^n\mathcal{N}/\hbar^{n+1}\mathcal{N}$ are coherent and

$$\hbar^n\mathcal{N}/\hbar^{n+1}\mathcal{N} \simeq p_2^*(\hbar^n\mathcal{M}/\hbar^{n+1}\mathcal{M}). \tag{3.1}$$

Let u_0 be the image of u via the map $\mathcal{N} \to \mathcal{N}/\hbar\mathcal{N}$. It follows from the assumptions that for every $t \in p_1(U)$, $u_0(t) = 0$ and form the isomorphism (3.1) that $u_0 \in p_2^*(\mathcal{M}/\hbar\mathcal{M})$. Then by Proposition 3.4, $u_0 = 0$. That is $u \in \hbar\mathcal{N}$.

Let us show by recursion that $u \in \bigcap_{n\geq 0} \hbar^n\mathcal{N}$. We just proved that $u \in \hbar\mathcal{N}$. Assume that $u \in \hbar^n\mathcal{N}$ and denote by u_n the image of u via the map $\mathcal{N} \to \hbar^n\mathcal{N}/\hbar^{n+1}\mathcal{N}$. By the isomorphism (3.1) we identify u_n with a section of the coherent $\mathcal{O}_{\mathbb{C}\times X}$-module $p_2^*(\hbar^n\mathcal{M}/\hbar^{n+1}\mathcal{M})$ such that for every t, $u_n(t) = 0$. Thus by Proposition 3.4, $u_n = 0$ that is $u \in \hbar^{n+1}\mathcal{N}$. It follows that $u \in \bigcap_{n\geq 0} \hbar^n\mathcal{N}$ and $\bigcap_{n\geq 0} \hbar^n\mathcal{N} = (0)$ by [KS12, Corollary 1.2.8] which proves the claim. ∎

Corollary 3.6. *Let $\mathcal{M}$ be a coherent $\mathcal{A}_X^{loc}$-module and let $\mathcal{N} = p_2^*\mathcal{M}$. Let U an open subset of $\mathbb{C} \times X$ and $u \in \mathcal{N}(U)$ such that for every $t \in p_1(U)$, $u(t) = 0$. Then $u = 0$.*

Proof. Let $(t,x) \in \mathbb{C} \times X$. There exists an open neighbourhood $V \times U \subset \mathbb{C} \times X$ of (t,x) and finitely many $u_i \in \mathcal{M}|_U$ such that $\mathcal{M}|_U = \sum_i \mathcal{A}_X^{loc} u_i$. We consider the $\mathcal{A}_U$-module $\mathcal{M}' = \sum_i \mathcal{A}_X u_i$. It is a finitely generated $\mathcal{A}_U$-submodule of the coherent $\mathcal{A}_U^{loc}$-module $\mathcal{M}$. Thus, $\mathcal{M}'$ is coherent. Shrinking $V \times U$ if necessary and multiplying u by $\hbar^n$ with $n \in \mathbb{N}$ sufficiently big, we can assume that $\hbar^n u \in \mathcal{A}_{V\times U} \otimes_{\mathcal{A}_U} \mathcal{M}'$. The section $\hbar^n u$ satisfies the hypothesis of the Proposition 3.5. It follows that $\hbar^n u = 0$. But, the action of $\hbar$ on $\mathcal{N}$ is invertible. It follows that $u = 0$. □

Corollary 3.7. *Let $\mathcal{M}$ be a coherent $\mathcal{A}_X$-module (resp. $\mathcal{A}_X^{loc}$-module) and set $\mathcal{N} = p_2^* \mathcal{M}$. Let U an open subset of $\mathbb{C} \times X$ and $u \in \mathcal{N}(U)$ such that for every $t \in p_1(U)$, $u'(t) = 0$. Then $u \in p_2^{-1}\mathcal{M}$.*

Proof. Since $\mathcal{M}$ is coherent, locally it has a presentation

$$0 \to \mathcal{I} \to \mathcal{A}_X^m \to \mathcal{M} \to 0.$$

Since $\mathcal{A}_{\mathbb{C}\times X}$ is flat over $\mathcal{A}_X$, the module $\mathcal{N}$ has the following presentation

$$0 \to \mathcal{A}_{\mathbb{C}\times X}\mathcal{I} \to \mathcal{A}_{\mathbb{C}\times X}^m \to \mathcal{N} \to 0. \tag{3.2}$$

Let $(t_0, x_0) \in \mathbb{C} \times X$. There exists an open neighbourhood V of (t_0, x_0) and a section $s = \sum_{i=1}^n a_i e_i \in \mathcal{A}_{\mathbb{C}\times X}^m|_U$ such that its image in $\mathcal{N}$ is u.

By hypothesis $u'(t) = 0$, it follows from the Proposition 3.5 (resp. the Corollary 3.6) that $u' = 0$ which implies that we can write

$$s' = \sum_j b_j v_j$$

with $b_j \in \mathcal{A}_{\mathbb{C}\times X}$ and $v_j \in \mathcal{I}$. Let c_j be a primitive of b_j in a neighbourhood of t_0 and set $w = \sum_j c_j \otimes v_j$. Thus $(s-w)' = 0$ in $\mathcal{A}_{\mathbb{C}\times X}^m$ which implies that $s - w \in p_2^{-1}\mathcal{A}_X^m$. Finally since $s - w$ and s have the same image in $\mathcal{N}$, it follows that u does not depend on t i.e $u \in p_2^{-1}\mathcal{M}$. □

4 Holomorphic Frobenius actions

In this section, we precise certain aspects of the definition of a F-action on a DQ-algebra or on a DQ-module. This notion was introduced in [KR08, Definition 2.2 and Definition 2.4].

Let $(X, \{\cdot\,,\,\cdot\})$ be a complex Poisson manifold. We assume that it comes equipped with a torus action, $\mathbb{C}^\times \to \mathrm{Aut}(X)$, $t \mapsto \mu_t$ such that $\mu_t^*\{f, g\} = t^{-m}\{\mu_t^* f, \mu_t^* g\}$ with $m \in \mathbb{Z}^*$.

Notations 4.1.
- We denote by $\sigma\colon \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times$ the group law of $\mathbb{C}^\times$,
- $\mu\colon \mathbb{C}^\times \times X \to X$ the action of $\mathbb{C}^\times$ on X.
- $\widetilde{\mu}\colon \mathbb{C}^\times \times X \to \mathbb{C}^\times \times X$, $(t,x) \mapsto (t, \mu(t,x))$

- for $t \in \mathbb{C}^\times$, the morphism

$$i_t \colon X \to \mathbb{C}^\times \times X,\ x \mapsto (t, x).$$

- We write μ_t (resp. $\widetilde{\mu}_t$) for the composition $\mu \circ i_t$ (resp. $\widetilde{\mu} \circ i_t$).

- Consider the product of manifolds $\mathbb{C}^\times \times X$. We denote by p_i the i-th projection.

- Consider the product of manifolds $\mathbb{C}^\times \times \mathbb{C}^\times \times X$. We denote by q_i the i-th projection, and by q_{ij} the (i, j)-th projection (*e.g.,* q_{13} is the projection from $\mathbb{C}^\times \times \mathbb{C}^\times \times X$ to $\mathbb{C}^\times \times X$, $(t_1, t_2, x_3) \mapsto (t_1, x_3)$).

- Recall that in all this paper, $\mathcal{A}_X$ is DQ-algebra and we write $\mathcal{A}_{\mathbb{C}^\times \times X}$ for the DQ-algebra $\mathcal{O}^\hbar_{\mathbb{C}^\times} \boxtimes \mathcal{A}_X$.

Lemma 4.2. *Let $\widetilde{\theta} : \widetilde{\mu}^{-1}\mathcal{A}_{\mathbb{C}^\times \times X} \to \mathcal{A}_{\mathbb{C}^\times \times X}$ be a morphism of sheaves of $p_1^{-1}\mathcal{O}_{\mathbb{C}^\times}$-algebras such that the adjoint morphism $\psi : \mathcal{A}_{\mathbb{C}^\times \times X} \to \widetilde{\mu}_*\mathcal{A}_{\mathbb{C}^\times \times X}$ is a continuous morphism of Fréchet $\mathbb{C}$-algberas. Then the dashed arrow in the below diagram is filled by a unique morphism $\widetilde{\lambda}$ of $q_{12}^{-1}\mathcal{O}_{\mathbb{C}^\times \times \mathbb{C}^\times}$-algebras. If $\widetilde{\theta}$ is an isomorphism then $\widetilde{\lambda}$ also.*

$$\begin{array}{ccc}
(\mathrm{id}_{\mathbb{C}^\times} \times \widetilde{\mu})^{-1}(\mathcal{O}_{\mathbb{C}^\times} \boxtimes \mathcal{A}_{\mathbb{C}^\times \times X}) & \xrightarrow{\mathrm{id} \times \widetilde{\theta}} & \mathcal{O}_{\mathbb{C}^\times} \boxtimes \mathcal{A}_{\mathbb{C}^\times \times X} \\
\big\downarrow & & \big\downarrow \\
(\mathrm{id}_{\mathbb{C}^\times} \times \widetilde{\mu})^{-1}\mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X} & \overset{\widetilde{\lambda}}{\dashrightarrow} & \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}
\end{array}$$

Proof. By adjunction, it is equivalent to show that the dashed arrow in the below diagram is filled by a unique map of $q_{12}^{-1}\mathcal{O}_{\mathbb{C}^\times \times \mathbb{C}^\times}$-algberas.

$$\begin{array}{ccc}
(\mathcal{O}_{\mathbb{C}^\times} \boxtimes \mathcal{A}_{\mathbb{C}^\times \times X}) & \xrightarrow{\mathrm{id} \times \widetilde{\theta}} & (\mathrm{id}_{\mathbb{C}^\times} \times \widetilde{\mu})_*(\mathcal{O}_{\mathbb{C}^\times} \boxtimes \mathcal{A}_{\mathbb{C}^\times \times X}) \\
\big\downarrow & & \big\downarrow \\
\mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X} & \dashrightarrow & (\mathrm{id}_{\mathbb{C}^\times} \times \widetilde{\mu})_*\mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}
\end{array}$$

Denoting by $\overset{\mathrm{p}}{\boxtimes}$ the external product of presheaves, there is a morphism

$$\mathrm{id} \overset{\mathrm{p}}{\boxtimes} \widetilde{\theta} \colon \mathcal{O}_{\mathbb{C}^\times} \overset{\mathrm{p}}{\boxtimes} \mathcal{A}_{\mathbb{C}^\times \times X} \to \mathcal{O}_{\mathbb{C}^\times} \overset{\mathrm{p}}{\boxtimes} \widetilde{\mu}_*\mathcal{A}_{\mathbb{C}^\times \times X} \tag{4.1}$$

DQ-algebras, the sheaf $\mathcal{O}_{\mathbb{C}^\times}$ as well as $\widetilde{\mu}_*\mathcal{A}_{\mathbb{C}^\times \times X}$ are sheaves of nuclear Fréchet $\mathbb{C}$-algebras. Moreover, there exists a countable basis $\mathfrak{B}$ of open set of $\mathbb{C}^\times \times \mathbb{C}^\times \times X$ of the form $U_i \times V_j$ such that $\mathcal{A}_{\mathbb{C}^\times X}|_{V_j}$ is isomorphic to a star-algebra. Evaluating the morphism (4.1), on the $U_i \times V_j \in \mathfrak{B}$, we get the continuous morphism of topological $\mathbb{C}$-algberas $\mathrm{id}_{U_i} \overset{\mathrm{p}}{\boxtimes} \widetilde{\theta}_{V_j}$ (As the spaces we consider are nuclear, the choice of a topology on the tensor products does not matter. For instance, we endow all the tensor product of nuclear spaces with the projective tensor product topology).

$$(\mathrm{id} \otimes \widetilde{\theta})_{U_i \times V_j} \colon \mathcal{O}_{\mathbb{C}^\times}(U_i) \otimes_\pi \mathcal{A}_{\mathbb{C}^\times X}(V_j) \to \mathcal{O}_{\mathbb{C}^\times}(U_i) \otimes_\pi \widetilde{\mu}_* \mathcal{A}_{\mathbb{C}^\times \times X}(V_j)$$

By definition the morphisms $\mathrm{id}_{U_i} \overset{\mathrm{p}}{\boxtimes} \widetilde{\theta}_{V_j}$ are compatible with restrictions and applying the completion functor to the above morphisms, we obtain the following diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{C}^\times}(U_i) \otimes_\pi \mathcal{A}_{\mathbb{C}^\times X}(V_j) & \xrightarrow{\mathrm{id} \otimes \widetilde{\theta}} & \mathcal{O}_{\mathbb{C}^\times}(U_i) \otimes_\pi \widetilde{\mu}_* \mathcal{A}_{\mathbb{C}^\times \times X}(V_j) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{C}^\times}(U_i) \widehat{\otimes}_\pi \mathcal{A}_{\mathbb{C}^\times \times X}(V_j) & \xrightarrow{\mathrm{id} \widehat{\otimes} \widetilde{\theta}} & \mathcal{O}_{\mathbb{C}^\times}(U_i) \widehat{\otimes}_\pi \widetilde{\mu}_* \mathcal{A}_{\mathbb{C}^\times \times X}(V_j). \end{array}$$

We have obtained a family of morphisms of Fréchet algebras $\{\mathrm{id} \widehat{\otimes} \widetilde{\theta}\}_{U_i \times V_j \in \mathfrak{B}}$.

