Almost Global Consensus on the n-Sphere

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***Abstract*—This paper establishes novel results regarding the global convergence properties of a large class of consensus protocols for multi-agent systems that evolve on the n-dimensional unit sphere or n-sphere. For any connected undirected graph and all $n \in \mathbb{N}\backslash\{1\}$, each protocol in said class is shown to yield almost global consensus. The feedback laws are intrinsic gradients of a Lyapunov function and include the canonical gradient descent protocol as a special case. This convergence result sheds new light on the general problem of consensus on Riemannian manifolds; the case of the n-sphere for $n \in \mathbb{N}\backslash\{1\}$ differs from those of the circle and $\mathsf{SO}(3)$ where the corresponding protocols fail to achieve almost global consensus. Finally, we derive a novel consensus protocol on $\mathsf{SO}(3)$ by combining two almost globally convergent protocols on the n-sphere for $n \in \{1, 2\}$. Theoretical and simulation results suggest that the combined protocol yields almost global consensus on $\mathsf{SO}(3)$.**

***Index Terms*—Consensus, agents and autonomous systems, cooperative control, aerospace, nonlinear systems.**

I. Introduction

CONSIDER a network of N agents whose states are points on an n-dimensional manifold. Each agent has a limited capability to sense certain information that pertain to some of the other agents. Distributed control protocols allow multi-agent systems to synchronize, *i.e.*, to reach a consensus as information propagates over time by means of local interactions [1]. There are a number of results pertaining to the case when the initial states of all agents belong to a geodesically convex subset of the manifold [2]–[4], but the likelihood of encountering such a scenario by chance decreases exponentially with N. The problem of almost global consensus on non-Euclidean manifolds is largely unexplored and requires further study [5], [6]. This paper establishes almost global convergence for a large class of consensus protocols on all n-spheres except the circle, a rather unexpected finding. Consensus problems on the circle and the sphere arise in a number of engineering applications, including cooperative reduced rigid-body attitude control [7], [8], planetary scale mobile sensing networks [9], and self synchronizing chemical and biological oscillators described by the Kuramoto model [10], [11].

The reduced attitude provides a model for the orientation of objects that for various reasons, such as task redundancy, cylindrical symmetry, actuator failure, *etc.*, lack one degree of rotational freedom in three-dimensional space. The orientation of such objects corresponds to a pointing direction with the rotation about the axis of pointing being of little to no importance [12]. The reduced attitude synchronization problem is equivalent to the consensus problem on the 2-sphere. The problem of cooperative control on the n-sphere in $\mathbb{R}^{n+1}$, denoted $\mathcal{S}^n$, has received some attention in the literature [13]–[19] but comparatively less than equivalent problems on $\mathsf{SO}(3)$ for which there is a vast literature including [2], [6], [20]–[28].

The problem of almost global consensus has been studied on $\mathcal{S}^1$ [14]–[16], on $\mathsf{SO}(3)$ [29], [30], on $\mathcal{S}^n$ in the special case of a complete graph [13], [17], and on other Riemannian manifolds [5]. Tron *et al.* [29] apply an optimization based method to characterize the stability of all equilibria on $\mathsf{SO}(3)$ for a particular discrete-time consensus protocol. Their result is akin to almost global consensus over any connected graph topology. The algorithm makes use of a reshaping function which depends on a parameter that must exceed a bound whose value cannot be calculated from local information. Moreover, the overall convergence speed of the algorithm decreases with increasing values of the parameter. In contrast to [29], this paper shows that almost global convergence of a large class of consensus protocols on $\mathcal{S}^n$ for $n \in \mathbb{N}\backslash\{1\}$ can be established without the use of a reshaping function or any non-local knowledge of the graph.

Tron *et al.* [29] divide the literature on discrete-time attitude consensus into two categories: extrinsic and intrinsic algorithms. An algorithm is said to be extrinsic if it calculates iterates in a Euclidean space and then projects them on $\mathsf{SO}(3)$. There are extrinsic algorithms that provide consensus on a global level [5]. The contribution of [29] is to provide the first intrinsic almost globally convergent discrete-time consensus protocol on $\mathsf{SO}(3)$. By intrinsic they refer to an algorithm that evolves on $\mathsf{SO}(3)$ without relying on the embedding of $\mathsf{SO}(3)$ in some ambient space. To further the line of inquiry developed in [29], this paper formulates a consensus protocol for systems that evolve on $\mathsf{SO}(3)$ in continuous time. This protocol only depends on the number of agents and displays stability properties to rival those of [29]. In this paper we use the canonical embedding of $\mathcal{S}^n$ in $\mathbb{R}^{n+1}$ to study control algorithms for systems that evolve continuously on $\mathcal{S}^n$. The advantage of not using any other parametrization is to avoid the so-called unwinding phenomenon [31], where an equilibrium set is attractive but unstable.

The 2-sphere is akin to the quotient space $\mathsf{SO}(3)/\mathsf{SO}(2)$ and, as such, many results obtained for $\mathsf{SO}(3)$ also apply to $\mathcal{S}^2$. Special cases sometimes allow for stronger results. This paper shows that the conditions for achieving almost global consensus are more favorable on $\mathcal{S}^n$ for $n \in \mathbb{N}\backslash\{1\}$ than what is implied by previously known results concerning $\mathcal{S}^1$ and $\mathsf{SO}(3)$. A large class of intrinsic consensus protocols over connected, undirected graph topologies renders all equilibria but the consensus set unstable on $\mathcal{S}^n$. By contrast, analysis of the corresponding consensus protocols on $\mathcal{S}^1 \simeq \mathsf{SO}(2)$ [15], [16] and a simulation study on $\mathsf{SO}(3)$ [29] show that certain graph topologies yield equilibrium sets aside from consensus

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that are asymptotically stable on $\mathsf{SO}(n)$, $n \in \{2, 3\}$.

The literature on continuous-time cooperative control on the n-sphere has largely concerned special cases. Previous work either concerns the case of a specific graph topology [8], [13], [17], [30], a specific sphere [5], [8], [13], [14], [16], [18], or a specific control law [8], [13], [29], [30], [32]. Many of them also lack a rigorous proof of almost global convergence [5], [14], [29], [30], [32], [33]. They only show that all equilibrium sets except the consensus set are unstable, which is a weaker result in general [34]. We provide a rigorous proof of almost global convergence for a large class of analytic consensus protocols over any connected graph by showing that the region of attraction of any exponentially unstable equilibrium set have measure zero on $(\mathcal{S}^n)^N$.

In the literature survey [35], it is observed that almost global convergence of consensus protocols on nonlinear spaces (in particular $\mathcal{S}^1$) is graph dependent and discusses three control design procedures to circumvent this problem: reshaping functions [14], [15], [29], gossip algorithms [15], and dynamic feedback [5]. The main contribution of this paper is to show that consensus on $\mathcal{S}^n$ is *not* graph dependent for any $n \in \mathbb{N} \backslash \{1\}$, and that almost global consensus can be achieved without utilizing any of the three designs in [35]. This leads to the contra-intuitive but intriguing notion that almost global consensus is more difficult to achieve on $\mathcal{S}^1$ than any other sphere. Preliminary results are found in [32], conjectures are made in [7], [13].

II. Problem Description

The following notation is used in this paper. The inner and outer product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\mathbf{x} \otimes \mathbf{y}$, respectively. The inner product of $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ is $\langle \mathbf{A}, \mathbf{B} \rangle = \operatorname{tr} \mathbf{A}^\top \mathbf{B}$. Let $\|\cdot\|$ denote the Euclidean norm of a vector and $\|\cdot\|_F$ the Frobenius norm of a matrix. The ordinary gradient is denoted $\nabla : f(\mathbf{x}) \mapsto \nabla f(\mathbf{x}) \in \mathbb{R}^n$, the intrinsic gradient on a manifold $\mathcal{M}$ is denoted $\square : f(\mathbf{x}) \mapsto \square f(\mathbf{x}) \in \mathsf{T}_\mathbf{x}\mathcal{M}$. The special orthogonal group is $\mathsf{SO}(n) = \{\mathbf{R} \in \mathbb{R}^{n\times n} \,|\, \mathbf{R}^{-1} = \mathbf{R}^\top, \det \mathbf{R} = 1\}$. The Lie algebra of $\mathsf{SO}(n)$ is $\mathsf{so}(n) = \{\mathbf{S} \in \mathbb{R}^{n\times n} \,|\, \mathbf{S}^\top = -\mathbf{S}\}$. The n-sphere in Euclidean space is $\mathcal{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \,|\, \|\mathbf{x}\| = 1\}$, where $n \in \mathbb{N}$. An undirected, simple graph is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} \subset \mathbb{N}$ is the node set and $\mathcal{E} \subset \{e \subset \mathcal{V} \,|\, |e| = 2\}$ is the edge set. A graph $\mathcal{G}$ is said to be connected if it contains a tree subgraph with $\|\mathcal{V}\| - 1$ edges.

A. Distributed Control Design on the n-Sphere

Consider a multi-agent system where each agent corresponds to an index $i \in \mathcal{V}$, and has a state $\mathbf{x}_i \in \mathcal{S}^n$ expressed in a world coordinate frame $\mathcal{W}$. Agent i uses a body-fixed frame $\mathcal{B}_i$ that relates to $\mathcal{W}$ by a rotation matrix $\mathbf{R}_i(t) \in \mathsf{SO}(n)$ for all $t \in [0, \infty)$. Define $\mathbf{R}_i : [\mathbf{v}]_{\mathcal{B}_i} \mapsto [\mathbf{v}]_\mathcal{W}$, where the bracket $[\,\cdot\,]_\mathcal{F}$ denote that its content is expressed in a frame $\mathcal{F}$. If the frame is omitted, then $\mathcal{W}$ is presupposed. Chose the reduced attitude $\mathbf{x}_i$ of agent i to satisfy $[\mathbf{x}_i]_{\mathcal{B}_i} = \mathbf{e}_1$. Thus $[\mathbf{x}_i]_\mathcal{W} = \mathbf{R}_i[\mathbf{x}_i]_{\mathcal{B}_i} = \mathbf{R}_i\mathbf{e}_1$, *i.e.*, $[\mathbf{x}_i]_\mathcal{W}$ is given by the first column of $\mathbf{R}_i$. The agents are capable of limited local sensing. The topology of the communication network is described by an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{i \in \mathbb{N} \,|\, i \leq N\}$, and $\{i, j\} \in \mathcal{E}$ implies that two neighboring agents i and j can sense the so-called relative information $[\mathcal{I}_{ij}]_{\mathcal{B}_i}$, $[\mathcal{I}_{ji}]_{\mathcal{B}_j}$ (see below) regarding the displacement of their states $\mathbf{x}_i$ and $\mathbf{x}_j$.

Control is based on relative information. The information that agent i has access to regarding its neighbor agent j could be defined to include

$$[\operatorname{pos}\{\mathbf{x}_j - \mathbf{x}_i\}]_{\mathcal{B}_i} \subseteq [\mathcal{I}_{ij}]_{\mathcal{B}_i}, \tag{1}$$

which is the relative information customary to the ambient space $\mathbb{R}^{n+1}$. The set of neighbors of agent i is $\mathcal{N}_i = \{j \in \mathcal{V} \,|\, \{i, j\} \in \mathcal{E}\}$. The set of relative information known to agent $i \in \mathcal{V}$ is $[\mathcal{I}_i]_{\mathcal{B}_i} = \operatorname{pos} \cup_{j\in\mathcal{N}_i} [\mathcal{I}_{ij}]_{\mathcal{B}_i}$. The dynamics (2) of agent i projects the input of agent i on the tangent space $\mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$, *i.e.*, on a hyperplane orthogonal to $\mathbf{x}_i$.

