Associative and Quasitrivial Operations on Finite Sets

Characterizations and Enumeration

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Part I: Single-peaked orderings

Single-peaked orderings

Motivating example (Romero, 1978)

Suppose you are asked to order the following six objects in decreasing preference:

$$\begin{aligned} a_1 &: \quad \text{0 sandwich} \\ a_2 &: \quad \text{1 sandwich} \\ a_3 &: \quad \text{2 sandwiches} \\ a_4 &: \quad \text{3 sandwiches} \\ a_5 &: \quad \text{4 sandwiches} \\ a_6 &: \quad \text{more than 4 sandwiches} \end{aligned}$$

We write $a_i \prec a_j$ if a_i is preferred to a_j

Single-peaked orderings

a_1 : 0 sandwich
a_2 : 1 sandwich
a_3 : 2 sandwiches
a_4 : 3 sandwiches
a_5 : 4 sandwiches
a_6 : more than 4 sandwiches

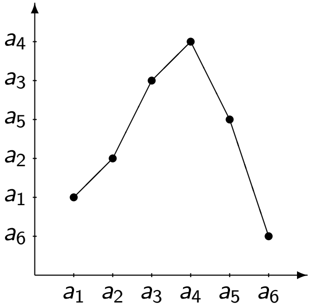
- after a good lunch: $a_1 \prec a_2 \prec a_3 \prec a_4 \prec a_5 \prec a_6$
- if you are starving: $a_6 \prec a_5 \prec a_4 \prec a_3 \prec a_2 \prec a_1$
- a possible intermediate situation: $a_4 \prec a_3 \prec a_5 \prec a_2 \prec a_1 \prec a_6$
- a quite unlikely preference: $a_6 \prec a_5 \prec a_2 \prec a_1 \prec a_3 \prec a_4$

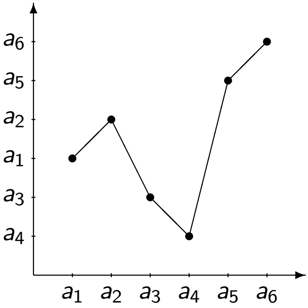
Single-peaked orderings

Let us represent graphically the latter two preferences with respect to the reference ordering $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$

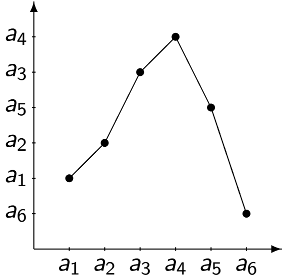
$a_4 \prec a_3 \prec a_5 \prec a_2 \prec a_1 \prec a_6$

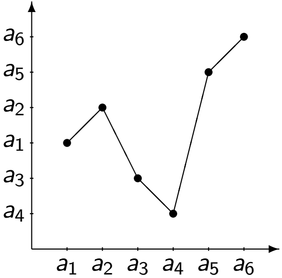
$a_6 \prec a_5 \prec a_2 \prec a_1 \prec a_3 \prec a_4$





Single-peaked orderings

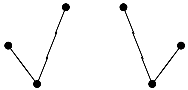




Single-peakedness

$$\boxed{a_i < a_j < a_k \quad \Longrightarrow \quad a_j \prec a_i \quad \text{or} \quad a_j \prec a_k}$$

Forbidden patterns



Single-peaked orderings

Definition (Black, 1948)

Let $\leq$ and $\preceq$ be total orderings on $X_n = \{a_1, \ldots, a_n\}$.
Then $\preceq$ is said to be *single-peaked for* $\leq$ if for any $a_i, a_j, a_k \in X_n$ such that $a_i < a_j < a_k$ we have $a_j \prec a_i$ or $a_j \prec a_k$.

Let us assume that $X_n = \{a_1, \ldots, a_n\}$ is endowed with the ordering $a_1 < \cdots < a_n$

For $n = 4$

$$\begin{array}{ll} a_1 \prec a_2 \prec a_3 \prec a_4 & a_4 \prec a_3 \prec a_2 \prec a_1 \\ a_2 \prec a_1 \prec a_3 \prec a_4 & a_3 \prec a_2 \prec a_1 \prec a_4 \\ a_2 \prec a_3 \prec a_1 \prec a_4 & a_3 \prec a_2 \prec a_4 \prec a_1 \\ a_2 \prec a_3 \prec a_4 \prec a_1 & a_3 \prec a_4 \prec a_2 \prec a_1 \end{array}$$

There are 2^{n-1} total orderings $\preceq$ on X_n that are single-peaked for $\leq$

Single-peaked orderings

Recall that a *weak ordering* (or *total preordering*) on X_n is a binary relation $\precsim$ on X_n that is total and transitive.

