ARITHMETIC BILLIARDS

Mathematical billiards are an idealisation of what we can experience on a regular pool table: for example, there is no friction and the billiard ball can bounce infinitely many times on the billiard sides. One fascinating aspect of these billiards is that they provide a geometrical method to determine the least common multiple and the greatest common divisor of two positive natural numbers [1, 2]. The reader can look at the first picture and try to figure out for themselves why the construction works. Indeed, the exploration of this construction is a source of activities for Math Circles [6, 7, 8, 4] and for schools [3, 5].

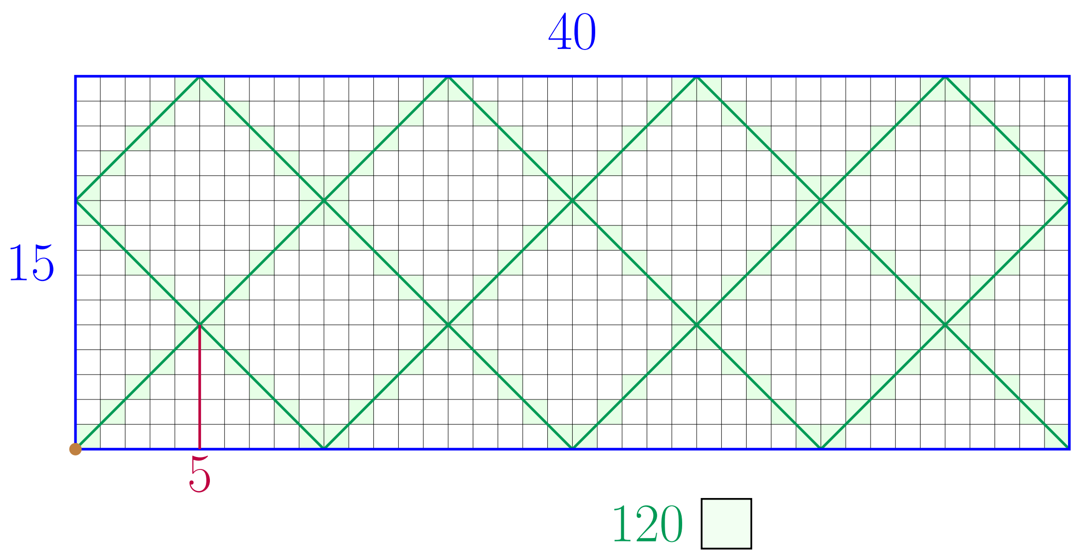

FIGURE 1. The least common multiple of 40 and 15 equals 120, the greatest common divisor is 5.

For the billiard table we take a rectangle whose side lengths are two positive natural numbers: we shoot a pointwise billiard ball from one corner making a $45°$ angle with the sides; the billiard ball bounces on the rectangle sides until it hits a corner. The billiard ball does not lose speed and each time it gets reflected exactly at a $45°$ angle (thus the path only makes left or right $90°$ turns). The trajectory of the billiard ball consists of line segments.

The *least common multiple* of the two given numbers is the total length of the path divided by $\sqrt{2}$. If one decomposes the billiard table into unit squares, the least common multiple is equal to the number of unit squares which are crossed by the path.

Now suppose that none of the two given numbers is a multiple of the other (this easy case is left to the reader). Because of our assumption, the path crosses itself. More precisely, the first segment of the path contains the point of self-intersection which is closest to the starting point. The *greatest common divisor* of the two given numbers is, equivalently: the distance from the starting point to the closest point of self-intersection divided by $\sqrt{2}$; the number of unit squares crossed by the first segment of the path up to the closest point of self-intersection.

Notation: Let a, b be the two given positive natural numbers and suppose that none of the two is a multiple of the other. We set coordinates in the billiard table such that the starting point of the arithmetic billiard path is $(0, 0)$ and the opposite corner is (a, b).

REFLECTING THE BILLIARD

While looking at an object in a mirror, you have the impression that the object is behind the mirror. Notice that three points are aligned: your position, the reflection of the object on the mirror and the (imaginary) object behind the mirror. We are going to exploit this simple idea, the mirror being one side of the billiard table.

Starting with a square whose sides are the least common multiple $\mathrm{lcm}(a, b)$, first decompose this square into rectangles with sides a and b. Next, mark the rectangle in one of the corners as the arithmetic billiard (the starting point for the billiard ball being the corner of the square), and draw the diagonal of the square from that corner.

