MULTI-ORIENTED PROPS AND HOMOTOPY ALGEBRAS WITH BRANES

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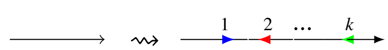
Abstract. We introduce a new category of differential graded *multi-oriented* props whose representations (called homotopy algebras with branes) in a graded vector space require a choice of a collection of k linear subspaces in that space, k being the number of extra directions (if $k = 0$ this structure recovers an ordinary prop); symplectic vector spaces equipped with k Lagrangian subspaces play a distinguished role in this theory. Manin triples is a classical example of an algebraic structure (concretely, a Lie bialgebra structure) given in terms of a vector space and its subspace; in the context of this paper Manin triples are precisely symplectic Lagrangian representations of the *2-oriented* generalization of the classical operad of Lie algebras. In a sense, the theory of multi-oriented props provides us with a far reaching strong homotopy generalization of Manin triples type constructions.

The homotopy theory of multi-oriented props can be quite non-trivial (and different from that of ordinary props). The famous Grothendieck-Teichmüller group acts faithfully as homotopy non-trivial automorphisms on infinitely many multi-oriented props, a fact which motivated much the present work as it gives us a hint to a non-trivial deformation quantization theory in every geometric dimension $d \geq 4$ generalizing to higher dimensions Drinfeld-Etingof-Kazhdan's quantizations of Lie bialgebras (the case $d = 3$) and Kontsevich's quantizations of Poisson structures (the case $d = 2$).

1. Introduction

1.1. Why bother with multi-oriented props? A short answer to this question: "Because of the Grothendieck-Teichmüller group GRT_1". It is the latter beautiful and mysterious structure which is the main motivation for introducing and study of a new category of multi-oriented props as well as their representations ("homotopy algebras with branes"). In geometric dimensions 2 and 3 the group GRT_1 acts on some ordinary props of odd/even strong homotopy Lie bialgebras [MW1] and plays thereby the classifying role in the associated transcendental deformation quantizations of Poisson and, respectively, ordinary Lie bialgebra structures. In geometric dimension $d \geq 4$ the group GRT_1 survives in the form of symmetries of some *multi-oriented* props of even/odd homotopy Lie bialgebras so that deformation quantizations in higher dimensions (in which GRT_1 retains its fundamental classifying role) should involve a really new class of algebro-geometric structures — the homotopy algebras *with branes*. It is an attempt to understand what could be a higher ($d \geq 4$) analogue of two famous formality theorems, one for Poisson structures [Ko] (the case $d = 2$), and another for ordinary Lie bialgebra ones [EK, Me2] (the case $d = 4$), that lead the author to the category of multi-oriented props after reading the paper [Z] by Marko Živković and its predecessor [W2] by Thomas Willwacher (see §5 for a brief but self-contained description of their remarkable results).

It is not that hard to define multi-oriented props in general, and multi-oriented generalizations of some concrete classical operads and props in particular, at the purely combinatorial level: the rules of the game with multi-oriented decorated graphs are more or less standard (see §2) — for any given $k \geq 1$ one just adds k extra directions to each edge/leg of a 1-oriented graph,



and defines rules for multi-oriented prop compositions via contractions along admissible multi-oriented subgraphs. However, it is much less evident how to transform that more or less standard rules into non-trivial and interesting representations (i.e. examples) — the intuition from the theory of ordinary (wheeled) props does not help much. Adding new k directions to each edges of a decorated graph of an ordinary prop can be naively understood as extending that ordinary prop into a 2^k-coloured one, but then the requirement that the new directions on graph edges do not create "wheels" (that is, closed directed paths of edges with respect to *any* of the new orientations) kills that naive picture immediately — the elements of the set of 2^k new colours start interacting with each other in a non-trivial way. We know which structure distinguishes ordinary props (the ones with no wheels in the given single orientation, i.e. the ones which are *1-oriented* in the terminology of this paper) from the ordinary wheeled props (that is, *0-oriented 1-directed* props in the terminology of this paper) in terms of representations in, say, a graded vector space V — it is the *dimension of* V. In general, a wheeled prop can have well-defined representations in V only in the case $\dim V < \infty$ as graphs with wheels generate the trace operation $V \times V^* \to \mathbb{K}$ which explodes in the case $\dim V = \infty$; this phenomenon explains the need for *1-oriented* props.

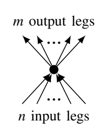
How to explain the need for all (or some) of the *extra* k directions to be oriented? Which structure on a graded vector space V can be used to separate (in the sense of representations) 0-oriented $(k+1)$-directed props from $(k+1)$-oriented ones? Perhaps the main result of this paper is a rather surprising answer to that question: one has to work again with a certain class of infinite-dimensional vector spaces V, but now equipped with k linear subspaces $W_1, \ldots, W_k \subset V$ together with their complements, and interpret a *single* element of a $(k+1)$-oriented prop $\mathcal{P}$ as a *collection of* k *linear maps* from various intersections of subspaces $W_\bullet$ and their complements and their duals to themselves; then indeed graphs with wheels in one or another extra "coloured direction" get exploded under generic representations and hence must be prohibited. Representations when V has a symplectic structure and the subspaces $W_1, \ldots, W_k$ are Lagrangian play a special role in this story, a fact which becomes obvious once all the general definitions are given (see §4).

A well-known example of such a "brane" algebraic structure in finite dimensions is provided by Manin triples [D]. In the context of this paper Manin triples construction emerges as a (reduced symplectic Lagrangian) representation of the 2-oriented *operad* of Lie algebras (note that one can not describe Manin triples using *ordinary operads* — one needs an *ordinary prop* of Lie bialgebras for that purpose). In a sense, multi-oriented props provide us with a far reaching strong homotopy generalization of Manin triples type constructions; they are really a new kind of "Cheshire cat smiles" controlling (via representations) homotopy algebras *with branes* and admitting in some interesting cases a highly non-trivial action of the Grothendieck-Teichmüller group [A, MW1].

This is the first of a sequence of papers on multi-oriented props. In the following paper [Me3] we study several transcendental constructions with multi-oriented props (elucidating their role as the construction material for building new highly non-trivial representations of *ordinary props*) and use them to prove several concrete deformation quantization theorems. This paper attempts to be as simple as possible and aims for more general audience: we explain here the main notion, illustrate it with examples, prove some theorems on multi-oriented resolutions, and, most importantly, discuss in full details the most surprising part of the story — the representation theory of multi-oriented props in the category of dg vector spaces with branes.

1.2. Finite dimensionality versus infinite one in the context of ordinary props. The theory of (wheeled) operads and props originated in 60s and 70s in algebraic topology, and has seen since an explosive development (see, e.g., the books [LV, MSS] or the articles [M, MMS, V] for details and references). Operads and props provide us with effective tools to discover surprisingly deep and unexpected links between different theories and even branches of mathematics.