Remark 1. *It can be argued that*

$$\operatorname{pos}\{\mathbf{P}_i(\mathbf{x}_j - \mathbf{x}_i)\} \subseteq \mathcal{I}_{ij},$$

where $\mathbf{P}_i : \mathbb{R}^{n+1} \to \mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$ *is an orthogonal projection matrix, is preferable to* (1) *since it confines* $\mathcal{I}_{ij}$ *to an intrinsic rather than on ambient space. However, we argue that the constraints on* $\mathcal{I}_{ij}$ *come from limited sensing capabilities rather than rigid-body dynamics, and that most applications on* $\mathcal{S}^2$ *involve sensors that measure features of ambient rather than intrinsic space.*

While agent i may not be able to calculate some $[\mathbf{u}_i]_{\mathcal{B}_i} \in [\mathcal{I}_i]_{\mathcal{B}_i}$ based on the information (1) obtained from all its neighbors, that agent may still be able to calculate an input $[\mathbf{v}_i]_{\mathcal{B}_i} \in [\mathcal{I}_{ij}]_{\mathcal{B}_i}$ whose projection on $\mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$ by the dynamics of $\mathbf{x}_i$ is identical to that of $\mathbf{u}_i$. This holds for inputs that belongs to $\operatorname{span} \cup_{j\in\mathcal{N}_i} \mathbf{x}_j$, and in particular for elements of the positive cone $\operatorname{pos} \cup_{j\in\mathcal{N}_i} \mathbf{x}_j$. Intuitively speaking, it is reasonable to assume that agent i should be able to sense the bearing and distance to any of its neighbors, and we therefore set $[\mathcal{I}_i]_{\mathcal{B}_i} = [\operatorname{pos} \cup_{j\in\mathcal{N}_i} \{\mathbf{x}_j\}]_{\mathcal{B}_i}$.

System 2. *The system is given by* N *agents, an undirected and connected graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$*, agent states* $\mathbf{x}_i \in \mathcal{S}^n$*, where* $n \in \mathbb{N}$*, and dynamics*

$$\dot{\mathbf{x}}_i = \mathbf{u}_i - \langle \mathbf{u}_i, \mathbf{x}_i \rangle \mathbf{x}_i = (\mathbf{I} - \mathbf{X}_i)\mathbf{u}_i = \mathbf{P}_i\mathbf{u}_i, \tag{2}$$

where $\mathbf{u}_i : \mathcal{I}_i \to \mathbb{R}^{n+1}$ *is the input signal of agent* i*,* $\mathbf{X}_i = \mathbf{x}_i \otimes \mathbf{x}_i$*, and* $\mathbf{P}_i = \mathbf{I} - \mathbf{X}_i$ *for all* $i \in \mathcal{V}$.

The results and proofs in this paper are carried out using the world frame $\mathcal{W}$. To implement the control law in a distributed fashion, $\mathbf{u}_i$ must be transfered to $\mathcal{B}_i$ for all $i \in \mathcal{V}$. Let a control law in $\mathcal{W}$ be given by $[\mathbf{u}_i]_\mathcal{W} = \sum_{j\in\mathcal{N}_i} f_{ij}[\mathbf{x}_j]_\mathcal{W}$. Hence $[\mathbf{u}_i]_{\mathcal{B}_i} = \sum_{j\in\mathcal{N}_i} f_{ij}\mathbf{R}_i^\top[\mathbf{x}_j]_\mathcal{W} = \sum_{j\in\mathcal{N}_i} f_{ij}[\mathbf{x}_j]_{\mathcal{B}_i}$. Moreover,

$$\begin{aligned}[\dot{\mathbf{x}}_i]_{\mathcal{B}_i} &= \mathbf{R}_i[\dot{\mathbf{x}}_i]_\mathcal{W} = \mathbf{R}_i^\top[\mathbf{u}_i]_\mathcal{W} - \langle [\mathbf{u}_i]_\mathcal{W}, [\mathbf{x}_i]_\mathcal{W} \rangle \mathbf{R}_i^\top [\mathbf{x}_i]_\mathcal{W} \\ &= [\mathbf{u}_i]_{\mathcal{B}_i} - \langle [\mathbf{u}_i]_{\mathcal{B}_i}, [\mathbf{x}_i]_{\mathcal{B}_i} \rangle [\mathbf{x}_i]_{\mathcal{B}_i},\end{aligned} \tag{3}$$

since inner products are invariant under orthogonal changes of coordinates. It is clear from (3) that (2) can be implemented in a distributed fashion.

The problem of multi-agent consensus on $\mathcal{S}^n$ concerns the design of distributed control protocols $(\mathbf{u}_i)_{i=1}^N$ based on relative information, as discussed in the above paragraphs, that stabilize the consensus set

$$\begin{aligned}\mathcal{C} &= \{(\mathbf{y}_i)_{i=1}^N \in (\mathcal{S}^n)^N \,|\, \mathbf{y}_i = \mathbf{y}_j,\, \forall\, i,j \in \mathcal{V}\} \\ &= \{(\mathbf{y}_i)_{i=1}^N \in (\mathcal{S}^n)^N \,|\, \mathbf{y}_i = \mathbf{y}_j,\, \forall\, \{i,j\} \in \mathcal{E}\} \end{aligned} \tag{4}$$

of System 2, where the second equality hinges on the assumption that $\mathcal{G}$ is connected. If the states of all agents assume the same value on the n-sphere, then they are said to reach consensus. Terms such as consensus, synchronization, rendezvous, and state-aggregation are used interchangeably in this paper, but note that some authors, see *e.g.*, [5], [17], assign the definition of these concepts subtle nuances.

B. Problem Statement

This paper concerns some aspects of control design but the main focus is stability analysis. Algorithm 3 is arguably the most basic conceivable feedback for consensus on $\mathcal{S}^n$ by virtue of its correspondence with the linear consensus protocol on $\mathbb{R}^{n+1}$ for single integrator dynamics given by $\dot{\mathbf{x}}_i = \mathbf{u}_i$ for all $i \in \mathcal{V}$. As such, it is of interest to determine the limits of Algorithm 3's performance, *i.e.*, the global level stability of the consensus set $\mathcal{C}$ as an equilibrium set of System 2. It is important to establish that the region of attraction of the undesired equilibria is thin, *i.e.*, meager in the sense of Baire and of measure zero [34].

Algorithm 3. *The feedback is given by* $\mathbf{u}_i = \sum_{j\in\mathcal{N}_i} f_{ij}\mathbf{x}_j$, *where the constants* $f_{ij} \in (0,\infty)$ *satisfy* $f_{ij} = f_{ji}$ *for all* $\{i,j\} \in \mathcal{E}$.

Definition 4 (Measure zero)**.** *A set* $\mathcal{N} \subset (\mathcal{S}^n)^N$ *has measure zero if for every chart* $\phi : \mathcal{D} \to \mathbb{R}^{N(n+1)}$ *in some atlas of* $(\mathcal{S}^n)^N$*, it holds that* $\phi(\mathcal{D}\cap\mathcal{N})$ *has Lebesgue measure zero.*

Definition 5 (Almost global attractiveness)**.** *Consider a system that evolves on* $(\mathcal{S}^n)^N$*. A set of equilibria* $\mathcal{D} \subset (\mathcal{S}^n)^N$ *is said to be almost globally attractive if for all initial conditions* $(\mathbf{x}_{i,0})_{i=1}^N \in (\mathcal{S}^n)^N \backslash \mathcal{N}$*, where* $\mathcal{N}$ *is some set of zero measure, it holds that* $\lim_{t\to\infty}(\mathbf{x}(t))_{i=1}^N \in \mathcal{D}$*.*

Problem 6. *Show that there is a large class of consensus protocols for System 2, including Algorithm 3, such that the consensus set* $\mathcal{C}$ *is stable and almost globally attractive.*

Problem 6 concerns the global behavior of System 2. Under certain assumptions regarding the connectivity of $\mathcal{G}$, local consensus on $\mathcal{S}^n$ can be established with the region of attraction being the largest geodesically convex sets on $\mathcal{S}^n$, *i.e.*, open hemispheres [19]. See also [25] in the case of an undirected graph and [27] in the case of a directed and time-varying graph. A global stability result for discrete-time consensus on $\mathsf{SO}(3)$ is provided in [29]. Almost global asymptotical stability of the consensus set on the n-sphere is known to hold when the graph is a tree [25] or is complete in the case of first- and second-order models [13], [17]. The author of [13] conjectures that global stability also holds for a larger class of topologies whereas [15], [16] provides counter-examples of basic consensus protocols that fail to generate consensus on $\mathcal{S}^1$.

Remark 7. *Global consensus on* $\mathcal{S}^n$ *cannot be achieved by means of a continuous feedback due to topological constraints [31]. It is however possible to achieve almost global asymptotical stability, as has been demonstrated on the circle [15], [16]. To prove almost global convergence to the consensus set is difficult since basic tools such as the Hartman-Grobman theorem or stable-unstable manifold theorems are unavailable due to the equilibria being nonhyperbolic [36]. Feasible approaches include dual Lyapunov stability theory [37] and a technique based on the stability in the first approximation [34] that applies to convergent systems.*

III. Stability of the Consensus Manifold

This section and the next concern System 2 governed by Algorithm 8 which is an extension of Algorithm 3. Algorithm 8 provides a large class of smooth continuous-time consensus protocol on the n-sphere. The stability properties of all equilibria are fully determined, as is that of the overall system.

A. Known Results

Consider a class of consensus protocols that formalizes the idea of increasing system cohesion by moving an agent into the convex hull of its state and those of its neighbors.

Algorithm 8. *The input is given by*

$$\mathbf{u}_i = \sum_{j\in\mathcal{N}_i} f_{ij}(s_{ij})\mathbf{x}_j,$$

where $s_{ij} = 1 - \langle \mathbf{x}_i, \mathbf{x}_j\rangle$ *and the feedback gains* $f_{ij} : \mathbb{R} \to \mathbb{R}$ *are real analytic functions that satisfy*

(i) $f_{ij} : [0,2] \to [0,\infty)$,
(ii) $f_{ij} = f_{ji}$,
(iii) $(n-2+s_{ij})s_{ij}f_{ij} - (2-s_{ij})s_{ij}^2 f_{ij}' > 0$,

for all $s_{ij} \in (0,2]$ *and all* $\{i,j\} \in \mathcal{E}$.

