Clones of pivotally decomposable operations

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Motivation

Shannon decomposition of operations $f\colon \{0,1\}^n \to \{0,1\}$:

$$f(\mathbf{x}) = x_k f(\mathbf{x}_k^1) + (1 - x_k) f(\mathbf{x}_k^0),$$

where

- $\mathbf{x}_k^a$ is obtained from $\mathbf{x}$ by replacing its k^{th} component by a.

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Median decomposition of polynomial operations over bounded DL:

$$f(\mathbf{x}) = \mathrm{med}(x_k, f(\mathbf{x}_k^1), f(\mathbf{x}_k^0)),$$

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Goal: Uniform approach of these decomposition schemes.

Pivotal decomposition

A set and $0, 1 \in A$

Let $\Pi\colon A^3 \to A$ an operation

Definition. An operation $f\colon A^n \to A$ is Π-*decomposable* if

$$f(\mathbf{x}) \;=\; \Pi(x_k, f(\mathbf{x}_k^1), f(\mathbf{x}_k^0))$$

for all $\mathbf{x} \in A^n$ and all $k \leq n$.

Pivotal decomposition

A set and $0, 1 \in A$

Let $\Pi\colon A^3 \to A$ an operation that satisfies the equation

$$\Pi(x, y, y) = y.$$

Such a Π is called a *pivotal operation*. In this talk, all Π are pivotal.

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Examples

$$f(\mathbf{x}) \;=\; \Pi(x_k, f(\mathbf{x}_k^1), f(\mathbf{x}_k^0))$$

Shannon decomposition: $\Pi(x, y, z) = xy + (1 - x)z$

Median decomposition: $\Pi(x, y, z) = \mathrm{med}(x, y, z)$

Benefits:

- uniformly isolate the marginal contribution of a factor
- repeated applications lead to normal form representations
- lead to characterization of operation classes

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$$\Pi(x, 1, 0) = x \tag{P}$$

$$\Pi(\Pi(x, y, z), u, v) = \Pi(x, \Pi(y, u, v), \Pi(z, u, v)) \tag{AD}$$

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Proposition. If $\Pi \models$ (AD), the following are equivalent

(i) Λ_Π is a clone

(ii) $\Lambda_\Pi \models$ (P)

Clones of pivotally decomposable Boolean operations

$$\text{(P)} \; + \; \text{(AD)} \implies \Lambda_\Pi \text{ is a clone} \qquad (\star)$$

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(i) There is Π such that $C = \Lambda_\Pi$

(ii) C is the clone of (monotone) Boolean functions

What about the converse of $(\star)$?

The case of Π-decomposable Π

$$\left.\begin{aligned} \Pi(\Pi(1,0,1),0,1) &= \Pi(1,\Pi(0,0,1),\Pi(1,0,1)) \\ \Pi(\Pi(0,0,1),0,1) &= \Pi(0,\Pi(0,0,1),\Pi(1,0,1)) \end{aligned}\right\} \qquad \text{(WAD)}$$

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Theorem. If $\Pi \in \Lambda_\Pi$ and $\Pi \models$ (WAD), then

$$\text{(P)} \ + \ \text{(AD)} \iff \Lambda_\Pi \text{ is a clone,}$$

and Λ_Π is the clone generated by Π and the constant maps.

What happens if Π is not Π-decomposable?

We have seen that if Λ_Π is a Boolean clone then $\Pi \in \Lambda_\Pi$.

There are some Π such that Λ_Π is a clone but $\Pi \notin \Lambda_\Pi$.

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Example. Let $A = \{0, 1/2, 1\}$ and Π be the pivotal operation s.t.

$$\begin{aligned} \Pi(x,1,0) &= x & \qquad \Pi(x,0,1/2) &= 1 \\ \Pi(x,0,1) &= 1-x & \qquad \Pi(x,1/2,1) &= 0 \\ \Pi(x,1,1/2) &= 1 & \qquad \Pi(x,1/2,0) &= 0 \end{aligned}$$

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$$\Pi \models \text{(P), (AD)} \quad \text{but} \quad \Pi \notin \Lambda_\Pi$$

since

$$\Pi(x,1/2,1/2) = 1/2 \quad \text{and} \quad \Pi(1/2, \Pi(x,1,1/2), \Pi(x,0,1/2)) = 1$$

Symmetry

Theorem. If $\Pi \in \Lambda_{\Pi}$ and $\Pi \models$ (P), then the following are equivalent

(i) Π is symmetric

(ii) $\Pi(0,0,1) \;=\; \Pi(0,1,0) \quad \text{and} \quad \Pi(1,0,1) \;=\; \Pi(1,1,0)$

Summary

- If $\Pi \in \Lambda_\Pi$ and $\Pi \models$ (WAD), then

$$\text{(P)} \;+\; \text{(AD)} \iff \Lambda_\Pi \text{ is a clone}$$

- There is a clone Λ_Π such that $\Pi \notin \Lambda_\Pi$

Summary

- If $\Pi \in \Lambda_\Pi$ and $\Pi \models$ (WAD), then

$$(\text{P}) \;+\; (\text{AD}) \iff \Lambda_\Pi \text{ is a clone}$$

- There is a clone Λ_Π such that $\Pi \notin \Lambda_\Pi$

Problems.

Find a characterization of those Λ_Π which are clones when $\Pi \notin \Lambda_\Pi$.

Structure of the family of decomposable classes of operations?

M. Couceiro, and B. Teheux. Pivotal decomposition schemes inducing clones of operations. *Beitr. Algebra Geom.*, 59:25 – 40, 2018.