We describe the completion of the topological vector spaces

$$\mathcal{O}_{\mathbb{C}^\times}(U_i) \otimes_\pi \mathcal{A}_{\mathbb{C}^\times X}(V_j) \qquad \mathcal{O}_{\mathbb{C}^\times}(U_i) \otimes_\pi \widetilde{\mu}_* \mathcal{A}_{\mathbb{C}^\times \times X}(V_j).$$

Observe that, on V_j, there is an isomorphism of Fréchet algebra

$$\mathcal{A}_{\mathbb{C}^\times \times X}(V_j) \simeq \mathcal{O}^{\hbar}_{\mathbb{C}^\times \times X}(V_j).$$

Hence, we obtain a continuous inclusion with dense image

$$\mathcal{O}_{\mathbb{C}^\times}(U_i) \otimes_\pi \mathcal{O}^{\hbar}_{\mathbb{C}^\times X}(V_j) \hookrightarrow \prod \mathcal{O}_{\mathbb{C}^\times}(U_i) \otimes_\pi \mathcal{O}_{\mathbb{C}^\times \times X}(V_j).$$

Applying the completion functor and using the fact that it commutes with products, we obtain the following isomorphisms algebras

$$\mathcal{O}_{\mathbb{C}^\times}(U_i) \widehat{\otimes}_\pi \mathcal{O}^{\hbar}_{\mathbb{C}^\times X}(V_j) \xrightarrow{\sim} \prod \mathcal{O}_{\mathbb{C}^\times}(U_i) \widehat{\otimes}_\pi \mathcal{O}_{\mathbb{C}^\times \times X}(V_j) \simeq \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}(U_i \times V_j).$$

Similarly, we have that

$$\mathcal{O}_{\mathbb{C}^\times}(U_i) \widehat{\otimes}_\pi \widetilde{\mu}_* \mathcal{A}_{\mathbb{C}^\times \times X}(V_j) \simeq (\mathrm{id}_{\mathbb{C}^\times} \times \widetilde{\mu})_* \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}(U_i \times V_j).$$

Hence, we have obtained a family of morphism of $\mathbb{C}$-algebras $\{\mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}(U_i \times V_j) \to (\mathrm{id} \times \widetilde{\mu})_* \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}(U_i \times V_j)\}_{U_i \times V_j \in \mathfrak{B}}$ which extends to a unique morphism of sheaves on $\mathbb{C}^\times \times \mathbb{C}^\times \times X$, $\widetilde{\lambda} \colon \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X} \to (\mathrm{id} \times \widetilde{\mu})_* \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}$. □

By [KS12, Lemma 2.2.9], there is a canonical morphism

$$q_{23}^{\sharp} \colon q_{23}^{-1} \mathcal{A}_{\mathbb{C}^\times \times X} \to \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}.$$

We obtain the morphism λ as the composition

$$\lambda \colon (\mathrm{id} \times \mu)^{-1} \mathcal{A}_{\mathbb{C}^\times \times X} \xrightarrow{(\mathrm{id} \times \widetilde{\mu})^{-1} q_{23}^{\sharp}} (\mathrm{id} \times \widetilde{\mu})^{-1} \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X} \xrightarrow{\widetilde{\lambda}} \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}. \tag{4.2}$$

We introduce the functor

$$\begin{aligned}\mathrm{Ev_t}\colon\ &\mathrm{Mod}(p_1^{-1}\mathcal{O}_{\mathbb{C}^\times}) \to \mathrm{Mod}(\mathbb{C}_X)\\ &\mathcal{M} \mapsto a_X^{-1}(\mathcal{O}_{\mathbb{C}^\times,t}/\mathfrak{m}_t) \underset{a_X^{-1}\mathcal{O}_{\mathbb{C}^\times,t}}{\otimes} i_t^{-1}\mathcal{M} \simeq i_t^{-1}\mathcal{M}/a_X^{-1}\mathfrak{m}_t\, i_t^{-1}\mathcal{M}.\end{aligned}$$

In particular, $\mathrm{Ev}_t(\mathcal{A}_{\mathbb{C}^\times \times X}) \simeq \mathcal{A}_X$ and $\mathrm{Ev}_t(\widetilde{\mu}^{-1}\mathcal{A}_{\mathbb{C}^\times \times X}) \simeq \mu_t^{-1}\mathcal{A}_X$.

The following definition should be compared with [KR08, Defintion 2.2] and with [BLPB12, p.15].

Definition 4.3. A F-action with exponent m on $\mathcal{A}_X$ is the data of an isomorphism of $p_1^{-1}\mathcal{O}_{\mathbb{C}^\times}$-algebras $\widetilde{\theta}\colon \widetilde{\mu}^{-1}\mathcal{A}_{\mathbb{C}^\times \times X} \to \mathcal{A}_{\mathbb{C}^\times \times X}$ such that

(a) the morphism $\theta_t := \mathrm{Ev_t}(\widetilde{\theta})$ satisifies $\theta_1 = \mathrm{id}$,

(b) for every $t \in \mathbb{C}^\times$, $\theta_t(\hbar^n) = t^{mn}\hbar^n$,

(c) the adjoint morphism of $\widetilde{\theta}$, $\widetilde{\psi} : \mathcal{A}_{\mathbb{C}^\times \times X} \to \widetilde{\mu}_* \mathcal{A}_{\mathbb{C}^\times \times X}$ is a continuous morphism of Fréchet $\mathbb{C}$-algberas,

(d) setting

$$\theta\colon \mu^{-1}\mathcal{A}_X \xrightarrow{\widetilde{\mu}^{-1}p_2^\sharp} \widetilde{\mu}^{-1}\mathcal{A}_{\mathbb{C}^\times \times X} \xrightarrow{\widetilde{\theta}} \mathcal{A}_{\mathbb{C}^\times \times X}$$

the below diagram commutes,

$$\begin{array}{ccc}(\mathrm{id}_{\mathbb{C}^\times} \times \mu)^{-1}\mu^{-1}\mathcal{A}_X & \xrightarrow{(\mathrm{id}_{\mathbb{C}^\times} \times \mu)^{-1}\theta} & (\mathrm{id}_{\mathbb{C}^\times} \times \mu)^{-1}\mathcal{A}_{\mathbb{C}^\times \times X}\\ \Vert & & \downarrow \lambda\\ \Vert & & \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}\\ \Vert & & \uparrow\\ (\sigma \times \mathrm{id}_X)^{-1}\mu^{-1}\mathcal{A}_X & \xrightarrow{(\sigma \times \mathrm{id}_X)^{-1}\theta} & (\sigma \times \mathrm{id}_X)^{-1}\mathcal{A}_{\mathbb{C}^\times \times X}\end{array}$$

where λ is provided by Lemma (4.2).

Definition 4.4. A F-action on $\mathcal{A}_X^{loc}$ is the localization with respect to $\hbar$ of a F-action on $\mathcal{A}_X$.

Remark 4.5. It would be possible to define directly the notion of F-action on $\mathcal{A}_X^{loc}$ but the definition would be slightly more involved. Moreover, any such action would be induced by a F-action on $\mathcal{A}_X$. This justify the choice of our previous definition.

The pair

$$(i_t, ev_t)\colon (X, \mathcal{A}_X^{loc}) \to (\mathbb{C}^\times \times X, \mathcal{A}_{\mathbb{C}^\times \times X}^{loc}) \tag{4.3}$$

is a morphism of ringed spaces. The F-action on $\mathcal{A}_X$ induces another morphism of ringed spaces

$$(\mu, \theta)\colon (\mathbb{C}^\times \times X, \mathcal{A}_{\mathbb{C}^\times \times X}^{loc}) \to (X, \mathcal{A}_X^{loc}). \tag{4.4}$$

Remark 4.6. A word of caution about Morphism (4.4). This morphism is a morphism of $\mathbb{C}$-ringed spaces but not of $\mathbb{C}^{\hbar}$-ringed spaces.

The morphism

$$\lambda\colon (\mathrm{id}\times\mu)^{-1}\mathcal{A}^{loc}_{\mathbb{C}^\times\times X} \to \mathcal{A}^{loc}_{\mathbb{C}^\times\times\mathbb{C}^\times\times X} \tag{4.5}$$

provided by Lemma 4.2 and the data of the F-action θ on $\mathcal{A}^{loc}_X$ allows to define a morphism of ringed space

$$(\mathrm{id}\times\mu,\lambda)\colon (\mathbb{C}^\times\times\mathbb{C}^\times\times X, \mathcal{A}^{loc}_{\mathbb{C}^\times\times\mathbb{C}^\times\times X}) \to (\mathbb{C}^\times\times X, \mathcal{A}^{loc}_{\mathbb{C}^\times\times X}).$$

The morphism of sheaves

$$\sigma^\sharp\colon \sigma^{-1}\mathcal{O}_{\mathbb{C}^\times} \to \mathcal{O}_{\mathbb{C}^\times\times\mathbb{C}^\times}$$

induces a map

$$\alpha\colon (\sigma\times\mathrm{id}_X)^{-1}\mathcal{A}_{\mathbb{C}^\times\times X} \xrightarrow{\sigma^\sharp\widehat{\otimes}\mathrm{id}} \mathcal{A}_{\mathbb{C}^\times\times\mathbb{C}^\times\times X}$$

which provides a morphism of ringed spaces

$$(\sigma\times id_X,\alpha)\colon (\mathbb{C}^\times\times\mathbb{C}^\times\times X, \mathcal{A}_{\mathbb{C}^\times\times\mathbb{C}^\times\times X}) \to (\mathbb{C}^\times\times X, \mathcal{A}_{\mathbb{C}^\times\times X}).$$

Lemma 4.7. *The morphisms of sheaves of rings θ, λ and α are flat.*

Proof. The proof for θ and λ are similar. Hence, we only provide the proof for θ. Since $\widetilde{\theta}$ is an isomorphism it is flat and $\mu^{-1}p_2^\sharp$ is flat by [KS12, Lemma 2.3.2]. Thus, $\theta = \widetilde{\theta}\circ\mu^{-1}p_2^\sharp$ is flat.

We now prove the flatness of α. Since σ is a submersion, for every $(t_1,t_2)\in\mathbb{C}^\times\times\mathbb{C}^\times$, there exist an open neighbourhood W of (t_1,t_2) and a biholomorphism $g: U\times V\to W$ such that $p_1(z_1,z_2) = \sigma\circ g(z_1,z_2) = z_1$. Since flatness is a local question, we can restrict α to an open neighbourhood of the form $W\times W'$ with W' an open subset of X. Hence, we obtain the following commutative diagram

$$\begin{array}{ccc}
(p_1\times\mathrm{id}_{W'})^{-1}\mathcal{A}_{U\times X} & \xrightarrow{(g\times\mathrm{id}_{W'})^{-1}\alpha|_{W\times W'}} & \mathcal{A}_{U\times V\times W'} \\
\downarrow\wr & & \downarrow\wr \\
(\sigma|_W\times\mathrm{id}_{W'})^{-1}\mathcal{A}_{\mathbb{C}^\times\times X} & \xrightarrow{\alpha|_{W\times W'}} & \mathcal{A}_{\mathbb{C}^\times\times\mathbb{C}^\times\times X}|_{W\times W'}
\end{array}$$

where the top morphism $(g\times\mathrm{id}_{W'})^{-1}\alpha|_{W\times W'} = q_{13}^\sharp$ is flat by [KS12, Lemma 2.3.2]. This implies that α is flat. □

The following definition is an adpation of [KR08, Definition 2.4] along the line of [MFK94, ch.1 §3 Definition 1.6].