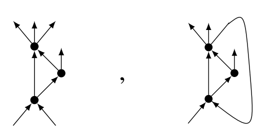
A building block of a prop(erad) $\mathcal{P} = \{\mathcal{P}(m,n)\}_{m,n\geq 0}$ is a graph (often called *corolla*)



consisting of one vertex (decorated with an element of some $\mathbb{S}_m^{op} \times \mathbb{S}_n$ module $\mathcal{P}(m,n)$) which has n incoming legs and m outgoing legs. Upon a representation of $\mathcal{P}$ in a graded vector space V this (m,n)-corolla gets transformed into a linear map from $V^{\otimes n} \to V^{\otimes m}$, i.e. every leg corresponds, roughly speaking, to V

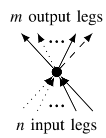
$$\longrightarrow \quad \Leftrightarrow \quad V\, .$$

Such linear maps can be composed which leads us to the idea of considering all possible graphs, for example these ones



composed from corollas by connecting some output legs of one corolla with input legs of another corolla and so on. The graphs shown above when translated into linear maps upon some representation of $\mathcal{P}$ in V give us two very different situations: if the left graph makes sense for representations in both finite- and infinite-dimensional vector spaces V, the right graph gives us a well-defined linear map only for *finite-dimensional* vector spaces V as it contains a closed path of directed edges ("wheel") and hence involves a trace map $V \otimes V^* \to \mathbb{K}$ which is not a well-defined operation in infinite dimensions in general. Hence to be able to work in infinite dimensions[1] one has to prohibit certain graphs — the graphs with wheels — and work solely with *oriented* (from bottom to the top) graphs. Similarly, to be able to work with certain completions (defined in **4.6**) of various intersections of of linear subspaces $W_1 \subset V, \ldots, W_k \subset V$ ("branes"), one has to prohibit certain divergent multi-directed graphs (which already have no wheels with respect to the basic direction!) — and this leads us to the new notion of $(k+1)$-*oriented* prop which takes care about more sophisticated divergences associated with branes (the case $k = 0$ recovers the ordinary props); this important "divergency handling" part of our story is discussed in detail in §4.

There is a nice generalization of the notion of prop which takes care about collections of vector spaces $W_1, \ldots, W_N$. The corresponding props are called *coloured props* and, say, N-coloured (wheeled) prop $\mathcal{P}$ is generated by corollas



whose input and output legs are "colored" (say, the unique vertex has a_1 input legs in "straight colour" , a_2 input legs in "dotted colour", etc) and correspond to linear maps of the form

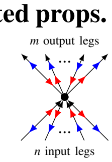
$$W_1^{\otimes a_1} \otimes W_2^{\otimes a_2} \otimes \ldots \otimes W_N^{\otimes a_N} \longrightarrow W_1^{\otimes b_1} \otimes W_2^{\otimes b_2} \otimes \ldots \otimes W_N^{\otimes b_N}, \quad a_1 + \ldots + a_N = n, \;\; b_1 + \ldots + b_N = m.$$

Again it makes sense to talk about wheeled (i.e. 0-*oriented* 1-*directed*) and ordinary (i.e. 1-*oriented*) coloured props. In this theory an oriented leg in "colour" $i \in \{1, \ldots, N\}$ corresponds to the i-th vector space

 $\Leftrightarrow \;\; W_i$

from the collection $\{W_1, \ldots, W_i, \ldots, W_N\}$.

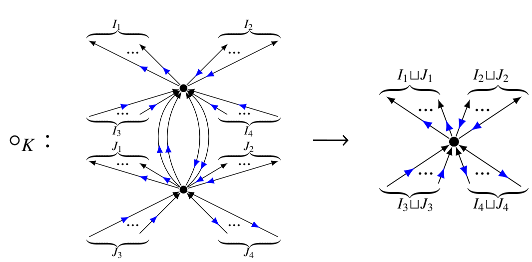
1.3. From branes to multidirected props. A $(k+1)$-directed prop $\mathcal{P}^{k+1} = \{\mathcal{P}^{k+1}(m,n)\}$ is generated (modulo, in general, some relations) by corollas



whose vertex is decorated with an element of some module $\mathcal{P}^{k+1}(m,n)$ (see §2 for details) and whose input and output legs are decorated with extra (labelled by integers from 1 to k or by some colours – blue, red, etc — as in the picture above) directions. The "original" (or basic) direction is always shown in pictures in black colour as in the case of ordinary props; it is this basic direction which permits us to call this creature an (m,n)-corolla (it can have different numbers of input and output legs with respect to directions in other colours). Comparing this picture to the definition of an N-coloured prop, one can immediately see that a $(k+1)$-directed prop is just a special case of a coloured prop when the number of colours is a power of 2,

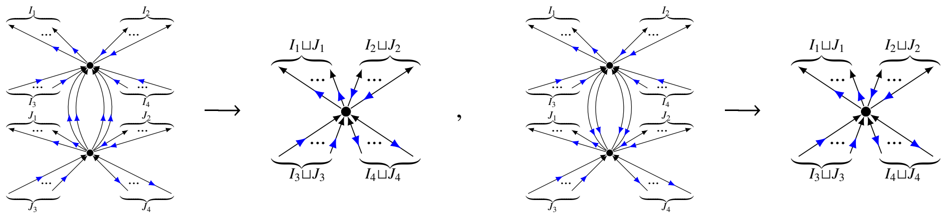
$$N = 2^k. \tag{1}$$

If we allow all possible compositions of such multi-oriented corollas along legs with identical extra directions, then we get nothing but a 2^k-coloured prop indeed (called 0-*oriented* $k+1$-*directed prop*). Keeping in mind the key distinction between ordinary and wheeled props, one might contemplate the possibility of prohibiting compositions along graphs which have closed wheels along any of the extra orientations (but no wheels along the basic one), i.e. prohibiting compositions of the form



[1]As the symmetric monoidal category of *infinite-dimensional* vector spaces is not closed, one must be careful about the definition of the *endomorphism prop* $\mathcal{E}nd_V$ in this category, see **4.1** for details.

and allowing only compositions



along the subgraphs with *no wheels with respect to any of the directions*; let us call a prop generated by such multi-oriented corollas and equipped with such compositions laws (see §2 for the full list of axioms) a $(k+1)$-*oriented* one.

Which structure on graded vector spaces V can be used to separate (in the sense of representations) 0-oriented $(k+1)$-directed props from $(k+1)$-oriented ones (or, more generally, $(l+1)$ oriented $(k+1)$-directed with $l \geq 0$)? Note that the compositions prohibited in the $(k+1)$-oriented prop are still nicely oriented with respect to the basic direction, so the answer can not be *dimension of* V only.