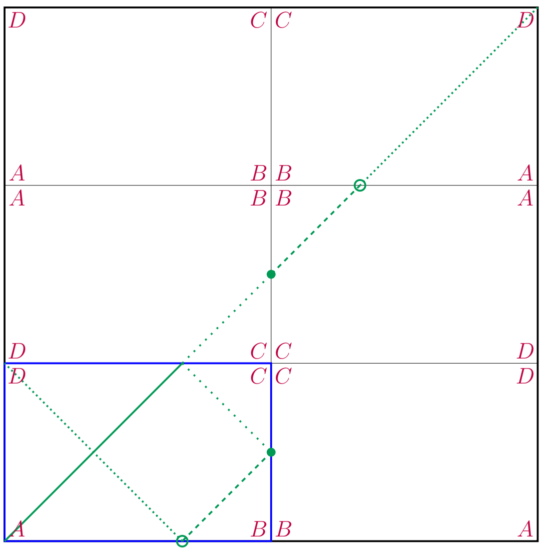

FIGURE 2. Reflecting the billiard, if the ratio between the two sides is $2/3$.

The path of the billiard ball corresponds precisely to the diagonal of the square if we reflect the billiard when a path segment hits a side: indeed, the segment and its reflection in the reflected billiard are aligned.

On the diagonal of the square, only its two endpoints are also corners for some of the rectangles, because there is no positive common multiple of a and b which is smaller than $\mathrm{lcm}(a,b)$. Since at most one of the two numbers $\mathrm{lcm}(a,b)/a$ and $\mathrm{lcm}(a,b)/b$ is even, the two endpoints of the diagonal correspond to distinct billiard corners.

We have thus shown:

- The billiard ball eventually lands in a billiard corner, and this is different from the starting point.
- The least common multiple $\mathrm{lcm}(a,b)$ is the length of the path divided by $\sqrt{2}$.

The diagonal of the square crosses $\mathrm{lcm}(a,b)/a-1$ rectangle sides parallel to a and $\mathrm{lcm}(a,b)/b-1$ rectangle sides parallel to b. This gives:

- There are exactly $\mathrm{lcm}(a,b)/a-1$ bouncing points on the two sides of length a and $\mathrm{lcm}(a,b)/b-1$ on those of length b.

THE UNIT SQUARES CROSSED BY THE PATH

The path of the arithmetic billiard is not periodic because the billiard ball never goes back to the starting point. More precisely, the path does not contain any loop and crosses itself only at isolated points:

- The path does not go more than once along the diagonal of a unit square, neither in the same direction nor in the opposite direction.

The reason beyond this is nicely explained in [6]: the motion of the billiard ball is "time reversible", meaning that if the billiard ball is currently traversing one particular unit square (in a particular direction), then there is absolutely no doubt from which unit square and from which direction it just came.

If some unit square diagonal gets repeated (being crossed in the same direction) then the previous unit square is also repeated: since the very first square cannot be repeated (the billiard ball never goes back to the starting point), there can be no repeated square. It is also impossible that some unit square diagonal gets repeated in the opposite direction because the billiard ball would then continue 'backwards' and reach the starting point.

The path consists of diagonals of unit squares and hence the parity of the sum of the coordinates is the same for every point of the path which has integer coordinates: since we start from the point $(0,0)$, the sum is even. In particular, the path contains at most two corners of a unit square and hence it can cover at most one of its two diagonals. (Another consequence is that the self-intersection points of the path have integer coordinates.) So we have:

- Unit squares are crossed by the path at most once.

Recalling the relation between $\mathrm{lcm}(a,b)$ and the length of the path, we then have:

- The least common multiple $\mathrm{lcm}(a, b)$ is the number of unit squares which are crossed by the path.

Since there are $a \cdot b$ unit squares in total, the last assertion implies:

- The path crosses each unit square if and only if a and b are coprime.

RESCALING

If a and b are coprime, then we know all points of the arithmetic billiard path which have integer coordinates: they are those such that the sum of the coordinates is even. In particular, these points are equally distributed along the perimeter of the billiard table, namely at distance 2 (along the perimeter).

The self-intersection point of the path which is closest to the starting point then lies on the first segment of the path, at distance $\sqrt{2}$ from the starting corner; the first segment of the path crosses 1 unit square up to the closest point of self-intersection.

We may then draw the path by marking first all bouncing points on the perimeter and then drawing from each of them the two segments which make a $45°$ angle with the sides. This construction also explains why the path is the intersection of the billiard table with a grid of diagonally oriented squares whose diagonal has length 2.

