On bisymmetric and quasitrivial operations

56th ISFE

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Connectedness and Contour Plots

Let X be a nonempty set and let $F\colon X^2 \to X$

Definition

- The points $(x, y), (u, v) \in X^2$ are *F-connected* if

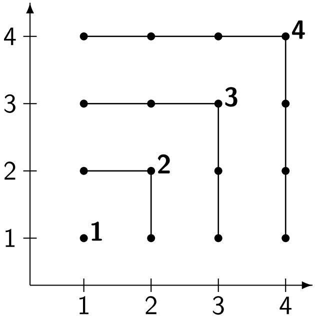
$$F(x, y) \;=\; F(u, v)$$

- The point $(x, y) \in X^2$ is *F-isolated* if it is not F-connected to another point in X^2

Connectedness and Contour Plots

For any integer $n \geq 1$, let $X_n = \{1, ..., n\}$ endowed with $\leq$

Example. $F(x, y) = \max\{x, y\}$ on $(X_4, \leq)$



Bisymmetry and quasitriviality

Definition

$F\colon X^2 \to X$ is said to be

- *bisymmetric* if

$$F(F(x,y),F(u,v)) \;=\; F(F(x,u),F(y,v)) \qquad x,y,u,v \in X$$

- *quasitrivial* if

$$F(x,y) \;\in\; \{x,y\} \qquad x,y \in X$$

Lemma (Kepka, 1981)

If F is bisymmetric and quasitrivial then it is associative

Weak orderings

Recall that a binary relation R on X is said to be

- *total* if $\forall x, y$: xRy or yRx
- *transitive* if $\forall x, y, z$: xRy and yRz implies xRz

A *weak ordering on X* is a binary relation $\precsim$ on X that is total and transitive.

- symmetric part: $\sim$
- asymmetric part: $\prec$

Recall that $\sim$ is an equivalence relation on X and that $\prec$ induces a linear ordering on the quotient set $X/\sim$

Motivation

Theorem (Länger, 1980)

F is associative and quasitrivial iff there exists a weak ordering $\precsim$ on X such that

$$F|_{A\times B} = \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \qquad \forall A, B \in X/\sim$$

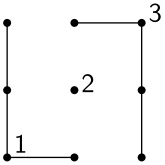
If $X = X_n = \{1, \ldots, n\}$, then

$$\boxed{x \precsim y \qquad \Longleftrightarrow \qquad |F^{-1}(\{x\})| \leq |F^{-1}(\{y\})|}$$

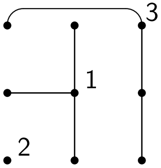
Motivation

$$F|_{A\times B} \;=\; \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \qquad \forall A, B \in X/\sim$$

$$\boxed{x \;\precsim\; y \qquad \Longleftrightarrow \qquad |F^{-1}(\{x\})| \;\leq\; |F^{-1}(\{y\})|}$$

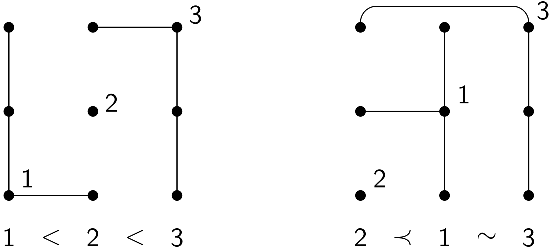


$1 \;<\; 2 \;<\; 3$



$2 \;\prec\; 1 \;\sim\; 3$

Motivation



$$F(F(2,3),F(1,2)) \;=\; 3 \;\neq\; 1 \;=\; F(F(2,1),F(3,2))$$

$\Longrightarrow$ F is not bisymmetric

Quasilinearity

Let $\precsim$ be a weak ordering on X

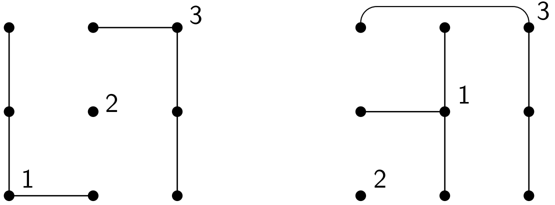
Definition
$\precsim$ is *quasilinear* if there exist no pairwise distinct $a, b, c \in X$ such that $a \prec b \sim c$

Example. On $X = \{1, 2, 3, 4\}$, consider the $\precsim$

$$2 \sim 3 \prec 1 \prec 4$$

Quasilinearity

$$F|_{A\times B} \;=\; \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \qquad \forall A, B \in X/\sim$$

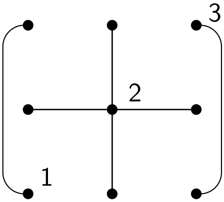


$1 \;<\; 2 \;<\; 3$ $\qquad$ $2 \;\prec\; 1 \;\sim\; 3$

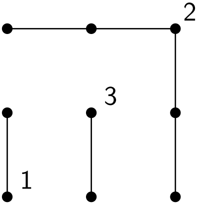
$\precsim$ is not quasilinear and F is not bisymmetric

Quasilinearity

$$x \precsim y \quad \Longleftrightarrow \quad |F^{-1}(\{x\})| \leq |F^{-1}(\{y\})|$$



$1 < 2 < 3$



$1 \sim 3 \prec 2$

$\precsim$ is quasilinear and F is bisymmetric

A characterization

$$F|_{A\times B} \; = \; \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A\neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A=B, \end{cases} \quad \forall A,B\in X/\sim \quad (*)$$

Theorem

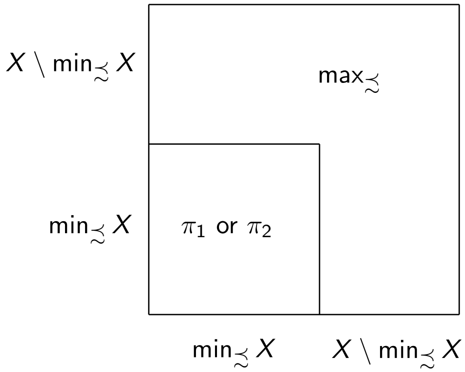
For any $F\colon X^2\to X$, the following are equivalent.

