On associative, idempotent, symmetric, and nondecreasing operations

ISAS 2018

Jimmy Devillet

in collaboration with Bruno Teheux

University of Luxembourg

Motivation

For any integer $n \geq 1$, let $X_n = \{1, \ldots, n\}$ endowed with $\leq$

Definition

$F\colon X_n^2 \to X_n$ is

- *idempotent* if

$$F(x, x) \;=\; x$$

- *nondecreasing for $\leq$* if

$$F(x, y) \;\leq\; F(x', y') \quad \text{whenever} \quad x \;\leq\; x' \;\text{ and }\; y \;\leq\; y'$$

Motivation

Definition

$e \in X_n$ is a *neutral element of F* if

$$F(x, e) \;=\; F(e, x) \;=\; x \qquad x \in X_n$$

Motivation

Definition

$F\colon X_n^2 \to X_n$ is a *discrete uninorm* if it is associative, symmetric, nondecreasing, and has a neutral element.

Idempotent discrete uninorms have been investigated for a few decades by several authors.

In the previous talk, the following characterization was provided.

Motivation

Definition. (Black, 1948)

$\leq'$ is said to be *single-peaked for* $\leq$ if for all $a, b, c \in X_n$,

$$a \leq b \leq c \quad \Longrightarrow \quad b \leq' a \quad \text{or} \quad b \leq' c$$

Theorem (Couceiro et al., 2018)

For any $F\colon X_n^2 \to X_n$, the following are equivalent.

(i) F is an idempotent discrete uninorm

(ii) $F = \max_{\leq'}$ for some $\leq'$ that is single-peaked for $\leq$

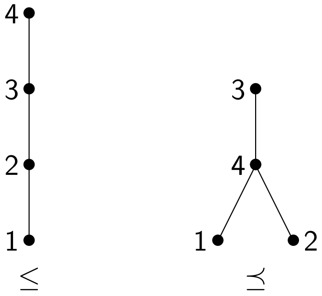
How can we generalize this result by removing the existence of a neutral element?

Towards a generalization

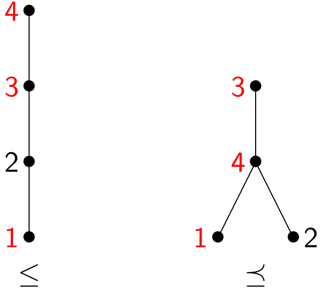
$\preceq$ will denote a join-semilattice order on X_n

F is associative, idempotent, and symmetric iff there exists $\preceq$ such that $F = \curlyvee$.

Example. On $X_4 = \{1, 2, 3, 4\}$, consider $\leq$ and $\preceq$



Towards a generalization



$\curlyvee(1,4) = 4$ and $\curlyvee(3,4) = 3 \quad \Rightarrow \quad \curlyvee$ is not nondecreasing

What are the $\preceq$ for which $\curlyvee$ are nondecreasing?

CI-property

Definition. We say that $\preceq$ has the *convex-ideal property* (*CI-property* for short) *for* $\leq$ if for all $a, b, c \in X_n$,

$$a \leq b \leq c \quad \Longrightarrow \quad b \preceq a \curlyvee c$$

Proposition

The following are equivalent.

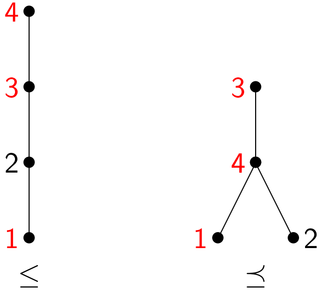
(i) $\preceq$ has the CI-property for $\leq$

