A GENERALIZATION OF THE CONCEPT OF DISTANCE BASED ON THE SIMPLEX INEQUALITY

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ABSTRACT. We introduce and discuss the concept of n-*distance*, a generalization to n elements of the classical notion of distance obtained by replacing the triangle inequality with the so-called simplex inequality

$$d(x_1,\ldots,x_n) \;\leq\; K\,\sum_{i=1}^{n} d(x_1,\ldots,x_n)_i^z\,, \qquad x_1,\ldots,x_n,z\in X,$$

where $K = 1$. Here $d(x_1,\ldots,x_n)_i^z$ is obtained from the function $d(x_1,\ldots,x_n)$ by setting its ith variable to z. We provide several examples of n-distances, and for each of them we investigate the infimum of the set of real numbers $K \in \,]0,1]$ for which the inequality above holds. We also introduce a generalization of the concept of n-distance obtained by replacing in the simplex inequality the sum function with an arbitrary symmetric function.

1. INTRODUCTION

The notion of metric space, as first introduced by Fréchet [13] and later developed by Hausdorff [14], is one of the key ingredients in many areas of pure and applied mathematics, particularly in analysis, topology, geometry, statistics, and data analysis.

Denote the half-line $[0,+\infty[$ by $\mathbb{R}_+$. Recall that a *metric space* is a pair (X,d), where X is a nonempty set and d is a distance on X, that is, a function $d\colon X^2\to\mathbb{R}_+$ satisfying the following conditions:

- $d(x_1,x_2)\leq d(x_1,z)+d(z,x_2)$ for all $x_1,x_2,z\in X$ (triangle inequality),
- $d(x_1,x_2)=d(x_2,x_1)$ for all $x_1,x_2\in X$ (symmetry),
- $d(x_1,x_2)=0$ if and only if $x_1=x_2$ (identity of indiscernibles).

Generalizations of the concept of distance in which $n\geq 3$ elements are considered have been investigated by several authors (see, e.g., [5, Chapter 3] and the references therein). The three conditions above may be generalized to n-variable functions $d\colon X^n\to\mathbb{R}_+$ in the following ways. For any integer $n\geq 1$, we set $[n]=\{1,\ldots,n\}$. For any $i\in[n]$ and any $z\in X$, we denote by $d(x_1,\ldots,x_n)_i^z$ the function obtained from $d(x_1,\ldots,x_n)$ by setting its ith variable to z. Let also denote by S_n the set of all permutations on $[n]$. A function $d\colon X^n\to\mathbb{R}_+$ is said to be an $(n-1)$-*semimetric* [7] if it satisfies

(i) $d(x_1,\ldots,x_n)\leq\sum_{i=1}^n d(x_1,\ldots,x_n)_i^z$ for all $x_1,\ldots,x_n,z\in X$,
(ii) $d(x_1,\ldots,x_n)=d(x_{\pi(1)},\ldots,x_{\pi(n)})$ for all $x_1,\ldots,x_n\in X$ and all $\pi\in S_n$,

and it is said to be an $(n-1)$-*hemimetric* [5, 6] if additionally it satisfies

(iii') $d(x_1,\ldots,x_n)=0$ if and only if $x_1,\ldots,x_n$ are not pairwise distinct.

Date: January 11, 2018.

2010 *Mathematics Subject Classification.* Primary 39B72; Secondary 26D99.

Key words and phrases. n-distance, simplex inequality, Fermat point, smallest enclosing sphere.

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Condition (i) is referred to as the *simplex inequality* [5,7]. For $n = 3$, this inequality can be interpreted as follows: the area of a triangle face of a tetrahedron does not exceed the sum of the areas of the remaining three faces.

The following variant of condition (iii') can also be naturally considered:

(iii) $d(x_1,\ldots,x_n) = 0$ if and only if $x_1 = \cdots = x_n$.

For $n = 3$, functions satisfying conditions (i), (ii), and (iii) were introduced by Dhage [8] and called D-*distances*. Their topological properties were investigated subsequently [9–11], but unfortunately most of the claimed results are incorrect, see [23]. Moreover, it turned out that a stronger version of D-distance is needed for a sound topological use of these functions [16, 23, 24].

In this paper we introduce and discuss the following simultaneous generalization of the concepts of distance and D-distance by considering functions with $n \geq 2$ arguments.

Definition 1.1 (see [17]). Let $n \geq 2$ be an integer. We say that (X,d) is an n-*metric space* if X is a nonempty set and d is an n-*distance* on X, that is, a function $d\colon X^n \to \mathbb{R}_+$ satisfying conditions (i), (ii), and (iii).

We observe that for any n-distance $d\colon X^n \to \mathbb{R}_+$, the set of real numbers $K \in \left]0,1\right]$ for which the condition

$$d(x_1,\ldots,x_n) \;\leq\; K\sum_{i=1}^{n} d(x_1,\ldots,x_n)_i^z\,, \qquad x_1,\ldots,x_n,z \in X, \tag{1}$$

holds has an infimum K^*. We call it the *best constant* associated with the n-distance d. Determining the value of K^* for a given n-distance is an interesting problem that might be mathematically challenging. It is the purpose of this paper to provide natural examples of n-distances and to show how elegant the investigation of the values of the best constants might be.

It is worth noting that determining the best constant K^* is not relevant for nonconstant $(n-1)$-hemimetrics because we always have $K^* = 1$ for those functions. Indeed, we have

$$0 \;<\; d(x_1,\ldots,x_n) \;=\; \sum_{i=1}^{n} d(x_1,\ldots,x_n)_i^{x_n}$$

for any pairwise distinct elements $x_1,\ldots,x_n$ of X.

The paper is organized as follows. In Section 2 we provide some basic properties of n-metric spaces as well as some examples of n-distances together with their corresponding best constants. In Section 3 we investigate the values of the best constants for Fermat point based n-distances and discuss the particular case of median graphs. In Section 4 we consider some geometric constructions (smallest enclosing sphere and number of directions) to define n-distances and study their corresponding best constants. In Section 5 we introduce a generalization of the concept of n-distance by replacing in condition (i) the sum function with an arbitrary symmetric n-variable function. Finally, in Section 6 we conclude the paper by proposing topics for further research.

Remark 1. A *multidistance* on X, as introduced by Martín and Mayor [19], is a function $d\colon\bigcup_{n\geqslant 1} X^n \to \mathbb{R}_+$ such that, for every integer $n \geq 1$, the restriction of d to X^n satisfies conditions (ii), (iii), and

(i') $d(x_1,\ldots,x_n) \leqslant \sum_{i=1}^{n} d(x_i,z)$ for all $x_1,\ldots,x_n,z \in X$.

Properties of multidistances as well as instances including the Fermat multidistance and smallest enclosing ball multidistances have been investigated for example in [2, 18–20]. Note that multidistances have an indefinite number of arguments whereas n-distances have a fixed number of arguments. In particular, an n-distance can be defined without referring to any given 2-distance. Interestingly, some of the n-distances we present in this paper cannot be constructed from the concept of multidistance (see Section 6).