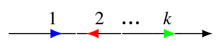
To make sense of these new restrictions (which have no analogue in the theory of coloured props) we suggest to define a *representation* of a $(k+1)$-oriented prop in a graded vector space V as follows. Assume V contains a collection of linear subspaces (satisfying certain restrictions in the infinite-dimensional case, see §4)

$$W_1^+ \subset V,\ W_2^+ \subset V,\ \ldots,\ W_k^+ \subset V \tag{2}$$

with chosen complements

$$V/W_1^+ \simeq W_1^- \subset V, \quad V/W_2^+ \simeq W_2^- \subset V, \quad \ldots \quad, V/W_k^+ \simeq W_k^- \subset V.$$

Then to a $(k+1)$-directed outgoing leg we associate (roughly) an intersection[2]

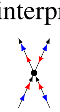


$$\Leftrightarrow \quad W_1^+ \cap W_2^- \cap \ldots \cap W_k^+$$

obtained by the intersection of "branes" according to the rule:

- to the basic direction we always associate the "full" vector space V;
- to the i-th direction we associate the vector subspace W_i^+ if that direction is in agreement with the basic one, or its complement W_i^- if it is not.

Then any multi-oriented corolla gets interpreted as a collection of k linear maps, one map for each coloured orientation.

For example, a 2-directed corolla  gets represented in a graded vector space V equipped with two branes $W_1{}^{\pm}, W_2{}^{\pm} \subset V$ as two linear maps, one corresponding to three blue inputs and one blue output of the corolla,

$$W_1{}^+ \cap W_2{}^+ \longrightarrow (W_1{}^+ \cap W_2{}^+) \otimes (W_1{}^+ \cap W_2{}^-) \otimes (W_1{}^- \cap W_2{}^+)^*,$$

and another to three red inputs and one red output of the same corolla,

$$(W_1{}^+ \cap W_2{}^+) \otimes (W_1{}^- \cap W_2{}^+) \otimes (W_1{}^+ \cap W_2{}^-)^* \longrightarrow W_1{}^+ \cap W_2{}^+$$

In finite dimensions both maps are just re-incarnations of one and the same linear map

$$(W_1{}^+ \cap W_2{}^+) \otimes (W_1{}^- \cap W_2{}^+) \longrightarrow (W_1{}^+ \cap W_2{}^+) \otimes (W_1{}^+ \cap W_2{}^-),$$

which is far from being the case in *infinite* dimensions. Most importantly, this approach to the representation theory of multi-directed props explains nicely why compositions along graphs with wheels in at least one extra orientation must be prohibited (we show explicit examples of the associated divergences in §4). This approach also explains the formula (1) for the associated number of "colours" on legs.

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[2]Strictly speaking, this is true only in finite dimensions. In infinite dimensions the subspaces W_i^+ are defined as *direct limits* of systems of finite-dimensional system while their complements W_i^- always come as *projective limits*, so their intersection makes sense only at the level of finite-dimensional systems first (it is here where the interpretation of W^+ and W^- as subspaces of *one and the same* vector space plays its role), and then taking either the direct or projective limit in accordance with the rule explained in §4.

1.4. Structure of the paper. In §2 we give a detailed (combinatorial type) definition of multi-oriented props. In §3 we consider concrete examples. In particular, we introduce and study multi-oriented analogues, $\mathcal{A}ss^{(k+1)}$ and $\mathcal{L}ie^{(k+1)}$, of the classical operads of associative algebras and Lie algebras, and explicitly describe their minimal resolutions $\mathcal{A}ss_\infty^{(k+1)}$ and $\mathcal{L}ie_\infty^{(k+1)}$; we also construct surprising "forgetful the basic direction" maps from $\mathcal{A}ss^{(2)}$ to the dioperad of infinitesimal bialgebras, and from $\mathcal{L}ie^{(2)}$ to the dioperad of Lie bialgebras (proving that among representations of multi-oriented props we can recover sometimes classical structures); we also introduce a family of $(k+1)$-oriented props of homotopy Lie bialgebras $\mathcal{H}olieb_{c,d}^{(c+d-1)}$ on which the Grothendieck-Teichmüller group acts faithfully (see §5). In §4, the main section of this paper, we define the notion of a representation of a multi-oriented prop in the category of graded vector spaces *with branes*, and show, as an illustration, that Manin triples give us a class of symplectic Lagrangian representations of $\mathcal{L}ie^{(2)}$.

Acknowledgement. It is a pleasure to thank Assar Andersson, Anton Khoroshkin, Thomas Willwacher and Marko Živković for valuable discussions.

2. Multi-oriented props

2.1. $\mathbb{S}$-bimodules reinterpreted. For a finite set I let $S_I^{(1)}$ be the set of all possible maps

$$\mathfrak{s} : I \to \{out, in\}$$

from I to the set consisting of two elements called *out* and *in*. A finite set I together with a fixed function $\mathfrak{s} \in S_I^{(1)}$ is called *1-oriented*. The collection of 1-oriented sets forms a groupoid $\mathcal{S}^{(1)}$ with isomorphisms

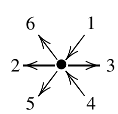
$$(I, \mathfrak{s}) \longrightarrow (I', \mathfrak{s}')$$

being bijections $\sigma : I \to I'$ of finite sets such that $\mathfrak{s}' = \mathfrak{s} \circ \sigma^{-1}$. The latter condition says that the groupoid $\mathcal{S}^{(1)}$ can be identified with the groupoid of cartesian products, $\{I_{in} := \mathfrak{s}^{-1}(in) \times I_{out} := \mathfrak{s}^{-1}(out)\}$, of finite sets.

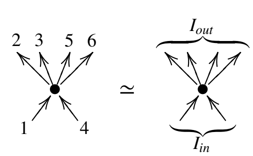
Let C be a symmetric monoidal category. A functor

$$\begin{array}{rccc} \mathcal{P}: & \mathcal{S}^{(1)} & \longrightarrow & C \\ & (I, \mathfrak{s}) & \longrightarrow & \mathcal{P}(I, \mathfrak{s}) \end{array}$$

is called an $\mathcal{S}^{(1)}$*-module*. An element $a \in \mathcal{P}(I, \mathfrak{s})$ can be represented pictorially as a corolla with $\#I$ legs labelled by elements of I and oriented via the rule: if $\mathfrak{s}(i) = out$ (resp., $\mathfrak{s}(i) = in$) we orient the i-labelled leg by putting the direction " > " *from* (resp., *towards*) the vertex; the vertex itself is decorated with a. For example, an element $a \in \mathcal{P}([6], \mathfrak{s})$ can have a pictorial representation of the form



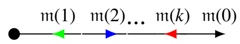
The category of finite sets has a skeleton whose objects are sets $[N] = \{1, 2, \ldots, N\}$ for some $N \geq 0$ (with $[0] = \emptyset$). For $I = [N]$, we often abbreviate $\mathcal{P}_\mathfrak{s}(N, \mathfrak{s}) := \mathcal{P}_\mathfrak{s}([N], \mathfrak{s})$. Note that the above corolla is not assumed to be planar so that it can be equivalently represented in a more standard way,



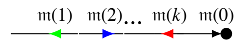
which respects the flow of orientations going from bottom to the top.

