MATH AROUND THE CLOCK

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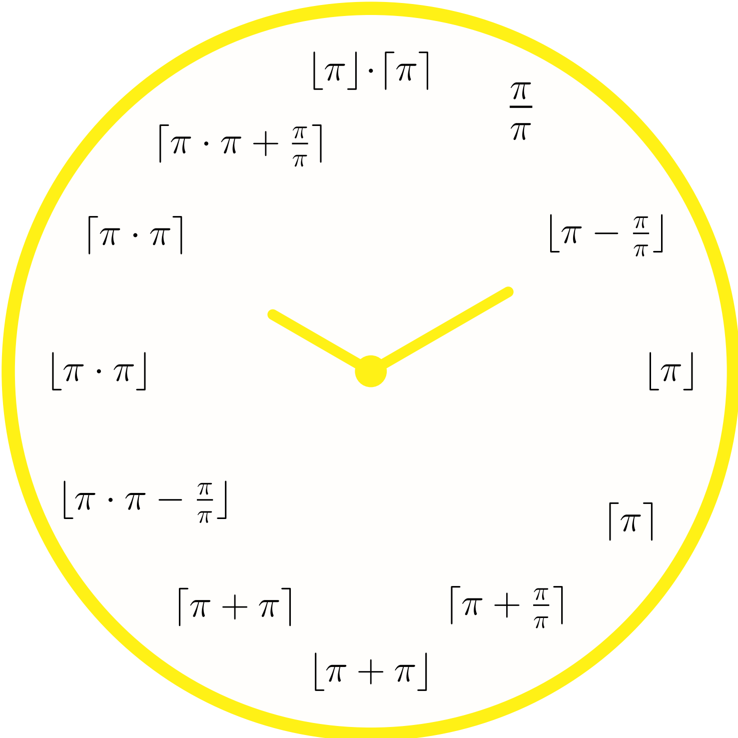

The numbers 1 to 12 are written using only the number pi, the basic arithmetic operations and the floor/ceiling functions.

On clock dials you find the numbers from 1 to 12: for a bit of change, can you write those numbers in a different way? For example, you can write 4 as $2 + 2$ or 2×2 or 2^2 by using the sum, the product, or by raising to a power. If you like the number pi ($\pi = 3.14159\ldots$) you could also write 4 as $\lceil \pi \rceil$ by using the ceiling function, i.e. by approximating π from above with the closest integer. There are infinitely many possibilities for expressing the number 4, and which one is the best is a matter of personal preference! In general, you can produce your own exclusive mathematical clock by writing 1 to 12 in your favourite way...

If you are looking for something extremely fancy, you can use Euler's identity [2] to express 1 with the Euler's number, pi, and the imaginary unit:

$$1 = -e^{\pi i}\,.$$

Or, if you like the Basel problem [1], you could display 6 as the sum of all reciprocals of the squares of the positive multiples of pi:

$$6 = \frac{1}{(\pi)^2} + \frac{1}{(\pi 2)^2} + \frac{1}{(\pi 3)^2} + \frac{1}{(\pi 4)^2} + \frac{1}{(\pi 5)^2} + \ldots$$

For something more concrete, you can pick up any digit from 1 to 9 and express the numbers on the clock dial only with this digit and mathematical symbols. Much more generally, *it is possible to write the integers from 1 to 12 by using only any given real number and mathematical symbols.* The reason is that we can always find a suitable expression for the number 1. Indeed, for a positive real number which is at most 1, it suffices to take the ceiling function (i.e. taking the integer approximation from above) to produce 1. For any real number x greater than 1 we can write 1 as the floor function of $\sqrt[x]{x}$ (i.e. taking the integer approximation from below of the x-th root of the number x), as can be checked with the logarithmic identity

$$\sqrt[x]{x} = e^{\frac{\log x}{x}} .$$

Finally, negative numbers may be turned positive with the absolute value, and for 0 one may use the factorial identity $0! = 1$.

There are plenty of mathematical clocks in circulation, with the most various expressions on the clock dial. Occasionally, rather than having the numbers from 1 to 12, you find the first twelve terms of a sequence. For example, you may have the first twelve Fibonacci numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 .$$

In the very same way, you could use the letters from A to L.

Another option is writing down an equation such that the desired number is the only solution, for example conveying 5 with

$$x^2 + 7 = 10x - 18 .$$

Some mathematical clocks show equations with more than one solution, but with exactly one solution among the integers from 1 to 12. In general, beware of mathematical inaccuracies in mathematical clocks (for example 3 is not really $\pi - 0.14$)!

Nevertheless, everything is allowed to have fun playing with numbers: the reader is challenged to produce its own mathematical clock!