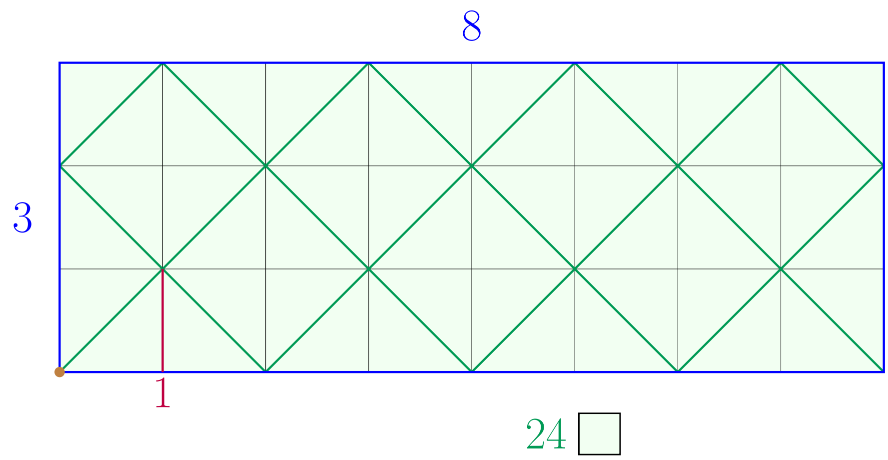

FIGURE 3. The least common multiple of 8 and 3 equals 24, the greatest common divisor is 1.

If a and b are not necessarily coprime, we can rescale the whole figure by a factor $\gcd(a, b)$: dividing a and b by their greatest common divisor, we obtain two coprime positive integers.

Notice that all geometric properties not related to the length (the shape of the path, in which corner the billiard ball lands, how many bouncing points there are...) are unaffected by the rescaling. We then have the following general assertions:

- The contact points and the points of self-intersection are exactly the points in the billiard table whose coordinates are multiples of $\gcd(a, b)$, and such that the sum of the coordinates is an even multiple of $\gcd(a, b)$.
- The path is the intersection of the billiard table with a grid of diagonally oriented squares with diagonal length $2 \cdot \gcd(a, b)$.
- The self-intersection point closest to the starting point lies on the first segment of the path at distance $\sqrt{2} \cdot \gcd(a, b)$ from the starting corner; the $\gcd(a, b)$ is the number of unit squares crossed by the first segment of the path from the starting point to the closest point of self-intersection.

One generalisation

The path of the arithmetic billiard starts in a corner and lands in a different corner. By applying a symmetry to the billiard table, we obtain a similar path between the remaining two corners. The points with integer coordinates which lie on either one of these two paths are exactly all points in the billiard table whose coordinates are multiples of the $\gcd(a, b)$.

Suppose [6] that the billiard ball can now start in any corner of some unit square inside the billiard table (again moving at $45°$ w.r.t. the billiard sides and bouncing on them). By the above consideration, we have:

- The starting positions for which the billiard ball lands in a corner are exactly the points in the billiard table whose coordinates are multiples of the $\gcd(a, b)$.

If the billiard ball does not land in a corner, the path must be periodic.

- The length of any periodic path is $2\sqrt{2} \cdot \operatorname{lcm}(a, b)$.

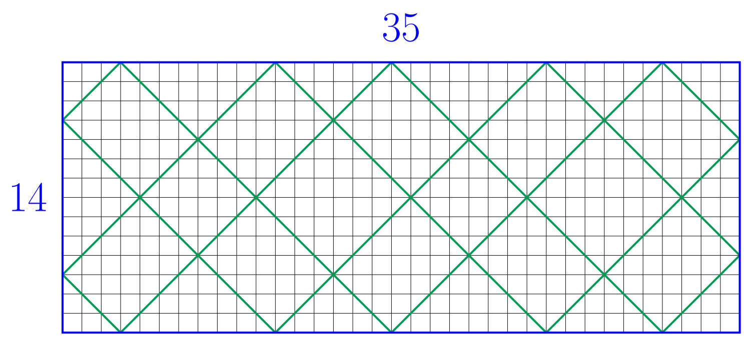

FIGURE 4. A periodic path in the billiard with sides 14 and 35.

We conclude by proving the last assertion. We may suppose w.l.o.g. that we cover a length $\sqrt{2}$ in each time unit. Considering the reflections of the billiard table, it is evident that $2 \cdot \mathrm{lcm}(a, b)$ is a possible period, and we just have to prove that it is the smallest period.