Defining a weak ordering on X_n amounts to defining an ordered partition of X_n

$$C_1 \prec \cdots \prec C_k$$

where $C_1, \ldots, C_k$ are the equivalence classes defined by $\sim$

For $n = 3$, we have 13 weak orderings

$$\begin{array}{lll}
a_1 \prec a_2 \prec a_3 & a_1 \sim a_2 \prec a_3 & a_1 \sim a_2 \sim a_3 \\
a_1 \prec a_3 \prec a_2 & a_1 \prec a_2 \sim a_3 & \\
a_2 \prec a_1 \prec a_3 & a_2 \prec a_1 \sim a_3 & \\
a_2 \prec a_3 \prec a_1 & a_3 \prec a_1 \sim a_2 & \\
a_3 \prec a_1 \prec a_2 & a_1 \sim a_3 \prec a_2 & \\
a_3 \prec a_2 \prec a_1 & a_2 \sim a_3 \prec a_1 &
\end{array}$$

Single-peaked orderings

Definition

Let $\leq$ be a total ordering on X_n and let $\precsim$ be a weak ordering on X_n. We say that $\precsim$ is *weakly single-peaked for* $\leq$ if for any $a_i, a_j, a_k \in X_n$ such that $a_i < a_j < a_k$ we have $a_j \prec a_i$ or $a_j \prec a_k$ or $a_i \sim a_j \sim a_k$.

Let us assume that X_n is endowed with the ordering $a_1 < \cdots < a_n$

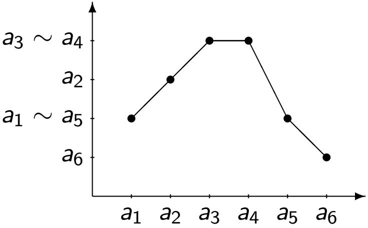
For $n = 3$

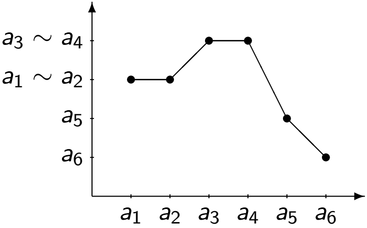
$a_1 \prec a_2 \prec a_3$	$a_1 \sim a_2 \prec a_3$	$a_1 \sim a_2 \sim a_3$
$a_1 \prec a_3 \prec a_2$	$a_1 \prec a_2 \sim a_3$	
$a_2 \prec a_1 \prec a_3$	$a_2 \prec a_1 \sim a_3$	
$a_2 \prec a_3 \prec a_1$	$a_3 \prec a_1 \sim a_2$	
$a_3 \prec a_1 \prec a_2$	$a_1 \sim a_3 \prec a_2$	
$a_3 \prec a_2 \prec a_1$	$a_2 \sim a_3 \prec a_1$	

Single-peaked orderings

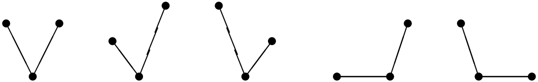
Examples

$\boxed{a_3 \sim a_4 \prec a_2 \prec a_1 \sim a_5 \prec a_6}$ $\boxed{a_3 \sim a_4 \prec a_2 \sim a_1 \prec a_5 \prec a_6}$





Forbidden patterns



Part II: Associative and quasitrivial operations

Connectedness and Contour plots

Let $F\colon X_n^2 \to X_n$ be an operation on $X_n = \{1, \ldots, n\}$

Definition

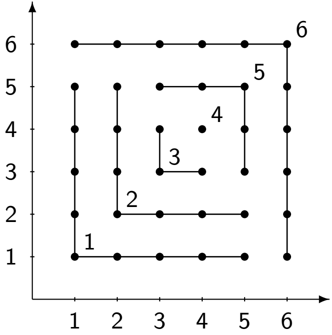
- The points (u, v) and (x, y) of X_n^2 are said to be *F-connected* if

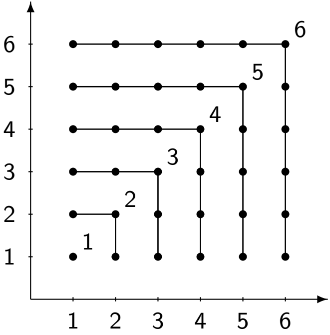
$$F(u, v) \ = \ F(x, y)$$

- The point (x, y) of X_n^2 is said to be *F-isolated* if it is not F-connected to another point