Mathematical Clock Themes

In the image gallery below you find various mathematical clocks, with alternative ways to write the numbers from 1 to 12.

To have your own mathematical clock, you can print out one of those images and use it as a clock dial (e.g. you buy a clock with a custom dial), or you can simply place selected numbers' expressions above or around your clock. Notice that there are also boards combined with clocks, or clocks where you can write upon (this allows you to change your mathematical clock at leisure).

The themes' description is as follows:

- **1-Clock to 9-Clock:** One can fix any decimal digit from 1 to 9 and then display all integers 1 to 12 with short expressions involving only that digit and some arithmetic operations. We use the basic arithmetic operations, taking powers, and taking the square-root (notice that with the digits 5,6,7 we only use the basic arithmetic operations).
- **123-Theme:** It is possible to write all integers from 1 to 12 using only the digits 1,2,3 exactly once and in this order. We use the basic arithmetic operations, taking powers, taking the square-root, taking the factorial (the factorial of a natural number n, denoted by $n!$, is defined as the product of all natural numbers from 1 to n), and applying the floor function $\lfloor \rfloor$ (this function approximates a real number with the closest integer from below).
- **pi-Clock:** It is possible to write all integers from 1 to 12 using only the number pi, the basic arithmetic operations, and the floor/ceiling functions $\lfloor \rfloor$ and $\lceil \rceil$ (these functions approximate a real number with the closest integer from below and from above respectively).

- e**-Clock:** It is possible to write all integers from 1 to 12 using only Euler's number e, the basic arithmetic operations, taking powers, taking the square-root, and applying the floor/ceiling functions $\lfloor\,\rfloor$ and $\lceil\,\rceil$ (these functions approximate a real number with the closest integer from below and from above respectively).
- **Binary-Clock:** Here we write all numbers from 1 to 12 in the binary system, which only uses the digits 0 and 1.
- **PrimeNumbers-Clock:** Here we only write those numbers from 1 to 12 which are prime numbers.
- **ChineseNumerals-Clock:** Here we write all numbers from 1 to 12 with Chinese numerals: you may notice how the numbers 11 and 12 are composed by the symbol for 10 and the numbers 1 and 2 respectively.
- **MayaNumerals-Clock:** Here we write all numbers from 1 to 12 with the Maya numerals: a dot stands for the number 1, while a bar stands for five.

The pi-Clock, the e-Clock, the 1-Clock to the 9-Clock, and the 123-Clock have been developed by the author (for the 123-theme, we got inspired from [3]).

References

[1] C. J. Sangwin, *An infinite series of surprises*, Plus Magazine, 1 December 2001, `https://plus.maths.org/content/infinite-series-surprises`

[2] P. J. Nahin, *Dr. Euler's Fabulous Formula: Cures Many Mathematical Ills*, Princeton University Press, 2006, 416p.