2. Basic examples and general properties of n-distances

Let us illustrate the concept of n-distance by giving a few elementary examples. Other classes of n-distances will be investigated in the next sections. We denote by $|E|$ the cardinalily of any set E.

Example 2.1 (Drastic n-distance)**.** For every integer $n \geqslant 2$, the map $d\colon X^n \to \mathbb{R}_+$ defined by $d(x_1,\ldots,x_n) = 0$, if $x_1 = \cdots = x_n$, and $d(x_1,\ldots,x_n) = 1$, otherwise, is an n-distance on X for which the best constant is $K_n^* = \frac{1}{n-1}$. Indeed, let $x_1,\ldots,x_n,z \in X$ and assume that $d(x_1,\ldots,x_n) = 1$. If there exists $k \in [n]$ such that $x_i = x_j \neq x_k$ for all $i,j \in [n]\setminus\{k\}$, then we have

$$\sum_{i=1}^{n} d(x_1,\ldots,x_n)_i^z \;=\; \begin{cases} n-1, & \text{if } z \in \{x_1,\ldots,x_n\}\setminus\{x_k\},\\ n, & \text{otherwise.}\end{cases}$$

In all other cases we have $\sum_{i=1}^{n} d(x_1,\ldots,x_n)_i^z = n$. □

Example 2.2 (Cardinality based n-distance)**.** For every integer $n \geqslant 2$, the map $d\colon X^n \to \mathbb{R}_+$ defined by

$$d(x_1,\ldots,x_n) \;=\; |\{x_1,\ldots,x_n\}| - 1$$

is an n-distance on X for which the best constant is $K_n^* = \frac{1}{n-1}$. Indeed, let $x_1,\ldots,x_n,z \in X$ and assume that $d(x_1,\ldots,x_n) \geq 1$. The case $n = 2$ is trivial. So let us further assume that $n \geq 3$. For every $i \in [n]$, set $m_i = |\{j \in [n] \mid x_j = x_i\}|$. If $|\{x_1,\ldots,x_n\}| < n$ (which means that there exists $j \in [n]$ such that $m_j \geq 2$), then it is straightforward to see that

$$\begin{aligned}\sum_{i=1}^{n} d(x_1,\ldots,x_n)_i^z \;&\geq\; n\, d(x_1,\ldots,x_n) - |\{i \in [n] \mid m_i = 1\}|\\ &\geq\; (n-1)\, d(x_1,\ldots,x_n),\end{aligned}$$

where the first inequality is an equality if and only if $z = x_j$ for some $j \in [n]$ such that $m_j \geq 2$, and the second inequality is an equality if and only if there is exactly one $j \in [n]$ such that $m_j \geq 2$. If $|\{x_1,\ldots,x_n\}| = n$, then

$$\sum_{i=1}^{n} d(x_1,\ldots,x_n)_i^z \;\geq\; (n-1)\, d(x_1,\ldots,x_n),$$

with equality if and only if $z \in \{x_1,\ldots,x_n\}$. □

Example 2.3 (Diameter)**.** Given a metric space (X,d) and an integer $n \geqslant 2$, the map $d_{\max}\colon X^n \to \mathbb{R}_+$ defined by

$$d_{\max}(x_1,\ldots,x_n) \;=\; \max_{\{i,j\}\subseteq[n]} d(x_i,x_j)$$

is an n-distance on X for which we have $K_n^* = \frac{1}{n-1}$. Indeed, let $x_1,\ldots,x_n,z \in X$ and assume without loss of generality that $d_{\max}(x_1,\ldots,x_n) = d(x_1,x_2)$. For every $i \in [n]$ we have

$$d_{\max}(x_1,\ldots,x_n)_i^z \;\geq\; \begin{cases} d(x_2,z), & \text{if } i = 1,\\ d(x_1,z), & \text{if } i = 2,\\ d(x_1,x_2), & \text{otherwise.}\end{cases}$$

Using the triangle inequality, we then obtain

$$\begin{aligned}\sum_{i=1}^{n} d_{\max}(x_1,\ldots,x_n)_i^z &\geq (n-2)\, d(x_1,x_2) + d(x_1,z) + d(x_2,z) \\ &\geq (n-1)\, d(x_1,x_2) = (n-1)\, d_{\max}(x_1,\ldots,x_n),\end{aligned}$$

which proves that $K_n^* \leq \frac{1}{n-1}$. To prove that $K_n^* = \frac{1}{n-1}$, note that if $x_1 = \cdots = x_{n-1} = z$ and $x_n \neq z$, then $\sum_{i=1}^{n} d_{\max}(x_1,\ldots,x_n)_i^z = (n-1)\, d_{\max}(x_1,\ldots,x_n)$. □

Example 2.4 (Sum based n-distance)**.** Given a metric space (X,d) and an integer $n \geq 2$, the map $d_\Sigma \colon X^n \to \mathbb{R}_+$ defined by

$$d_\Sigma(x_1,\ldots,x_n) = \sum_{\{i,j\}\subseteq[n]} d(x_i,x_j)$$

is an n-distance on X for which we have $K_n^* = \frac{1}{n-1}$. Indeed, for fixed $x_1,\ldots,x_n,z \in X$, we have

$$\sum_{i=1}^{n} d_\Sigma(x_1,\ldots,x_n)_i^z = (n-2) \sum_{\{i,j\}\subseteq[n]} d(x_i,x_j) + (n-1) \sum_{i=1}^{n} d(x_i,z).$$

Using the triangle inequality we obtain

$$(n-1) \sum_{i=1}^{n} d(x_i,z) = \sum_{\{i,j\}\subseteq[n]} \big(d(x_i,z) + d(x_j,z)\big) \geq \sum_{\{i,j\}\subseteq[n]} d(x_i,x_j).$$

Therefore, we finally obtain

$$\sum_{i=1}^{n} d_\Sigma(x_1,\ldots,x_n)_i^z \geq (n-1) \sum_{\{i,j\}\subseteq[n]} d(x_i,x_j) = (n-1)\, d_\Sigma(x_1,\ldots,x_n),$$

which proves that $K_n^* \leq \frac{1}{n-1}$. To prove that $K_n^* = \frac{1}{n-1}$, note that if $x_1 = \cdots = x_{n-1} = z$ and $x_n \neq z$, then $\sum_{i=1}^{n} d_\Sigma(x_1,\ldots,x_n)_i^z = (n-1)\, d_\Sigma(x_1,\ldots,x_n)$. □