Connectedness and Contour plots

Examples





Connectedness and Contour plots

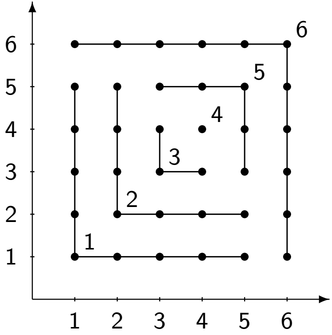
Definition

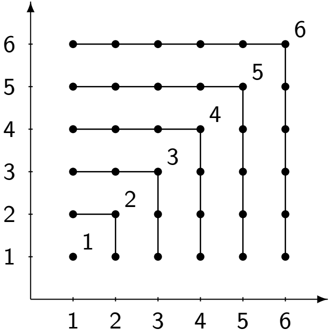
For any $x \in X_n$, the *F-degree of x*, denoted $\deg_F(x)$, is the number of points $(u, v) \neq (x, x)$ such that $F(u, v) = F(x, x)$

Remark. The point (x, x) is F-isolated iff $\deg_F(x) = 0$

Connectedness and Contour plots

Examples





Quasitriviality

Definition

$F\colon X_n^2 \to X_n$ is said to be

- *quasitrivial* (or *conservative*) if

$$F(x,y) \ \in \ \{x,y\} \qquad (x,y \in X_n)$$

- *idempotent* if

$$F(x,x) \ = \ x \qquad (x \in X_n)$$

Fact. If F is quasitrivial, then it is idempotent

Fact. If F is idempotent and if (x,y) is F-isolated, then $x=y$

$$F(x,y) \ = \ F(F(x,y),F(x,y))$$

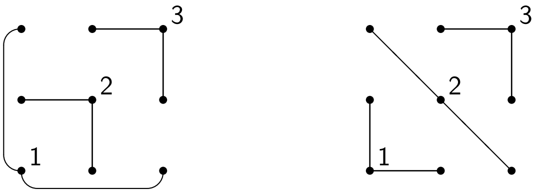
Quasitriviality

Let $\Delta_{X_n} = \{(x,x) \mid x \in X_n\}$

Fact

$F\colon X_n^2 \to X_n$ is quasitrivial iff

- it is idempotent
- every point $(x,y) \notin \Delta_{X_n}$ is F-connected to either (x,x) or (y,y)



Corollary. If F is quasitrivial, then it has at most one F-isolated point

Neutral and annihilator elements

Definition

- $e \in X_n$ is said to be a *neutral element* of $F\colon X_n^2 \to X_n$ if

$$F(x,e) \;=\; F(e,x) \;=\; x, \qquad x \in X_n$$

- $a \in X_n$ is said to be an *annihilator element* of $F\colon X_n^2 \to X_n$ if

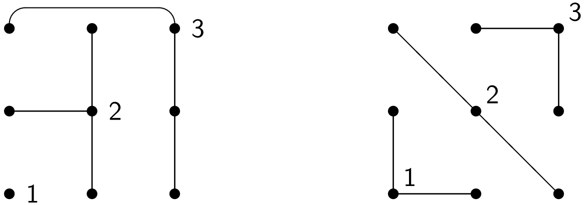
$$F(x,a) \;=\; F(a,x) \;=\; a, \qquad x \in X_n$$

Neutral and annihilator elements

Proposition

Assume that $F\colon X_n^2 \to X_n$ is quasitrivial.

- $e \in X_n$ is a neutral element of F iff $\deg_F(e) = 0$
- $a \in X_n$ is an annihilator element of F iff $\deg_F(a) = 2n - 2$.



Associative, quasitrivial, and commutative operations

Theorem

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

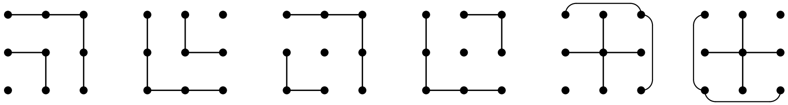
(i) F is associative, quasitrivial, and commutative

(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on X_n

The total ordering $\preceq$ is uniquely determined as follows:

$$x \preceq y \quad \Longleftrightarrow \quad \deg_F(x) \leq \deg_F(y)$$

Fact. There are exactly $n!$ such operations



Associative, quasitrivial, and commutative operations

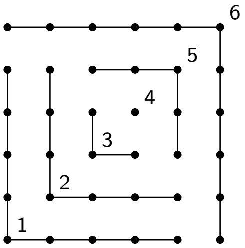
Theorem

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative, quasitrivial, and commutative

(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on X_n

(iii) F is quasitrivial and $\{deg_F(x) \mid x \in X_n\} = \{0, 2, 4, \ldots, 2n-2\}$



Associative, quasitrivial, and commutative operations

Definition.