Any $\mathbb{S}$-bimodule $E = \{E(m, n)\}_{m,n\geq 0}$, each $E(m, n)$ being an $\mathbb{S}_m^{op} \times \mathbb{S}_n$-module, gives rise to a $\mathcal{S}^{(1)}$-module in the obvious way (and vice versa).

2.2. Multi-oriented modules. For a natural number $k \geq 0$ let $\mathcal{O}r_{k^+}$ be the set of all maps $\mathfrak{m} : [k^+] := \{0, 1, \ldots, k\} \to \{out, in\}$; the value $\mathfrak{m}(\tau) \subset \{out, in\}$ on $\tau \in [k^+]$ is called τ-th *orientation*; the zero-th orientation $\mathfrak{m}(0)$ is called the *basic* one; the map $\mathfrak{m}$ is called itself a *multi-direction*. The elements of $[k^+]$ are often called (in shown in our pictures) as *colours*. One can represent pictorially a multi-direction $\mathfrak{m} \in \mathcal{O}r_{k^+}$ as an "outgoing leg" if $\mathfrak{m}(0) = out$,

$\mathfrak{m} \leftrightarrow$ 

or "ingoing leg" if $\mathfrak{m}(0) = in$

$\mathfrak{m} \leftrightarrow$ 

using the obvious rule: for any $\tau \in [k]$ the value $\mathfrak{m}(\tau)$ is represented by the τ-coloured symbol " $>_\tau$ " oriented in the same direction as $\mathfrak{m}(0) =$ " $>$ " if $\mathfrak{m}(\tau) = \mathfrak{m}(0)$, or in the opposite direction, " $<_\tau$ ", if $\mathfrak{m}(\tau) \neq \mathfrak{m}(0)$.

For a finite set I consider the associated set $S_I^{(k+1)}$ of all maps

$$\begin{array}{rccc} \mathfrak{s}: & I & \longrightarrow & \mathcal{O}r_{k^+} \\ & i & \longrightarrow & \mathfrak{s}_i := \mathfrak{s}(i) : [k^+] \to \{out, in\} \end{array}$$

For $i \in I$ the value $\mathfrak{s}_i$ on $\tau \in [k^+]$ is called *τ-th orientation (or τ-th direction) of the element i*. For any such a function $\mathfrak{s}$ there is associated the *opposite* function $\mathfrak{s}^{opp} : I \to \mathcal{O}r_{k^+}$ which is uniquely defined by the following condition: for each $i \in I$ and each $\tau \in [k^+]$ the value of $\mathfrak{s}_i^{opp}$ on τ is different from the value of $\mathfrak{s}_i$ on τ. Thus the set $S_I^{(k+1)}$ comes equipped with an involution. The restriction of the function $\mathfrak{s}_i$ to the subset $[k] \subset [k^+]$ is denoted by $\bar{\mathfrak{s}}_i$; hence we can write

$$\bar{\mathfrak{s}}_i \in \mathcal{O}r_k := \{[k] \to \{out, in\}\}, \quad \forall i \in I.$$

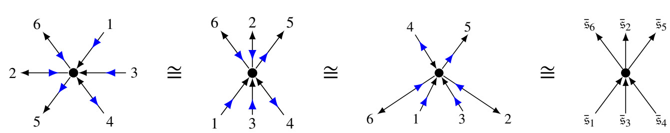
This function takes care about *extra* (i.e. non-basic) orientation assigned to an element $i \in I$. In some pictorial representations of multi-oriented sets $(I, \mathfrak{s})$ we show explicitly only the basic orientation while compressing all the extra ones into this symbol $\bar{\mathfrak{s}}_i$ (see below).

Note that for any given multi-oriented set $(I, \mathfrak{s})$ and a fixed colour $\tau \in [k^+]$ there is an associated map

$$\begin{array}{rccc} \check{\mathfrak{s}}_\tau : & I & \longrightarrow & \{out, in\} \\ & i & \longrightarrow & \check{\mathfrak{s}}_\tau(i) := \mathfrak{s}_i(\tau) \end{array} \tag{3}$$

which we use in several constructions below.

Using the above pictorial interpretation of elements of $\mathcal{O}r_{k^+}$ as multi-oriented legs, one can uniquely represent any element $\mathfrak{s} \in S_I^{(k+1)}$ as a *multi-directed (or multi-oriented) corolla*, that is, as a (non-planar) graph with one vertex $\bullet$ and $\# I$ legs such that each leg is (i) distinguished by an element i from I and (ii) decorated with the multi-direction $\mathfrak{s}_i \in \mathcal{O}r_{k^+}$ as explained just above. For example, a corolla

(4) 

represents non-ambiguously some element $\mathfrak{s} \in \mathcal{S}_{[6]}^{(2)}$. In the theory of ordinary props corollas are often depicted in such a way that the orientation flow runs from the bottom to the top. In the multi-directed case such a respecting flow representation (now non-unique — one for each coloured direction) also makes sense in applications.

A finite set I together with a fixed function $\mathfrak{s} \in S_I^{(k+1)}$ is called $(k+1)$-*oriented*. The collection of $(k+1)$-oriented sets forms a groupoid $\mathcal{S}^{(k+1)}$ with isomorphisms

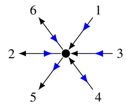
$$(I, \mathfrak{s}) \longrightarrow (I', \mathfrak{s}')$$

being isomorphisms $\sigma : I \to I'$ of finite sets such that $\mathfrak{s}' = \mathfrak{s} \circ \sigma^{-1}$. For example, the automorphism group of the object $([6], \mathfrak{s})$ given by corolla (4) is $\mathbb{S}_2 \times \mathbb{S}_2$ as we can permute only labels $(1, 3)$ and independently $(5, 6)$ using morphisms in the $\mathcal{S}^{(2)}$.