Example 2.5 (Arithmetic mean based n-distance)**.** For any integer $n \geq 2$, the map $d\colon \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$d(x_1,\ldots,x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i - x_{(1)} = \frac{1}{n}\sum_{i=1}^{n}(x_i - x_{(1)}),$$

where $x_{(1)} = \min\{x_1,\ldots,x_n\}$, is an n-distance on $\mathbb{R}$ for which $K_n^* = \frac{1}{n-1}$. Indeed, let $x_1,\ldots,x_n,z \in \mathbb{R}$. By symmetry of d we may assume that $x_1 \leq \cdots \leq x_n$. We then obtain

$$d(x_1,\ldots,x_n) = \frac{1}{n}\Big(\sum_{i=1}^{n} x_i\Big) - x_1$$

and

$$\sum_{i=1}^{n} d(x_1,\ldots,x_n)_i^z = \Big(1 - \frac{1}{n}\Big)\Big(\sum_{i=1}^{n} x_i\Big) + z - (n-1)\, \min\{x_1,z\} - \min\{x_2,z\}.$$

It follows that condition (1) holds for $K_n = \frac{1}{n-1}$ if and only if

$$(n-1)(x_1 - \min\{x_1,z\}) + (z - \min\{x_2,z\}) \geq 0.$$

We then observe that this inequality is trivially satisfied, which proves that $K_n^* \leq \frac{1}{n-1}$. To prove that $K_n^* = \frac{1}{n-1}$, just take $x_1,\ldots,x_n,z \in \mathbb{R}$ so that $x_1 < z < x_2 = \cdots = x_n$. □

In the next result, we show how to construct an $(n-1)$-hemimetric from an n-distance.

Proposition 2.6. *Let (X,d) be an n-metric space for some integer $n \geq 2$. The function $d'\colon X^n \to \mathbb{R}_+$ defined as*

$$d'(x_1,\ldots,x_n) = \begin{cases} 0, & \text{if } x_1,\ldots,x_n \text{ are not pairwise distinct,} \\ d(x_1,\ldots,x_n), & \text{otherwise,} \end{cases}$$

is an $(n-1)$-hemimetric.

Proof. It is easy to see that d' satisfies conditions (ii) and (iii'). To see that condition (i) holds, let $x_1,\ldots,x_n,z \in X$ and assume that $d'(x_1,\ldots,x_n) > 0$. If $d'(x_1,\ldots,x_n)_i^z = d(x_1,\ldots,x_n)_i^z$ for every $i \in [n]$, then the simplex inequality holds for d'. Otherwise, we must have $z \in \{x_1,\ldots,x_n\}$ and then $\sum_{i=1}^n d'(x_1,\ldots,x_n)_i^z = d'(x_1,\ldots,x_n)$. This shows that condition (i) holds. □

The next proposition shows that two of the standard constructions of distances from existing ones are still valid for n-distances. The proof uses the following lemma.

Lemma 2.7. *For any $a_1,\ldots,a_n,a \in \mathbb{R}_+$ such that $a \leq \sum_{i=1}^n a_i$, we have*

$$\frac{a}{1+a} \leq \sum_{i=1}^n \frac{a_i}{1+a_i}\,.$$

Proof. We proceed by induction on $n \geq 1$. The result is easily obtained for $n \in \{1,2\}$. Assume that the result holds for $k \in \{1,\ldots,n-1\}$ for some $n \geq 3$, and that $a \leq \sum_{i=1}^n a_i$ for some $a,a_1,\ldots,a_n \in \mathbb{R}_+$. Letting $b = \max\{0, a-a_n\}$, we obtain

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{a_n}{1+a_n} \leq \sum_{i=1}^n \frac{a_i}{1+a_i}\,,$$

where the first inequality is obtained by the induction hypothesis applied to $a \leq b + a_n$, and the second to $b \leq \sum_{i=1}^{n-1} a_i$. □

Proposition 2.8. *Let d and d' be n-distances on X and let $\lambda > 0$. The following assertions hold.*

(a) *$d + d'$ and $\lambda\, d$ are n-distances on X.*
(b) *$\frac{d}{1+d}$ is an n-distance on X, with values in $[0,1]$.*

Proof. (a) is a simple verification. For (b) we note that condition (i) holds for $\frac{d}{1+d}$ by Lemma 2.7. □

Remark 2. In the same spirit as Proposition 2.8 we observe that if $d\colon X \to \mathbb{R}_+$ is an n-distance and $d_0\colon X \to \mathbb{R}_+$ is an $(n-1)$-hemimetric, then $d + d_0$ is an n-distance.

3. Fermat point based n-distances

Recall that, given a metric space (X,d) and an integer $n \geq 2$, the *Fermat set* F_Y of any n-element subset $Y = \{x_1,\ldots,x_n\}$ of X is defined as

$$F_Y = \Big\{ x \in X \;\Big|\; \sum_{i=1}^n d(x_i,x) \leq \sum_{i=1}^n d(x_i,z) \text{ for all } z \in X \Big\}.$$

Elements of F_Y are the *Fermat points of* Y. The problem of finding the Fermat point of a triangle in the Euclidean plane was formulated by Fermat in the early 17th century, and was first solved by Torricelli around 1640. The general problem stated for $n \geq 2$ in any metric space was considered by many authors, and applications were found for instance in geometry, combinatorial optimization, and facility location. We refer to [3, Chapter II] and [12] for an account of the history of this problem. Also, in [15], the location problem is extended in various directions and studied also for very general metrics – more general than those of normed spaces.

We observe that F_Y need not be nonempty in a general metric space. However, it follows from the continuity of the function $h\colon X \to \mathbb{R}_+$ defined by $h(x) = \sum_{i=1}^{n} d(x_i, x)$ that F_Y is nonempty whenever (X, d) is a proper metric space. (Recall that a metric space is proper if every closed ball is compact.) In this section we will therefore assume that (X, d) is a proper metric space.

Proposition 3.1. *For any proper metric space (X,d) and any integer $n \geq 2$, the map $d_F\colon X^n \to \mathbb{R}_+$ defined as*

$$d_F(x_1, \ldots, x_n) = \min_{x \in X} \sum_{i=1}^{n} d(x_i, x),$$

is an n-distance on X and we call it the Fermat n-distance.

Proof. The map d_F clearly satisfies conditions (ii) and (iii). Let us show that it satisfies condition (i). Assume first that $n = 2$ and let $y_1, y_2 \in X$ be such that

$$d_F(z, x_2) = d(z, y_1) + d(x_2, y_1) \quad \text{and} \quad d_F(x_1, z) = d(x_1, y_2) + d(z, y_2).$$

By applying the triangle inequality, we obtain

$$\begin{aligned} d_F(z, x_2) + d_F(x_1, z) &= \big(d(x_1, y_2) + d(z, y_2)\big) + \big(d(z, y_1) + d(x_2, y_1)\big) \\ &\geq d(x_1, x_2) = d(x_1, x_1) + d(x_1, x_2) \geq d_F(x_1, x_2). \end{aligned}$$

Assume now that $n \geq 3$ and let $y_1, \ldots, y_n \in X$ be such that

$$d_F(x_1, \ldots, x_n)_i^z = \sum_{j \neq i} d(x_j, y_i) + d(z, y_i), \qquad i = 1, \ldots, n.$$

It follows that

$$\begin{aligned} \sum_{i=1}^{n} d_F(x_1, \ldots, x_n)_i^z &\geq \sum_{i=1}^{n} \sum_{j \neq i} d(x_j, y_i) \\ &\geq \big(d(x_1, y_n) + d(x_2, y_n)\big) + \sum_{i=2}^{n-1} \big(d(x_1, y_i) + d(x_{i+1}, y_i)\big), \end{aligned}$$

that is, by applying the triangle inequality,

$$\sum_{i=1}^{n} d_F(x_1, \ldots, x_n)_i^z \geq \sum_{i=2}^{n} d(x_1, x_i) = \sum_{i=1}^{n} d(x_1, x_i) \geq d_F(x_1, \ldots, x_n),$$

where the last inequality follows from the definition of d_F. □

In the next proposition we use rough counting arguments to obtain bounds for the best constant K_n^* associated with the Fermat n-distance.