$F\colon X_n^2 \to X_n$ is said to be $\leq$-preserving for some total ordering $\leq$ on X_n if for any $x, y, x', y' \in X_n$ such that $x \leq x'$ and $y \leq y'$, we have $F(x,y) \leq F(x',y')$

Theorem

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative, quasitrivial, and commutative

(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on X_n

(iii) F is quasitrivial and $\{deg_F(x) \mid x \in X_n\} = \{0, 2, 4, \ldots, 2n-2\}$

(iv) F is quasitrivial, commutative, and $\leq$-preserving for some total ordering $\leq$ on X_n

Associative, quasitrivial, and commutative operations

Definition.

A *uninorm on X_n* is an operation $F\colon X_n^2 \to X_n$ that

- has a neutral element $e \in X_n$

and is

- associative
- commutative
- $\leq$-preserving for some total ordering $\leq$ on X_n

Associative, quasitrivial, and commutative operations

Theorem

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative, quasitrivial, and commutative

(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on X_n

(iii) F is quasitrivial and $\{deg_F(x) \mid x \in X_n\} = \{0, 2, 4, \ldots, 2n-2\}$

(iv) F is quasitrivial, commutative, and $\leq$-preserving for some total ordering $\leq$ on X_n

(v) F is an idempotent uninorm on X_n for some total ordering $\leq$ on X_n

Associative, quasitrivial, and commutative operations

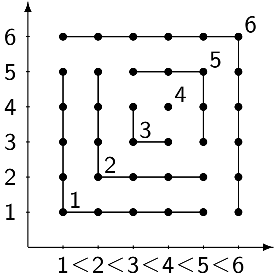
Assume that $X_n = \{1, \ldots, n\}$ is endowed with the usual total ordering $\leq_n$ defined by $1 <_n \cdots <_n n$

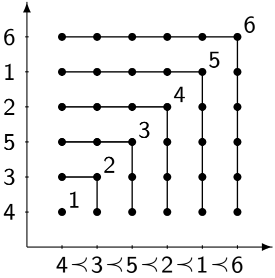
Theorem

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is quasitrivial, commutative, and $\leq_n$-preserving ($\Rightarrow$ associative)

(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on X_n that is single-peaked for $\leq_n$

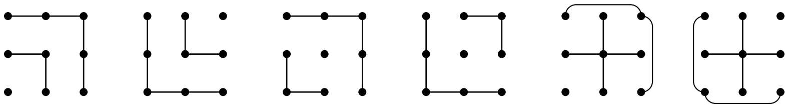




Associative, quasitrivial, and commutative operations

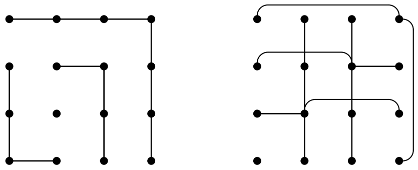
Remark.

- There are $n!$ operations $F\colon X_n^2 \to X_n$ that are associative, quasitrivial, and commutative.
- There are 2^{n-1} of them that are $\leq_n$-preserving



Associative and quasitrivial operations

Examples of noncommutative operations



Associative and quasitrivial operations

Definition.

The *projection operations* $\pi_1\colon X_n^2 \to X_n$ and $\pi_2\colon X_n^2 \to X_n$ are respectively defined by

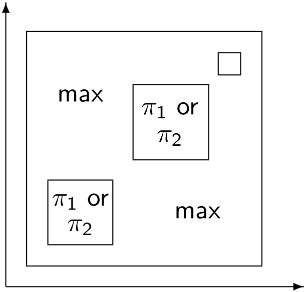
$$\begin{aligned} \pi_1(x,y) &= x, \qquad x,y \in X_n \\ \pi_2(x,y) &= y, \qquad x,y \in X_n \end{aligned}$$

Associative and quasitrivial operations

Assume that $X_n = \{1, \ldots, n\}$ is endowed with a weak ordering $\precsim$

Ordinal sum of projections

$$\mathrm{osp}_{\precsim} \colon X_n^2 \to X_n$$



Permuting the elements related to a box does not change the graph of F

Associative and quasitrivial operations

Theorem (Länger 1980)

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative and quasitrivial

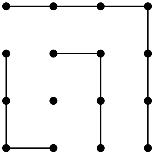
(ii) $F = \mathrm{osp}_{\precsim}$ for some weak ordering $\precsim$ on X_n

The weak ordering $\precsim$ is uniquely determined as follows:

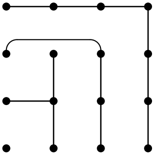
$$x \precsim y \quad \Longleftrightarrow \quad \deg_F(x) \leq \deg_F(y)$$

Associative and quasitrivial operations

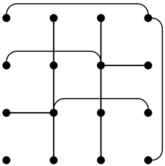
Examples



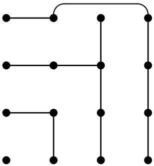
$1 < 2 < 3 < 4$



$2 \prec 1 \sim 3 \prec 4$



$1 < 2 < 3 < 4$



$1 \prec 4 \prec 2 \sim 3$

Associative and quasitrivial operations

How to check whether a quasitrivial operation $F\colon X_n^2 \to X_n$ is associative?