Note that f_{ij} depends on $s_{ij} : \mathcal{S}^n \times \mathcal{S}^n \to [0,2]$ given by

$$s_{ij} = \tfrac{1}{2}\|\mathbf{x}_j - \mathbf{x}_i\|^2 = 1 - \langle \mathbf{x}_i, \mathbf{x}_j\rangle, \tag{5}$$

which is invariant under orthogonal changes of coordinates. Algorithm 8 therefore complies with the requirements of Section II-A regarding distributed feedback laws over the n-sphere. Various forms of the closed loop dynamics of System 2 under Algorithm 8 is stated on the readers behalf and for the sake of completeness

$$\begin{aligned}\dot{\mathbf{x}}_i &= \sum_{j\in\mathcal{N}_i} f_{ij}(s_{ij})\mathbf{x}_j - \left\langle \sum_{j\in\mathcal{N}_i} f_{ij}(s_{ij})\mathbf{x}_j, \mathbf{x}_i \right\rangle \mathbf{x}_i \\ &= \sum_{j\in\mathcal{N}_i} f_{ij}(s_{ij})(\mathbf{x}_j - (1-s_{ij})\mathbf{x}_i) \\ &= \mathbf{P}_i \sum_{j\in\mathcal{N}_i} f_{ij}(s_{ij})\mathbf{x}_j. \end{aligned} \tag{6}$$

Remark 9. *Algorithm 8 comprises a class of algorithms that includes those of Algorithm 3 for all* $n \in \mathbb{N}\backslash\{1\}$*. If* $f_{ij} = k \in$

$(0,\infty)$ for all $\{i,j\}\in\mathcal{E}$, then (iii) evaluates to $k(n-2+s_{ij})s_{ij}\geq ks_{ij}^2>0$ for all $s_{ij}\in(0,2]$ when $n\geq 2$ but for $n=1$ we obtain

$$(-1+s_{ij})s_{ij}\cdot k+(2-s_{ij})s_{ij}^2\cdot 0=-k(1-s_{ij})s_{ij}\leq 0$$

for all $s_{ij}\in[0,1]$. Note that the class grows with n. For example, if $f_{ij}=s_{ij}^k$ for some $k\in\mathbb{N}$ then (iii) evaluates to

$$(n-(k+1)(2-s_{ij}))s_{ij}^{k+1},$$

which is positive on $(0,2]$ when $k\leq n/2-1$. To see that the class is empty for $n=1$, note that (iii) can be rewritten as

$$\frac{f_{ij}'}{f_{ij}}<\frac{-1+s_{ij}}{(2-s_{ij})s_{ij}} \tag{7}$$

for all $\{i,j\}\in\mathcal{E}$ and all $s_{ij}\in(0,2)$. This implies $\lim_{s_{ij}\to 0}f_{ij}'/f_{ij}=-\infty$. Since f_{ij}' is continuous, it is bounded on $[0,2]$ whereby $f_{ij}(0)=0$ and $f_{ij}'(0)\leq 0$. Even if $f_{ij}'(0)=0$, the inequality (7) still implies that $f_{ij}'(s)<0$ for all $s\in(0,\delta)$ for some $\delta\in(0,\infty)$. By continuity there exists an $\varepsilon\in(0,\infty)$ such that $f_{ij}(s_{ij})<0$ for all $s_{ij}\in(0,\varepsilon)$, which contradicts requirement (i) of Algorithm 8.

Remark 10. *For some feedback gains f_{ij} there is a ball in the space $\mathcal{C}^\omega$ of real analytic functions consisting entirely of feedback gains of other elements of Algorithm 8. For instance, Algorithm 3 still converges if instead of a constant f_{ij} agent i and j use $f_{ij}+g_{ij}$, where $g_{ij}\in\mathcal{C}^\omega$ is of sufficiently small norm. This could be interpreted as a form of robustness against analytic radial errors, e.g., constant measurement errors that are due to biased sensors.*

Algorithm 8 can be derived by taking the gradient of the candidate Lyapunov function

$$V(s_{ij}\,|\,\{i,j\}\in\mathcal{E})=\sum_{\{i,j\}\in\mathcal{E}}\int_0^{s_{ij}}f_{ij}(r)\mathrm{d}r. \tag{8}$$

Denote $\nabla V=(\nabla_i V)_{i=1}^N$, where $\nabla_i=\nabla_{\mathbf{x}_i}$. Then

$$\nabla_i V=\sum_{j\in\mathcal{N}_i}\frac{\mathrm{d}V}{\mathrm{d}s_{ij}}\nabla_i s_{ij}=-\sum_{j\in\mathcal{N}_i}f_{ij}(s_{ij})\mathbf{x}_j. \tag{9}$$

It follows that $\mathbf{u}_i=-\nabla_i V$ and $\dot{\mathbf{x}}_i=-\mathbf{P}_i\nabla_i V$ for all $i\in\mathcal{V}$.

Proposition 11. *System 2 under Algorithm 8 converges to an equilibrium set in $(\mathcal{S}^n)^N$. At any equilibrium point, each input is parallel to the state of its agent.*

Proof. Consider the potential function (8). It holds that

$$\begin{aligned}\dot{V}&=\sum_{\{i,j\}\in\mathcal{E}}f_{ij}\dot{s}_{ij}=-\sum_{\{i,j\}\in\mathcal{E}}f_{ij}(\langle\dot{\mathbf{x}}_i,\mathbf{x}_j\rangle+\langle\mathbf{x}_i,\dot{\mathbf{x}}_j\rangle)\\&=-\sum_{i\in V}\left\langle\dot{\mathbf{x}}_i,\sum_{j\in\mathcal{N}_i}f_{ij}\mathbf{x}_j\right\rangle-\sum_{j\in V}\left\langle\sum_{i\in\mathcal{N}_j}f_{ij}\mathbf{x}_i,\dot{\mathbf{x}}_j\right\rangle\\&=-2\sum_{i\in V}\langle\mathbf{u}_i-\langle\mathbf{u}_i,\mathbf{x}_i\rangle\mathbf{x}_i,\mathbf{u}_i\rangle\\&=-2\sum_{i\in V}\|\mathbf{u}_i\|^2-\langle\mathbf{u}_i,\mathbf{x}_i\rangle^2\end{aligned} \tag{10}$$

System 2 converges to the set $\{(\mathbf{x}_i)_{i=1}^N\,|\,\dot{V}=0\}$ by LaSalle's theorem. The Cauchy-Schwarz inequality applied to (10) shows that the input and state of each agent align up to sign asymptotically. This implies $\dot{\mathbf{x}}_i=\mathbf{0}$ for all $i\in\mathcal{V}$, *i.e.*, that the system is at an equilibrium by inspection of (2). □

The equilibria that are characterized by Proposition 11 can be divided into three categories:

$$(\mathbf{x}_i,\mathbf{u}_i)\in\left\{\left(-\frac{\mathbf{u}_i}{\|\mathbf{u}_i\|},\mathbf{u}_i\right),\left(\frac{\mathbf{u}_i}{\|\mathbf{u}_i\|},\mathbf{u}_i\right),(\mathbf{x}_i,\mathbf{0})\right\}, \tag{11}$$

where $\mathbf{u}_i=\sum_{j\in\mathcal{N}_i}f_{ij}\mathbf{x}_j$ for all $i\in\mathcal{V}$. The case of $\mathbf{u}_i=\mathbf{0}$ for all $i\in\mathcal{V}$ is illustrated by Figure 1. The agent states in Figure 1 correspond to the six corners of an octahedron, which is one of the five platonic solids. Likewise, the tetrahedral graph (*i.e.*, the complete graph over four nodes) has the tetrahedron as an equilibrium with $\mathbf{x}_i=-\mathbf{u}_i/\|\mathbf{u}_i\|$ for all $i\in\mathcal{V}$; whereas the cube, icosahedral, and dodecahedral graphs have respectively the cube, icosahedron, and dodecahedron as equilibria with $\mathbf{x}_i=\mathbf{u}_i/\|\mathbf{u}_i\|$ for all $i\in\mathcal{V}$.

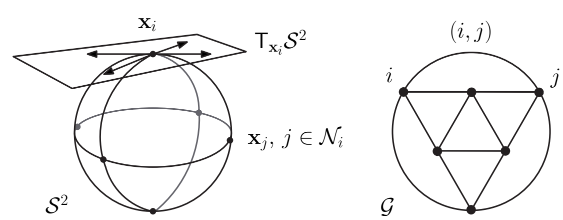


Fig. 1. An equilibrium of a system on $\mathcal{S}^2$ (left) with an octahedral graph (right). The sum of neighbor states projected on the tangent plane $\mathsf{T}_{\mathbf{x}_i}\mathcal{S}^2$ is zero (left).

Analogues of the following result, Proposition 12 concerning consensus over the largest geodesically convex sets on $\mathcal{S}^n$, *i.e.*, open hemispheres, and various generalizations thereof, are known to the control community. For example, [4] uses invariant convex hulls in a manner that was preceded in [2], [38] to prove local convergence of time switched consensus protocols on $\mathsf{SE}(3)$. To solve Problem 6, this paper provides a companion to Proposition 12, Theorem 13, that characterize all equilibrium sets of System 2 under Algorithm 8 in terms of attractiveness and stability. Although Proposition 12 is used in the proof of Theorem 13, its full power is not needed. Rather, it is included as a contrast to highlight the greater generality achieved by our analysis.

Proposition 12. *Consider System 2 under Algorithm 8. The consensus set $\mathcal{C}$ is asymptotically stable. Moreover, the system reaches consensus asymptotically if there is some finite time such that all agents belong to an open hemisphere.*

Proof. Let $\mathcal{H}$ denote the open hemisphere. Since $f_{ij}\in[0,\infty)$ for all $j\in\mathcal{N}_i$, $\dot{\mathbf{x}}_i=\mathbf{P}_i\sum_{j\in\mathcal{N}_i}f_{ij}\mathbf{x}_j$ points towards the geodesically convex hull of $\{\mathbf{x}_j\,|\,j\in\mathcal{N}_i\}$ on $\mathcal{S}^n$ along the tangent space $\mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$. This shows $\mathcal{H}$ to be invariant and $\mathcal{C}$ to be stable. It remains to show attractiveness. Proposition 11 establishes that System 2 under Algorithm 8 converges to an equilibrium set. Since $\mathcal{H}$ is invariant the desired result follows if the only equilibrium configuration on $\mathcal{H}$ is a consensus.

There must be at least one agent k that minimizes the distance to the boundary of $\mathcal{H}$. At any equilibrium, it holds that $\mathbf{x}_i$ is parallel $\mathbf{u}_i$ for all $i \in \mathcal{V}$ by Proposition 11. Since all agents belong to an open hemisphere it follows that $\mathbf{0}, -\mathbf{x}_i \notin \operatorname{pos}\{\mathbf{x}_i \,|\, i \in \mathcal{V}\}$. By (11), only $\mathbf{x}_i = {}^{\mathbf{u}_i}/_{\|\mathbf{u}_i\|}$ remains. Agent k belongs to an extreme ray of the convex cone $\operatorname{pos}\{\mathbf{x}_i \,|\, i \in \mathcal{V}\}$. But then $\mathbf{x}_k = {}^{\mathbf{u}_k}/_{\|\mathbf{u}_k\|}$ if and only if $\mathbf{x}_j = \mathbf{x}_k$ for all $j \in \mathcal{N}_i$. An induction argument can be applied to show that the system is at a consensus due to $\mathcal{G}$ being connected. □

B. Main Result

In light of the results of the previous section, we hereby state our main result.

