Trigonometry for the multiples of 3 degrees

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When you study trigonometry, and more precisely sine and cosine, you encounter square roots: for example, you have

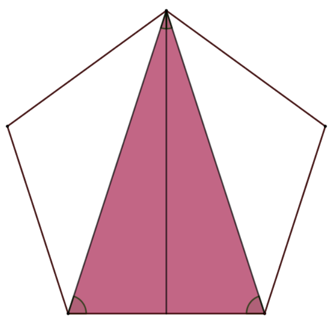
$$\sin(60°) = \frac{\sqrt{3}}{2}\,.$$

It turns out that there are pretty nice expressions for the values of sine and cosine for all angles that are integer multiples of 3°, see the table below [2]. Indeed, those values can be expressed with the help of the following square roots:

$$\sqrt{2} \quad , \quad \sqrt{3} \quad , \quad \sqrt{5} \quad , \quad \sqrt{6} \quad , \quad \sqrt{2 \pm \sqrt{3}} \quad , \quad \sqrt{5 \pm \sqrt{5}}\,.$$

In the table you only find angles between $0°$ and $45°$. The reason is that, up to a sign, you can reduce to consider the multiples of 3° that are between $0°$ and $90°$. Moreover, to deal with the complementary angle, you only need to swap sine and cosine. Also notice that from the expressions of sine and cosine we may easily deduce expressions for the tangent and cotangent, so there was no need to write those down as well.

How did we find the expressions in the table? For $0°$ the expressions are obvious. For $45°$ and for $30°$ we may easily invoke Pythagora's Theorem because the corresponding right triangles are half a square and half an equilateral triangle respectively.
For $18°$ we can make use of a regular pentagon. First of all, the isosceles triangle consisting of two diagonals and a pentagon side has apex angle $36°$.



θ	$\cos\theta$	$\sin\theta$
$0°$	1	0
$3°$	$\frac{2(\sqrt{3}+1)\sqrt{5+\sqrt{5}}+\sqrt{2}(\sqrt{3}-1)(\sqrt{5}-1)}{16}$	$\frac{\sqrt{2}(\sqrt{3}+1)(\sqrt{5}-1)-2(\sqrt{3}-1)\sqrt{5+\sqrt{5}}}{16}$
$6°$	$\frac{\sqrt{2}\sqrt{5-\sqrt{5}}+\sqrt{3}(\sqrt{5}+1)}{8}$	$\frac{\sqrt{6}\sqrt{5-\sqrt{5}}-(\sqrt{5}+1)}{8}$
$9°$	$\frac{\sqrt{2}(\sqrt{5}+1)+(\sqrt{5}-1)\sqrt{5+\sqrt{5}}}{8}$	$\frac{\sqrt{2}(\sqrt{5}+1)-(\sqrt{5}-1)\sqrt{5+\sqrt{5}}}{8}$
$12°$	$\frac{\sqrt{6}\sqrt{5+\sqrt{5}}+\sqrt{5}-1}{8}$	$\frac{\sqrt{2}\sqrt{5+\sqrt{5}}-\sqrt{3}(\sqrt{5}-1)}{8}$
$15°$	$\frac{\sqrt{2+\sqrt{3}}}{2}$	$\frac{\sqrt{2-\sqrt{3}}}{2}$
$18°$	$\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$
$21°$	$\frac{2(\sqrt{3}-1)\sqrt{5-\sqrt{5}}+\sqrt{2}(\sqrt{3}+1)(\sqrt{5}+1)}{16}$	$\frac{2(\sqrt{3}+1)\sqrt{5-\sqrt{5}}-\sqrt{2}(\sqrt{3}-1)(\sqrt{5}+1)}{16}$
$24°$	$\frac{\sqrt{5}+1+\sqrt{6}\sqrt{5-\sqrt{5}}}{8}$	$\frac{\sqrt{3}(\sqrt{5}+1)-\sqrt{2}\sqrt{5-\sqrt{5}}}{8}$
$27°$	$\frac{2\sqrt{5+\sqrt{5}}+\sqrt{2}(\sqrt{5}-1)}{8}$	$\frac{2\sqrt{5+\sqrt{5}}-\sqrt{2}(\sqrt{5}-1)}{8}$
$30°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$33°$	$\frac{2(\sqrt{3}+1)\sqrt{5+\sqrt{5}}-\sqrt{2}(\sqrt{3}-1)(\sqrt{5}-1)}{16}$	$\frac{2(\sqrt{3}-1)\sqrt{5+\sqrt{5}}+\sqrt{2}(\sqrt{3}+1)(\sqrt{5}-1)}{16}$
$36°$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}$
$39°$	$\frac{\sqrt{2}(\sqrt{3}-1)(\sqrt{5}+1)+2(\sqrt{3}+1)\sqrt{5-\sqrt{5}}}{16}$	$\frac{\sqrt{2}(\sqrt{3}+1)(\sqrt{5}+1)-2(\sqrt{3}-1)\sqrt{5-\sqrt{5}}}{16}$
$42°$	$\frac{\sqrt{2}\sqrt{5+\sqrt{5}}+\sqrt{3}(\sqrt{5}-1)}{8}$	$\frac{\sqrt{6}\sqrt{5+\sqrt{5}}-(\sqrt{5}-1)}{8}$
$45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