Proposition 3.2. *For every $n \geq 2$, the best constant K_n^* associated with the Fermat n-distance satisfies the inequalities $\frac{1}{n-1} \leq K_n^* \leq \frac{1}{\lfloor n/2 \rfloor}$.*

Proof. Let $x_1, \ldots, x_n \in X$ and let z be a Fermat point of $\{x_1, \ldots, x_n\}$. For every $i \in [n]$, denote by y_i a Fermat point of $\{z\} \cup \{x_1, \ldots, x_n\} \setminus \{x_i\}$. We then have

$$\begin{aligned} d_F(x_1, \ldots, x_n)_i^z &= \sum_{j \neq i} d(x_j, y_i) + d(z, y_i) \\ &\leq \sum_{j \neq i} d(x_j, z) + d(z, z) = \sum_{j \neq i} d(x_j, z). \end{aligned} \tag{2}$$

By summing over $i = 1,\ldots,n$, we obtain

$$\sum_{i=1}^{n} d_F(x_1,\ldots,x_n)_i^z \;\leq\; (n-1)\sum_{i=1}^{n} d(x_i,z) \;=\; (n-1)\, d_F(x_1,\ldots,x_n),$$

which shows that $K_n^* \geq 1/(n-1)$.

Now, if z denotes any element of X and if $y_1,\ldots,y_n$ are defined as in the first part of the proof, the identity (2) holds for every $i \in [n]$. Then, for $i = 1,\ldots,n-1$, we have

$$\begin{aligned} d_F(x_1,\ldots,x_n)_i^z + d_F(x_1,\ldots,x_n)_{i+1}^z \;&\geq\; d(z,y_i) + d(z,y_{i+1}) + d(x_i,y_{i+1}) + \sum_{j\neq i} d(x_j,y_i)\\ &\geq\; d(x_i,y_i) + \sum_{j\neq i} d(x_j,y_i) \qquad (3)\\ &\geq\; d_F(x_1,\ldots,x_n), \qquad (4)\end{aligned}$$

where (3) is obtained by a double application of the triangle inequality and (4) is obtained by definition of d_F.

It follows from (4) that $\sum_{i=1}^{n} d_F(x_1,\ldots,x_n)_i^z \geq \lfloor n/2\rfloor\, d_F(x_1,\ldots,x_n)$, which proves that $K_n^* \leq \lfloor n/2\rfloor^{-1}$. □

The next proposition uses a more refined counting argument to provide an improvement of the upper bound obtained for K_n^* in Proposition 3.2. Let us first state an immediate generalization of the hand-shaking lemma, which is folklore in graph theory.

Lemma 3.3. *Let $G = (V,E,w)$ be a weighted simple graph, where $w\colon E \to \mathbb{R}_+$ is the weighting function. If $f\colon V \to \mathbb{R}_+$ is such that $f(x)+f(y) \geq w(e)$ for every $e = \{x,y\} \in E$, then*

$$\sum_{x\in V} f(x)\,\deg_G(x) \;\geq\; \sum_{e\in E} w(e),$$

where $\deg_G(x)$ is the degree of x in G.

Proposition 3.4. *For every $n \geq 2$, the best constant K_n^* associated with the Fermat n-distance satisfies $K_n^* \leq (4n-4)/(3n^2-4n)$.*

Proof. Let $z,x_1,\ldots,x_n,y,y_1,\ldots,y_n \in X$ be such that y is a Fermat point of $\{x_1,\ldots,x_n\}$ and such that equation (2) holds for every $i \in [n]$. For any distinct $i,j \in [n]$, by the triangle inequality we have

$$d(z,y_i) + d(z,y_j) + d(x_i,y_j) \;\geq\; d(x_i,y_i). \qquad (5)$$

By summing (5) over all $j \in [n]\setminus\{i\}$ we obtain

$$(n-1)\,d(z,y_i) + \sum_{j\neq i}\big(d(z,y_j) + d(x_i,y_j)\big) \;\geq\; (n-1)\,d(x_i,y_i). \qquad (6)$$

By summing (6) over all $i \in [n]$ we then obtain

$$2\,(n-1)\sum_{i=1}^{n} d(z,y_i) + \sum_{i=1}^{n}\sum_{j\neq i} d(x_j,y_i) \;\geq\; (n-1)\sum_{i=1}^{n} d(x_i,y_i). \qquad (7)$$

Let us set $S = \sum_{i=1}^{n} \sum_{j \neq i} d(x_j, y_i)$. We then have

$$2\,(n-1) \sum_{i=1}^{n} d_F(x_1, \ldots, x_n)_i^z = (2\,n-3)\,S + S + 2\,(n-1) \sum_{i=1}^{n} d(z, y_i) \tag{8}$$

$$\geq (2\,n-3)\,S + (n-1) \sum_{i=1}^{n} d(x_i, y_i) \tag{9}$$

$$= (n-2)\,S + (n-1) \sum_{i=1}^{n} \sum_{j=1}^{n} d(x_j, y_i), \tag{10}$$

where (8) follows by the definitions of S and d_F, (9) follows by (7), and (10) by the definition of S.

Now, on the one hand, by the definition of d_F we have

$$(n-1) \sum_{i=1}^{n} \sum_{j=1}^{n} d(x_j, y_i) \geq n\,(n-1)\,d_F(x_1, \ldots, x_n). \tag{11}$$

On the other hand, let us fix $i \in [n]$ and set $V = \{x_1, \ldots, x_n\} \smallsetminus \{x_i\}$. Define the function $f\colon V \to \mathbb{R}_+$ by $f(x_j) = d(x_j, y_i)$ for any $j \neq i$, and consider the complete weighted graph $G = (V, \binom{V}{2}, w)$ defined by $w(\{x_\ell, x_j\}) = d(x_\ell, x_j)$ for any distinct $x_\ell, x_k \in V$. It follows from Lemma 3.3 that

$$(n-2) \sum_{j \neq i} d(x_j, y_i) \geq \sum_{\{x_k, x_\ell\} \in \binom{V}{2}} d(x_k, x_\ell). \tag{12}$$