Theorem 13. *Consider System 2 under Algorithm 8 in the case of $n \in \mathbb{N}\backslash\{1\}$. The consensus set $\mathcal{C}$ given by (4) is almost globally asymptotically stable. Moreover, each trajectory of the system converges to some point. The set of unstable equilibria is meager. The rate of convergence is locally exponential if the feedback gains f_{ij} are nonzero over $\mathcal{C}$ for all $\{i,j\} \in \mathcal{E}$.*

The proof of Theorem 13 is given in Section IV-D. Let us briefly sketch the main ideas. That the consensus set is asymptotically stable follows from Proposition 12. To prove the exponential instability of the undesired equilibria we use the indirect method of Lyapunov. The system is linearized around an equilibrium on the n-sphere. Perturbing all agents towards the north pole increases cohesion in the north hemisphere while depleting it in the south. One such perturbation corresponds to a direction of instability for the linearized system. Finally, a known result connects exponential instability with a measure zero and meager region of attraction.

IV. Instability of Undesired Equilibrium Sets

The global behavior of the system is determined by the stability and attractiveness of all its equilibria, which often can be characterized locally by linearization. However, to get the global picture we must determine the measure of the see of attraction for all unstable equilibria. It is possible for a set of exponentially unstable equilibria to have a non-zero region of attraction, but only if the system fails to be convergent [34]. Our control design guarantees System 2 under Algorithm 8 to be convergent as is shown in Proposition 20. Let us therefore study the signs of the real part of the linearization of System 2 under Algorithm 8 in order to establish the instability of all undesired equilibria and equilibrium sets.

A. Linearization on the N-Fold n-Sphere

Proposition 14. *The $(n+1)\times(n+1)$ blocks of the $N(n+1)\times N(n+1)$ matrix $\mathbf{A}$ that describes the linearization on $\mathcal{S}^n$ of System 2 under Algorithm 8 are given by*

$$\mathbf{A}_{ii} = -(\langle \mathbf{u}_i, \mathbf{x}_i\rangle \mathbf{I} + \mathbf{x}_i \otimes \mathbf{u}_i)\mathbf{P}_i - \sum_{j\in\mathcal{N}_i} f'_{ij}\mathbf{P}_i\mathbf{X}_j\mathbf{P}_i,$$

for $i \in \mathcal{V}$,

$$\mathbf{A}_{ij} = \mathbf{P}_i\left(f_{ij}\mathbf{I} - f'_{ij}\mathbf{x}_j \otimes \mathbf{x}_i\right)\mathbf{P}_j$$

for $\{i,j\} \in \mathcal{E}$, and $\mathbf{A}_{ij} = \mathbf{0}$ otherwise. The matrix $\mathbf{A}$ is symmetric at all equilibria.

Proof. For systems evolving on manifolds, a perturbation technique is used to obtain the linearized dynamics. Let $\mathbf{x}_i$ for all $i \in \mathcal{V}$ be a solution to (2). Consider a perturbed solution $\mathbf{x}_i(\varepsilon, \mathbf{v}_i)$ given by

$$\mathbf{x}_i(\varepsilon, \mathbf{v}_i) = \frac{\mathbf{x}_i + \varepsilon\mathbf{v}_i}{\|\mathbf{x}_i + \varepsilon\mathbf{v}_i\|},$$

where $\mathbf{v}_i$ is a nonzero constant vector for all $i \in \mathcal{V}$. The perturbed solution is required to satisfy the differential equation

$$\dot{\mathbf{x}}_i(\varepsilon, \mathbf{v}_i) = (\mathbf{I} - \mathbf{X}_i(\varepsilon, \mathbf{v}_i))\mathbf{u}_i(\varepsilon, (\mathbf{x}_j(\varepsilon, \mathbf{v}_j))_{j\in\mathcal{N}_i}),$$

where $\mathbf{X}_i(\varepsilon, \mathbf{v}_i) = \mathbf{I} - \mathbf{x}_i(\varepsilon, \mathbf{v}_i) \otimes \mathbf{x}_i(\varepsilon, \mathbf{v}_i)$ is a perturbed projection matrix and $\mathbf{u}_i(\varepsilon, (\mathbf{x}_j(\varepsilon, \mathbf{v}_j))_{j\in\mathcal{N}_i})$ is the input of the perturbed solution. The linearized dynamics on $\mathcal{S}^n$ can be derived by studying the linear effect of $\mathbf{v}_i$ on $\dot{\mathbf{x}}_i(\varepsilon, \mathbf{v}_i)$. Define

$$\begin{aligned}\mathbf{w}_i = \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{x}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} &= \frac{\mathbf{v}_i}{\|\mathbf{x}_i + \varepsilon\mathbf{v}_i\|}\bigg|_{\varepsilon=0} - \\ &\quad \frac{\mathbf{x}_i + \varepsilon\mathbf{v}_i}{\|\mathbf{x}_i + \varepsilon\mathbf{v}_i\|^3}\langle \mathbf{x}_i, \mathbf{v}_i\rangle\bigg|_{\varepsilon=0} \\ &= \mathbf{v}_i - \mathbf{x}_i \otimes \mathbf{x}_i\mathbf{v}_i = (\mathbf{I} - \mathbf{X}_i)\mathbf{v}_i = \mathbf{P}_i\mathbf{v}_i. \end{aligned} \quad (12)$$

The role of the matrix $\mathbf{P}_i$ is to project the perturbation onto the tangent space $\mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$. Note that

$$\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{X}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} = \mathbf{w}_i \otimes \mathbf{x}_i + \mathbf{x}_i \otimes \mathbf{w}_i$$

by the product rule. Let $f_{ij}(\varepsilon, \mathbf{v}_i, \mathbf{v}_j)$ denote the perturbed feedback gain. Then,

$$\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}f_{ij}(\varepsilon, \mathbf{v}_i, \mathbf{v}_j)|_{\varepsilon=0} = -f'_{ij}(s_{ij})(\langle \mathbf{w}_i, \mathbf{x}_j\rangle + \langle \mathbf{x}_i, \mathbf{w}_j\rangle).$$

Finally,

$$\begin{aligned}\dot{\mathbf{w}}_i &= \tfrac{\mathrm{d}^2}{\mathrm{d}t\mathrm{d}\varepsilon}\mathbf{x}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} = \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\dot{\mathbf{x}}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}(\mathbf{I} - \mathbf{X}_i(\varepsilon, \mathbf{v}_i))\sum_{j\in\mathcal{N}_i} f_{ij}(\varepsilon, \mathbf{v}_i, \mathbf{v}_j)\,\mathbf{x}_j(\varepsilon, \mathbf{v}_j)\big|_{\varepsilon=0} \\ &= -\left(\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{X}_i(\varepsilon, \mathbf{v}_i)\right)\sum_{j\in\mathcal{N}_i} f_{ij}(\varepsilon, \mathbf{v}_i, \mathbf{v}_j)\,\mathbf{x}_j(\varepsilon, \mathbf{v}_j)\big|_{\varepsilon=0} + \\ &\quad (\mathbf{I} - \mathbf{X}_i(\varepsilon, \mathbf{v}_i))\sum_{j\in\mathcal{N}_i}\left(\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}f_{ij}(\varepsilon, \mathbf{v}_i, \mathbf{v}_j)\right)\mathbf{x}_j(\varepsilon, \mathbf{v}_j)\big|_{\varepsilon=0} + \\ &\quad (\mathbf{I} - \mathbf{X}_i(\varepsilon, \mathbf{v}_i))\sum_{j\in\mathcal{N}_i} f_{ij}(\varepsilon, \mathbf{v}_i, \mathbf{v}_j)\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{x}_j(\varepsilon, \mathbf{v}_j)\big|_{\varepsilon=0} \\ &= -(\mathbf{w}_i \otimes \mathbf{x}_i + \mathbf{x}_i \otimes \mathbf{w}_i)\sum_{j\in\mathcal{N}_i} f_{ij}\mathbf{x}_j - \\ &\quad (\mathbf{I} - \mathbf{X}_i)\sum_{j\in\mathcal{N}_i} f'_{ij}(\langle \mathbf{w}_i, \mathbf{x}_j\rangle + \langle \mathbf{x}_i, \mathbf{w}_j\rangle)\mathbf{x}_j + \\ &\quad (\mathbf{I} - \mathbf{X}_i)\sum_{j\in\mathcal{N}_i} f_{ij}\mathbf{w}_j \\ &= -(\mathbf{w}_i \otimes \mathbf{x}_i + \mathbf{x}_i \otimes \mathbf{w}_i)\mathbf{u}_i - \\ &\quad (\mathbf{I} - \mathbf{X}_i)\sum_{j\in\mathcal{N}_i} f'_{ij}(\mathbf{X}_j\mathbf{w}_i + \mathbf{x}_j \otimes \mathbf{x}_i\mathbf{w}_j) + \\ &\quad (\mathbf{I} - \mathbf{X}_i)\sum_{j\in\mathcal{N}_i} f_{ij}\mathbf{w}_j \end{aligned}$$

$$
\begin{aligned}
&= -\left(\langle \mathbf{u}_i, \mathbf{x}_i\rangle \mathbf{I} + \mathbf{x}_i \otimes \mathbf{u}_i + \mathbf{P}_i \sum_{j\in\mathcal{N}_i} f'_{ij}\mathbf{X}_j\right)\mathbf{w}_i +\\
&\quad \mathbf{P}_i \sum_{j\in\mathcal{N}_i}\left(-f'_{ij}\mathbf{x}_j\otimes\mathbf{x}_i + f_{ij}\mathbf{I}\right)\mathbf{w}_j\\
&= -\left(\langle \mathbf{u}_i, \mathbf{x}_i\rangle \mathbf{I} + \mathbf{x}_i \otimes \mathbf{u}_i + \mathbf{P}_i \sum_{j\in\mathcal{N}_i} f'_{ij}\mathbf{X}_j\right)\mathbf{P}_i\mathbf{v}_i +\\
&\quad \mathbf{P}_i \sum_{j\in\mathcal{N}_i}\left(f_{ij}\mathbf{I} - f'_{ij}\mathbf{x}_j\otimes\mathbf{x}_i\right)\mathbf{P}_j\mathbf{v}_j, \qquad (13)
\end{aligned}
$$

where the relation $\langle \mathbf{x}, \mathbf{y}\rangle \mathbf{z} = (\mathbf{z}\otimes\mathbf{x})\,\mathbf{y} = (\mathbf{z}\otimes\mathbf{y})\mathbf{x}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{n+1}$ is used. The vector $\mathbf{w} = [\mathbf{w}_1^\top \ldots \mathbf{w}_N^\top]^\top$ has $N(n+1)$ components whereas the linearized system actually evolves on an Nn-dimensional space that lies embedded in $\mathbb{R}^{N(n+1)}$. The dimension reduction is given implicitly by the definition of $\mathbf{w}_i$ which requires $\mathbf{w}_i \in \mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$. This constraint is removed by using variables that are premultiplied by the projection matrices $\mathbf{P}_i : \mathbb{R}^{n+1} \to \mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$, *i.e.*, the variables $\mathbf{v}_i$ in (12). The matrix $\mathbf{A}$ is obtained by inspection of (13).