Definition 4.8. A F-action on a $\mathcal{A}^{loc}_X$-module $\mathcal{M}$ is the data of an isomorphism of $\mathcal{A}^{loc}_{\mathbb{C}^\times\times X}$-modules

$$\phi\colon \mu^*\mathcal{M} \xrightarrow{\sim} p_2^*\mathcal{M} \tag{4.6}$$

such that the diagram

$$\begin{array}{ccccccc}
(\mathrm{id}_{\mathbb{C}^\times} \times \mu)^* \mu^* \mathcal{M} & \xrightarrow{(\mathrm{id}_{\mathbb{C}^\times} \times \mu)^* \phi} & (\mathrm{id}_{\mathbb{C}^\times} \times \mu)^* p_2^* \mathcal{M} & \xrightarrow{\sim} & q_{23}^* \mu^* \mathcal{M} & \xrightarrow{q_{23}^* \phi} & q_{23}^* p_2^* \mathcal{M} \\
\| & & & & & & \downarrow \wr \\
\| & & & & & & q_3^* \mathcal{M} \\
\| & & & & & & \uparrow \wr \\
(\sigma \times \mathrm{id}_X)^* \mu^* \mathcal{M} & & & \xrightarrow{(\sigma \times \mathrm{id}_X)^* \phi} & & & (\sigma \times \mathrm{id}_X)^* p_2^* \mathcal{M}.
\end{array} \tag{4.7}$$

commutes.

Following [KR08], we denote by $\mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc})$ the category of $(\mathcal{A}_X^{loc}, \theta)$-modules whose morphisms are the morphisms of $\mathcal{A}_X^{loc}$-modules compatible with the action of $\mathbb{C}^\times$. This category is a $\mathbb{C}$-linear abelian category. We write $\mathrm{Mod}_{\mathrm{F,coh}}(\mathcal{A}_X^{loc})$ for the full subcategory of $\mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc})$ the objects of which are coherent modules in $\mathrm{Mod}(\mathcal{A}_X^{loc})$.

Let $\mathcal{M}$ be an $\mathcal{A}_X^{loc}$-module endowed with a F-action $\phi\colon \mu^*\mathcal{M} \to p_2^*\mathcal{M}$ and $t \in \mathbb{C}^\times$. There is the following commutative diagram defining the morphism ϕ_t

$$\begin{array}{ccc}
i_t^* \mu^* \mathcal{M} & \xrightarrow{i_t^* \phi} & i_t^* p_2^* \mathcal{M} \\
\wr \downarrow & & \downarrow \wr \\
\mu_t^{-1} \mathcal{M} & \xrightarrow{\phi_t} & \mathcal{M}
\end{array}$$

where the vertical map are isomorphism of $\mathbb{C}_X$-modules. Hence, we have obtained a map of $\mathbb{C}_X$-module

$$\phi_t \colon \mu_t^{-1}\mathcal{M} \to \mathcal{M}$$

such that

(a) ϕ_t depends holomorphically of t,

(b) $\phi_{tt'} = \phi_{t'} \circ \mu_{t'}^{-1}\phi_t$ for t, $t' \in \mathbb{C}^\times$,

(c) $\phi_t(am) = \theta_t(a)\phi_t(m)$ for $a \in \mathcal{A}_X^{loc}$ and $m \in \mathcal{M}$.

Remark 4.9. (a) We will usually write $\phi_{tt'} = \phi_{t'} \circ \phi_t$ instead of $\phi_{tt'} = \phi_{t'} \circ \mu_{t'}^{-1}\phi_t$.

(b) This implies that a F-action in our sense give rise to a F-action in the sense of [KR08]. In practice, the examples of F-action in the sense of [KR08] are also F-action in our sense.

Let $\mathcal{M}$ be an $\mathcal{A}_X^{loc}$-module endowed with a F-action $\phi\colon \mu^*\mathcal{M} \to p_2^*\mathcal{M}$. The F-action provides a derivation of $\mathcal{M}$. Indeed, notice that $p_2^*\mathcal{M}$ has a structure of left $p_2^{-1}\mathcal{D}_{\mathbb{C}^\times}$-module. Hence, we have

$$\mu^*\mathcal{M} \xrightarrow{\phi} p_2^*\mathcal{M} \xrightarrow{\partial_t} p_2^*\mathcal{M} \tag{4.8}$$

Let $t_0 \in \mathbb{C}^\times$. Consider the morphism

$$i_{t_0} \colon X \to X \times \mathbb{C}^\times,\ x \mapsto (x, t_0).$$

Applying the functor $i_{t_0}^{-1}$ to the morphism (4.8), we obtain

$$\frac{d\phi_t(\cdot)}{dt}|_{t=t_0} \colon \mu_{t_0}^{-1}\mathcal{M} \longrightarrow i_{t_0}^{-1}\mu^*\mathcal{M} \xrightarrow{i_{t_0}^{-1}\phi} i_{t_0}^{-1}p_2^*\mathcal{M} \xrightarrow{i_{t_0}^{-1}\partial_t} i_{t_0}^{-1}p_2^*\mathcal{M} \xrightarrow{ev_{t_0}} \mathcal{M}.$$

In particular, when $t_0 = 1$, we get

$$v \colon \mathcal{M} \longrightarrow i_1^{-1}\mu^*\mathcal{M} \xrightarrow{i_1^{-1}\phi} i_1^{-1}p_2^*\mathcal{M} \xrightarrow{i_1^{-1}\partial_t} i_1^{-1}p_2^*\mathcal{M} \xrightarrow{ev_1} \mathcal{M}.$$

In other words,

$$\begin{aligned} v \colon \mathcal{M} &\to \mathcal{M} \\ s &\mapsto \frac{d\phi_t(s)}{dt}|_{t=1}. \end{aligned}$$

The morphism v is a $\mathbb{C}$-linear derivation of the module $\mathcal{M}$.

5 Invariant sections

5.1 Generalities

We start by defining the notion of locally invariant and invariant sections.

Definition 5.1. Let $(\mathcal{M}, \phi) \in \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc})$, $U \subset X$ and $s \in \mathcal{M}(U)$.

(i) The section s is locally invariant at x' if there exists an open neighbourhood $U' \times V \subset \mathbb{C}^\times \times X$ of $(1, x')$ such that for every $(t, x) \in V \times U'$, $\mu(t, x) \in U$ and $\phi_t(s_{\mu(t,x)}) = s_x$.

(ii) The section s is locally invariant on U, if it is locally invariant at every $x' \in U$.

(iii) Assume that $U \subset X$ is stable by the action of $\mathbb{C}^\times$. A section $s \in \mathcal{M}(U)$ is invariant if for every $t \in \mathbb{C}^\times$, $\phi_t(s) = s$.

Lemma 5.2. *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc})$, $U \subset X$ an open subset and $s \in \mathcal{M}(U)$ such that $v(s) = 0$. Then, for every $x' \in U$ there is a neighbourhood $V \times U'$ of $(1, x') \in \mathbb{C}^\times \times U$ such that for every $t' \in V$*

1. $\frac{d\phi_t(s|_{t'U'})}{dt}|_{t=t'} = 0.$
2. $\phi_{t'}(s|_{t'U'}) = s|_{U'}$

Proof. (i) Since μ is continuous, there exists a neighbourhhod $V \times U' \subset U$ of $(1, x')$ such that $\mu(V \times U') \subset U$. Hence for every $t' \in V$, we have the following equalities.

$$\begin{aligned}\frac{d\phi_t(s|_{t'U'})}{dt}|_{t=t'} &= \frac{1}{t'}\frac{d\phi_{tt'}(s|_{t'U'})}{dt}|_{t=1} \\ &= \frac{1}{t'}\phi_{t'}\left(\frac{d\phi_t(s|_{t'U'})}{dt}|_{t=1}\right) \\ &= \frac{1}{t'}\phi_{t'}(v(s|_{t'U'})) = 0.\end{aligned}$$

(ii) Shrinking V and U' if necessary Corollary 3.7 implies that locally for every $t' \in V$ $\phi_{t'}(s|_{t'U'}) = s|_{U'}$. □

Proposition 5.3. *Let $(\mathcal{M}, \phi) \in \mathrm{Mod}_{\mathrm{F,coh}}(\mathcal{A}_X^{loc})$ and U be an open subset of X stable by the action of $\mathbb{C}^\times$ and let $s \in \mathcal{M}(U)$. The following conditions are equivalent*

(1) for every $t \in \mathbb{C}^\times$, $\phi_t(s) = s$,

(2) $v(s) = 0$.

Proof. (1) $\Rightarrow$ (2) is clear.

Let us prove that (2) $\Rightarrow$ (1). It follows from Lemma 5.2 that for every $t' \in \mathbb{C}^\times$, $\frac{d\phi(s)}{dt}|_{t=t'} = 0$. Then, Corollary 3.7 implies that $\phi(s) = s$ □

Lemma 5.4. *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F,coh}}(\mathcal{A}_X^{loc})$. Locally, there exists a coherent $\mathcal{A}_X$-module $\mathcal{M}_0 \subset \mathcal{M}$ such that $\mathcal{M} \simeq \mathcal{M}_0^{loc}$ and $v(\mathcal{M}_0) \subset \mathcal{M}_0$.*

Proof. The question is local and $\mathcal{M}$ is coherent. Hence we can assume that $\mathcal{M}$ is finitely generated. Thus, there exists $s_1, \dots, s_n \in \mathcal{M}$ such that $\mathcal{M} = \sum_{i=1}^n \mathcal{A}_X^{loc} s_i$. This implies that there exists $l \in \mathbb{N}$ such for every $1 \leq i \leq n$, there exists $a_{ij} \in \hbar^{-l}\mathcal{A}_{\mathbb{C}^\times \times X}$ such that

$$\phi_t(s_i) = \sum_{j=1}^n a_{ij}(t) s_j.$$

Setting $t = e^u$, this implies that

$$v^k(s) = \sum_{i=j}^n \frac{d^k}{du^k} a_{ij}(e^u)\Big|_{u=0} s_i.$$

Setting $\mathcal{N} = \sum_{i=1}^n \mathcal{A}_X s_i$, it follows that for every $k \in \mathbb{N}, v^k(s) \in \hbar^{-l}\mathcal{N}$. We consider the submodules

$$\mathcal{M}_0 = \sum_{k,\, i} \mathcal{A}_X\, v^k(s_i) \qquad \mathcal{M}_{0,\leq p} = \sum_{\substack{1 \leq i \leq n \\ 0 \leq k \leq p}} \mathcal{A}_X\, v^k(s_i)$$

of $\hbar^{-l}\mathcal{N}$. It is clear that $\mathcal{M}_0$ is stable by v, that $\mathcal{M}_{0,\leq p}$ is a coherent $\mathcal{A}_X$-module, that for every $p \in \mathbb{N}$, $\mathcal{M}_{0,\leq p} \subset \mathcal{M}_{0,\leq p+1}$ and $\mathcal{M}_0 = \bigcup_{p \geq 0} \mathcal{M}_{0,\leq p}$. Since $\mathcal{A}_X$ is a noetherian sheaf of algebras and $\hbar^{-l}\mathcal{N}$ is a coherent $\mathcal{A}_X$-module, the sequence $(\mathcal{M}_{0,\leq p})_{p\in\mathbb{N}}$ is locally stationary. Thus, there exists a covering $(U_j)_{j\in J}$ of X such that for every $j \in J$ there exists $p_j \in \mathbb{N}$ such that $\mathcal{M}_0|_{U_j} = \mathcal{M}_{0,\leq p_j}$. Hence, $\mathcal{M}_0$ is coherent.

□

Theorem 5.5. *Assume that the action μ is free and let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F,coh}}(\mathcal{A}_X^{loc})$. Then $\mathcal{M}$ is locally finitely generated by locally invariant sections.*

Notations 5.6. We introduce the following notation. Let $\underline{\delta} = (\delta_1, \dots, \delta_n)$ (resp. $\underline{\eta} = (\eta_1, \dots, \eta_n)$) with $\delta_i > 0$ for $1 \leq i \leq n$ (resp. $\eta_i > 0$ for $1 \leq i \leq n$). We denote by $R(x, \underline{\delta}, \underline{\eta})$ the subset of $\mathbb{C}^n$ the elements of which are the $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ such that for every $1 \leq i \leq n$

$$\Re(x_i) - \delta_i < \Re(z_i) < \Re(x_i) + \delta_i \text{ and } \Im(x_i) - \eta_i < \Im(z_i) < \Im(x_i) + \eta_i.$$

We call such a subset of $\mathbb{C}^n$ a rectangle centered in x.