(i) F is bisymmetric and quasitrivial

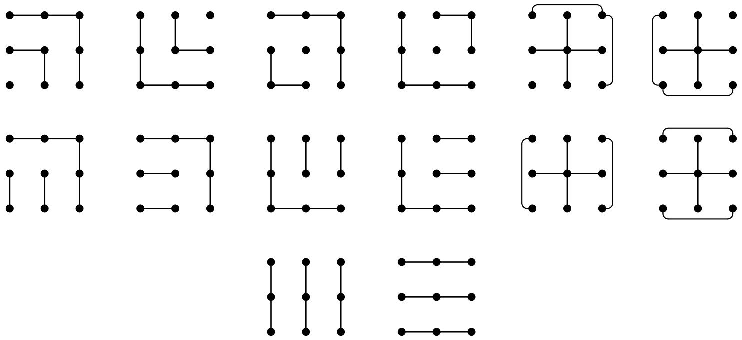
(ii) F is of the form $(*)$ for some quasilinear $\precsim$

A characterization

We denote the *set of minimal elements of X for $\precsim$* by $\min_{\precsim} X$



Bisymmetric and quasitrivial operations



The nondecreasing case

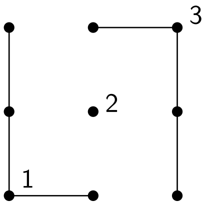
Let $\leq$ be a linear ordering on X

Definition. $F\colon X^2 \to X$ is *nondecreasing for* $\leq$ if

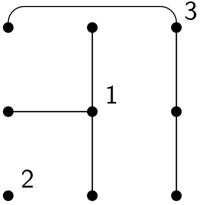
$$F(x,y) \leq F(x',y') \quad \text{whenever} \quad x \leq x' \text{ and } y \leq y'$$

The nondecreasing case

$$F|_{A\times B} = \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \quad \forall A, B \in X/\sim$$



$1 < 2 < 3$



$2 \prec 1 \sim 3$

Weakly single-peaked weak orderings

Definition (Couceiro et al., 2018)
$\precsim$ is said to be *weakly single-peaked for* $\leq$ if for any $a, b, c \in X$,

$$a < b < c \quad \Longrightarrow \quad b \prec a \quad \text{or} \quad b \prec c \quad \text{or} \quad a \sim b \sim c$$

Example. The weak ordering $\precsim$ on

$$X_4 = \{1 < 2 < 3 < 4\}$$

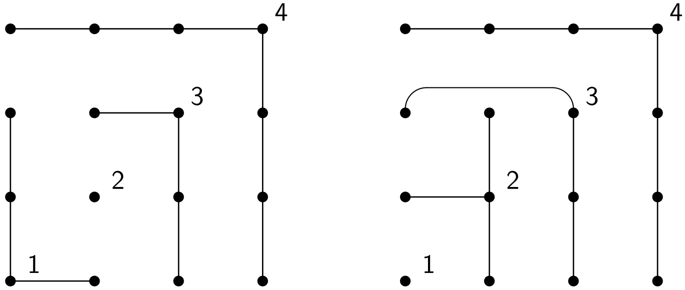
defined by

$$2 \prec 1 \sim 3 \prec 4$$

is weakly single-peaked for $\leq$

Weakly single-peaked weak orderings

$$x \precsim y \iff |F^{-1}(\{x\})| \leq |F^{-1}(\{y\})|$$



$1 < 2 < 3 < 4$ $\qquad$ $2 \prec 1 \sim 3 \prec 4$

Associative quasitrivial and nondecreasing operations

$$F|_{A\times B} \;=\; \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \qquad \forall A, B \in X/\sim \qquad (*)$$

Theorem (Couceiro et al., 2018)

For any $F\colon X^2 \to X$, the following are equivalent.

(i) F is associative, quasitrivial, and nondecreasing

(ii) F is of the form $(*)$ for some $\precsim$ that is weakly single-peaked for $\leq$

Bisymmetric quasitrivial and nondecreasing operations

$$F|_{A\times B} \;=\; \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A\neq B,\\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A=B, \end{cases} \qquad \forall A,B\in X/\sim \qquad (*)$$

Theorem

For any $F\colon X^2\to X$, the following are equivalent.

(i) F is bisymmetric, quasitrivial, and nondecreasing

(ii) F is of the form $(*)$ for some $\precsim$ that is quasilinear and weakly single-peaked for $\leq$

Final remarks

In arXiv: 1712.07856

1. Characterizations of the class of bisymmetric and quasitrivial operations as well as the subclass of those operations that are nondecreasing

2. New integer sequences (http://www.oeis.org)
    - Number of bisymmetric and quasitrivial operations: A296943
    - Number of bisymmetric, quasitrivial, and nondecreasing operations: A296953
    - . . .

Selected references

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