By summing (12) over all $i \in [n]$, we get

$$(n-2)\,S \geq (n-2) \sum_{\{k,\ell\} \in \binom{n}{2}} d(x_k, x_\ell) = \frac{n-2}{2} \sum_{k=1}^{n} \sum_{\ell=1}^{n} d(x_k, x_\ell)$$

$$\geq n\,\frac{n-2}{2}\,d_F(x_1, \ldots, x_n), \tag{13}$$

where (13) is obtained by definition of d_F. By substituting (11) and (13) into (10), we finally obtain

$$\sum_{i=1}^{n} d_F(x_1, \ldots, x_n)_i^z \geq \frac{n\,(3\,n-4)}{4\,(n-1)}\,d_F(x_1, \ldots, x_n),$$

which proves that $K_n^* \leq (4\,n-4)/(3\,n^2 - 4\,n)$. □

We observe that Proposition 3.4 provides a better upper bound than Proposition 3.2 for every $n \geq 2$, but the difference between these bounds converges to zero as n tends to infinity. The high number of inequalities involved in the proof of Proposition 3.4 suggests that it is in general very difficult to obtain the exact value of K_n^* (we have to find $x_1, \ldots, x_n, z \in X$ that turn these inequalities into equalities). However, we will now show that we can determine the value of K_n^* when d_F is the Fermat n-distance associated with the distance function in median graphs.

Recall that a *median graph* is a connected undirected simple graph in which, for any triplet of vertices u, v, w, there is one and only one vertex $\mathbf{m}(u, v, w)$ that is at the intersection of shortest paths between any two elements among u, v, w. Cubes and trees are instances of median graphs. In a median graph $G = (V, E)$, the Fermat 3-distance is the function $d_{\mathbf{m}}\colon V^3 \to \mathbb{R}_+$ defined by

$$d_{\mathbf{m}}(u, v, w) = \min_{y \in V} \big(d(u, y) + d(v, y) + d(w, y)\big), \tag{14}$$

where d denotes the usual distance function between vertices in a connected graph.

Proposition 3.5. *If $G = (V, E)$ is a median graph, then the best constant K^* associated with its Fermat 3-distance $d_{\mathbf{m}}$ is equal to $\frac{1}{2}$. Moreover, the only Fermat point of $\{u, v, w\}$ is $\mathbf{m}(u, v, w)$.*

Proof. The minimum in (14) is realized by any $y_0 \in V$ that realizes the minimum of the values

$$\big(d(u,y) + d(v,y)\big) + \big(d(w,y) + d(u,y)\big) + \big(d(v,y) + d(w,y)\big) \tag{15}$$

for $y \in V$. By definition, the vertex $y_0 = \mathbf{m}(u, v, w)$ is on shortest paths between any two elements among u, v, w, which shows that it realizes the minimum of each of the three terms in (15), and hence the minimum in (14).

It follows that

$$\begin{aligned} d_{\mathbf{m}}(u, v, z) &= d(u, y_0) + d(v, y_0) + d(z, y_0) \\ &= \frac{1}{2}(d(u, y_0) + d(v, y_0) + d(z, y_0) + d(u, y_0) + d(v, y_0) + d(z, z_0)) \\ &= \frac{1}{2}(d(u, v) + d(u, z) + d(v, z)), \end{aligned}$$

which shows that $\min_{z \in V} d_{\mathbf{m}}(u, v, z)$ is equal to $d(u, v)$, and is realized by any element z_0 on a shortest path between u and v. We conclude that the minimum of

$$d_{\mathbf{m}}(z, v, w) + d_{\mathbf{m}}(u, z, w) + d_{\mathbf{m}}(u, v, z)$$

for $z \in V$ is realized by $z_0 = \mathbf{m}(u, v, w)$, and is equal to $d(v, w) + d(u, w) + d(u, v) = 2\, d_{\mathbf{m}}(u, v, w)$. We have proved that the best constant K^* associated with $d_{\mathbf{m}}$ is $\frac{1}{2}$. □

4. EXAMPLES OF n-DISTANCES BASED ON GEOMETRIC CONSTRUCTIONS

In this section we introduce n-distances defined from certain geometric constructions and investigate their corresponding best constants. In what follows, we denote by d the Euclidean distance on $\mathbb{R}^k$ for some integer $k \geq 2$.

The first n-distances we investigate are based on the following construction.

Definition 4.1. For any $n \geq 2$ and any $x_1, \ldots, x_n \in \mathbb{R}^k$, we denote by $S(x_1, \ldots, x_n)$ the smallest $(k-1)$-dimensional sphere enclosing $\{x_1, \ldots, x_n\}$. For any $i \in [n]$ and any $z \in \mathbb{R}^k$, we denote by $S(x_1, \ldots, x_n)_i^z$ the smallest $(k-1)$-dimensional sphere enclosing $\{x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n\}$.

The sphere introduced in Definition 4.1 always exists and is unique. Moreover, it can be computed in linear time [21, 22] or expected linear time [26].

When $k = 2$, we have the following fact.

Fact 4.2. *Let A, B, C be the vertices of a triangle in $\mathbb{R}^2$.*

(a) *If ABC forms an acute triangle with angles α, β and γ, respectively, then $S(A, B, C)$ is the circumcircle $\mathcal{C}$ of ABC whose radius R satisfies*

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}, \tag{16}$$

where $a = d(B, C)$, $b = d(A, C)$, and $c = d(A, B)$. Let A^* be one of the two points of the circle $\mathcal{C}$ that is on the bisector of BC. Then the perimeter of the triangle ABC strictly decreases as A moves along $\mathcal{C}$ from A^* to B (or from A^* to C).

(b) *If ABC is obtuse in A, then $S(A,B,C)$ contains B and C, and its diameter is equal to a.*

(c) *It follows from (a) and (b) that the radius R of $S(A,B,C)$ satisfies*

$$R \geq \max\Big\{\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\Big\}. \tag{17}$$

Proposition 4.3 (Radius of $S(x_1,\ldots,x_n)$ in $\mathbb{R}^2$). *For any $n \geq 2$, the map $d_r\colon (\mathbb{R}^2)^n \to \mathbb{R}_+$ that associates with any $(x_1,\ldots,x_n) \in (\mathbb{R}^2)^n$ the radius of $S(x_1,\ldots,x_n)$ is an n-distance for which we have $K_n^* = \frac{1}{n-1}$.*

Proof. Let us show that the map d_r satisfies the simplex inequality for $K_n = \frac{1}{n-1}$. Since d_r is a continuous function, we can assume that its arguments are pairwise distinct.

Consider first the case where $n = 2$. For any distinct $A, B \in \mathbb{R}^2$, we have $d_r(A,B) = \frac{1}{2}\, d(A,B)$, which proves that the simplex inequality holds for $n = 2$.