Proof of Theorem 5.5. By lemma 5.4, $\mathcal{M}$ has a coherent $\mathcal{A}_X$-lattice $\mathcal{M}_0$ stable by v. Let us show that this lattice is generated by locally invariant sections. This problem is local. So, it is sufficient to work in an open neighbourhood U of a point $x \in X$. Since $\mathcal{M}_0$ is coherent, we can assume it is finitely generated on U by a family $(e_i))_{1\leq i\leq l}$. For every $1 \leq i \leq l$ there exists $a_{ij} \in \mathcal{A}_U$ with $1 \leq j \leq l$ such that

$$v(e_i) = \sum_{j=i}^{l} a_{ij} e_j.$$

We form the matrix $A = (a_{ij})_{1\leq i,j\leq l} \in \mathrm{M}_l(\mathcal{A}_U)$ and set

$$e = \begin{pmatrix} e_1 \\ \vdots \\ e_l \end{pmatrix} \qquad\qquad v(e) = \begin{pmatrix} v(e_1) \\ \vdots \\ v(e_l) \end{pmatrix}.$$

It follows that $v(e) = Ae$.

If $B \in \mathrm{GL}_l(\mathcal{A}_U)$ is such that $v(Be) = 0$ then, Be will provide a generating family of $\mathcal{M}_0$, formed of locally invariant sections. We have the following equalities.

$$\begin{aligned} v(Be) &= v(B)e + Bv(e) \\ &= v(B)e + BAe. \end{aligned}$$

Therefore, we are looking for B in $\mathrm{GL}_l(\mathcal{A}_U)$ such that

$$v(B)e + BAe = 0.$$

Hence, it is sufficient to prove the existence of $B \in \mathrm{GL}_l(\mathcal{A}_U)$ such that

$$v(B) + BA = 0. \tag{5.1}$$

Shrinking U if necessary, we can further assume that $\mathcal{A}_U$ is isomorphic to a star-algebra $(\mathcal{O}_U^\hbar, \star)$ the star-product of which is given by

$$f \star g = \sum_{i \geq 0} P_i(f, g)$$

where the P_i are bidifferential operators. Writting $A = \sum_{k\geq 0} A_k \hbar^k$ and $B = \sum_{j \geq 0} B_j \hbar^j$ with A_k and B_j in $\mathrm{M}_l(\mathcal{O}_U)$, we obtain

$$\begin{aligned} BA &= (\sum_{j\geq 0} \hbar^j B_j) \star (\sum_{k \geq 0} \hbar^k A_k) \\ &= \sum_{n \geq 0} \hbar^n (\sum_{k+j+m=n} P_m(B_j, A_k)) \end{aligned}$$

and

$$\begin{aligned} v(B) &= \sum_{n\geq 0} \hbar^n \left(\sum_{i+k=n} v_k(B_i) + m\, n\, B_n \right) \\ &= \sum_{n \geq 0} \hbar^n \left(\sum_{\substack{i+k=n \\ k \neq 0}} v_k(B_i) + v_0(B_n) + m\, n\, B_n \right). \end{aligned}$$

Thus, Equation (5.1) is equivalent to the recursive system of equations

$$\forall n \in \mathbb{N},\ v_0(B_n) + B_n(mn \,\mathrm{id} + A_0) \ + \sum_{\substack{i+k=n \\ i\neq n}} v_k(B_i) + \sum_{\substack{k+j+m=n \\ j \neq n}} P_m(B_j, A_k) = 0. \tag{5.2}$$

Setting $C_0 = 0$ and $C_n = -\left(\sum_{\substack{i+k=n \\ i \neq n}} v_k(B_i) + \sum_{\substack{k+j+m=n \\ j\neq n}} P_m(B_j, A_k) \right)$, the system (5.2) is rewritten as

$$\forall n \in \mathbb{N},\ v_0(B_n) + B_n(mn \,\mathrm{id} + A_0) = C_n \tag{5.3}$$

Notice that C_n depends only of the B_j with $j < n$.

Since the action of $\mathbb{C}^\times$ on X is free, v_0 does not vanish and we can find a local coordinate system $(z_1, \ldots, z_n)$ on an open neighborhood of x (that we still denote U) such that in this coordinate system

$$v_0 \equiv \frac{\partial}{\partial z_1}.$$

Thus, the system (5.3) becomes

$$\forall n \in \mathbb{N},\ \frac{\partial B_n}{\partial z_1} + B_n(mn \,\mathrm{id} + A_0) = C_n. \tag{5.4}$$

If $(z;u)$ is the coordinnate system on T^*U associated to the coordinate system $(z) = (z_1, \dots, z_n)$ on U, it follows from [SK75] that the characteristic variety of the system (5.4) is $\{u_1 = 0\}$.

Since U is open, there exists a rectangle $R(x, \underline{\delta}, \underline{\eta}) \subset U$. Set $Y = \{z \in U | z_1 = p_1\}$ and $Y' = Y \cap R(x, \underline{\delta}, \underline{\eta})$. Consider a function $g \in \mathcal{O}_U(R(x, \underline{\delta}, \underline{\eta}))$. We have the following Cauchy problem

$$\begin{cases} \dfrac{\partial f}{\partial z_1} + f(mn\,\mathrm{id} + A_0) = g \\ f|_{Y'} = \mathrm{I}_l. \end{cases} \tag{5.5}$$

The hypersurface Y' is non-characteristic. Thus, it follows from the Cauchy-Kowaleski Theorem (see for instance [Sch85, Theorem 3.1.1]) that the equation (5.5) admits a solution f in a open neighbourhood Ω of Y'. Moreover, any hyperplan the normal of which is the limit of characteristic directions in at least one point of $R(x, \underline{\delta}, \underline{\eta})$ intersects Y' since the characteristic variety of the system (5.5) is $\{u_1 = 0\}$. Then it follows from [BS72, Theroem 2.1] that f extends to $R(x, \underline{\delta}, \underline{\eta})$. This proves that for every $n \in \mathbb{N}$, the equation (5.2) has a solution B_n with $B_n|_Y = \mathrm{I}_l$ and defined on $R(x, \underline{\delta}, \underline{\eta})$. This implies in particular that $B_0(p) = \mathrm{I}_l$ is invertible. Hence, B_0 is invertible in a neighbourhood V of x. This ensures that $B = \sum_{j \geq 0} \hbar^j B_j$ is invertible on V which proves the claim. □

5.2 The case of free and proper actions

We now assume that the action of $\mathbb{C}^\times$ on X is **free and proper**. We set $Y = X/\mathbb{C}^\times$ and denote by $p\colon X \to Y$ the canonical projection. The morphism p is a $\mathbb{C}^\times$-principal bundle. We say that an open subset V of X is equivariant if $V = p^{-1}p(V)$.

Lemma 5.7. *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc})$.*

(i) If $s \in \mathcal{M}$ is a locally invariant section, then s is locally the restriction of a globally invariant section.

(ii) If $\mathcal{M}$ is locally finitely generated by locally invariant sections, then $\mathcal{M}$ is locally finitely generated by invariant sections.

Proof. (i) Let U be an open subset of X and assume that $s \in \mathcal{M}(U)$. Let $x \in U$. Since s is locally invariant on U there exist a neigbourhoud $U' \subset U$ of x and a neighbourhood V of $1 \in \mathbb{C}^\times$ such that for every $t \in V$, $x' \in U'$, $\mu(t, x') \in U$ and $\phi_t(s) = s$.

(a) Shrinking U' and V if necessary we can assume that the pair U' and V satisfies the following property. Let x_0, $x_1 \in U'$ such that there exist $t \in \mathbb{C}^\times$ such that $x_1 = \mu(t, x_0)$ then $t \in V$.

(b) Let $y \in W = p^{-1}(p(U'))$. There exists $(t_0, x_0) \in \mathbb{C}^\times \times U'$ such that $y = \mu(t_0, x_0)$. We set

$$\tilde{s}_y = \phi_{t_0}^{-1}(s_{x_0}).$$

Let us show that the section $\tilde{s}$ is well define. Assume that there exist $(t_1, x_1) \in \mathbb{C}^\times \times U'$ such that $y = \mu(t_1, x_1)$. Then there exists $t \in V$ such that $x_1 = \mu(t, x_0)$. It follows that

$$\phi_{t_1}^{-1}(s_{x_1}) = \phi_{t_0}^{-1}(\phi_t^{-1}(s_{x_1})) = \phi_{t_0}^{-1}(s_{x_0}) = \tilde{s}_y.$$

(ii) Since $\mathcal{M}$ is locally finitely generated by locally invariant sections, there exists an open subset U of X and locally invariant sections $(s_1, \ldots, s_n)$ generating $\mathcal{M}|_U$. Keeping the notation and applying the construction of (i) to the family $(s_1, \ldots, s_n)$, we obtain some open sets U' and V satisfying the same propreties as in (i) and a family of invariant sections $(\tilde{s}_1, \ldots, \tilde{s}_n)$ defined on the open subset $W = p^{-1}(p(U'))$ of X. It remains to prove that $(\tilde{s}_1, \ldots, \tilde{s}_n)$ is a generating family of $\mathcal{M}|_W$. Since W is stable by the action, we can assume for the sake of simplicity that $X = W$. The problem is now equivalent to check that the morphism of sheaves of $\mathcal{A}_X^{loc}$-modules $u\colon (\mathcal{A}_X^{loc})^n \to \mathcal{M},\ e_i \mapsto \tilde{s}_i$ is an epimorphism. There is the following commutative diagram of $p_1^{-1}\mathcal{O}_{\mathbb{C}^\times}$-modules.

$$\begin{array}{ccc} \mu^*(\mathcal{A}_X^{loc})^n & \xrightarrow{\mu^* u} & \mu^*\mathcal{M} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\phi} \\ p_2^*(\mathcal{A}_X^{loc})^n & \xrightarrow{p_2^* u} & p_2^*\mathcal{M}. \end{array} \tag{5.6}$$

Let $x_1 \in X$. Then there exists $x_0 \in U'$ and $t \in V$ such that $x_1 = \mu(t, x_0)$. We consider the map

$$i_{(t,x_0)}\colon \{\mathrm{pt}\} \to \mathbb{C}^\times \times X,\ \mathrm{pt} \mapsto (t, x_0)$$

and the evalutation map

$$i^\sharp_{(t,x_0)}\colon i^{-1}_{(t,x_0)} p_2^{-1}\mathcal{O}_{\mathbb{C}^\times} \simeq \mathcal{O}_{\mathbb{C}^\times, t} \to \mathbb{C}.$$

These two maps allow us to define the morphism of ringed spaces

$$(i_{(t,x_0)}, i^\sharp_{(t,x_0)})\colon (\{\mathrm{pt}\}, \mathbb{C}) \to (\mathbb{C}^\times \times X, p_2^{-1}\mathcal{O}_{C^\times})$$

Applying the functor $i^*_{(t,x_0)}$ to the diagram (5.6), we obtain

$$\begin{array}{ccc} (\mathcal{A}_X^{loc})^n_{x_1} & \xrightarrow{u_{x_1}} & \mathcal{M}_{x_1} \\ {\scriptstyle\theta_{(x_0,t)}}\downarrow & & \downarrow{\scriptstyle\phi_{(x_0,t)}} \\ (\mathcal{A}_X^{loc})^n_{x_0} & \xrightarrow{u_{x_0}} & \mathcal{M}_{x_0} \end{array}$$

where the two vertical arrows are isomorphisms and the map u_{x_0} is an epimorphism. This implies that u_{x_1} is an epimorphism which proves the claim. □

Corollary 5.8. *Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F,coh}}(\mathcal{A}_X^{loc})$. Then there exists a covering $(V_i)_{i \in I}$ of X by equivariant open subsets of X such that for each $i \in I$, $\mathcal{M}|_{V_i}$ admits, in $\mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc}|_{V_i})$, a presentation of length one by free modules of finite rank.*

Proof. This follows from Theorem 5.5 and Lemma 5.7 (ii). □

6 From DQ-modules to modules over the ring of invariant sections

From now on we assume that the action of $\mathbb{C}^\times$ on X is **free and proper**.