It remains to show that $\mathbf{A}$ is symmetric at all equilibria. Proposition 11 reveals that $\mathbf{u}_i$ is parallel to $\mathbf{x}_i$ at any equilibria. Thus $\mathbf{A}_{ii}$ is symmetric by inspection. Moreover,

$$
\begin{aligned}
\mathbf{A}_{ji}^\top - \mathbf{A}_{ij} = &\left(f_{ji}\mathbf{P}_j\mathbf{P}_i - f'_{ji}\mathbf{P}_j\mathbf{x}_i\otimes\mathbf{x}_j\mathbf{P}_i\right)^\top - \\
&\left(f_{ij}\mathbf{P}_i\mathbf{P}_j - f'_{ij}\mathbf{P}_i\mathbf{x}_j\otimes\mathbf{x}_i\mathbf{P}_j\right) = \mathbf{0},
\end{aligned}
$$

since $f_{ji} = f_{ij}$ for all $\{i,j\} \in \mathcal{E}$. □

Remark 15. *Note that the matrix* $\mathbf{A}$ *is the negative intrinsic Hessian matrix of the candidate Lyapunov function* V *given by* (8)*, i.e.,* $\mathbf{H} = \Box^2 V = -\mathbf{A}$. *The approach of this paper, which is based on studying the linearization of the dynamics* (6)*, is therefore similar to that of [29], which study the nature of the critical points of a potential function. Of course, our results and many details differ from those of [29].*

B. Instability of Undesired Equilibria

Consider an equilibrium such that all agents belong to the intersection of $\mathcal{S}^n$ and a hyperplane in $\mathbb{R}^{n+1}$. Perturb all agents into an open hemisphere by an arbitrarily small movement along a direction orthogonal to the hyperplane. By Proposition 12, the perturbed system converges to a consensus. Proposition 16 expands on this idea; perturbing all agents towards the north pole increases cohesion in the north hemisphere while depleting it in the south. We show that one such perturbation corresponds to a direction of exponential instability for the linearized system.

Proposition 16. *Any equilibrium* $(\mathbf{x}_i)_{i=1}^N \notin \mathcal{C}$ *of System 2 under Algorithm 8 is exponentially unstable.*

Proof. The proof makes use of the linearization provided by Proposition 14. The Courant-Fischer-Weyl min-max principle bounds the range of the Rayleigh quotient of a symmetric matrix by its minimal and maximal eigenvalues [39]. If the Rayleigh quotient is positive for some argument, then the maximal eigenvalue is positive. Recall that if $\mathbf{A}$ has a positive eigenvalue at an equilibrium, then that equilibrium is unstable by the indirect method of Lyapunov [40].

Proposition 11 establishes that the states and input are parallel at any equilibrium. The matrix $\mathbf{A}$ can then be expressed as

$$
\mathbf{A}_{ij} = \begin{cases} -\langle \mathbf{u}_i, \mathbf{x}_i\rangle \mathbf{P}_i - \sum_{j\in\mathcal{N}_i} f'_{ij}\mathbf{P}_i\mathbf{X}_j\mathbf{P}_i & \text{if } j = i,\\ f_{ij}\mathbf{P}_i\mathbf{P}_j - f'_{ij}\mathbf{P}_i\mathbf{x}_j\otimes\mathbf{x}_i\mathbf{P}_j & \text{if } \{i,j\}\in\mathcal{E}, \end{cases}
$$

and $\mathbf{A}_{ij} = \mathbf{0}$ otherwise. The matrix $\mathbf{A}$ is symmetric at all equilibria by Proposition 14.

Let $\mathbf{v} = [\mathbf{y}^\top \ldots \mathbf{y}^\top]^\top \in \mathsf{T}_{\mathcal{C}}(\mathcal{S}^n)^N$, *i.e.*, $\mathbf{y} \in \mathbb{R}^{n+1}$ since $\cup_{\mathbf{x}\in\mathcal{S}^n}\mathsf{T}_{\mathbf{x}}\mathcal{S}^n \simeq \mathbb{R}^{n+1}$, and consider

$$
\begin{aligned}
\langle \mathbf{v}, \mathbf{A}\mathbf{v}\rangle &= \sum_{i\in\mathcal{V}}\langle \mathbf{y}, \mathbf{A}_{ii}\mathbf{y}\rangle + \sum_{j\in\mathcal{N}_i}\langle \mathbf{y}, \mathbf{A}_{ij}\mathbf{y}\rangle\\
&= \left\langle \mathbf{y}, \left(\sum_{i\in\mathcal{V}}\mathbf{A}_{ii} + \sum_{j\in\mathcal{N}_i}\mathbf{A}_{ij}\right)\mathbf{y}\right\rangle.
\end{aligned}
$$

Denote $\mathbf{B} = \sum_{i\in\mathcal{V}}\mathbf{A}_{ii} + \sum_{j\in\mathcal{N}_i}\mathbf{A}_{ij}$. The matrix $\mathbf{B}$ is symmetric due to $\mathbf{A}$ being symmetric whereby $\sigma(\mathbf{B}) \subset \mathbb{R}$ by the spectral theorem. If $\mathbf{B}$ has a strictly positive eigenvalue, then for the corresponding eigenvector $\mathbf{z} \in \mathbb{R}^{n+1}$ it holds that $\langle \mathbf{z}, \mathbf{B}\mathbf{z}\rangle > 0$ whereby setting $\mathbf{y} = \mathbf{z}$ yields $\langle \mathbf{v}, \mathbf{A}\mathbf{v}\rangle > 0$. The min-max principle then implies that $\mathbf{A}$ has a strictly positive eigenvalue, *i.e.*, the equilibrium is exponentially unstable.

Let us prove that $\mathbf{B}$ has a positive eigenvalue. Consider

$$
\begin{aligned}
\operatorname{tr}\mathbf{B} &= \sum_{i\in\mathcal{V}} -n\langle \mathbf{u}_i, \mathbf{x}_i\rangle + \sum_{j\in\mathcal{N}_i}\Big(-f'_{ij}(1 - \langle \mathbf{x}_i, \mathbf{x}_j\rangle^2) +\\
&\qquad f_{ij}(n - 1 + \langle \mathbf{x}_i, \mathbf{x}_j\rangle^2) - f'_{ij}\langle \mathbf{x}_i, \mathbf{x}_j\rangle(\langle \mathbf{x}_i, \mathbf{x}_j\rangle^2 - 1)\Big)\\
&= \sum_{i\in\mathcal{V}} -n\langle \mathbf{u}_i, \mathbf{x}_i\rangle + \sum_{j\in\mathcal{N}_i}\Big(f_{ij}(n - 1 + \langle \mathbf{x}_i, \mathbf{x}_j\rangle^2) -\\
&\qquad f'_{ij}(2 - s_{ij})s_{ij}^2\Big)\\
&= n\left(\sum_{i\in\mathcal{V}} -\langle \mathbf{u}_i, \mathbf{x}_i\rangle + \sum_{j\in\mathcal{N}_i} f_{ij}\right) -\\
&\qquad \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} f_{ij}(2 - s_{ij})s_{ij} + f'_{ij}(2 - s_{ij})s_{ij}^2\\
&= n\left(\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} -f_{ij}\langle \mathbf{x}_i, \mathbf{x}_j\rangle + \sum_{j\in\mathcal{N}_i} f_{ij}\right) -\\
&\qquad \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} f_{ij}(2 - s_{ij})s_{ij} + f'_{ij}(2 - s_{ij})s_{ij}^2\\
&= n\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} f_{ij}s_{ij} -\\
&\qquad \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} f_{ij}(2 - s_{ij})s_{ij} + f'_{ij}(2 - s_{ij})s_{ij}^2\\
&= \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} f_{ij}(n - 2 + s_{ij})s_{ij} - f'_{ij}(2 - s_{ij})s_{ij}^2,
\end{aligned}
$$

where we used that $\operatorname{tr}\mathbf{X}_i = \|\mathbf{x}_i\|^2 = 1$ and $s_{ij} = 1 - \langle \mathbf{x}_i, \mathbf{x}_j\rangle$. Recall that

$$
f_{ij}(n - 2 + s_{ij})s_{ij} - f'_{ij}(2 - s_{ij})s_{ij}^2 > 0
$$

for all $s_{ij} \in (0,2]$ and all $\{i,j\} \in \mathcal{E}$ by condition (iii) of Algorithm 8. Since $\operatorname{tr} \mathbf{B} \geq 0$ with strict inequality unless $s_{ij} = 0$ for all $\{i,j\} \in \mathcal{E}$, *i.e.*, unless $(\mathbf{x}_i)_{i=1}^N \in \mathcal{C}$, it follows that $\mathbf{B}$ has a strictly positive eigenvalue. □

Remark 17. *Requirement (iii) in Algorithm 8 arises from the lower bound of* $\operatorname{tr} \mathbf{B}$. *This lower bound is clearly conservative with respect to the requirements on f_{ij} for all $\{i,j\} \in \mathcal{E}$ that results in $\mathbf{A}$ having a positive eigenvalue. The class of control signals that yield almost global consensus on $\mathcal{S}^n$ is hence larger than that given in Algorithm 8.*

Proposition 18 is used to prove Theorem 13. The version presented here is particularized for our purposes; a more general result and its proof may *e.g.*, be found in [34].

Proposition 18 (R.A. Freeman [34])**.** *Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ that evolves on a state-space $\mathcal{X}$, where $\mathbf{f} \in \mathcal{C}^1$. Let $\mathcal{S} \subset \mathcal{X}$ be a set consisting entirely of exponentially unstable equilibria. If each trajectory of the system converges to some equilibrium, then the region of attraction of $\mathcal{S}$ is of zero measure and meager in $\mathcal{X}$.*

C. Point-Wise Convergence

The instability requirements of Proposition 18 are satisfied by Proposition 16. However, to show that every trajectory of the system converges to a point, *i.e.*, that the system is so-called pointwise convergent [41], requires some additional analysis. Point-wise convergence is of importance since Proposition 11 only establishes convergence to some equilibrium sets, all of which have n degrees of rotational invariance. In theory, it would be possible for each agent to traverse its sphere indefinitely while the system as a whole approaches some equilibrium set. The use of Proposition 19, a corollary of the Łojasiewicz gradient inequality [42], may be not be necessary but suffices to establish point-wise convergence. It is the reason that we assume that the feedback gains $f_{ij} \in \mathcal{C}^\omega[0,2]$ for all $\{i,j\} \in \mathcal{E}$ rather than $\mathcal{C}^1[0,2]$.

Proposition 19 (S. Łojasiewicz [41], [42])**.** *Let $\mathcal{M}$ be a real analytic Riemannian manifold and $f : \mathcal{M} \to \mathbb{R}$ be a real analytic function. For the intrinsic gradient flow $\dot{\mathbf{x}} = -\square f$ it either holds that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{y}$ for some $\mathbf{y} \in \mathcal{M}$ or the set of ω-limit points is empty.*

Proposition 20. *Each trajectory of System 2 under Algorithm 8 converges to an equilibrium.*

Proof. The n-sphere is a real analytic manifold, and so is $(\mathcal{S}^n)^N$. Sums, composite functions, integrals, and derivatives of multivariate analytic functions are analytic [43]. By analyticity of the feedback gains in Algorithm 8, it follows that the candidate Lyapunov function V given by (8) is analytic.