Since a periodic path touches all billiard sides (and up to symmetry), we may suppose that the starting point has coordinates $(x, 0)$ for some $1 < x < a$ and that at the beginning of the path both coordinates increase. We need to go back to the starting point which means, considering the reflections of the billiard table, to a point of the form $(\pm x + 2an, 2bm)$ for some integers n, m. With the first possibility, the smallest period t satisfies

$$(x + 2an, 2bm) = (x + t, t)\,.$$

We deduce that t is a common multiple of $2a$ and $2b$, and hence of $2 \cdot \mathrm{lcm}(a, b)$.

The second possibility gives

$$(-x + 2an, 2bm) = (x + t, t)\,,$$

from which we deduce the equality $x = an - bm$. Any such combination of a and b is a multiple of $\gcd(a, b)$, which means that the billiard ball starting at $(x, 0)$ lands in a corner, case which we are excluding.

As a final remark notice that, if we would allow the billiard ball to bounce also at the billiard corners, then for any starting point with integer coordinates we would find a periodic path with length $2\sqrt{2} \cdot \mathrm{lcm}(a, b)$.

Questions for the reader

(1) If one of the two given numbers is a multiple of the other, what is the shape of the arithmetic billiard path?

(2) For which numbers does the arithmetic billiard path end in the corner opposite to the starting point?

(3) What are the symmetries of the arithmetic billiard path (as a geometrical figure)?

(4) How many self-intersection points does the arithmetic billiard path have?

References

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[8] Joshua Zucker, *The Bouncing Ball Problem*, San Jose Math Circle, 2007, http://sanjosemathcircle.org/handouts/2007-2008/20071003.pdf. (See page 1.)

SOLUTIONS TO THE QUESTIONS FOR THE READER

(1) If a is a multiple of b, the arithmetic billiard path is a zigzag (in particular, it has no self-intersections) and consists of a/b segments of length $b\sqrt{2}$. The billiard ball lands in the billiard corner opposite to the starting point if a/b is odd, else the starting point and the ending point are connected by a side of length a.

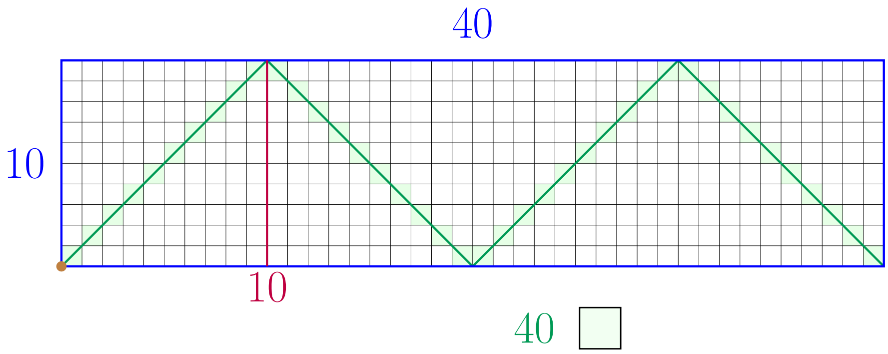

FIGURE 5. The least common multiple of 40 and 10 equals 40, the greatest common divisor is 10.

(2) If a and b are coprime, we have three cases: (1) a and b are odd; (2) a is even and b is odd; (3) a is odd and b is even. Recall that if a (respectively, b) is even/odd then there is an odd/even number of bouncing points on the two sides parallel to b (respectively, a). We then obtain the following classification: (1) starting point and ending point are opposite corners; (2) starting point and ending point are connected by a side of length a; (3) starting point and ending point are connected by a side of length b. If a and b are not necessarily coprime, we obtain the same classification with the following case distinction: (1) a and b have the same number of factors 2 in their prime factorisation; (2) a has more factors 2 than b; (3) b has more factors 2 than a.

(3) If the starting point and the ending point of the arithmetic billiard path are opposite corners, then the path is symmetric with respect to the middle of the rectangle, else the path is symmetric with respect to the bisector of the side connecting the starting point and the ending point. Indeed, we can think of mirroring the billiard table instead: this means starting at the ending point of the path and moving backwards, so the points on the path remain the same.

(4) The answer is
$$\frac{(a/\gcd(a,b)-1)\cdot(b/\gcd(a,b)-1)}{2}\,.$$
By rescaling, we may reduce to the prove this formula when the two numbers are coprime. W.l.o.g. let a be odd: there are $b+1$ lines of the unit grid in the

billiard table and parallel to the sides of length a, and on each of them we have $(a+1)/2$ points of the arithmetic billiard path. By removing the contact points with the billiard sides we obtain

$$\frac{(a+1)(b+1)}{2} - (a+b) = \frac{(a-1)(b-1)}{2}$$

self-intersection points for the arithmetic billiard path.