6.1 Equivariant extension and invariant sections functors

We define the sheaf of locally invariant sections of $\mathcal{A}_X$ as the sheaf on X such that for every open set U

$$\mathcal{A}_X^{\mathbb{C}^\times}(U) = \{s \in \mathcal{A}_X(U) | v(s) = 0\}$$

and we also set

$$\mathcal{A}_X^{loc,\,\mathbb{C}^\times}(U) = \{s \in \mathcal{A}_X^{loc}(U) | v(s) = 0\}.$$

The sheaf of invariant sections of $\mathcal{A}_X$ is defined as the subsheaf $\mathcal{B}_Y(0)$ of $p_*\mathcal{A}_X$ which is given by, for every open set $V \subset Y$,

$$\mathcal{B}_Y(0)(V) = \{s \in p_*\mathcal{A}_X(V) | v(s) = 0\} = \{s \in p_*\mathcal{A}_X(V) | \theta_t(s) = s\}.$$

This is a sheaf of $\mathbb{C}$-algebra. We define $\mathcal{B}_Y$ similarly i.e.

$$\mathcal{B}_Y(V) = \{s \in p_*\mathcal{A}_X^{loc}(V) | v(s) = 0\} = \{s \in p_*\mathcal{A}_X^{loc}(V) | \theta_t(s) = s\}.$$

By definition of $\mathcal{B}_Y(0)$ and $\mathcal{B}_Y$, there are morphisms of algebras

$$p^{-1}\mathcal{B}_Y(0) \to \mathcal{A}_X^{\mathbb{C}^\times}, \tag{6.1}$$

$$p^{-1}\mathcal{B}_Y \to \mathcal{A}_X^{loc,\,\mathbb{C}^\times}. \tag{6.2}$$

Lemma 6.1. *The morphisms* (6.1) *and* (6.2) *are isomorphims of* $\mathbb{C}$*-algebras.*

Proof. This follows from Lemma 5.7. □

We define the functor of locally invariant sections as follows

$$(\cdot)^{\mathbb{C}^\times} : \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc}) \to \mathrm{Mod}(\mathcal{A}_X^{loc,\mathbb{C}^\times})$$
$$\mathcal{M} \mapsto \mathcal{M}^{\mathbb{C}^\times}$$

where $\mathcal{M}^{\mathbb{C}^\times}$ is the subsheaf of $\mathcal{M}$ such that for every open $U \subset X$

$$\mathcal{M}^{\mathbb{C}^\times}(U) = \{s \in \mathcal{M}(U) | v(s) = 0\}$$

The functor of globally invariant sections is defined by

$$p_*^{\mathbb{C}^\times} : \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc}) \to \mathrm{Mod}(\mathcal{B}_Y)$$
$$\mathcal{M} \mapsto p_*^{\mathbb{C}^\times}\mathcal{M} := p_*(\mathcal{M}^{\mathbb{C}^\times}).$$

Note that by definition of $p_*^{\mathbb{C}^\times}$, there is a natural transformation $i\colon p_*^{\mathbb{C}^\times} \to p_*$, such that for $(\mathcal{M}, \psi) \in \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc})$

$$i_{\mathcal{M}}\colon p_*^{\mathbb{C}^\times}\mathcal{M} \hookrightarrow p_*\mathcal{M}$$

is given by the inclusion.

Consider the subsheaf $\widetilde{\mathcal{M}}$ of $p_*\mathcal{M}$ defined by

$$\widetilde{\mathcal{M}}(V) := \{s \in p_*\mathcal{M}(V) \,|\, \phi_t(s) = s\}$$

By definition of $p_*^{\mathbb{C}^\times}\mathcal{M}$ and $\widetilde{\mathcal{M}}$, there is a canonical map

$$\widetilde{\mathcal{M}} \to p_*^{\mathbb{C}^\times}\mathcal{M}. \tag{6.3}$$

This is an isomorphism by Proposition 5.3. Hence, we do not make any distinction between $p_*^{\mathbb{C}^\times}\mathcal{M}$ and $\widetilde{\mathcal{M}}$ and denote both of them by $p_*^{\mathbb{C}^\times}\mathcal{M}$.

By definition of $\mathcal{B}_Y$ there is a morphism of sheaves of algebras

$$p^{-1}\mathcal{B}_Y \to \mathcal{A}_X^{loc}. \tag{6.4}$$

This allows us to define the functor

$$p^*\colon\ \mathrm{Mod}(\mathcal{B}_Y) \to \mathrm{Mod}(\mathcal{A}_X^{loc}) \qquad \mathcal{N} \mapsto p^*\mathcal{N} = \mathcal{A}_X^{loc} \otimes_{p^{-1}\mathcal{B}_Y} p^{-1}\mathcal{N}.$$

There is the following adjunction

$$p^*\colon\ \mathrm{Mod}(\mathcal{B}_Y) \rightleftarrows \mathrm{Mod}(\mathcal{A}_X^{loc})\colon p_*.$$

The module $p^*\mathcal{N}$ is canonically equipped with a F-action. Since $p \circ \mu = p \circ p_2$, there is an isomorphism of algebras $\mu^{-1}p^{-1}\mathcal{B}_Y \simeq p_2^{-1}p^{-1}\mathcal{B}_Y$. As $p^{-1}\mathcal{B}_Y \simeq \mathcal{A}_X^{loc,\mathbb{C}^\times}$, we get a monomorphism

$$p^{-1}\mathcal{B}_Y \to \mathcal{A}_X^{loc}$$

and obtain the following commutative diagram

$$\begin{array}{ccccc}
\mu^{-1}p^{-1}\mathcal{B}_Y & \longrightarrow & \mu^{-1}\mathcal{A}_X^{loc} & \xrightarrow{\theta} & \mathcal{A}_{\mathbb{C}^\times \times X}^{loc} \\
\wr\downarrow & & & & \| \\
p_2^{-1}p^{-1}\mathcal{B}_Y & \longrightarrow & p_2^{-1}\mathcal{A}_X^{loc} & \longrightarrow & \mathcal{A}_{\mathbb{C}^\times \times X}^{loc}.
\end{array}$$

This implies that $\mu^*p^* \simeq p_2^*p^*$. Hence, there is an isomorphism $\phi\colon \mu^*p^*\mathcal{N} \xrightarrow{\sim} p_2^*p^*\mathcal{N}$. That is $(\mathcal{N}, \phi) \in \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc})$. We obtain a functor $p_{\mathbb{C}^\times}^*$

$$p_{\mathbb{C}^\times}^*\colon\ \mathrm{Mod}(\mathcal{B}_Y) \to \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc}), \quad \mathcal{N} \mapsto p_{\mathbb{C}^\times}^*\mathcal{N} := (p^*\mathcal{N},\, \phi).$$

Proposition 6.2. *The functors*

$$p_{\mathbb{C}^\times}^*\colon\ \mathrm{Mod}(\mathcal{B}_Y) \rightleftarrows \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc})\colon p_*^{\mathbb{C}^\times}$$

form the adjoint pair $(p_{\mathbb{C}^\times}^*, p_*^{\mathbb{C}^\times})$.

Proof. (i) Let $\mathcal{N} \in \mathrm{Mod}(\mathcal{B}_Y)$. We start by constructing the unit of the adjunction $(p^*_{\mathbb{C}^\times}, p_*^{\mathbb{C}^\times})$. Consider the unit $\eta' \colon \mathrm{id} \to p_* p^*$ of the adjunction

$$p^* \colon \mathrm{Mod}(\mathcal{B}_Y) \rightleftarrows \mathrm{Mod}(\mathcal{A}_X^{loc}) \colon p_*.$$

The stalk of the map $\eta'_{\mathcal{N}}$ in $y \in Y$ is given by

$$\mathcal{N}_y \to \varinjlim_{y \in V} \mathcal{A}_X^{loc}(p{-}1(V)) \underset{\mathcal{B}_{Y,y}}{\otimes} \mathcal{N}_y,\ n \mapsto 1 \otimes n.$$

Sections of the form $1 \otimes n$ are invariant section of $p^*_{\mathbb{C}^\times}\mathcal{N}$. Thus, the preceding map factorizes through $p_*^{\mathbb{C}^\times} p^*_{\mathbb{C}^\times}\mathcal{N}$ and we obtain the following commutative diagram.

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\eta'_{\mathcal{N}}} & p_* p^* \mathcal{N} \\ & {\scriptstyle \eta_{\mathcal{N}}} \searrow \quad \nearrow {\scriptstyle i_{p^*_{\mathbb{C}^\times}\mathcal{N}}} & \\ & p_*^{\mathbb{C}^\times} p^*_{\mathbb{C}^\times}\mathcal{N} & \end{array} \tag{6.5}$$

We have obtained a natural transformation $\eta : \mathrm{id} \to p_*^{\mathbb{C}^\times} p^*_{\mathbb{C}^\times}$.

Let $(\mathcal{M}, \psi) \in \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc})$. Let $\varepsilon' \colon p^* p_* \to \mathrm{id}$ be the counit of the adjunction

$$p^* \colon \mathrm{Mod}(\mathcal{B}_Y) \rightleftarrows \mathrm{Mod}(\mathcal{A}_X^{loc}) \colon p_*.$$

We define the following natural transfromation

$$\varepsilon = \varepsilon' \circ i. \tag{6.6}$$

By construction the folowing diagram commutes.

$$\begin{array}{ccc} p^* p_* \mathcal{M} & \xrightarrow{\varepsilon'_{\mathcal{M}}} & \mathcal{M} \\ & {\scriptstyle p^* i_{\mathcal{M}}} \nwarrow \quad \nearrow {\scriptstyle \varepsilon_{\mathcal{M}}} & \\ & p^*_{\mathbb{C}^\times} p_*^{\mathbb{C}^\times} \mathcal{M} & \end{array} \tag{6.7}$$

Let x in X. The stalk at x of Morphism (6.6) is given by

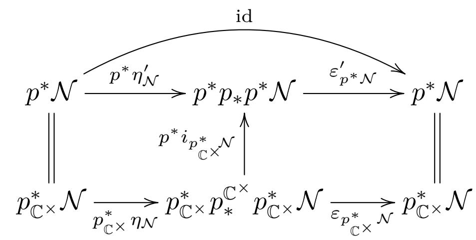
$$\mathcal{A}^{loc}_{X,x} \underset{\mathcal{B}_{Y,p(x)}}{\otimes} (p_*^{\mathbb{C}^\times}\mathcal{M})_{p(x)} \to \mathcal{M}_x,\ a \otimes m \mapsto am$$

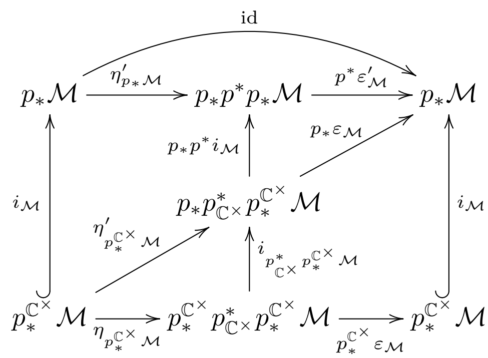
This implies that the following diagram

$$\begin{array}{ccc} \mu^* p^* p_*^{\mathbb{C}^\times} \mathcal{M} & \xrightarrow{\mu^* \varepsilon_{\mathcal{M}}} & \mu^* \mathcal{M} \\ {\scriptstyle \phi} \downarrow & & \downarrow {\scriptstyle \psi} \\ p_2^* p^* p_*^{\mathbb{C}^\times} \mathcal{M} & \xrightarrow{p_2^* \varepsilon_{\mathcal{M}}} & p_2^* \mathcal{M} \end{array}$$

where ϕ is provided by the functor $p^*_{\mathbb{C}^\times}$ is commutative. Hence, Morphism (6.6) is equivariant i.e. ε is a morphism of $\mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{loc})$.

In view of diagrams (6.5) and (6.7), the following diagrams are commutative





which proves that $(p^*_{\mathbb{C}^\times}, p^{\mathbb{C}^\times}_*)$ is an adjoint pair.

□

6.2 Coherence of the sheaf of invariant sections

In this subsection, we prove the coherence of $\mathcal{B}_Y$. The following result is elementary.

Lemma 6.3. *Let $x \in X$. Let v_0 be an holomorphic derivation of $\mathcal{O}_X$ that does not vanishes at x. Then there exist a neighbourhood U of x such that for every $c \in \mathbb{C}$ the map*

$$v_0 + c\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U), \quad f \mapsto v_0(f) + cf$$

is surjective.