By halving this isosceles triangle through the apex, we find the right triangle that we need. It then suffices to recall that the ratio between diagonal and side of a regular pentagon is the golden ratio [1]

$$\varphi = \frac{1+\sqrt{5}}{2},$$

whose inverse equals $\frac{\sqrt{5}-1}{2}$. We leave the conclusion of the calculation of $\cos(18°)$ as an exercise: the value of $\sin(18°)$ then follows from the identity $\sin^2(18°)+\cos^2(18°) = 1$. All other values in the table can be deduced with the help of the fundamental identity

$\cos^2 x + \sin^2 x = 1$, and trigonometric rules such as the addition/subtraction formulas

$$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y \qquad \cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y\,,$$

the duplication formulas

$$\sin(2x) = 2\sin x \cdot \cos x \qquad \cos(2x) = 2\cos^2 x - 1\,,$$

and the half-angle formulas

$$\sin^2 \frac{x}{2} = \frac{1-\cos x}{2} \qquad \cos^2 \frac{x}{2} = \frac{1+\cos x}{2}\,.$$

You can find different (but equivalent) expressions for sine and cosine, according to which trigonometric formula you use and how you simplify an expression. Sometimes choosing the "simplest" expression is really a matter of personal preference: which of the following three expressions

$$\frac{(-1+\sqrt{5})\sqrt{5+\sqrt{5}}+\sqrt{2}+\sqrt{10}}{8}; \quad \frac{1}{2}\sqrt{2+\sqrt{\frac{5+\sqrt{5}}{2}}}; \quad \frac{\sqrt{2}(\sqrt{5}+1)+(\sqrt{5}-1)\sqrt{5+\sqrt{5}}}{8}$$

would you choose to express the cosine of $9°$?

Disclaimer: For most angles you cannot write down an exact expression for the values of sine and cosine. For example, you would keep $\cos(1°)$ in your calculations and, if needed, you can use an approximated value for it. In fact, for multiples of $1°$ which are not multiples of $3°$ there is no exact expression for the values of sine and cosine involving real radicals (it is necessary to make use of the complex numbers). To know more about such results you can have a look at [3].

Exercises

1. Compute with geometrical methods the exact values of sine and cosine for $45°$, $30°$, and $18°$.
2. Show (without doing the actual calculations) that it is possible to compute the values of sine and cosine for all integer multiples of $3°$.
3. Compute the exact values of sine and cosine for $3°$.
4. Prove that if an angle can be expressed with real radicals, then the same holds for its half. In particular this shows that there are infinitely many angles that can be expressed with real radicals.

References

[1] Fernando Corbalán. *The Golden Ratio: The Beautiful Language of Mathematics*, Everything is mathematical, RBA Coleccionables, 2012.

[2] Deborah Stranen. *Exact trigonometric table for multiples of 3 degrees.* Available on Wikimedia Commons, `https://commons.wikimedia.org/wiki/File:Exact_trigonometric_table_for_multiples_of_3%C2%B0.jpg` (public domain).

[3] Wikipedia contributors. *Trigonometric constants expressed in real radicals.* Wikipedia, The Free Encyclopedia. November 2, 2017, `https://en.wikipedia.org/w/index.php?title=Trigonometric_constants_expressed_in_real_radicals&oldid=808339981`, retrieved January 1, 2018.