[3] The Math Clock — 1-2-3 Edition, SBCrafts, `http://www.sbcrafts.net/clocks/`

	1	2	3	4	5	6	7	8	9	10	11	12
1 – Theme	1	$1+1$	$\sqrt{11-1-1}$	$(1+1)^{1+1}$	$\frac{11-1}{1+1}$	$\frac{11+1}{1+1}$	$\sqrt{\frac{111-11}{1+1}-1}$	$\frac{(11+1)(1+1)}{1+1+1}$	$\frac{111-11-1}{11}$	$11-1$	11	$11+1$
2 – Theme	2^{2-2}	2	$2+\frac{2}{2}$	$2\cdot 2$	$2^2+\frac{2}{2}$	2^2+2	$2^2+2+\frac{2}{2}$	$2\cdot 2^2$	$(2+\frac{2}{2})^2$	$2(2^2+\frac{2}{2})$	$\frac{2^{2^2}+2^2+2}{2}$	$2^{2^2}-2^2$
3 – Theme	3^{3-3}	$3-\frac{3}{3}$	3	$\frac{3\cdot 3+3}{3}$	$\frac{3^3+3}{3+3}$	$3+3$	$\frac{3^3+\frac{3}{3}}{3+\frac{3}{3}}$	$\frac{33-\frac{3}{3}}{3+\frac{3}{3}}$	$3\cdot 3$	$\frac{3^3+3}{3}$	$\frac{(3+3)(3+3)-3}{3}$	$3\cdot 3+3$
4 – Theme	4^{4-4}	$\sqrt{4}$	$\frac{4\cdot 4-4}{4}$	4	$4+\frac{4}{4}$	$\frac{44+4}{4+4}$	$\frac{44-4\cdot 4}{4}$	$4+4$	$(4-\frac{4}{4})^{\sqrt{4}}$	$(4+\frac{4}{4})\sqrt{4}$	$\frac{44}{4}$	$4^{\sqrt{4}}-4$
5 – Theme	$\frac{5}{5}$	$\frac{5+5}{5}$	$\frac{5\cdot 5+5}{5+5}$	$\frac{5\cdot 5-5}{5}$	5	$5+\frac{5}{5}$	$\frac{5\cdot 5+5+5}{5}$	$\frac{55+5\cdot 5}{5+5}$	$\frac{5\cdot 5+5\cdot 5-5}{5}$	$5+5$	$\frac{55}{5}$	$\frac{5\cdot 5\cdot 5-5}{5+5}$
6 – Theme	$\frac{6}{6}$	$\frac{6\cdot 6}{6+6+6}$	$\frac{6\cdot 6}{6+6}$	$\frac{6\cdot 6+6\cdot 6}{6+6+6}$	$\frac{6\cdot 6-6}{6}$	6	$6+\frac{6}{6}$	$\frac{66+6\cdot 6-6}{6+6}$	$\frac{66+6\cdot 6+6}{6+6}$	$\frac{66-6}{6}$	$\frac{66}{6}$	$6+6$
7 – Theme	$\frac{7}{7}$	$\frac{77+7}{7\cdot 7-7}$	$\frac{77+7}{77-7\cdot 7}$	$\frac{77-7\cdot 7}{7}$	$\frac{77-7}{7+7}$	$\frac{7\cdot 7-7}{7}$	7	$7+\frac{7}{7}$	$\frac{77+7\cdot 7}{7+7}$	$\frac{77-7}{7}$	$\frac{77}{7}$	$\frac{77+7}{7}$
8 – Theme	$\frac{88-8\cdot 8}{8+8+8}$	$\sqrt{\sqrt{8+8}}$	$\frac{88-8\cdot 8}{8}$	$\sqrt{8+8}$	$\frac{8\cdot 8+8\cdot 8-88}{8}$	$\sqrt{\frac{8\cdot 8+8\cdot 8+8+8}{\sqrt{8+8}}}$	$8-\frac{8}{8}$	8	$8+\frac{8}{8}$	$\sqrt{\frac{888-88}{8}}$	$\frac{88}{8}$	$\frac{8(8\cdot 8+8\cdot 8-8)}{88-8}$
9 – Theme	$.\bar{9}$	$\frac{99-9\cdot 9}{9}$	$\sqrt{9}$	$\sqrt{9}+\frac{9}{9}$	$\frac{9\sqrt{9}+\sqrt{9}}{9-\sqrt{9}}$	$9-\sqrt{9}$	$\frac{9\sqrt{9}+\frac{9}{9}}{\sqrt{9}+\frac{9}{9}}$	$\frac{9\cdot 9-\frac{9}{9}}{9+\frac{9}{9}}$	9	$\frac{99-9}{9}$	$\frac{99}{9}$	$9+\sqrt{9}$
π – Theme	$\lfloor\sqrt[\pi]{\pi}\rfloor$	$\lceil\sqrt{\pi}\rceil$	$\lfloor\pi\rfloor$	$\lceil\pi\rceil$	$\lfloor\pi\sqrt{\pi}\rfloor$	$\lfloor\pi+\pi\rfloor$	$\lfloor\pi^{\sqrt{\pi}}\rfloor$	$\lfloor\pi\cdot\pi-\sqrt{\pi}\rfloor$	$\lfloor\pi\cdot\pi\rfloor$	$\lfloor\pi\cdot\pi+\frac{\pi}{\pi\cdot\pi}\rfloor$	$\lfloor\frac{\pi^\pi}{\pi}\rfloor$	$\lfloor\pi\rfloor\cdot\lceil\pi\rceil$
e – Theme	$\lceil e^{-e}\rceil$	$\lceil e-\frac{e}{e}\rceil$	$\lceil e\rceil$	$\lfloor e\sqrt{e}\rfloor$	$\lfloor e+e\rfloor$	$\lceil\frac{e^e}{e}\rceil$	$\lfloor e\cdot e\rfloor$	$\lceil e\cdot e\rceil$	$\lfloor\frac{e^e}{\sqrt{e}}\rfloor$	$\lfloor e\cdot e+e\rfloor$	$\lfloor\frac{e^e}{\sqrt{e}}+e\rfloor$	$\lfloor e^e-e\rfloor$
123 – Theme	$\frac{1+2}{3}$	$1-2+3$	$(-1+2)\cdot 3$	1^2+3	$1\cdot 2+3$	$1+2+3$	$1+2\cdot 3$	$1\cdot 2^3$	$1+2^3$	$\lfloor 12-\sqrt{3}\rfloor$	$-1+2\cdot 3!$	$1!\cdot 2!\cdot 3!$