Let C be a symmetric monoidal category. A functor

$$\begin{array}{rccc} \mathcal{P}^{(k+1)}: & \mathcal{S}^{(k+1)} & \longrightarrow & C \\ & (I, \mathfrak{s}) & \longrightarrow & \mathcal{P}^{(k+1)}(I, \mathfrak{s}) \end{array}$$

is called an $\mathcal{S}^{(k+1)}$-*module*. Thus an element of $\mathcal{P}^{(k+1)}(I, \mathfrak{s})$ is a of pair of the form

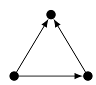
$$\left(c = \right.$$ 

$$\left. , \quad V := \mathcal{P}^{(k+1)}(c) \in Objects(C) \right)$$

Note that V carries a representation of the group $Aut(c)$ (in this particular case, of $\mathbb{S}_2 \times \mathbb{S}_2$). We shall work in this paper in category of topological vector spaces so that it make sense to talk about elements v of V. The pairs (c, v) are called *decorated corollas* and are often represented pictorially by the corolla c with its vertex decorated (often tacitly) by the vector v. Such decorated corollas span $\mathcal{P}^{(k+1)}(I, \mathfrak{s})$.

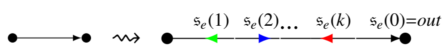
When k is clear, we often abbreviate $\mathcal{P} = \mathcal{P}^{(k+1)}$. The case $k = 0$ corresponds to the ordinary $\mathbb{S}$-bimodule.

2.3. Directed and multidirected graphs. By a graph Γ we understand a 1-dimensional CW complex whose zero-cells are called *vertices*, and whose 1-cells are called *edges*. The set of vertices Γ is denoted by $V(\Gamma)$ and the set of its edges by $E(\Gamma)$. A graph is called *directed* if each edge comes equipped with a fixed orientation (which we show in pictures as an arrow and call it the *basic orientation*). Here is an example of a directed graph

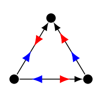


with three vertices and three edges.

By a *multi-directed*, more precisely, $(k+1)$-*directed* graph we understand a pair $\left(\Gamma, s \in \mathcal{S}^{(k+1)}_{E(\Gamma)}\right)$ consisting of a directed graph and a function $\mathfrak{s} : E(\Gamma) \to Or_{k^+}$ such that for each $e \in E(\Gamma)$ the value of the associated function $\mathfrak{s}_e : [k^+] \to \{out, in\}$ takes value out at the "zero-th colour" 0 always equals "out" (or, equivalently, "in") and is identified pictorially with the original (basic) direction of e,

$$e =$$ 

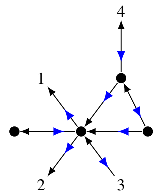
Thus the data $\left(\Gamma, s \in \mathcal{S}^{(k+1)}_{E(\Gamma)}\right)$ can be represented pictorially as a graph whose edges are equipped with $(k+1)$-directions. Here is an example of a 3-directed graph.



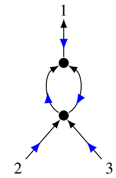
Let $G^{0\uparrow k+1}$ denote set of all such $(k+1)$-directed graphs. The permutation group $\mathbb{S}_{k+1}$ acts on this set via its canonical action on the set of colours $[k^+]$.

Let A be a subset of $[k^+]$. A $(k+1)$-directed graph $\Gamma \in G^{0\uparrow k+1}$ is called A-*oriented* if Γ contains no closed directed paths of edges ("wheels" or "loops") in every colour $c \in A$. The subset of A-oriented graphs is denoted by $G^{A\uparrow k+1}$. If A is non-empty, then applying a suitable element of the automorphism group $\mathbb{S}_{k+1}$ we can (and will) assume without loss of generality that $A = \{0, 1, 2, \ldots, l\}$ for some $l \geq 0$ and re-denote $G^{l+1\uparrow k+1} := G^{A\uparrow k+1}$. If $l = k$, we further abbreviate $G^{(k+1)\text{-or}} := G^{k+1\uparrow k+1}$ and call its elements *multi-oriented* graphs.

2.4. From multi-directed graphs to endofunctors on $\mathcal{S}^{(k+1)}$-modules. Fix an integer $k \geq 0$ and an integer l in the range $-1 \leq l \leq k$. For a finite set I define $G^{l+1\uparrow k+1}(I)$ to be the set of $(k+1)$-directed $(l+1)$-oriented graphs Γ equipped with an injection $\mathfrak{i} : I \to V_1(\Gamma)$, where $V_1(\Gamma) \subset V(\Gamma)$ is the subset of univalent vertices. The univalent vertices lying in the image $L(\Gamma) := \mathfrak{i}(I)$ of this map are called $(k+1)$-directed *legs* of Γ; each such leg is labelled therefore by an element i of I and is called an i-leg (in pictures we show it as a leg indeed with the 1-valent vertex *erased* and the index i put on its place); vertices in $V_{int}(\Gamma) := V(\Gamma) \setminus L(\Gamma)$ are called *internal*. Edges connecting internal vertices are called *internal*; there is a decomposition $E(\Gamma) = L(\Gamma) \sqcup E_{int}(\Gamma)$. Here are examples of 2-directed graphs, one with 4 internal vertices and 4 legs, the other with two internal vertices and 3 legs,



$\in G^{2\uparrow 2}(4)$,



$\in G^{1\uparrow 2}(3)$

Given an internal vertex $v \in V_{int}(\Gamma)$, there is an associated set H_v of edges attached to v and an obvious function ("the multi-oriented corolla at v")

$$\mathfrak{s}_v : H_v \longrightarrow Or_{k^+}$$

There is also an induced function

$$\mathfrak{s} : I = L(\Gamma) \longrightarrow Or_{k^+}$$

on the set of legs defined uniquely by the pictorial rule explained in the the first paragraph of §**2.2**. Let $G^{l+1\uparrow k+1}(I, \mathfrak{s}) \subset G^{l+1\uparrow k+1}(I)$ be the subset of multi-directed (partially oriented, in general) graphs which have one and the same orientation function $\mathfrak{s}$ on the set of legs I.