Equation (9) only provides the extrinsic gradient $\nabla V : (\mathcal{S}^n)^N \to (\mathbb{R}^{n+1})^N$ of (8) without regard to the fact that $\{\mathbf{x}_i\} \in (\mathcal{S}^n)^N$. The intrinsic gradient $\square V : (\mathcal{S}^n)^N \to \mathsf{T}(\mathcal{S}^n)^N$ is given by

$$\square V = (\square_i V)_{i=1}^N = (\nabla_i V - \langle \nabla_i V, \mathbf{x}_i \rangle \mathbf{x}_i)_{i=1}^N,$$

where $\square_i = \square_{\mathbf{x}_i}$. The intrinsic gradient $\square V$ is hence the projection of ∇V on the tangent space $\mathsf{T}_{(\mathbf{x}_i)_{i=1}^N} (\mathcal{S}^n)^N$ [44]. Equation (9) gives $\nabla_i V = -\mathbf{u}_i$ whereby

$$\square V = -\left(\mathbf{u}_i - \langle \mathbf{u}_i, \mathbf{x}_i \rangle \mathbf{x}_i\right)_{i=1}^N.$$

The closed-loop dynamics of System 2 under Algorithm 8 can be written

$$\dot{\mathbf{x}}_i = -\square_i V \tag{14}$$

for all $i \in \mathcal{V}$, *i.e.*, it is a gradient descent flow on $(\mathcal{S}^n)^N$.

The conditions of Proposition 19 are satisfied by $(\mathcal{S}^n)^N$ and (14). Since $(\mathcal{S}^n)^N$ is compact, every sequence has a convergent subsequence by the Bolzano-Weierstrass theorem wherefore the set of limit points is nonempty. It follows that $(\mathbf{x}_i)_{i=1}^N$ converges to a single point, and by Proposition 11 that point is an equilibrium. □

D. Proof of Main Theorem

Recall that it remains to prove Theorem 13. Proposition 12, 16, 18, and 20 provide the sufficient tools to do so.

Proof of Theorem 13. The requirements of Proposition 18 are satisfied by Proposition 20 and Proposition 16. Since all system trajectories converge to equilibria by Proposition 20, and the set of initial conditions $(\mathbf{x}_0)_{i=1}^N$ resulting in trajectories that converge to any equilibrium that does not belong to the consensus set is of zero measure and meager by Proposition 18, it follows that the set of trajectories converging to the consensus set is almost all of $(\mathcal{S}^n)^N$. This establish almost global attractiveness. Stability follows from Proposition 12.

It remains to show local exponential stability. The linearized system dynamics expressed in the variables $(\mathbf{y}_i)_{i=1}^N$ are hence

$$\dot{\mathbf{y}}_i = \mathbf{P} \sum_{j \in \mathcal{N}_i} f_{ij}(0)(\mathbf{y}_j - \mathbf{y}_i), \tag{15}$$

where $\mathbf{P} = \mathbf{I} - \mathbf{c} \otimes \mathbf{c}$ and $\mathbf{x}_i = \mathbf{c} \in \mathcal{S}^n$ for all $i \in \mathcal{V}$. Each vector $\mathbf{y}_i$ of the linearized system evolves along a hyperplane of codimension 1 given by $\mathcal{H} = \mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n = \operatorname{Im}\mathbf{P}$ for all $i \in \mathcal{V}$. Since the graph is connected, and $f_{ij}(0)$ is strictly positive for all $\{i,j\} \in \mathcal{E}$, it follows that (15) reaches consensus exponentially if $\mathbf{y}_{i,0} \in \mathcal{H}$ for all $i \in \mathcal{V}$ [1]. □

V. Perspectives

Let us compare what is known with regard to consensus on $\mathcal{S}^1$ and $\mathsf{SO}(3)$ in relation to Theorem 13.

A. The Circle and the Sphere

Algorithm 3 does not satisfy property (iii) of Algorithm 8 in the case of $n = 1$. The requirement is however only sufficient for almost global consensus. A counter-example is provided by [15], [16]: the equilibrium set over cycle graphs where agents are spread out equidistantly over $\mathcal{S}^1$ such that the geodesic distance $d_\theta : \mathcal{S}^1 \times \mathcal{S}^1 \to [0,\pi]$ satisfies $d_\theta(\mathbf{x}_i, \mathbf{x}_j) = 2\pi/N$ for all $\{i,j\} \in \mathcal{E}$ is asymptotically stable. This section explores the difference between $\mathcal{S}^1$ and $\mathcal{S}^2$ with regard to the preconditions for achieving almost global consensus.

Example 21. *Consider six agents on $\mathcal{S}^2$ and a cycle graph*

$$\mathcal{G} = (\{i \in \mathbb{N} \,|\, i \leq 6\}, \{1, 6\} \cup \{\{i, j\} \in \mathcal{V} \times \mathcal{V} \,|\, i - j = 1\})$$

where we use the weights $f_{ij} = 1$ for all $\{i, j\} \in \mathcal{E}$. One equilibrium consists of the agents being equidistantly spread out over the equator at a distance $\pi/3$ from one another, see Figure 2. Then

$$\mathbf{A} = \mathbf{PC} \otimes \mathbf{I}_{3\times 3},$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_5 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_6 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

where $\otimes$ denote the Kronecker product, and $\mathbf{I}_{3\times 3}$ is the identity matrix of dimension 3. The block diagonal elements of $\mathbf{P}$ satisfy $\mathbf{P}_i = \mathbf{I}_{3\times 3} - \mathbf{R}^{i-1}\mathbf{e} \otimes \mathbf{e}(\mathbf{R}^\top)^{i-1}$ for some $\mathbf{e} \in \mathcal{S}^2$. Note that $\mathbf{C}$ is a circulant matrix for which all eigenpairs can be calculated explicitly [45].

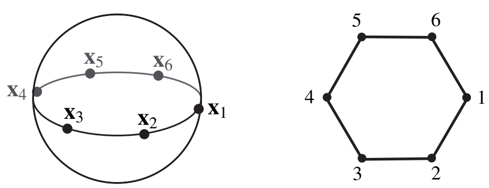


Fig. 2. An equilibrium of a system on $\mathcal{S}^2$ with a cycle graph.

There is no loss of generality in setting $\mathbf{e} = \mathbf{e}_1 = [1\ 0\ 0]^\top$ and positioning all agents on the equator to decouple $\mathbf{P}_i$ into a block diagonal matrix,

$$\mathbf{P}_i = \begin{bmatrix} \mathbf{Q}_i & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix},$$

$$\mathbf{Q}_i = \mathbf{I}_{2\times 2} - \mathbf{R}^{i-1}\mathbf{e}_1 \otimes \mathbf{e}_1(\mathbf{R}^\top)^{i-1},$$

$$\mathbf{R} = \tfrac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}.$$

The linearized dynamics are thereby decoupled into two independent subsystems corresponding to the variables $\mathbf{y}_i \in \mathbb{R}^2$ and $z_i \in \mathbb{R}$ for all $i \in \mathcal{V}$. For perturbations $(\mathbf{y}_i)_{i=1}^N$ that belong to the equatorial plane, it follows that

$$\dot{\mathbf{y}}_i = \mathbf{Q}_i(\mathbf{y}_{i-1} + \mathbf{y}_{i+1} - \mathbf{y}_i), \tag{16}$$

for all $i \in \mathcal{V}$, where the indices are added modulo 6. For perturbations that are normal to said plane, it holds that

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z}$$

where $\mathbf{z} = [z_1\, z_2, \ldots, z_6]^\top$.

The dynamics (16) *can be written on the form*

$$\dot{\mathbf{y}} = \mathbf{QC} \otimes \mathbf{I}_{2\times 2}\mathbf{y},$$

where $\mathbf{Q}$ is a block diagonal matrix with $\mathbf{Q}_i$ for $i \in \mathcal{V}$ as blocks, $\otimes$ denote the Kronecker product, $\mathbf{I}_{2\times 2}$ is the identity matrix of dimension 2, and $\mathbf{y} = [\mathbf{y}_1{}^\top, \ldots, \mathbf{y}_6{}^\top]^\top \in \mathbb{R}^{2N}$. Unlike $\sigma(\mathbf{PC} \otimes \mathbf{I}_{3\times 3})$, the spectrum of $\mathbf{PC} \otimes \mathbf{I}_{2\times 2}$ belongs to the closed left half complex plane. Note that $\sigma(\mathbf{PC} \otimes \mathbf{I}_{2\times 2})$ does not depend on $\mathbf{e}$. An eigenpair can be interpreted as a perturbation direction of the system resulting in an instantaneous response that is either aligned or negatively aligned with the perturbation. For example, $(0, [\mathbf{v}^\top\, (\mathbf{R}\mathbf{v})^\top\, (\mathbf{R}^2\mathbf{v})^\top\, (\mathbf{R}^3\mathbf{v})^\top\, (\mathbf{R}^4\mathbf{v})^\top\, (\mathbf{R}^5\mathbf{v})^\top]^\top)$ is an eigenpair of $\mathbf{QC} \otimes \mathbf{I}_{2\times 2}$ for all $\mathbf{v} \in \mathbb{R}^2$. It corresponds to the perturbation of moving each agent a fixed distance along its tangent space, thereby rotating the entire cyclic formation.

The dynamics of $\mathbf{z}$ are unstable since $(1, [1\,1\,1\,1\,1\,1]^\top)$ is an eigenpair of $\mathbf{C}$. This eigenpair can be interpreted as a perturbation that takes all agents into the north hemisphere, from where they reach consensus at the north pole. Another eigenpair is $(-3, [1\,{-1}\;\;1\,{-1}\;\;1\,{-1}]^\top)$. The corresponding perturbation lifts and drops agents above and below the equator, thereby distancing any agent from the convex hull of itself and its neighbors. The response is hence a recoil towards the equator, as demonstrated by the negative eigenvalue. The effects of both these perturbations on the original nonlinear system are illustrated in Figure 3. Note that effect of the first perturbation always is a consensus whereas the effect of the second is more sensitive; the nonlinear system will however return to the equator if $f_{ij} = f$ for all $\{i, j\} \in \mathcal{E}$.

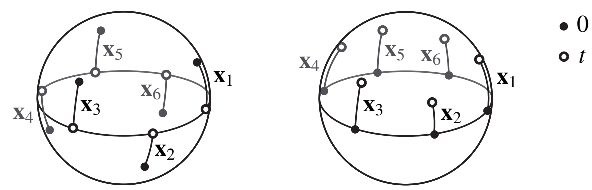


Fig. 3. The trajectories of two nonlinear systems which are perturbed from an equilibrium at the equator along the stable and unstable manifolds (left and right respectively).

The unstable directions of perturbations are all orthogonal to the equator. The stability of a cycle equilibrium of System 2 under Algorithm 3 on $\mathcal{S}^1$ is therefore not inherited by the embedding of $\mathcal{S}^1$ in higher dimensional spheres. Aside from the instability, it is important to note such a perturbation bring all agents into a hemisphere from where they reach consensus by Proposition 12, implying that the equator is also unattractive. The circle is also embedded on an infinite cylinder, but that case is not covered by this analysis.