(ii) Every ideal of $(X_n, \preceq)$ is a convex subset of $(X_n, \leq)$

CI-property

Definition. We say that $\preceq$ has the *CI-property for $\leq$* if for all $a, b, c \in X_n$,

$$a \leq b \leq c \quad \Longrightarrow \quad b \preceq a \curlyvee c$$

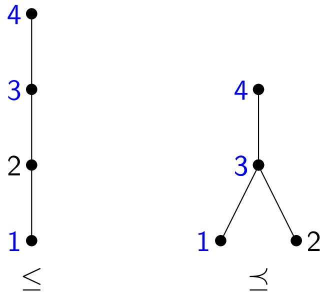


$\preceq$ does not have the CI-property for $\leq$

CI-property

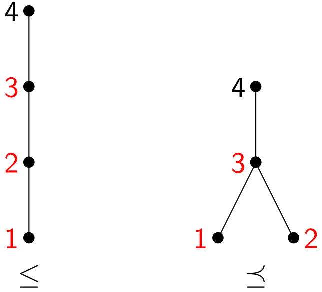
Definition. We say that $\preceq$ has the *CI-property for* $\leq$ if for all $a, b, c \in X_n$,

$$a \leq b \leq c \quad \Longrightarrow \quad b \preceq a \curlyvee c$$



$\preceq$ has the CI-property for $\leq$

CI-property



$\Upsilon(1,2) = 3$ and $\Upsilon(2,2) = 2$ $\Longrightarrow$ Υ is not nondecreasing

Internality

Definition. $F\colon X_n^2 \to X_n$ is said to be *internal* if $x \leq F(x,y) \leq y$ for every $x, y \in X_n$ with $x \leq y$

Definition. We say that $\preceq$ is *internal for* $\leq$ if for all $a, b, c \in X_n$,

$$a < b < c \quad \Longrightarrow \quad (a \neq b \curlyvee c \quad \text{and} \quad c \neq a \curlyvee b)$$

Proposition

The following are equivalent.

(i) $\preceq$ is internal for $\leq$

(ii) The join operation $\curlyvee$ of $\preceq$ is internal

A direct consequence

Corollary

For any $F\colon X_n^2 \to X_n$, the following are equivalent.

(i) F is associative, internal, and symmetric

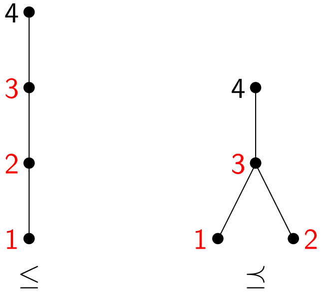
(ii) $F = \curlyvee$ for some $\preceq$ that is internal for $\leq$

Fact. If F is idempotent and nondecreasing then it is internal.

Internality

Definition. We say that $\preceq$ is *internal for $\leq$* if for all $a, b, c \in X_n$,

$$a < b < c \quad \Longrightarrow \quad (a \neq b \curlyvee c \quad \text{and} \quad c \neq a \curlyvee b)$$

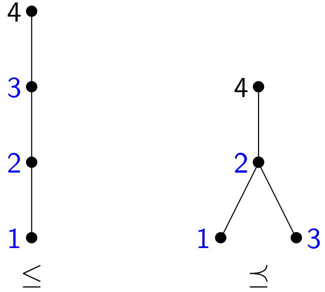


$\preceq$ has the CI-property but is not internal for $\leq$

Internality

Definition. We say that $\preceq$ is *internal for* $\leq$ if for all $a, b, c \in X_n$,

$$a < b < c \quad \Longrightarrow \quad (a \neq b \curlyvee c \quad \text{and} \quad c \neq a \curlyvee b)$$



$\preceq$ has the CI-property and is internal for $\leq$

Also, $\curlyvee$ is nondecreasing

Nondecreasingness

Definition. We say that $\preceq$ is *nondecreasing for $\leq$* if

- CI-property for $\leq$
- internal for $\leq$.

$$F \text{ is associative, idempotent, and symmetric iff } F = \curlyvee$$

Theorem

For any $F\colon X_n^2 \to X_n$, the following are equivalent.

(i) F is associative, idempotent, symmetric, and nondecreasing

(ii) $F = \curlyvee$ for some $\preceq$ that is nondecreasing for $\leq$

Enumeration

Proposition

The number of associative, idempotent, symmetric, and nondecreasing operations on X_n is the n^{th} Catalan number.

Final remarks

1. Our characterization is still valid on arbitrary chains (not necessarily finite)
2. In arXiv: 1805.11936, we provided an alternative characterization of the class of associative, idempotent, symmetric, and nondecreasing operations on X_n.

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