Lemma 6.4. *Let $s \in \mathcal{A}_X(X)$ and assume that there is a covering $(U_i)_{i\in I}$ of X and $u_i \in \mathcal{A}_X(U_i)$ such that $s|_{U_i} = \hbar u_i$. Then, there exist $u \in \mathcal{A}_X(X)$ such that $s = \hbar u$*

Proof. On $U_i \cap U_j$, $\hbar u_i|_{U_{ij}} = \hbar u_j|_{U_{ij}}$. As $\hbar$ is not a zero divisor, $u_i|_{U_{ij}} = u_j|_{U_{ij}}$. □

Lemma 6.5. *Let $x \in X$. There exists an equivariant neighbourhood V of x and a section $s \in \mathcal{A}_X(V)$ such that $v(s) = 0$ and $s = \hbar u$ with $u \in \mathcal{A}_X(V)$ an invertible section.*

Proof. Let $x \in X$ and $m \in \mathbb{Z}$. Since v_0 does not vanish in x, there exists an open neighbourhood U of x and an hypersurface Y in U which is non-characterisitic for $v_0 + m$ on U. Hence the following Cauchy problem

$$\begin{cases} v_0(f) + mf = 0 \\ f|_Y = 1 \end{cases}$$

has a solution f_0 in an open neighbourhood U' of x. Shrinking U' if necessary we can further assume that f_0 is invertible on it.

We now show that there exists $u = \sum_{i \geq 0} \hbar^i u_i \in \mathcal{A}_{X,x}$ such that

$$u_0 = f_0 \quad \text{and} \quad v(\hbar u) = 0.$$

It follows from equation (2.1) that this is equivalent to show that there exist $u = \sum_{i \geq 0} \hbar^i u_i \in \mathcal{A}_{X,x}$ such that for every $n \in \mathbb{N}$

$$\sum_{i+k=n} v_k(u_i) + m\,(n+1)\,u_n = 0.$$

Thus, it remains to show that the recursive system

$$\begin{cases} v_0(u_n) + m\,(n+1)\,u_n = \sum_{\substack{i+k=n \\ k \neq 0}} v_k(u_i) \\ u_0 = f_0 \end{cases}$$

has a solution u in an open neighbourhood V' of x. This follows from Lemma 6.3. Finally using Lemma 5.7, we extend the section $\hbar u$ to an invariant section s defined on an equivariant open set V.

Let us check that there exists an invertible section $\widetilde{u} \in \mathcal{A}_X(V)$ such that $s = \hbar\widetilde{u}$. Let $y \in V$. By construction (see the proof of Lemma 6.5), $s_y = \phi_{t_0}^{-1}(\hbar u_{x_0}) = \hbar\, t_0^{-m} \phi_{t_0}^{-1}(u_{x_0})$ and $t_0^{-m} \phi_{t_0}^{-1}(u_{x_0})$ is an invertible section. Finally, Lemma 6.4 implies the existence of $\widetilde{u}$. □

Assume that there exist a section $s \in \mathcal{A}_X(X)$ such that $v(s) = 0$ and $s = \hbar u$ with u invertible. Consider the exact sequence

$$0 \to s^n \mathcal{A}_X \to \mathcal{A}_X \xrightarrow{\pi} \mathcal{A}_X / s^n \mathcal{A}_X \to 0. \tag{6.8}$$

and apply the functor $(\cdot)^{\mathbb{C}^\times}$ to it. We obtain the left exact sequence

$$0 \to (s^n \mathcal{A}_X)^{\mathbb{C}^\times} \to \mathcal{A}_X^{\mathbb{C}^\times} \to (\mathcal{A}_X / s^n \mathcal{A}_X)^{\mathbb{C}^\times}. \tag{6.9}$$

Lemma 6.6. *The left exact sequence* (6.9) *is exact.*

Proof. The proof is similar to the one of Lemma 6.5. □

Lemma 6.7. *Assume that there exist a section $s \in \mathcal{A}_X(X)$ such that $v(s) = 0$ and $s = \hbar u$ with u invertible. Then*

$$p_*^{\mathbb{C}^\times}(\mathcal{A}_X/s^n\mathcal{A}_X) \simeq \mathcal{B}_Y(0)/s^n\mathcal{B}_Y(0).$$

Proof. Notice that $p_* s^n \mathcal{A}_X \simeq s^n p_* \mathcal{A}_X$. Hence, $p_*^{\mathbb{C}^\times}(s^n\mathcal{A}_X) \simeq s^n\mathcal{B}_Y(0)$.

Applying $p_*^{\mathbb{C}^\times}$ to the sequence (6.8), we obtain the following left exact sequence

$$0 \to s^n\mathcal{B}_Y(0) \to \mathcal{B}_Y(0) \xrightarrow{p_*^{\mathbb{C}^\times}\pi} p_*^{\mathbb{C}^\times}(\mathcal{A}_X/s^n\mathcal{A}_X). \tag{6.10}$$

Let us show that the above sequence is exact. Let $y \in Y$, $a \in p_*^{\mathbb{C}^\times}(\mathcal{A}_X/s^n\mathcal{A}_X)_y$. Thus, there exists an open subset $V \subset Y$ such that $y \in V$, $a \in \mathcal{A}_X/s^n\mathcal{A}_X(U)$ where $U = p^{-1}(V)$ and a is an invariant section. Choose $x \in U$ such that $y = p(x)$. Since the sequence (6.9) is exact, there exist an open set $W \subset U$ containing x and a locally invariant section $u \in \mathcal{A}_X(W)$ such that $\pi(u) = a|_W$. Hence, there exists an equivariant open subset U' and an invariant section $b \in \mathcal{A}_X(U')$ such that on a neighbourhood $W' \subset W \cap U'$ of x, $b|_{W'} = u|_{W'}$. Moreover, shrinking U' if necessary, we can assume that the orbit of W' under the action of $\mathbb{C}^\times$ is U'. As a and b are invariant sections, it follows that $\pi(b) = a|_{U'}$. This proves that $p^*_{\mathbb{C}^\times}\pi$ is an epimorphism of sheaves. □

The following theorem is a minor variation of [KS12, Theorem 1.2.5 (i)] and the proof is the same.

Theorem 6.8. *Let k be a field of characteristic zero, $\mathcal{R}$ a sheaf of k-algebras and s a section of $\mathcal{R}$ such that*

(i) $s\mathcal{R} = \mathcal{R}s$,

(ii) $\mathcal{R} \xrightarrow{\cdot s} \mathcal{R}$ *is a monomorphism,*

(iii) $\mathcal{R} \simeq \varprojlim_{n\in\mathbb{N}} \mathcal{R}/\mathcal{R}s^n$

(iv) $\mathcal{R}/\mathcal{R}s$ *is a left Noetherian ring.*

Then $\mathcal{R}$ is a left Noetherian k-algebra.

Theorem 6.9. *(i) The sheaf $\mathcal{B}_Y(0)$ is a Noetherian sheaf of $\mathbb{C}$-algebras.*

(ii) The sheaf $\mathcal{B}_Y$ is a Noetherian sheaf of $\mathbb{C}$-algebras.

(iii) The sheaf $\mathcal{A}_X^{\mathbb{C}^\times}$ is a Noetherian sheaf of $\mathbb{C}$-algebras.

(iv) The sheaf $\mathcal{A}_X^{loc,\mathbb{C}^\times}$ is a Noetherian sheaf of $\mathbb{C}$-algebras.

Proof. (i) We apply Theorem 6.8. Coherency is a local property. Hence, using Lemma 6.5, we can assume that that there exist there exist a section $s \in \mathcal{A}_X(X)$ such that $v(s) = 0$ and $s = \hbar u$ with u invertible. Then point (i) and (ii) of

Theorem 6.8 are immediately satisfied. Since $p_*^{\mathbb{C}^\times}$ is a a right adjoint it commutes with limits. Thus,

$$\mathcal{B}_Y(0) = p_*^{\mathbb{C}^\times}\mathcal{A}_X = p_*^{\mathbb{C}^\times}(\varprojlim_n \mathcal{A}_X/s^n\mathcal{A}_X) \simeq \varprojlim_n p_*^{\mathbb{C}^\times}(\mathcal{A}_X/s^n\mathcal{A}_X).$$

Moreover, Lemma 6.7 implies that $p_*^{\mathbb{C}^\times}(\mathcal{A}_X/s^n\mathcal{A}_X) \simeq \mathcal{B}_Y(0)/s^n\mathcal{B}_Y(0)$. Hence

$$\mathcal{B}_Y(0) \simeq \varprojlim_n \mathcal{B}_Y(0)/s^n\mathcal{B}_Y(0).$$

This proves that condition (iii) of Theorem 6.8 holds as well as condition (iv) since $\mathcal{B}_Y(0)/s\mathcal{B}_Y(0) \simeq \mathcal{O}_Y$. Thus, $\mathcal{B}_Y(0)$ is Noetherian sheaf of $\mathbb{C}$-algebras.

(ii) We keep the notation of (i) and consider a section $s = \hbar u$ as above. Consider the free algebra $\mathcal{B}_Y(0)\langle T\rangle$ and impose the relations

$$\forall a \in \mathcal{B}_Y(0),\ T \cdot a = \psi_u(a) \cdot T$$

where $\psi_u(a) = u^{-1}au$. We obtain the skew polynomial algebra $\mathcal{B}_Y(0)[T;\psi_u]$. It follows from (i) and [DK11, Theorem 5.1.1] that the ring $\mathcal{B}_Y(0)[T;\psi_u]$ is Noetherian. Using [Kas03, Proposition A.10], this implies that $\mathcal{B}_Y \simeq \mathcal{B}_Y(0)[T;\psi_u]/(Ts - 1)\mathcal{B}_Y(0)[T;\psi_u]$ is also Noetherian.

(iii) & (iv) We have that $\mathcal{A}_X^{\mathbb{C}^\times} \simeq p^{-1}\mathcal{B}_Y(0)$ (resp. $\mathcal{A}_X^{loc,\mathbb{C}^\times} \simeq p^{-1}\mathcal{B}_Y$). Hence the result follows from (i) (resp. (ii)) and [Kas03, Proposition A.14]. □

Proposition 6.10. *(i) The sheaf $\mathcal{A}_X$ is faithfully flat over $\mathcal{A}_X^{\mathbb{C}^\times}$,*

(ii) the sheaf $\mathcal{A}_X^{loc}$ is faithfully flat over $\mathcal{A}_X^{loc,\mathbb{C}^\times}$.

Proof. (i) The proof is similar to point (i) of the proof of [KS12, Lemma 6.1.2 (a)].

(ii) This follows from the isomorphism

$$\mathcal{A}_X^{loc} \simeq \mathcal{A}_X \otimes_{\mathcal{A}_X^{\mathbb{C}^\times}} \mathcal{A}_X^{loc,\mathbb{C}^\times}.$$

□

6.3 The equivalence of categories

The aim of this subsection is to prove the following theorem.

Theorem 6.11. *The adjoint pair $(p^*_{\mathbb{C}^\times}, p_*^{\mathbb{C}^\times})$ induces a well defined adjunction*

$$p^*_{\mathbb{C}^\times} \colon \operatorname{Mod}_{\mathrm{coh}}(\mathcal{B}_Y) \rightleftarrows \operatorname{Mod}_{\mathrm{F,\,coh}}(\mathcal{A}_X^{loc}) \colon p_*^{\mathbb{C}^\times}. \tag{6.11}$$

These functors are equivalence of categories inverse to each others.

For that purpose, we have to prove that the adjunction is well defined and that the unit and the counit of this adjunction are isomorphism.

Lemma 6.12. *Let* $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F,coh}}(\mathcal{A}_X^{loc})$*. The natural morphism* (6.6)

$$\varepsilon\colon p_{\mathbb{C}^\times}^{*} p_*^{\mathbb{C}^\times} \mathcal{M} \longrightarrow \mathcal{M}$$

is an isomorphism.

Proof. It follows from Corollary 5.8 that the morphism (6.6) is an epimorphism.