For an $\mathcal{S}^{(k+1)}$-module $\mathcal{E} = \{\mathcal{E}(I, \mathfrak{s})\}$ in a symmetric monoidal category C with countable coproducts and a graph $\Gamma \in G^{l+1\uparrow k+1}(I, \mathfrak{s})$ consider the unordered tensor product[3] (cf. [M, MSS])

$$\Gamma\langle\mathcal{E}\rangle(I, \mathfrak{s}) := \left(\bigotimes_{v \in V_{int}(G)} \mathcal{E}(H_v, \mathfrak{s}_v)\right)_{Aut(\Gamma)}$$

where $Aut(\Gamma)$ stands for the automorphism group of the graph Γ, and define an $\mathcal{S}^{(k+1)}$-module in C

$$\begin{array}{rccc} \mathcal{F}ree^{l+1\uparrow k+1}\langle\mathcal{E}\rangle : & \mathcal{S}^{(k+1)} & \longrightarrow & C \\ & (I, \mathfrak{s}) & \longrightarrow & \mathcal{F}ree^{l+1\uparrow k+1}\langle\mathcal{E}\rangle(\mathfrak{s}, I) := \bigoplus_{\Gamma \in G^{l+1\uparrow k+1}(I,\mathfrak{s})} \Gamma\langle\mathcal{E}\rangle(I, \mathfrak{s}) \end{array}$$

A we shall see below, that $\mathcal{F}ree^{l+1\uparrow k+1}\langle\mathcal{E}\rangle$ gives us an example of a $(k+1)$-directed $(l+1)$-oriented prop (called the *free* prop generated by the $\mathcal{S}^{(k+1)}$-module $\mathcal{E}$. For $l = k = 0$ this is precisely the ordinary free prop generated by the $\mathcal{S}^{(1)}$-module $\mathcal{E}$. For $l = -1, k = 0$ this is the free *wheeled prop* generated by $\mathcal{E}$ [Me1, MMS]. If $l = k$, i.e. if all directions are oriented, we abbreviate $\mathcal{F}ree^{k+1\uparrow k+1}\langle\mathcal{E}\rangle =: \mathcal{F}ree^{(k+1)\text{-or}}\langle\mathcal{E}\rangle$.

2.5. Multi-oriented prop(erad)s. A (possibly disconnected) subgraph γ of a (connected or disconnected) graph $\Gamma \in G^{l+1\uparrow l+1}(I, \mathfrak{s})$ is called *complete* if the complement $V_{int} \setminus V_{int}(\gamma)$ does not contain internal edges of Γ attached (on both ends) to vertices of γ. Let Γ/γ be the graph obtained from Γ by contracting all internal vertices and all internal edges of γ to a single new vertex; note that the legs of Γ/γ are the same as in Γ so that Γ/γ comes equipped with the same orientation function $\mathfrak{s} : L(\Gamma/\gamma) \to Or_{k^+}$. A complete subgraph $\gamma \subset \Gamma$ is called admissible if Γ/γ belongs to $G^{l+1\uparrow k+1}(I, \mathfrak{s})$, i.e. the contraction procedure does not create wheels in the first $l+1$ coloured directions. Note that by its very definition γ belongs to $\in G^{l+1\uparrow k+1}(I', \mathfrak{s}')$, where I' is the subset of $E(\Gamma)$ consisting of (non-loop) edges attached only to *one* vertex of γ, and the function $\mathfrak{s}' : I' \to Or_{k^+}$ is given by the restriction of $\mathfrak{s}$ to that subset (in accordance with the pictorial representation of multi-orientations as explained in the first paragraph of §**2.2**).

A $(k+1)$-*directed* $(l+1)$-*oriented prop in a symmetric monoidal category (with countable colimits)* C is, by definition, an $\mathcal{S}^{(k+1)}$-module $\mathcal{P} = \{\mathcal{P}(I, \mathfrak{s})\}$ in C together with a natural transformation of functors

$$\begin{array}{rccc} \mu : & \mathcal{F}ree^{l+1\uparrow k+1}\langle\mathcal{P}\rangle & \longrightarrow & \mathcal{P} \\ \mu_\Gamma : & \Gamma\langle\mathcal{P}\rangle(I, \mathfrak{s}) & \longrightarrow & \mathcal{P}(I, \mathfrak{s}) \end{array}$$

such that for any graph $\Gamma \in G^{l+1\uparrow k+1}(I, \mathfrak{s})$ and any admissible subgraph $\gamma \subset \Gamma$ one has

(5) $$\mu_\Gamma = \mu_{\Gamma/\gamma} \circ \mu'_\gamma,$$

where $\mu'_\gamma : \Gamma\langle\mathcal{P}\rangle(I, \mathfrak{s}) \to (\Gamma/\gamma)\langle\mathcal{P}\rangle(I, \mathfrak{s})$ stands for the obvious map which equals μ_γ on the (decorated) subgraph γ and which is identity on all other vertices of Γ.

[3]The (unordered) tensor product $\bigotimes_{i \in I} X_i$ of vector spaces X_i labelled by elements i of a finite set I of cardinality, say, n is defined as the space of $\mathbb{S}_n$-coinvariants $\left(\bigoplus_{\sigma:[n]\xrightarrow{\simeq} I} X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \ldots \otimes X_{\sigma(n)}\right)_{\mathbb{S}_n}$.

The most interesting case for us is $k = l$. The associated props are called *multi-oriented* (more precisely, $(k+1)$-*oriented*). Thus a multi-oriented prop is an $\mathcal{S}^{(k+1)}$-module $\mathcal{P}$ equipped with

(i) a *horizontal* composition

$$\boxtimes : \mathcal{P}(I_1, \mathfrak{s}_1) \otimes \mathcal{P}(I_2, \mathfrak{s}_2) \longrightarrow \mathcal{P}(I_1 \sqcup I_2, \mathfrak{s}_1 \sqcup \mathfrak{s}_2),$$

(ii) and, for any two injections of the same finite set $f_1 : K \to I_1$ and $f_2 : K \to I_2$, a *vertical* composition

$$\mathcal{P}(I_1, \mathfrak{s}_1) \circ_K \mathcal{P}(I_2, \mathfrak{s}_2) \longrightarrow \mathcal{P}_{\mathfrak{s}_{12}}((I \setminus f_1(K)) \sqcup (I_2 \setminus f_2(K)), \mathfrak{s}_{12})$$

which is non-zero if only if the compositions

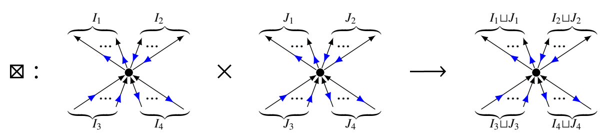
$$K \stackrel{\mathfrak{s}_1 \circ f_1}{\longrightarrow} \mathcal{O}r_{k^+} \quad , \quad K \stackrel{\mathfrak{s}_2 \circ f_2}{\longrightarrow} \mathcal{O}r_{k^+}$$

satisfy the condition $\mathfrak{s}_1 \circ f_1 = (\mathfrak{s}_2 \circ f_2)^{opp}$ (put another way, one can compose decorated corollas (4) along legs which have opposite orientations). Here

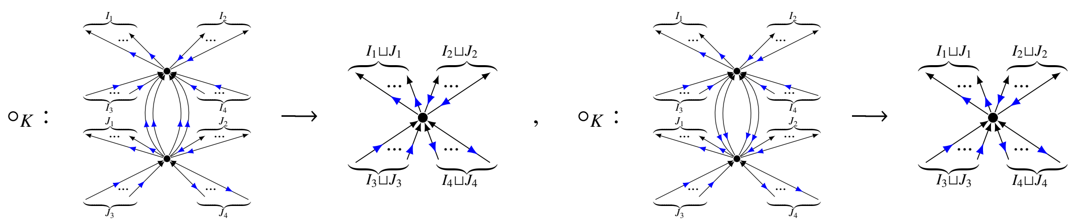
$$\mathfrak{s}_{12} : (I_1 \setminus f_1(K)) \sqcup (I_2 \setminus f_2(K)) \longrightarrow \mathcal{O}r_{k^+}$$

is defined by $\mathfrak{s}_{12}(i) = \mathfrak{s}_1(i)$ for $i \in I_1 \setminus f_1(K)$ and $\mathfrak{s}_{12}(i) = \mathfrak{s}_2(i)$ for $i \in I_2 \setminus f_2(K)$.