Suppose now that $n = 3$ and let us show that, for any $A,B,C,Z \in \mathbb{R}^2$, with A,B,C pairwise distinct, we have

$$2\, d_r(A,B,C) \leq d_r(Z,B,C) + d_r(A,Z,C) + d_r(A,B,Z). \tag{18}$$

Set $a = d(B,C)$, $b = d(A,C)$, and $c = d(A,B)$. By (17) we have

$$d_r(Z,B,C) \geq \frac{a}{2}\,, \quad d_r(A,Z,C) \geq \frac{b}{2}\,, \quad d_r(A,B,Z) \geq \frac{c}{2}\,, \tag{19}$$

and hence

$$d_r(Z,B,C) + d_r(A,Z,C) + d_r(A,B,Z) \geq \frac{a+b+c}{2} \geq \max\{a,b,c\}. \tag{20}$$

Suppose first that ABC is not acute, assuming for instance that $\beta \geq \frac{\pi}{2}$. Then $2\, d_r(A,B,C) = b$, and then (18) immediately follows from (20). Suppose now that ABC is acute, with circumcircle $\mathcal{C}$, and consider the triangle $A'BC$, with sides a, b', c', such that $A' \in \mathcal{C}$ and $\sphericalangle A'BC = \frac{\pi}{2}$. By Fact 4.2 (a) we have

$$\frac{a+b+c}{2} \geq \frac{a+b'+c'}{2} \geq b' = 2\, d_r(A',B,C) = 2\, d_r(A,B,C),$$

and then again (18) follows from (20). Finally, the equality is obtained in (18) by taking $A \neq B = C = Z$.

We now prove the general case where $n \geq 3$. Let $A_1,\ldots,A_n,Z \in \mathbb{R}^2$, with $A_1,\ldots,A_n$ pairwise distinct. It is a known fact [4] that either there are $j,k \in [n]$ such that A_j and A_k are distinct and

$$S(A_1,\ldots,A_n) = S(A_j,A_k)$$

or there are $j,k,\ell \in [n]$ such that A_j, A_k, and A_ℓ are distinct and

$$S(A_1,\ldots,A_n) = S(A_j,A_k,A_\ell).$$

Let us consider the latter case (the proof in the former case can be dealt with similarly). On the one hand, using (17) it is easy to see that

$$d_r(A_1,\ldots,A_n)_i^Z \geq d_r(A_1,\ldots,A_n), \qquad i \notin \{j,k,\ell\}. \tag{21}$$

On the other hand, the following inequalities hold:

$$\begin{aligned} d_r(A_1,\ldots,A_n)_j^Z &\geq d_r(Z,A_k,A_\ell), \\ d_r(A_1,\ldots,A_n)_k^Z &\geq d_r(A_j,Z,A_\ell), \\ d_r(A_1,\ldots,A_n)_\ell^Z &\geq d_r(A_j,A_k,Z). \end{aligned}$$

Indeed, $S(A_1,\ldots,A_n)_j^Z$ encloses the points Z, A_k, and A_ℓ and hence cannot have a radius strictly smaller than that of $S(Z,A_k,A_\ell)$.

Adding up these inequalities and then using (18), we obtain

$$\begin{aligned} & d_r(A_1,\ldots,A_n)_j^Z + d_r(A_1,\ldots,A_n)_k^Z + d_r(A_1,\ldots,A_n)_\ell^Z \\ \geq\ & d_r(Z,A_k,A_\ell) + d_r(A_j,Z,A_\ell) + d_r(A_j,A_k,Z) \\ \geq\ & 2\,d_r(A_j,A_k,A_\ell) \ = \ 2\,d_r(A_1,\ldots,A_n). \end{aligned} \tag{22}$$

Combining (21) with (22), we finally obtain

$$\sum_{i=1}^n d_r(A_1,\ldots,A_n)_i^Z \ \geq \ (n-1)\,d_r(A_1,\ldots,A_n),$$

which proves that $K_n^* \leq \frac{1}{n-1}$. To prove that $K_n^* = \frac{1}{n-1}$, just consider $A_2 = \cdots = A_n = Z$ and $A_1 \neq A_2$. □

Proposition 4.4 (Area bounded by $S(x_1,\ldots,x_n)$ in $\mathbb{R}^2$)**.** *For any $n \geq 3$, the map $d_s\colon (\mathbb{R}^2)^n \to \mathbb{R}_+$ that associates with any $(x_1,\ldots,x_n) \in (\mathbb{R}^2)^n$ the surface area bounded by $S(x_1,\ldots,x_n)$ is an n-distance for which we have $K_n^* = (n-\frac{3}{2})^{-1}$.*

Proof. Let us show that the map $d_s = \pi\,d_r^2$ satisfies the simplex inequality with constant $K_n = (n-\frac{3}{2})^{-1}$. Since d_r is continuous, we can assume that its arguments are pairwise distinct.

Consider first the case where $n = 3$ and let us show that, for any $A,B,C,Z \in \mathbb{R}^2$, with A,B,C pairwise distinct, we have

$$d_r(A,B,C)^2 \ \leq \ \frac{2}{3}\big(d_r(Z,B,C)^2 + d_r(A,Z,C)^2 + d_r(A,B,Z)^2\big). \tag{23}$$

If the triangle ABC is acute, then we may assume for instance that $\frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2}$, which implies $\frac{\sqrt{3}}{2} \leq \sin\alpha \leq 1$. Using (16), we then have

$$d_r(A,B,C)^2 \ \leq \ \frac{a^2}{3} \ \leq \ \frac{2}{3}\Big(\frac{a^2}{4}+\frac{a^2}{4}\Big) \ \leq \ \frac{2}{3}\Big(\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}\Big), \tag{24}$$

where the latter inequality holds by the law of cosines. We then obtain (23) by combining (19) with (24).

If ABC is obtuse in C, then $d_r(A,B,C) = \frac{c}{2}$. Using the triangle inequality and the square and arithmetic mean inequality, we also have

$$\frac{a^2+b^2}{2} \ \geq \ \Big(\frac{a+b}{2}\Big)^2 \ \geq \ \frac{c^2}{4}\,.$$

Combining these observations with (19), we obtain

$$\begin{aligned} & \frac{2}{3}\big(d_r(Z,B,C)^2 + d_r(A,Z,C)^2 + d_r(A,B,Z)^2\big) \\ \geq\ & \frac{2}{3}\Big(\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}\Big) \ \geq \ \frac{2}{3}\,\frac{3}{8}\,c^2 \ = \ \Big(\frac{c}{2}\Big)^2 \ = \ d_r(A,B,C)^2. \end{aligned}$$

To see that the general case where $n \geq 3$ also holds, it suffices to proceed as in the proof of Proposition 4.3. This shows that $K_n^* \leq (n-\frac{3}{2})^{-1}$. To prove that $K_n^* = (n-\frac{3}{2})^{-1}$, just consider $A_1 \neq A_2$ and $A_3 = \cdots = A_n = Z = (A_1+A_2)/2$, where $(A_1+A_2)/2$ is the midpoint of A_1 and A_2. □