Let us prove that it is a monomorphism. Let $x \in X$ and let $m = \sum_{i=1}^{n} w_i \otimes m_i \in \mathcal{A}_{X,x}^{loc} \underset{\mathcal{A}_{X,x}^{loc,\mathbb{C}^\times}}{\otimes} p_*^{\mathbb{C}^\times}(\mathcal{M})_{p(x)}$. We can assume that $m_1, \dots, m_n$ are invariant sections of $\mathcal{M}$ defined on an equivariant open subset V of X containing x. Consider the map

$$\phi\colon (\mathcal{A}_X^{loc})^n|_V \to \mathcal{M}|_V, \quad \phi(e_j) = m_j,\ 1 \leq j \leq n$$

where $(e_i)_{1 \leq i \leq n}$ is the canonical basis of $(\mathcal{A}_X^{loc})^n$. Then we get the following left exact sequence

$$0 \to \ker\phi \to (\mathcal{A}_X^{loc})^n|_V \xrightarrow{\phi} \mathcal{M}|_V.$$

The module $\ker\phi$ belongs to $\mathrm{Mod}_{\mathrm{F,\,coh}}(\mathcal{A}_X^{loc}|_V)$. It is locally finitely generated by invariant sections. Hence there exists an equivariant open set V' containing x and invariant sections $s_1, \dots, s_q$ where $s_i = (s_{i1}, \dots, s_{in})$ with $s_{ij} \in \mathcal{A}_X^{loc,\mathbb{C}^\times}(V')$ and generating $\ker\phi|_{V'}$.

Assume that $\varepsilon(m) = \sum_{j=1}^{n} w_j m_j = 0$. Thus, $w = (w_1, \dots, w_n) \in \ker\phi_x$ and we have

$$w = \sum_{i=1}^{q} \alpha_i s_i$$

with $\alpha_i \in \mathcal{A}_{X,x}^{loc}$. Then

$$\begin{aligned}\sum_{j=1}^{n} w_j \otimes m_j &= \sum_{j=1}^{n} (\sum_{i=1}^{q} \alpha_i s_{ij}) \otimes m_j \\ &= \sum_{i=1}^{q} (\alpha_i \otimes (\underbrace{\sum_{j=1}^{n} s_{ij} m_j}_{=0})) = 0.\end{aligned}$$

This proves that the map (6.6) is a monomorphism. □

Proposition 6.13. *The functor* $p_*^{\mathbb{C}^\times}$ *restricted to* $\mathrm{Mod}_{\mathrm{F,\,coh}}(\mathcal{A}_X^{loc})$ *is exact.*

Proof. As $p_*^{\mathbb{C}^\times}$ is a right adjoint it is left exact. Let us show it is right exact on $\mathrm{Mod}_{\mathrm{F,\,coh}}(\mathcal{A}_X^{loc})$. Consider the exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0.$$

and apply the functor $p_{\mathbb{C}^\times}^* p_*^{\mathbb{C}^\times}$ to the above exact short exact sequence. Since $p_{\mathbb{C}^\times}^* p_*^{\mathbb{C}^\times}$ is isomorphic to the identity functor on $\mathrm{Mod}_{\mathrm{F,\,coh}}(\mathcal{A}_X^{loc})$, we obtain the short exact sequence

$$0 \to p_{\mathbb{C}^\times}^* p_*^{\mathbb{C}^\times} \mathcal{M}' \to p_{\mathbb{C}^\times}^* p_*^{\mathbb{C}^\times} \mathcal{M} \to p_{\mathbb{C}^\times}^* p_*^{\mathbb{C}^\times} \mathcal{M}'' \to 0.$$

The ring $\mathcal{A}_X^{loc}$ is faithfully flat over $\mathcal{A}_X^{loc,\mathbb{C}^\times}$, this implies that the sequence

$$0 \to p^{-1} p_*^{\mathbb{C}^\times} \mathcal{M}' \to p^{-1} p_*^{\mathbb{C}^\times} \mathcal{M} \to p^{-1} p_*^{\mathbb{C}^\times} \mathcal{M}'' \to 0 \tag{6.12}$$

is exact. Moreover, $p\colon X \to Y$ is surjective. Hence taking the stalks of the short exact sequence (6.12) in every $x \in X$, we find that for every $y \in Y$ the sequence

$$0 \to (p_*^{\mathbb{C}^\times} \mathcal{M}')_y \to (p_*^{\mathbb{C}^\times} \mathcal{M})_y \to (p_*^{\mathbb{C}^\times} \mathcal{M}'')_y \to 0$$

is exact. This proves that $p_*^{\mathbb{C}^\times}$ is exact. □

Lemma 6.14. *The functors $p_{\mathbb{C}^\times}^*$ and $p_*^{\mathbb{C}^\times}$ preserve coherent modules.*

Proof. (i) Since $\mathcal{A}_X^{loc}$ is coherent, a $\mathcal{A}_X^{loc}$-modules is coherent if and only if it admits a presentation of length one by finitely generated free modules. This implies that $p_{\mathbb{C}^\times}^*$ preserves coherent modules.
(ii) Let $\mathcal{M}$ be a coherent $\mathcal{A}_X^{loc}$-module endowed with a F-action. It follows from Corollary 5.8 that there exists an equivariant open subset V of X such that $\mathcal{M}|_V$ has a presentation of length one by free modules of finite rank in $\mathrm{Mod}_{\mathrm{F,\,coh}}(\mathcal{A}_X^{loc})$, i.e

$$(\mathcal{A}_X^{loc}|_V)^n \to (\mathcal{A}_X^{loc}|_V)^m \to \mathcal{M}|_V \to 0.$$

Applying the exact functor $p_*^{\mathbb{C}^\times}$ to the above sequence, we obtain the right exact sequence

$$(\mathcal{B}_Y|_{p(V)})^n \to (\mathcal{B}_Y|_{p(V)})^m \to (p_*^{\mathbb{C}^\times} \mathcal{M})|_{p(V)} \to 0.$$

As $\mathcal{B}_Y$ is coherent, this implies that $p_*^{\mathbb{C}^\times} \mathcal{M}$ is a coherent $\mathcal{B}_Y$-module. □

We are now ready to prove the main result of this section.

Proof of Theorem 6.11. It follows from Proposition 6.2 and Lemma 6.14 that the adjunction (6.11) is well defined. Because of Lemma 6.12, it only remains to show that for every $\mathcal{N} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{B}_Y)$

$$\eta_{\mathcal{N}} \colon \mathcal{N} \to p_*^{\mathbb{C}^\times} p_{\mathbb{C}^\times}^* \mathcal{N} \tag{6.13}$$

is an isomorphism. The $\mathcal{B}_Y$-module $\mathcal{N}$ is coherent. Hence, there is a covering $(V_i)_{i \in I}$ of Y such that for each $i \in I$, $\mathcal{N}|_{V_i}$ has a free presentations of length one. It follows that the morphism (6.13) is an isomorphism since $p_{\mathbb{C}^\times}^*$ and $p_*^{\mathbb{C}^\times}$ are right exact functors and $\eta_{\mathcal{B}_Y}$ is an isomorphism. □

6.4 An example: the case of $\widehat{\mathcal{W}}_X$

In this section, we sketch the construction of the canonical weight one F-action on $\widehat{\mathcal{W}}_X$. Let M be a complex manifold and we set $X = T^*M$. We denote by $\widehat{\mathcal{W}}_X(0)$, the standard quantization of the cotangent bundle and $\widehat{\mathcal{W}}_X := \mathbb{C}^{\hbar,loc} \otimes_{\mathbb{C}^\hbar} \mathcal{W}_X(0)$ (see for instance [KS12, p.133]). We write $\widehat{\mathcal{E}}_X$ for the ring of formal microdifferentials operators on X, $\widehat{\mathcal{E}}_X(0)$ for the subsheaf of order zero microdifferential operators and set $\mathcal{A}_{\mathbb{C}^\times \times X} := \mathcal{O}^\hbar_{\mathbb{C}^\times} \boxtimes \mathcal{W}_X(0)$.

We endow $\mathbb{C}$ with a coordinate t, $T^*\mathbb{C}$ with the coordinate system (t,τ). Similarly, we equipped $\mathbb{C}^\times$ with the coordinate r and $T^*\mathbb{C}^\times$ with the coordinate system (r,λ). We consider the map

$$\gamma\colon T^*(\mathbb{C}^\times \times \mathbb{C}) \to T^*(\mathbb{C}^\times \times \mathbb{C}),\ (r,t;\lambda,\tau) \mapsto (r,t;\lambda,\tau/r).$$

and the isomorphism of sheaves

$$\gamma^\sharp\colon \left(\widehat{\mathcal{E}}_{T^*_{\tau\neq 0}(\mathbb{C}^\times\times\mathbb{C}),\,\hat{t},\hat{\partial}_r}(0)\right) \xrightarrow{\sim} \gamma_*\left(\widehat{\mathcal{E}}_{T^*_{\tau\neq 0}(\mathbb{C}^\times\times\mathbb{C}),\,\hat{t},\hat{\partial}_r}(0)\right),\ g(r;\tau) \mapsto g(r;\tau/r) \tag{6.14}$$

where

$$\widehat{\mathcal{E}}_{T^*_{\tau\neq 0}(\mathbb{C}^\times\times\mathbb{C}),\,\hat{t},\hat{\partial}_r}(0) = \{P \in \widehat{\mathcal{E}}_{T^*_{\tau\neq 0}\mathbb{C}^\times\times\mathbb{C}}(0) |\, [P,r]=0 \text{ and } [P,\partial_t]=0\}.$$

Remark 6.15. Here, we have used the fact that there are global coordinate systems on $T^*\mathbb{C}$ and $T^*(\mathbb{C}^\times \times \mathbb{C})$ which allows us to identify formal microdifferential operators with their total symbols.

Applying $(\cdot) \boxtimes \widehat{\mathcal{E}}_X(0)$ to the morphism (6.14), we obtain the isomorphism

$$\gamma^\sharp \boxtimes \mathrm{id}\colon \widehat{\mathcal{E}}_{T^*_{\tau\neq 0}(\mathbb{C}^\times\times\mathbb{C}),\,\hat{t},\hat{\partial}_r}(0) \boxtimes \widehat{\mathcal{E}}_X(0) \xrightarrow{\sim} (\gamma \times \mathrm{id})_* \widehat{\mathcal{E}}_{T^*_{\tau\neq 0}(\mathbb{C}^\times\times\mathbb{C}),\,\hat{t},\hat{\partial}_r}(0) \boxtimes \widehat{\mathcal{E}}_X(0).$$

Since $\widehat{\mathcal{E}}_{T^*_{\tau\neq 0}(\mathbb{C}^\times\times\mathbb{C}),\,\hat{t},\hat{\partial}_r}(0)$ and $\widehat{\mathcal{E}}_X(0)$ are sheaves of nuclear Fréchet algebras and $\gamma^\sharp$ is continuous, we get, by completing the tensor products, an isomorphism

$$\tilde{\gamma}^\sharp\colon \widehat{\mathcal{E}}_{T^*_{\tau\neq 0}(\mathbb{C}^\times\times\mathbb{C})\times X,\,\hat{t},\hat{\partial}_r}(0) \xrightarrow{\sim} (\gamma \times \mathrm{id})_* \widehat{\mathcal{E}}_{T^*_{\tau\neq 0}(\mathbb{C}^\times\times\mathbb{C})\times X,\,\hat{t},\hat{\partial}_r}(0). \tag{6.15}$$

Consider the morphism

$$\widetilde{\rho}\colon T^*_{\tau\neq 0}(\mathbb{C}^\times \times \mathbb{C}) \times X \to \mathbb{C}^\times \times X,\ (r,t,x;\lambda,\tau,\xi) \mapsto (r,x;\xi/\tau)$$

Applying $\widetilde{\rho}_*$ to (6.15) provides the continuous morphism

$$\widetilde{\psi}\colon \mathcal{A}_{\mathbb{C}^\times \times X}(0) \xrightarrow{\sim} \widetilde{\mu}_* \mathcal{A}_{\mathbb{C}^\times \times X}(0).$$

By adjunction, we get the F-action

$$\widetilde{\theta}\colon \widetilde{\mu}^{-1} \mathcal{A}_{\mathbb{C}^\times \times X} \xrightarrow{\sim} \mathcal{A}_{\mathbb{C}^\times \times X}.$$

Let V be an open subset of $\mathbb{C}^\times$ and $(U, x; u)$ a local symplectic coordinate system of X where U is a conical open subset of X. Then

$$\begin{aligned}\widetilde{\theta}_{V\times U}\colon \widetilde{\mu}^{-1}\widehat{\mathcal{W}}_{\mathbb{C}^\times\times X}(0)(V\times U) &\to \widehat{\mathcal{W}}_{\mathbb{C}^\times\times X}(0)(V\times U)\\ \sum_{i\geq 0} f_i(r,x;u)\hbar^i &\mapsto \sum_{i\geq 0} f_i(r,x;r\cdot u)r^i\hbar^i.\end{aligned}$$

This implies that

$$\begin{aligned}\theta_r\colon \mu_r^{-1}\widehat{\mathcal{W}}_X(0)(U) &\to \widehat{\mathcal{W}}_X(0)(U)\\ \sum_{i\geq 0} f_i(x;u)\hbar^i &\mapsto \sum_{i\geq 0} f_i(x;r\cdot u)r^i\hbar^i.\end{aligned}$$

A section $s = \sum_{i\geq 0} f_i(x;u)\hbar^i$ in $\widehat{\mathcal{W}}_X(0)(U)$ is invariant, if for every $r \in \mathbb{C}^\times$, $\theta_r(s) = s$. That is,

$$\text{for every } i \geq 0,\ f_i(x, ru) = r^{-i} f_i(x, u).$$

This implies that $s \in \iota(\widehat{\mathcal{E}}_X(0)(U))$ and in particular that $(\widehat{\mathcal{W}}_X(0))^{\mathbb{C}^\times} \simeq \widehat{\mathcal{E}}_X(0)$ and $\widehat{\mathcal{W}}_X^{\mathbb{C}^\times} \simeq \widehat{\mathcal{E}}_X$. Hence, applying Theorem 6.11, we obtain the proposition

Proposition 6.16. *The adjoint pair $(p^*_{\mathbb{C}^\times}, p_*^{\mathbb{C}^\times})$ induces a well defined adjunction*

$$p^*_{\mathbb{C}^\times}\colon \operatorname{Mod}_{\mathrm{coh}}(\widehat{\mathcal{E}}_Y) \rightleftarrows \operatorname{Mod}_{\mathrm{F,\,coh}}(\widehat{\mathcal{W}}_X)\colon p_*^{\mathbb{C}^\times}.$$

These functors are equivalence of categories inverse to each others.