The following corollary of Theorem 13 lack the generality of its precursor, but is nevertheless a result that we find to be interesting in its own right. It provides an exhaustive characterization of the stability properties of a particular dynamical system, both forwards and backwards in time. Recall that s_{ij} defined by (5) measure the extrinsic distance between two points on $\mathcal{S}^n$. Theorem 22 states that, under certain conditions, Algorithm 3 solves both the minimax and maximin problems of s_{ij} over all $\{i, j\} \in \mathcal{E}$ almost globally.

Corollary 22. *Consider System 2 on $\mathcal{S}^2$ under Algorithm 3 with $f_{ij} = 1$ for all $\{i,j\} \in \mathcal{E}$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a cycle graph. The α-limits of the flow from almost all initial conditions belong to*

$$\{(\mathbf{x}_i)_{i=1}^N \in (\mathcal{S}^n)^N \,|\, s_{ij} = \max_{(\mathcal{S}^n)^N} \min_{\{k,l\}\in\mathcal{E}} s_{kl}, \forall \{i,j\} \in \mathcal{E}\}$$

whereas the ω-limits belong to the consensus set, i.e.,

$$\{(\mathbf{x}_i)_{i=1}^N \in (\mathcal{S}^n)^N \,|\, s_{ij} = \min_{(\mathcal{S}^n)^N} \max_{\{k,l\}\in\mathcal{E}} s_{kl}, \forall \{i,j\} \in \mathcal{E}\}.$$

Proof. This is a direct consequence of Theorem 13 and the characterization of equilibria obtained by closing System 2 with the negation of Algorithm 3 provided in [7], [8]. □

B. Simulations

This section compares the global performance of two consensus protocols on $\mathcal{S}^n$ for $n \in \{1, 2\}$ and on $\mathsf{SO}(3)$ respectively in simulation. To that end, consider the following multi-agent system on the special orthogonal group $\mathsf{SO}(n)$.

System 23. *The system is given by N agents, an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, agent states $\mathbf{R}_i \in \mathsf{SO}(n)$, and dynamics $\dot{\mathbf{R}}_i = \mathbf{\Omega}_i \mathbf{R}_i$ where $\mathbf{\Omega}_i \in \mathsf{so}(n)$ for all $i \in \mathcal{V}$. It is assumed that $\mathcal{G}$ is connected and that the system can be actuated on a kinematic level, i.e., $\mathbf{\Omega}_i$ is the input signal of agent i.*

Recall that Algorithm 3 can be derived by taking the gradient of the potential function (8). A related consensus protocol on $\mathsf{SO}(n)$ can be derived by taking the gradient of the potential function $V : \mathsf{SO}(n) \times \mathsf{SO}(n) \to [0, \infty)$ given by

$$V(\mathbf{R}_i, \mathbf{R}_j) = \tfrac{1}{2} \sum_{\{i,j\}\in\mathcal{E}} \int_0^{s_{ij}} f_{ij}(r)\mathrm{d}r,$$

where $f_{ij} : [0, 2n] \to [0, \infty)$ and $s_{ij} = n - \langle \mathbf{R}_i, \mathbf{R}_j \rangle$ for all $\{i,j\} \in \mathcal{E}$. As such, Algorithm 3 is similar to the following algorithm on System 23.

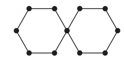
Algorithm 24. *The feedback is given by*

$$\mathbf{\Omega}_i = \sum_{j\in\mathcal{N}_i} f_{ij}(s_{ij})(\mathbf{R}_i^\top \mathbf{R}_j - \mathbf{R}_j^\top \mathbf{R}_i),$$

where $f_{ij} = f_{ji}$ for all $\{i,j\} \in \mathcal{E}$.

Table I displays the outcome of running 10^6 trials of Algorithm 3 on System 2 and 10^4 trials of Algorithm 24 on System 23 for three different graphs (we set $f_{ij} = 5$ for all $\{i,j\} \in \mathcal{E}$ for both algorithms). The initial conditions are drawn uniformly from the sphere using the fact that $\mathbf{x} \in \mathcal{N}(\mathbf{0}, \mathbf{I})$ implies that ${}^{\mathbf{x}}/_{\|\mathbf{x}\|} \in \mathcal{U}(\mathcal{S}^n)$ [46]. This method is also used to draw from $\mathcal{U}(\mathsf{SO}(3))$ by first generating a uniform distribution on the unit sphere in quaternion space, *i.e.*, drawing from $\mathcal{U}(\mathcal{S}^3)$, and then mapping the sample to $\mathsf{SO}(3)$. By inspection of Table I, note that Algorithm 3 fails to yield almost global consensus on $\mathcal{S}^1$. Likewise, almost global consensus does not hold for Algorithm 24 on System 23 over $\mathsf{SO}(3)$. These results agree with those of [15], [29]. As predicted by Theorem 13, there were no failures to reach consensus on $\mathcal{S}^2$ despite the high number of trials.

TABLE I. Number of failures to reach consensus on the space $\mathcal{X} \in \{\mathcal{S}^1, \mathcal{S}^2, \mathsf{SO}(3)\}$ over 10^4 random trials using Algorithm 3 and 8 with constant feedback gains.

$\mathcal{X}$			
$\mathcal{S}^1$	1504	2173	2126
$\mathcal{S}^2$	0	0	0
$\mathsf{SO}(3)$	711	66	86

C. Extension to the Special Orthogonal Group

In Section V-A we learn that a certain undesired equilibrium set of System 2 under Algorithm 3 on $\mathcal{S}^1$ is stable. Section V-B shows that the problem of multi-agent consensus on $\mathsf{SO}(3)$ poses similar challenges. In fact, if the reduced attitudes of all agents agree, then the remaining degree of rotational freedom of each agent is confined to a set that is diffeomorphic to $\mathcal{S}^1$. On the n-sphere, a perturbation that is orthogonal to the equator will allow a system in such a configuration to reach consensus. On $\mathsf{SO}(3)$, the destabilizing effect of such a perturbation is counter-acted by the reduced attitude which, figuratively speaking, serves as a ballast that stabilizes the two other axes of all agent to a single great circle.

Let us utilize what we have learned about consensus on $\mathcal{S}^1$ and $\mathcal{S}^2$ to attempt to design a control law on $\mathsf{SO}(3)$ that stabilizes the consensus set almost globally. To that end, rewrite the variables $\mathbf{R}_i$ of (23) as $\mathbf{R}_i = [\mathbf{x}_i\, \mathbf{y}_i\, \mathbf{z}_i]$, *i.e.*, $\mathbf{x}_i = \mathbf{R}_i \mathbf{e}_1$, $\mathbf{y}_i = \mathbf{R}_i \mathbf{e}_2$, and $\mathbf{z}_i = \mathbf{R}_i \mathbf{e}_3$ whereby

$$[\dot{\mathbf{x}}_i\, \dot{\mathbf{y}}_i\, \dot{\mathbf{z}}_i] = \mathbf{\Omega}_i [\mathbf{x}_i\, \mathbf{y}_i\, \mathbf{z}_i]$$

for all $i \in \mathcal{V}$. Let $\mathbf{S} : \mathbb{R}^3 \to \mathsf{so}(3)$ be the bijective linear map defined by $\mathbf{S}(\mathbf{x})\mathbf{y} \mapsto \mathbf{x} \times \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. Denote $\boldsymbol{\omega}_i = \mathbf{S}^{-1}(\mathbf{\Omega}_i)$ for all $i \in \mathcal{V}$. The following algorithm decouples the evolution of $\mathbf{x}_i$ from any dependence on $\mathbf{y}_i$ and $\mathbf{z}_i$ by utilizing a decomposition of $\boldsymbol{\omega}_i$ into a part that is orthogonal to $\mathbf{x}_i$ and a part that is parallel to $\mathbf{x}_i$.

Algorithm 25. *The feedback is given by*

$$\mathbf{\Omega}_i = \mathbf{S}\left(\mathbf{x}_i \times \mathbf{u}_i + \sum_{j\in\mathcal{N}_i} g_{ij} \mathbf{x}_i\right),$$

where $\mathbf{u}_i$ is the input signal of Algorithm 8 and the locally Lipschitz function $g_{ij} : \mathcal{I}_i \to \mathbb{R}$ is related to the feedback gain of a second almost globally convergent consensus protocol on $\mathcal{S}^1$ for all $\{i,j\} \in \mathcal{E}$. More specifically, we require that the feedback gains g_{ij} are such that the system

$$\dot{\mathbf{y}}_i = \sum_{j\in\mathcal{N}_i} g_{ij} \mathbf{z}_i, \quad \dot{\mathbf{z}}_i = -\sum_{j\in\mathcal{N}_i} g_{ij} \mathbf{y}_i, \tag{17}$$

reach consensus for almost all initial conditions such that $\mathbf{x}_i(0) = \mathbf{x}_j(0)$ for all $\{i,j\} \in \mathcal{E}$ (these dynamics evolve over a single great circle on $\mathcal{S}^2$ since $\mathbf{z}_i = \mathbf{x}_i \times \mathbf{y}_i$ and $\mathbf{x}_i$ is constant, i.e., $\mathbf{z}_i$ can be expressed in terms of $\mathbf{y}_i$ for all $i \in \mathcal{V}$).

Remark 26. *Note that any implementation of Algorithm 25 requires the use of an almost globally convergent consensus protocol on $\mathcal{S}^1$, e.g., that of [15], [16]. The protocol of [15],* *[16] requires an upper bound on the total number of agents, which is a weaker form of graph dependence than that of the protocol in [29]. To see that Algorithm 25 can be implemented by only using local and relative information, note that*

$$
\begin{aligned}
[\boldsymbol{\Omega}_i]_{\mathcal{B}_i} &= \mathbf{R}_i^\top[\boldsymbol{\Omega}_i]_{\mathcal{W}}\mathbf{R}_i = \mathbf{S}\left(\mathbf{R}_i^\top(\mathbf{x}_i\times\mathbf{u}_i)+\sum_{j\in\mathcal{N}_i}g_{ij}\mathbf{R}_i^\top\mathbf{x}_i\right)\\
&= \mathbf{S}\left((\mathbf{R}_i^\top\mathbf{x}_i)\times(\mathbf{R}_i^\top\mathbf{u}_i)+\sum_{j\in\mathcal{N}_i}g_{ij}\mathbf{R}_i^\top\mathbf{x}_i\right)\\
&= \mathbf{S}\left(\mathbf{e}_1\times\sum_{j\in\mathcal{N}_i}\mathbf{R}_i^\top\mathbf{R}_j\mathbf{e}_1+\sum_{j\in\mathcal{N}_i}g_{ij}\mathbf{e}_1\right).
\end{aligned}
$$

The feedback hence only only depends on the relative information $(\mathbf{R}_i^\top\mathbf{R}_j)_{j\in\mathcal{N}_i}\in\mathcal{I}_i$ *on* $\mathsf{SO}(3)$.