Remark 3. The map d_s defined in Proposition 4.4 can be naturally extended to the case where $n = 2$. However, in this case d_s no longer satisfies condition (i) and hence is not a 2-distance. Indeed, for any $A, B, Z \in \mathbb{R}^2$, with A, B distinct, we have

$$d_s(A,B) \ \le\ 2\big(d_s(A,Z) + d_s(Z,B)\big),$$

or equivalently,

$$d(A,B)^2 \ \le\ 2\,d(A,Z)^2 + 2\,d(Z,B)^2,$$

where the constant 2 is optimal (take A and B distinct and $Z = (A+B)/2$). To see that this inequality holds, set $A = (0,0)$, $B = (b,0)$, and $Z = (x,y)$. Then, the inequality becomes

$$b^2 \ \le\ 2\,(x^2+y^2) + 2\,(x-b)^2 + 2\,y^2,$$

which always holds because it is algebraically equivalent to

$$(2x-b)^2 + 4y^2 \ \ge\ 0.$$

Remark 4. In an attempt to generalize the previous two propositions to $\mathbb{R}^k$ ($k \ge 2$), we may consider the following open questions:

(a) Prove (or disprove) that Proposition 4.3 still holds in $\mathbb{R}^k$.
(b) Prove (or disprove) that, for any $n \ge 3$, the map $d_v\colon (\mathbb{R}^k)^n \to \mathbb{R}_+$ that associates with any $(x_1,\ldots,x_n) \in (\mathbb{R}^k)^n$ the k-dimensional volume bounded by $S(x_1,\ldots,x_n)$ is an n-distance for which we have $K_n^* = (n-2+2^{1-k})^{-1}$.

Note that the problem in (b) above is motivated by the fact that the corresponding simplex inequality with $K_n = (n-2+2^{1-k})^{-1}$ holds when x_1 and x_2 are distinct and $x_3 = \cdots = x_n = z$ is the midpoint of x_1 and x_2.

We now show that counting the number of different directions defined by pairs of distinct elements among n points in the plane defines an n-distance.

For any distinct $x, y \in \mathbb{R}^2$, we denote by $\overline{xy}$ the direction $\pm(x-y)/\|x-y\|$. Here we assume that $\overline{xy}$ and $\overline{yx}$ represent the same direction.

Proposition 4.5 (Number of directions in $\mathbb{R}^2$)**.** *For any $n \ge 3$, the map $d_n\colon (\mathbb{R}^2)^n \to \mathbb{R}_+$ that associates with any $(x_1,\ldots,x_n) \in (\mathbb{R}^2)^n$ the cardinality $|\Delta|$ of the set*

$$\Delta \ =\ \big\{\overline{x_ix_j} \mid i,j \in [n] \text{ and } x_i \ne x_j\big\}$$

is an n-distance for which we have $\frac{1}{n-2+\frac{2}{n}} \le K_n^* < \frac{1}{n-2}$.

Proof. Let $x_1,\ldots,x_n,z \in \mathbb{R}^2$. For any $i \in [n]$, let

$$\Delta_i \ =\ \{\overline{x_jx_k} \mid j,k \in [n]\setminus\{i\} \text{ and } x_j \ne x_k\}.$$

On the one hand, we clearly have $|\Delta_i| \le d_n(x_1,\ldots,x_n)_i^z$ for every $i \in [n]$. On the other hand, it is easy to see that each direction in Δ is counted at least $(n-2)$ times in the sum $\sum_{i=1}^n |\Delta_i|$. From these observations it follows that

$$(n-2)\,d_n(x_1,\ldots,x_n) \ =\ (n-2)\,|\Delta| \ \le\ \sum_{i=1}^n |\Delta_i| \ \le\ \sum_{i=1}^n d_n(x_1,\ldots,x_n)_i^z, \tag{25}$$

which proves that $K_n^* \le \frac{1}{n-2}$.

We now show by contradiction that the latter inequality is strict. Assume that there exist $x_1,\ldots,x_n,z \in \mathbb{R}^2$ such that

$$(n-2)\,d_n(x_1,\ldots,x_n) \ =\ \sum_{i=1}^n d_n(x_1,\ldots,x_n)_i^z.$$

It follows that for these points we can replace both inequalities in (25) with equalities. The first equality then means that each direction in Δ is counted exactly $(n-2)$ times in the sum $\sum_{i=1}^n |\Delta_i|$. It is easy to see that this condition also means that no three of the points $x_1,\ldots,x_n$ are collinear. Let us now consider the second inequality. Since $|\Delta_i| \leq d_n(x_1,\ldots,x_n)_i^z$ for every $i\in[n]$, we must have $|\Delta_i| = d_n(x_1,\ldots,x_n)_i^z$ for every $i\in[n]$. Suppose first that $n\geq 4$. It follows from the latter condition that both sets $\{x_2,\ldots,x_n\}$ and $\{z,x_2,\ldots,x_n\}$ generate the same number of directions. Since no three of the points $x_2,\ldots,x_n$ are collinear, we should have $z=x_\ell$ for some $\ell\in\{2,\ldots,n\}$. But then we have $|\Delta_\ell| < d_n(x_1,\ldots,x_n)_\ell^z$, a contradiction. A similar contradiction can be easily reached when $n=3$.

Let us now establish the lower bound for K_n^*. Let $x_1,\ldots,x_n$ be pairwise distinct and placed clockwise on the unit circle. Let also $z=x_1$. Then we have

$$d_n(x_1,\ldots,x_n) = \binom{n}{2} \quad\text{and}\quad d_n(x_1,\ldots,x_n)_i^z = \begin{cases}\binom{n}{2} & \text{if } i=1,\\ \binom{n-1}{2} & \text{if } i\neq 1,\end{cases}$$

and hence

$$\begin{aligned}\sum_{i=1}^n d_n(x_1,\ldots,x_n)_i^z &= \binom{n}{2} + (n-1)\binom{n-1}{2} = \Big(n-2+\frac{2}{n}\Big)\binom{n}{2}\\ &= \Big(n-2+\frac{2}{n}\Big)\, d_n(x_1,\ldots,x_n),\end{aligned}$$

which completes the proof. □

Remark 5. An n-distance $d\colon(\mathbb{R}^k)^n\to\mathbb{R}_+$ is said to be *homogeneous of degree* $q\geq 0$ if, for any $t>0$, we have

$$d(tx_1,\ldots,tx_n) = t^q\, d(x_1,\ldots,x_n), \qquad x_1,\ldots,x_n\in\mathbb{R}^k.$$

This means that under any dilation $x\mapsto tx$, the n-distance d is magnified by the factor t^q. Since a distance on $\mathbb{R}^k$ usually represents a linear dimension, we could expect any n-distance on $\mathbb{R}^k$ to be homogeneous of degree 1. This is for instance the case for the n-distance defined in Proposition 4.3. Surprisingly enough, the n-distances defined in Examples 2.1, 2.2, and Proposition 4.5 are homogeneous of degree 0, that is, invariant under any dilation. Also, the n-distance defined in Proposition 4.4 is homogeneous of degree 2.

5. A GENERALIZATION OF THE CONCEPT OF n-DISTANCE

The concept of n-distance as defined in Definition 1.1 can naturally be generalized by relaxing condition (i) as follows.