These compositions are required to satisfy the "associativity" axioms which essentially say that when we iterate such compositions the order in which we do it does not matter. In terms of decorated corollas these composition correspond to contraction maps (for $k = 1$)

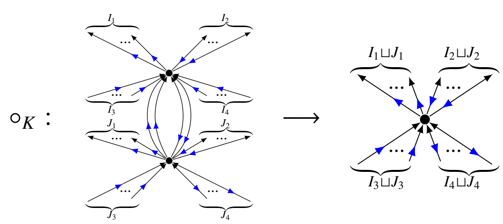


(6)



Note that compositions of the form

(7)



are prohibited in 2-oriented props (as they contain at lats one wheel in blue colour), but are allowed in 1- or 0-oriented 2-directed props.

If in the above definition of the natural transformation μ we restrict only to the subset $G_c^{l+1\uparrow k+1}(I, \mathfrak{s}) \subset G^{l+1\uparrow k+1}(I, \mathfrak{s})$ of *connected* graphs, we get the notion of an $(l+1)$-*oriented* $(k+1)$-*directed properad* $\mathcal{P}$ (cf. [V]). In this case we do not have horizontal compositions in $\mathcal{P}$, only vertical ones. There is an obvious exact functor from $(k+1)$-directed properads to $(k+1)$-directed props.

For any $\mathcal{S}^{(k+1)}$-module $\mathcal{E}$ the associated $\mathcal{S}^{(k+1)}$-module $\mathcal{F}ree^{l+1\uparrow k+1}\langle\mathcal{E}\rangle$ is a $(l+1)$-oriented $(k+1)$-directed prop with contraction maps μ_Γ being tautological. It is called the *free* multi-directed prop generated by $\mathcal{E}$. If $l = k$, it is called the free *multi-oriented* (more precisely, $(k+1)$-*oriented*) prop generated by $\mathcal{E}$.

2.6. Multi-oriented operads. If a multi-oriented properad $\mathcal{P} = \{\mathcal{P}(I, \mathfrak{s})\}$ is such that $\mathcal{P}(I, \mathfrak{s})$ vanishes unless $\check{\mathfrak{s}}_0^{-1}(out) \neq 1$ (i.e. the functor $\mathcal{P}$ is non-trivial only on multi-oriented corollas with precisely *one* outgoing leg with respect to the basic direction) it is called a *multi-oriented operad*. Note that there is no such a restriction on non-basic directions.

2.7. Multi-oriented dioperads. Restricting the functor $\mathcal{F}ree^{l+1\uparrow k+1}$ (and denoting it by $\mathcal{F}ree_0^{l+1\uparrow k+1}$) and the compositions μ_Γ above to connected graphs of genus zero only one gets the notion of a $(k+1)$-directed $(l+1)$-oriented *dioperad*. Note that a multi-oriented operad is a special case of a multi-oriented dioperad.

2.8. Multidirected wheeled prop(erad)s. The above definitions makes sense in the case $l = -1$ as well; the associated 0-oriented $(k+1)$-directed prop(erad) $\mathcal{P}$ is called a $(k+1)$*-directed wheeled prop(erad)* (cf. [Me1, MMS]). In this case graphs Γ can have an internal edge connecting one and the same vertex so that one has in addition to the above mentioned horizontal and vertical compositions in $\mathcal{P}$ one has to add a trace map

$$Tr_K : \mathcal{P}(I, \mathfrak{s}) \longrightarrow \mathcal{P}\left(I \setminus [f_1(K) \sqcup f_2(K), \mathfrak{s}']\right)$$

well-defined for any two injections $f_1 : K \to I$, $f_2 : K \to I$ such that $f_1(K) \cap f_2(K) = \emptyset$ and $\mathfrak{s} \circ f_1 = (\mathfrak{s} \circ f_2)^{opp}$; the orientation function $\mathfrak{s}'$ is obtained from $\mathfrak{s}$ by its restriction to the subset $I \setminus [f_1(K) \sqcup f_2(K)]$.

Thus in the the family of $(k+1)$-directed $(l+1)$ oriented props the special case $(l = -1, k \geq 0)$ corresponds to the ordinary 2^k-coloured wheeled prop while the case $(l = 0, k \geq 0)$ to the ordinary 2^k coloured prop. Thus the really new cases start must have $k \geq l \geq 1$.

In the next section we introduce multi-oriented versions of some classical operads and props at the combinatorial level (which is straightforward). In the next Section after we discuss their representations, i.e. explain what these multi-oriented graphical combinatorics gives us in practice (and this step is, perhaps, not that straightforward).

2.9. Ordinary props as multi-oriented ones. By the very definition, the category of ordinary props is precisely the the category of 1-oriented props. It is worth mentioning, however, that there is a canonical but very naive functor for any $k \geq 1$

$$\begin{array}{cccc} O^{(k+1)}: & \text{Category of ordinary props} & \longrightarrow & \text{Category of } (k+1)\text{oriented props} \\ & P = \{P(I, \mathfrak{s})\} & \longrightarrow & O^{(k+1)}(P) = \{P(I, \mathfrak{s}^{(k+1)})\} \end{array}$$

which simply associates to the $\mathcal{S}^{(1)}$-module $\{P(I, \mathfrak{s})\}$ an $\mathcal{S}^{(k+1)}$-module $\{P(I, \mathfrak{s}^{(k+1)}) = P(I, \mathfrak{s})\}$ which is non-trivial (and coincides with $\{P(I, \mathfrak{s})\}$) only for those multi-orientations $\mathfrak{s}^{(k+1)}$ in which *all* extra directions are aligned coherently with the basic one. More precisely, given an ordinary (1-oriented) prop

$$P\left(I, \mathfrak{s} : I \to Or_{0^+} \equiv \{out, in\}\right)$$

we define

$$P(I, \mathfrak{s}^{(k+1)}) := \begin{cases} P(I, \mathfrak{s}) & \text{if } \mathfrak{s}^{(k+1)} \text{ satisfies } \mathfrak{s}_i^{(k+1)}(\tau) := \mathfrak{s}(i)\ \forall\, i \in I,\ \forall \tau \in [k^+], \\ 0 \text{ or } \emptyset & \text{otherwise.} \end{cases}$$

We do not use this naive functor in this paper (as it gives nothing new), but it is worth keeping in mind that all classical props can be "embedded" into the category of $(k+1)$-oriented props; at least nothing is lost.