The closed loop dynamics of System 2 under Algorithm 25 are given by

$$\dot{\mathbf{x}}_i = (\mathbf{x}_i\times\mathbf{u}_i)\times\mathbf{x}_i = \mathbf{u}_i-\langle\mathbf{u}_i,\mathbf{x}_i\rangle\mathbf{x}_i = \mathbf{P}_i\mathbf{u}_i, \tag{18}$$

$$
\begin{aligned}
\dot{\mathbf{y}}_i &= (\mathbf{x}_i\times\mathbf{u}_i)\times\mathbf{y}_i+\sum_{j\in\mathcal{N}_i}g_{ij}\mathbf{z}_i,\\
&= -\langle\mathbf{u}_i,\mathbf{y}_i\rangle\mathbf{x}_i+\sum_{j\in\mathcal{N}_i}g_{ij}\mathbf{z}_i,
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\dot{\mathbf{z}}_i &= (\mathbf{x}_i\times\mathbf{u}_i)\times\mathbf{z}_i-\sum_{j\in\mathcal{N}_i}g_{ij}\mathbf{y}_i,\\
&= -\langle\mathbf{u}_i,\mathbf{z}_i\rangle\mathbf{x}_i-\sum_{j\in\mathcal{N}_i}g_{ij}\mathbf{y}_i,
\end{aligned} \tag{20}
$$

for all $i\in\mathcal{V}$. Note that the dynamics of $(\mathbf{x}_i)_{i=1}^N$ given by (18) are precisely those of System 2 under Algorithm 8. The consensus set for the reduced attitudes $(\mathbf{x}_i)_{i=1}^N$ is hence almost globally asymptotically stable by Theorem 13. We will utilize the triangular structure of the system given by (18)–(20) to establish a local convergence result. To this end, consider Proposition 27 from [47] which have been adapted to our setting.

Proposition 27 (M.I. El-Hawwary & M. Maggiore [47])**.** *Consider a system* $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$*, where* $\mathbf{f}$ *is locally Lipschitz, that evolves on a compact state-space* $\mathcal{X}$*. Let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*, where* $\mathcal{S}_1\subset\mathcal{S}_2\subset\mathcal{X}$*, be two closed, positively invariant sets. Then,* $\mathcal{S}_1$ *is asymptotically stable if the following conditions hold:*

(i) $\mathcal{S}_1$ *is asymptotically stable relative to* $\mathcal{S}_2$*,*
(ii) $\mathcal{S}_2$ *is asymptotically stable.*

Remark 28. *There exists a global version of Proposition 27 [47]. The case when convergence from* $\mathcal{S}_2$ *to* $\mathcal{S}_1$ *is global but convergence from* $\mathcal{X}$ *to* $\mathcal{S}_2$ *is almost global can be addressed by redefining* $\mathcal{X}$ *to be the region of attraction of* $\mathcal{S}_2$*, see [48]. It cannot be applied to our problem however. The problem is that* $\mathcal{S}_1$ *is only almost globally stable relative to* $\mathcal{S}_2$*. We cannot guarantee that the convergence from* $\mathcal{X}$ *to* $\mathcal{S}_2$ *would not bring the system to state at which convergence from* $\mathcal{S}_2$ *to* $\mathcal{S}_1$ *fails.*

Proposition 29. *The consensus set on* $\mathsf{SO}(3)$*,*

$$\mathcal{C}=\{(\mathbf{R}_i)_{i=1}^N\in(\mathsf{SO}(3))^N\,|\,\mathbf{R}_i=\mathbf{R}_j,\,\forall\{i,j\}\in\mathcal{E}\},$$

is an asymptotically stable equilibrium set of System 23 under Algorithm 25.

Proof. In terms of Proposition 27, let

$$
\begin{aligned}
\mathcal{S}_1 &= \{(\mathbf{R}_i)_{i=1}^N\in(\mathsf{SO}(3))^N\,|\,\mathbf{R}_i=\mathbf{R}_j,\,\forall\{i,j\}\in\mathcal{E}\},\\
\mathcal{S}_2 &= \{(\mathbf{R}_i)_{i=1}^N\in(\mathsf{SO}(3))^N\,|\,\mathbf{R}_i\mathbf{e}_1=\mathbf{R}_j\mathbf{e}_1,\,\forall\{i,j\}\in\mathcal{E}\},
\end{aligned}
$$

denote, respectively, the consensus set and reduced attitude consensus set. Clearly $\mathcal{S}_1$, $\mathcal{S}_2$ are closed, positively invariant, nested sets. Property (ii) follows by application of Theorem 13 to the dynamics (18). To establish property (i), consider the case of $(\mathbf{R}_i(0))_{i=1}^N\in\mathcal{S}_2$. Then $\{\mathbf{y}_1,\ldots\mathbf{y}_N,\mathbf{z}_1,\ldots,\mathbf{z}_N\}\subset\mathcal{P}$, where $\mathcal{P}$ is the plane that has $\mathbf{x}_i$ as normal for any $i\in\mathcal{V}$. The system (18)–(20) is hence on the form (17). The consensus set of the system (17) is almost globally asymptotically stable by our assumptions on g_{ij} for all $\{i,j\}\in\mathcal{E}$, which implies (i).

It remains to show that there exists at least one consensus protocol g_{ij} with the required properties. Let

$$g_{ij}=g\left(\operatorname{acos}\left(\frac{\langle\mathbf{y}_i,\mathbf{y}_j\rangle}{(\langle\mathbf{y}_i,\mathbf{y}_j\rangle^2+\langle\mathbf{z}_i,\mathbf{y}_j\rangle^2)^{\frac{1}{2}}}\right)\operatorname{sgn}\langle\mathbf{z}_i,\mathbf{y}_j\rangle\right),$$

where g is the almost globally convergent consensus protocol for the dynamics (21) in [15], [16], *i.e.*,

$$g(\vartheta)=\begin{cases}-\frac{1}{N-1}(\pi+\vartheta) & \text{if }\vartheta\in[-\pi,-\frac{1}{N}\pi),\\ \vartheta & \text{if }\vartheta\in[-\frac{1}{N}\pi,\frac{1}{N}\pi],\\ \frac{1}{N-1}(\pi-\vartheta) & \text{if }\vartheta\in(\frac{1}{N}\pi,\pi].\end{cases}$$

To see that g_{ij} is Lipschitz, note that the discontinuity of the sign function appears when $\langle\mathbf{y}_j,\mathbf{z}_i\rangle=0$ in which case the argument of g is $\operatorname{acos}\operatorname{sgn}\langle\mathbf{y}_i,\mathbf{y}_j\rangle\in\{0,\pi\}$ and $g(-\pi)=g(\pi)$.

Let $\{\mathbf{v}_1,\mathbf{v}_2\}$ be a basis of $\mathcal{P}$. If $(\mathbf{R}_i(0))_{i=1}^N\in\mathcal{S}_2$, then

$$
\begin{aligned}
\mathbf{y}_i &= \cos\vartheta_i\mathbf{v}_1+\sin\vartheta_i\mathbf{v}_2,\\
\mathbf{z}_i &= \cos(\vartheta_i+\tfrac{\pi}{2})\mathbf{v}_1+\sin(\varphi_i+\tfrac{\pi}{2})\mathbf{v}_2\\
&= -\sin\theta_i\mathbf{v}_1+\cos\theta_i\mathbf{v}_2.
\end{aligned}
$$

for some $\vartheta_i\in(-\pi,\pi]$ for all $i\in\mathcal{V}$. Moreover,

$$\dot{\mathbf{y}}_i=-\dot{\vartheta}_i\sin\vartheta_i\mathbf{v}_1+\dot{\vartheta}_i\cos\vartheta_i\mathbf{v}_2,$$

wherefore (19) yields,

$$\dot{\vartheta}_i=\langle-\sin\vartheta_i\mathbf{v}_1+\cos\vartheta_i\mathbf{v}_2,\dot{\mathbf{y}}_i\rangle=\sum_{j\in\mathcal{N}_i}g_{ij}. \tag{21}$$

Note that $\mathbf{y}_j=\langle\mathbf{y}_i,\mathbf{y}_j\rangle\mathbf{y}_i+\langle\mathbf{z}_i,\mathbf{y}_j\rangle\mathbf{z}_i$ on $\mathcal{P}$. The argument of g is hence $\operatorname{acos}\langle\mathbf{y}_i,\mathbf{y}_j\rangle\operatorname{sgn}\langle\mathbf{z}_i,\mathbf{y}_j\rangle=\vartheta_j-\vartheta_i$, which can be interpreted as a signed relative arc length on $\mathcal{S}^1$. As such, the dynamics (21) reduces to

$$\dot{\vartheta}_i=\sum_{j\in\mathcal{N}_i}g(\vartheta_j-\vartheta_i), \tag{22}$$

i.e., to the form of the almost globally convergent consensus protocol [15], [16] on $\mathcal{S}^1$. □

Let us return to the simulation problem of Section V-B. Generating uniformly distributed initial conditions on $\mathsf{SO}(3)$ and simulating Algorithm 25 where the algorithm of [15], [16] is used to generate consensus on $\mathcal{S}^1$ for the three graph topologies of Table I, we find no failures to reach consensus. Algorithm 25 hence outperforms Algortihm 24 and rivals the practical performance of the algorithm [29]. Moreover, the version of Algorithm 25 based on [15], [16] only requires each agent to know an upper bound on N. Algorithm 25 also rivals the theoretical performance of [29], as shown in Corollary 30 of Theorem 13. Note that we cannot conclude that the consensus manifold is almost globally stable from the result of Corollary 30 since System 23 under Algorithm 25 is not a gradient descent flow.

Corollary 30. *Suppose all feedback gains g_{ij}, for $\{i,j\} \in \mathcal{E}$, in Algorithm 25 are chosen such that all equilibria of system* (17) *are exponentially unstable except for those in $\mathcal{C}$. Then all equilibria of System 23 under Algorithm 25 are exponentially unstable except those in $\mathcal{C}$ which are asymptotically stable.*

Proof. Note that the linearization decouples like the dynamics (18)–(20). Theorem 13 establishes that the all equilibria except those belonging to the consensus set are unstable for the subsystem (18). Any candidate for a stable equilibrium must hence satisfy $\mathbf{x}_i = \mathbf{x}_j$ for all $\{i,j\} \in \mathcal{E}$. This requirement reduces the dynamics (18)–(20) to (17) for which all equilibria apart from those in $\mathcal{C}$ are exponentially unstable by assumption. That $\mathcal{C}$ is asymptotically stable follows from Proposition 29. □

VI. Conclusions

This paper establishes almost global consensus on the n-sphere for general $n \in \mathbb{N}\backslash\{1\}$, a class of intrinsic consensus protocols and all connected, undirected graph topologies. The term intrinsic refers to the feedback law, which is an intrinsic gradient. These results show that the conditions for achieving almost global consensus are more favorable on the n-sphere than known results regarding other Riemannian manifolds would suggest. In particular, almost global consensus on $\mathcal{S}^1$ [15] and $\mathsf{SO}(3)$ [29], [30] requires protocols that are tailored for this specific purpose. The case of $\mathcal{S}^1$ differs from that of the general n-sphere due to its low dimension. There are asymptotically stable equilibrium sets on $\mathcal{S}^1$ that are disjunct from the consensus set. If these sets are embedded on the n-sphere for $n \in \mathbb{N}\backslash\{1\}$ in the form of great circles then any normal to the corresponding equatorial plane is a direction of instability. The circle can also be embedded on $\mathsf{SO}(3)$, but there it gives rise to asymptotically stable undesired equilibria. By combing our understanding of almost global consensus on $\mathcal{S}^1$ and $\mathcal{S}^2$ we design a novel consensus protocol on $\mathsf{SO}(3)$ which is shown to avoid undesired equilibria in simulation.

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