Definition 5.1. Let $g\colon\mathbb{R}_+^n\to\mathbb{R}_+$ be a symmetric function, i.e., invariant under any permutation of its arguments. We say that a function $d\colon X^n\to\mathbb{R}_+$ is a *g-distance* if it satisfies conditions (ii), (iii), and

$$d(x_1,\ldots,x_n) \leqslant g\big(d(x_1,\ldots,x_n)_1^z,\ldots,d(x_1,\ldots,x_n)_n^z\big)$$

for all $x_1,\ldots,x_n,z\in X$.

In view of Proposition 2.8, it is natural to require $d+d'$, $\lambda\, d$, and $\frac{d}{1+d}$ to be g-distances whenever so are d and d'. The following proposition provides sufficient conditions on g for these properties to hold. Recall that a function $g\colon\mathbb{R}_+^n\to\mathbb{R}$ is *positively homogeneous* if $g(\lambda\mathbf{r})=\lambda\, g(\mathbf{r})$ for all $\mathbf{r}\in\mathbb{R}_+^n$ and all $\lambda>0$. It is said to be *superadditive* if $g(\mathbf{r}+\mathbf{s})\geq$

$g(\mathbf{r}) + g(\mathbf{s})$ for every $\mathbf{r}, \mathbf{s} \in \mathbb{R}_+^n$. Also, it is *additive* if $g(\mathbf{r}+\mathbf{s}) = g(\mathbf{r}) + g(\mathbf{s})$ for every $\mathbf{r}, \mathbf{s} \in \mathbb{R}_+^n$.

Proposition 5.2. *Let $g{:}\,\mathbb{R}_+^n \to \mathbb{R}_+$ be a symmetric function, and let $d, d'{:}\,X^n \to \mathbb{R}_+$ be g-distances. The following assertions hold.*

(a) *If g is positively homogeneous, then $\lambda\, d$ is a g-distance for every $\lambda > 0$.*
(b) *If g is superadditive, then $d + d'$ is a g-distance.*
(c) *If g is both positively homogeneous and superadditive, then it is concave.*
(d) *The function g is additive if and only if there exists $\lambda \geq 0$ such that*

$$g(\mathbf{r}) \;=\; \lambda \sum_{i=1}^{n} r_i\,, \qquad \mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}_+^n. \tag{26}$$

(e) *If g satisifies* (26) *for some $\lambda \geq 1$, then $\frac{d}{1+d}$ is a g-distance.*

Proof. (a) and (b) follow from the definitions.

(c) For any $\lambda \in [0,1]$, we have

$$g(\lambda\mathbf{r} + (1-\lambda)\mathbf{s}) \;\leq\; g(\lambda\mathbf{r}) + g((1-\lambda)\mathbf{s}) \;=\; \lambda g(\mathbf{r}) + (1-\lambda)g(\mathbf{s}),$$

where the inequality follows from superadditivity and the equality from positive homogeneity.

(d) The sufficiency is trivial. To see that the necessity holds, note that g is additive and bounded from below (since it ranges in $\mathbb{R}_+$) and hence it is continuous and there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $g(\mathbf{r}) = \sum_{i=1}^{n} \lambda_i r_i$; see [1, Cor. 2, p. 35]. The result then follows from the symmetry of g.

(e) Let $x_1, \ldots, x_n, z \in X$ and set $d = d(x_1, \ldots, x_n)$ and $d_i = d(x_1, \ldots, x_n)_i^z$ for every $i \in [n]$. Since $\lambda \geq 1$, we have $\lambda\, r/(1+\lambda\, r) \leq \lambda\, r/(1+r)$ for every $r \geq 0$. It then follows that

$$\frac{1}{1+d} \;\leq\; \sum_{i=1}^{n} \frac{\lambda\, d_i}{1+\lambda\, d_i} \;\leq\; \sum_{i=1}^{n} \frac{\lambda\, d_i}{1+d_i}\,,$$

where the first inequality follows from Lemma 2.7 and the fact that d is a g-distance. □

6. Conclusion and further research

In this paper we have introduced and discussed the concept of n-distance as a natural generalization of the concept of distance to functions of $n \geq 2$ variables. There are two key features in this generalization: one is an n-ary version of the identity of indiscernibles, and the other is the simplex inequality, which is a natural generalization of the triangle inequality. We have observed that any n-distance d has an associated best constant $K_n^* \in \left]0,1\right]$ satisfying inequality (1). Also, we have provided many natural examples of n-distances, and have shown that searching for their associated best constant may be mathematically challenging and may sometimes require subtle arguments. The examples we have discussed might suggest that we have $K_n^* < 1$ for any n-distance. The following example, which was communicated to us by Roberto Ghiselli Ricci [25], shows that this is not the case.

Example 6.1. Let $n \geq 3$ and $a \in \mathbb{R}$. Let also $\mathcal{A}(a,n)$ be the set of n-tuples whose components are consecutive elements of arithmetic progressions with common difference a. Consider the map $d_n{:}\,\mathbb{R}^n \to \mathbb{R}_+$ defined as

$$d_n(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } x_1 = \cdots = x_n, \\ 1 & \text{if } (x_1, \ldots, x_n) \in \mathcal{A}(a,n) \text{ for some } a \neq 0, \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

We prove that d_n is an n-distance for which we have $K_n^* = 1$. Conditions (ii) and (iii) are easily verified. To see that condition (i) holds, consider $x_1, \ldots, x_n, z \in \mathbb{R}$. First assume that $d_n(x_1, \ldots, x_n) = \frac{1}{n}$. There is at most one $i \in [n]$ such that $d_n(x_1, \ldots, x_n)_i^z = 0$. Thus, we obtain

$$\sum_{i=1}^{n} d_n(x_1, \ldots, x_n)_i^z \geq \frac{n-1}{n} \geq d_n(x_1, \ldots, x_n).$$

Assume now that $d(x_1, \ldots, x_n) = 1$. It follows that $d_n(x_1, \ldots, x_n)_i^z \geq \frac{1}{n}$ for all $i \in [n]$, which shows that the simplex inequality holds in that case as well. To prove that $K_n^* = 1$, just consider $x_1 = 1, x_2 = 2, \ldots, x_n = n$, and $z = -1$. □

We also observe that certain n-distances cannot be constructed from the concept of multidistance as defined by Martín and Mayor [19] (see Remark 1). Instances of such n-distances are given, e.g., in Propositions 4.4 and 4.5.

We conclude this paper by proposing a few topics for further research.

(a) Improve the bounds for the best constant associated with the Fermat n-distance (at least in some given proper metric spaces).
(b) Consider and solve the problems stated in Remark 4.
(c) Investigate properties of topological spaces based on n-metric spaces. On this issue we observe that in [24] the authors introduced a stronger version of 3-metric space called *G-metric space* (see also [16]). It is shown that there is a natural metric space associated with any G-metric space. Finding an appropriate generalization of the notion of G-metric space as a stronger version of n-metric space and investigating its topological properties seems to be an interesting topic of research.

Acknowledgments

This research is supported by the internal research project R-AGR-0500 of the University of Luxembourg.

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