7 The codimension three conjecture for formal micro-differential modules

In this subsection, we deduce the codimension-three conjecture for formal microdifferential modules from the codimension-three conjecture for DQ-modules. Let M be a complex manifold, X be an open subset of T^*M, $\dot{T}^*M = T^*M \setminus M$, P^*M the projective cotangent bundle and $p\colon \dot{T} * M \to P^*M$ be the canonical projection. We denote by d_X the dimension of X.

Let l be a non-negative integer, from now on, we set $X_l := (\mathbb{C}^\times)^l \times X$, similarly $Z_l := (\mathbb{C}^\times)^l \times Z$ and $\mathcal{A}_{X_l} := \mathcal{O}^\hbar_{(\mathbb{C}^\times)^l} \boxtimes \widehat{\mathcal{W}}_X(0)$. We will need the following proposition whose proof is similar to the one of [KK81, Theorem 1.2.2].

Proposition 7.1. *Let r and l be non-negative integers and $\mathcal{M}$ be a coherent $\mathcal{A}^{loc}_{X_l}$-module so that $\mathcal{E}xt^j_{\mathcal{A}^{loc}_{X_l}}(\mathcal{M}, \mathcal{A}^{loc}_{X_l}) = 0$ for any $j > r$. Then $\mathrm{H}^j_{Z_l}(\mathcal{M}) = 0$ for any closed analytic subset Z of X and any $j < \operatorname{codim} Z - r$.*

Lemma 7.2. *Assume X does not intersect the zero section of T^*M. Let Λ be a Lagrangian subvariety of X, Z be a closed analytic subset of Λ such that* $\operatorname{codim}_\Lambda Z \geq 2$*, $j\colon X \setminus Z \hookrightarrow X$ the inclusion and $\mathcal{M}$ a holonomic $\widehat{\mathcal{W}}_X$-module supported by Λ. Let $(f, f^\sharp)\colon (X_l, \mathcal{A}_{X_l}) \to (X_{l'}, \mathcal{A}_{X_{l'}})$ a morphism of $\mathbb{C}$-ringed space such that $f^\sharp$ is flat. Set $V = X \setminus Z$ and denote by f_V the restriction of f to V. Then*

$$(\mathrm{id}_{(\mathbb{C}^\times)^l} \times j)_* f_V^* \mathcal{M}|_V \simeq f^*\mathcal{M}.$$

Proof. Consider the following exact sequence

$$0 \to \mathrm{H}^0_{Z_l}(f^*\mathcal{M}) \to f^*\mathcal{M} \to (\mathrm{id}_{(\mathbb{C}^\times)^l} \times j)_*(\mathrm{id}_{(\mathbb{C}^\times)^l} \times j)^{-1} f^*\mathcal{M} \to \mathrm{H}^1_{Z_l}(f^*\mathcal{M}) \to 0.$$

Since $\mathcal{M}$ is holonomic and f^* is exact, it follows that for any $j > d_X/2$

$$\mathcal{E}xt^j_{\mathcal{A}^{loc}_{X_l}}(f^*\mathcal{M}, \mathcal{A}^{loc}_{X_l}) = 0.$$

Hence, by Proposition 7.1, $\mathrm{H}^j_{Z_l}(f^*\mathcal{M}) = 0$ for $0 \leq j < 2$. Then, the above exact sequence implies that

$$(\mathrm{id}_{(\mathbb{C}^\times)^l} \times j)_* f_V^* \mathcal{M}|_V \simeq (\mathrm{id}_{(\mathbb{C}^\times)^l} \times j)_*(\mathrm{id}_{(\mathbb{C}^\times)^l} \times j)^{-1} f^*\mathcal{M} \simeq f^*\mathcal{M}.$$

□

Lemma 7.3. *Assume X is a conical open subset of T^*M and does not intersect the zero section. Let Λ be a conical Lagrangian subvariety of X, let Z be a closed conical analytic subset of Λ such that* $\operatorname{codim}_\Lambda Z \geq 2$*, $j\colon X \setminus Z \hookrightarrow X$ the inclusion and $\mathcal{M}$ a holonomic $\widehat{\mathcal{W}}_X$- module supported in Λ such that $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F}}(\widehat{\mathcal{W}}_X|_{X \setminus Z})$. Then $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F}}(\widehat{\mathcal{W}}_X)$.*

Proof. Set $V = X \setminus Z$; On $\mathbb{C}^\times \times V$, the F-action is given by

$$\phi'\colon \mu_V^*\mathcal{M}|_V \to p_{2,V}^*\mathcal{M}|_V.$$

Applying $(\mathrm{id}_{\mathbb{C}^\times} \times j)_*$, we get

$$(\mathrm{id}_{\mathbb{C}^\times} \times j)_*\phi\colon (\mathrm{id}_{\mathbb{C}^\times} \times j)_*\mu_V^*\mathcal{M}|_V \to (\mathrm{id}_{\mathbb{C}^\times} \times j)_* p_{2,V}^*\mathcal{M}|V.$$

By Lemma 7.2,

$$(\mathrm{id}_{\mathbb{C}^\times} \times j)_*\mu_V^*\mathcal{M}|_V \simeq \mu^*\mathcal{M}, \qquad (\mathrm{id}_{\mathbb{C}^\times} \times j)_* p_{2,V}^*\mathcal{M}|_V \simeq p_2^*\mathcal{M}.$$

Thus, we obtain a morphism

$$\phi\colon \mu^*\mathcal{M} \to p_2^*\mathcal{M}.$$

Applying $(\mathrm{id}_{(\mathbb{C}^\times)^2} \times j)_*$ to the below diagram diagram

$$\begin{array}{ccccccc}
(\mathrm{id}_{\mathbb{C}^\times} \times \mu_V)^*\mu_V^*\mathcal{M}|_V & \xrightarrow{(\mathrm{id}_{\mathbb{C}^\times} \times \mu_V)^*\phi'} & (\mathrm{id}_{\mathbb{C}^\times} \times \mu_V)^* p_{2,V}^*\mathcal{M}|_V & \xrightarrow{\sim} & q_{12,V}^*\mu_V^*\mathcal{M}|_V & \xrightarrow{q_{12,V}^*\phi} & q_{12,V}^* p_{2,V}^*\mathcal{M}|V \\
\| & & & & & & \downarrow \wr \\
\| & & & & & & q_{3,V}^*\mathcal{M} \\
\| & & & & & & \uparrow \wr \\
(\sigma \times \mathrm{id}_V)^*\mu_V^*\mathcal{M}|_V & & \xrightarrow{\qquad (\sigma \times \mathrm{id}_V)^*\phi' \qquad} & & & & (\sigma \times \mathrm{id}_V)^* p_2^*\mathcal{M}|_V.
\end{array}$$

and using the isomorphisms provided by Lemma 7.2, we obtain a commutative diagram identical to Diagram (4.7). This shows that ϕ is a F-action. $\square$

We recall the DQ-module version of the codimension three conjecture.

Theorem 7.4 ([Pet17, Theorem 1.5])**.** *Let X be a complex manifold endowed with a DQ-algebra $\mathcal{A}_X$ such that the associated Poisson structure is symplectic. Let Λ be a closed Lagrangian analytic subset of X, Z a closed analytic subset of Λ such that* $\operatorname{codim}_\Lambda Z \geq 3$ *and $j : X \setminus Z \to X$ the open embedding. Let $\mathcal{M}$ be a holonomic $(\mathcal{A}_X^{loc}|_{X\setminus Z})$-module, whose support is contained in $\Lambda \setminus Z$. Assume that $\mathcal{M}$ has an $\mathcal{A}_X|_{X\setminus Z}$-lattice. Then $j_*\mathcal{M}$ is a holonomic module and is the unique holonomic extension of $\mathcal{M}$ to X whose support is contained in Λ.*

We now deduce the codimension three for formal microdifferential modules from the one for DQ-modules.

Theorem 7.5 ([KV14, Theorem 1.2])**.** *Let M be a complex manifold, X an open subset of T^*M, Λ a closed Lagrangian analytic subset of X, and Z a conical closed analytic subset of Λ such that* $\operatorname{codim}_\Lambda Z \geq 3$*. Let $\widehat{\mathcal{E}}_X$ the sheaf of formal microdifferential operators on X and $\mathcal{N}$ be a holonomic $(\widehat{\mathcal{E}}_X|_{X\setminus Z})$-module whose support is contained in $\Lambda \setminus Z$. Assume that $\mathcal{N}$ possesses an $(\widehat{\mathcal{E}}_X(0)|_{X\setminus Z})$-lattice $\mathcal{N}_0$. Then $\mathcal{N}$ extends uniquely to a holonomic module defined on X whose support is contained in Λ.*

Proof. (i) The unicity is proved as in [KV14]. Thus we do not repeat the proof here.
(ii) Proving the coherence of an extension of $\mathcal{N}$ is a local problem. By the dummy variable trick, we can assume that X does not intersect the zero section of T^*M. Hence, for any $x \in X$, there exists a neigbourhood V of x, a conical open subset U, of X such that $V \subset U$ and a coherent $\widehat{\mathcal{E}}_{P^*U}$-module $\mathcal{N}'$ together with a $\widehat{\mathcal{E}}_{P^*U}(0)$-lattice $\mathcal{N}_0$ such that $\mathcal{N}|_V \simeq (p^{-1}\mathcal{N}')|_V$ and $\mathcal{N}_0|_V \simeq (p^{-1}\mathcal{N}'_0)|_V$ where $p_U : U \to P^*U$ is the restriction to U of the canonical projection. We can further assume that Z and Λ are subset of U.

The module $\mathcal{M} := p^*_{\mathbb{C}^\times}\mathcal{N}'$ belongs to $\mathrm{Mod}_{\mathrm{F,coh}}(\widehat{\mathcal{W}}_U|_{U\setminus Z})$, has a $\widehat{\mathcal{W}}_U(0)|_{U\setminus Z}$-lattice and its support is contained in $\Lambda \setminus Z$. By [Pet17, Theorem 1.4], $j_*\mathcal{M}$ is an holonomic module supported by Λ and is also endowed with a F-action by Lemma 7.3. Moreover it follows from Proposition 6.16 that $p_*^{\mathbb{C}^\times} j_*\mathcal{M}$ is a coherent $\widehat{\mathcal{E}}_{P^*U}$-module such that

$$(p_*^{\mathbb{C}^\times} j_*\mathcal{M})|_{p(U)\setminus p(Z)} \simeq p_*^{\mathbb{C}^\times}((j_*\mathcal{M})|_{U\setminus Z}) \simeq \mathcal{N}'.$$

This proves that $p_*^{\mathbb{C}^\times} j_*\mathcal{M}$ is a coherent extension of $\mathcal{N}'$. This implies that $j_*\mathcal{N}$ is a coherent $\widehat{\mathcal{E}}_X$-module. $\square$

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