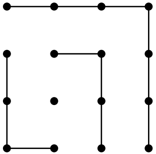
1. Order the elements of X_n according to the weak ordering $\precsim$ defined by

$$x \precsim y \quad \Longleftrightarrow \quad \deg_F(x) \leq \deg_F(y)$$

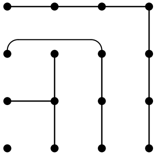
2. Check whether the resulting operation is one of the corresponding ordinal sums

Associative and quasitrivial operations

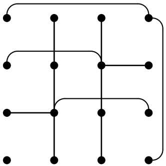
Which ones are $\leq$-preserving?



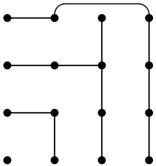
$1 < 2 < 3 < 4$



$2 \prec 1 \sim 3 \prec 4$



$1 < 2 < 3 < 4$



$1 \prec 4 \prec 2 \sim 3$

Associative and quasitrivial operations

Assume that $X_n = \{1, \ldots, n\}$ is endowed with the usual total ordering $\leq_n$ defined by $1 <_n \cdots <_n n$

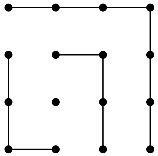
Theorem

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

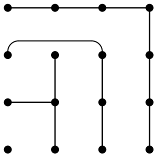
(i) F is associative, quasitrivial, and $\leq_n$-preserving

(ii) $F = \mathrm{osp}_{\precsim}$ for some weak ordering $\precsim$ on X_n that is weakly single-peaked for $\leq_n$

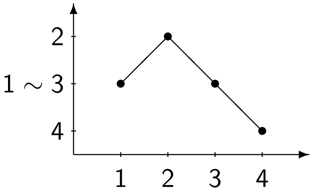
Associative and quasitrivial operations

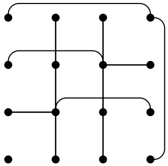


$1 < 2 < 3 < 4$

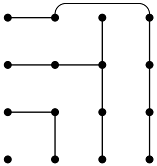


$2 \prec 1 \sim 3 \prec 4$

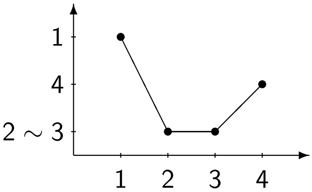




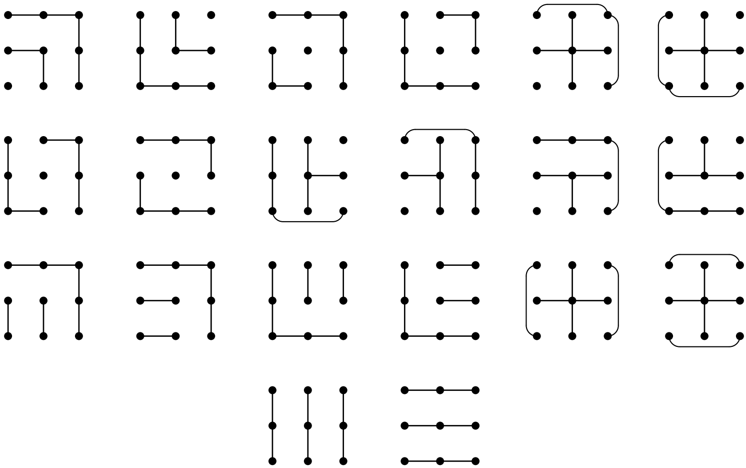
$1 < 2 < 3 < 4$



$1 \prec 4 \prec 2 \sim 3$



Associative and quasitrivial operations



Final remarks

1. We have introduced and identified a number of integer sequences in `http://oeis.org`
    - Number of associative and quasitrivial operations: A292932
    - Number of associative, quasitrivial, and $\leq_n$-preserving operations: A293005
    - Number of weak orderings on X_n that are weakly single-peaked for $\leq_n$: A048739
    - $\cdots$
2. Most of our characterization results still hold on arbitrary sets X (not necessarily finite)

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