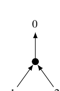
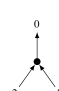
Similarly one can interpret a $(k+1)$-oriented prop as a $(k+l+1)$-oriented prop for any $l \geq 1$. In the next section we consider much less naive extensions of classical operads and props to the multi-oriented setting.

3. Multi-oriented versions of some classical operads and props

3.1. Multi-oriented operad of (strongly homotopy) associative algebras. Let us recall an explicit combinatorial description of the operad $\mathcal{A}ss$ of associative algebras in terms of planar 1-oriented (with orientation flow running implicitly from the bottom to the top) corollas. By definition, $\mathcal{A}ss$ is the quotient,

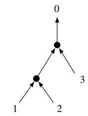
$$\mathcal{A}ss := \mathcal{F}ree^{\text{1-or}}\langle A\rangle/(R)$$

of the free operad $\mathcal{F}ree\langle A\rangle$ generated by the $\mathbb{S}$-module $A = \{A(n)\}_{n\geq 0}$ with

$A(n) :=$ $\mathbb{K}[\mathbb{S}_2] \equiv \text{span}\langle$  ,  $\rangle$ if $n = 2$,

0 otherwise

modulo the ideal generated by the relation (together with its $\mathbb{S}_3$ permutations),



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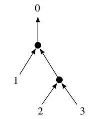
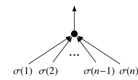


$= 0$

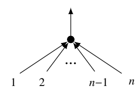
Its minimal resolution is a dg free operad, $\mathcal{A}ss_\infty := (\mathcal{F}ree\langle E\rangle, \delta)$ generated by the $\mathbb{S}$-module $E = \{E(n)\}_{\geq 2}$ (whose generators we represent pictorially as *planar* corollas of homological degree $2-n$)

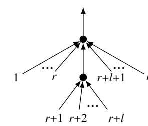
$$E(n) := \mathbb{K}[\mathbb{S}_n][n-2] = \text{span}\left(\quad \right)_{\sigma\in\mathbb{S}_n},$$



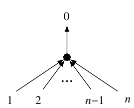
and equipped with the differential given on the generators by

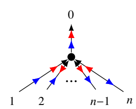
$$\delta \quad = \sum_{r=0}^{n-2}\sum_{l=2}^{n-r}(-1)^{rl+n-r-l+1}$$

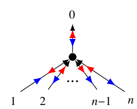


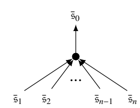


Let us first consider the most naive multi-oriented generalization of $\mathcal{A}ss_\infty$ in which we enlarge the set of generators by decorating each leg of each planar corolla with k extra orientations in all possible ways[4] while preserving its homological degree,



$$\longrightarrow \left\{ \quad , \quad , \quad \ldots \right\} = \left\{ \quad \right\}_{\forall \bar{\mathfrak{s}}_0,\bar{\mathfrak{s}}_1,\ldots\bar{\mathfrak{s}}_n\in Or_k}$$




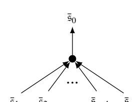


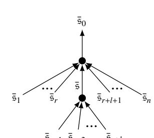
In some pictures we show explicitly only the basic direction while extra directions are indicated only by extra-orientation functions $\bar{\mathfrak{s}}_i$. Denote $\mathcal{A}svbig_\infty^{(k+1)} := \mathcal{F}ree^{(k+1)\text{-or}}\left\langle A^{(k+1)}\right\rangle$ be the free ("veri big") operad generated by these corollas, more precisely, by the associated $\mathcal{S}^{(k+1)}$-module $A^{(k+1)} = \{A^{(k+1)}(I,\mathfrak{s})\}$ which can be formally defined as follows,

$$A^{(k+1)}(I,\mathfrak{s}) = \begin{cases} 0 & \text{if } \#I \leq 2 \\ 0 & \text{if } \check{\mathfrak{s}}_0^{-1}(out) \neq 1 \\ \text{span}\langle ord(I')\rangle[\#I-3] & \text{otherwise.} \end{cases}$$

where $ord(I')$ is the set of total orderings on the finite set $I' = I \setminus \check{\mathfrak{s}}_0^{-1}(out)$. The differential in $\mathcal{A}ss_\infty$ can be extended to $\mathcal{A}svbig_\infty^{(k+1)}$ by summing over all possible ways of attaching extra directions $\bar{\mathfrak{s}} \in Or_k$ to the internal edge,

(8)
$$\delta \quad = \sum_{r=0}^{n-2}\sum_{l=2}^{n-r}\sum_{\bar{\mathfrak{s}}\in Or_k}(-1)^{rl+n-r-l+1}$$





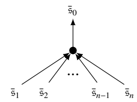
Note that the generating corollas in $\mathcal{A}svbig_\infty^{(k+1)}$ have at least one ingoing leg and at least one outgoing leg with respect to only basic direction (this condition kills "curvature terms" in that direction). As we shall see in the next chapter (where we introduce representations of multi-oriented props), it is actually extra directions (if present) which play the genuine role of inputs and outputs. Hence to avoid "curvature terms" with respect to *any* direction, we have to consider an ideal I_1 in the free operad $\mathcal{A}svbig_\infty^{(k+1)}$ generated by those corollas which have no at least one output or no at least one input leg with respect to at least one extra orientation. It is easy to see that the above differential δ respects this ideal so that the quotient

$$\mathcal{A}sbig_\infty^{(k+1)} := \mathcal{A}svbig_\infty^{(k+1)}/I_1$$

is a dg free operad again. It is generated by a "smaller" set of generators, but still that set can be further reduced. Note that once the basic direction is fixed, the set of extra orientations Or_k can be identified with the set of words of length

[4]In some pictures we show explicitly only the basic direction while extra directions are indicated only by "extra-orientation" functions $\bar{\mathfrak{s}}_i : [k] \to \{out, in\}$.

k in two letter, $>$ and $<$, and hence can be equipped with the lexicographic order $\leq$. Let us call a generating corolla



special if $\bar{\mathfrak{s}}_1 \leq \bar{\mathfrak{s}}_2 \leq \ldots \leq \bar{\mathfrak{s}}_n$ (i.e. if the planar order agrees with the lexicographic one), and let I_2 be the ideal in the free operad $\mathcal{A}sbig_\infty^{(k+1)}$ generated by non-special corollas. It is again easy to see that the differential δ respects that second ideal so that the quotient

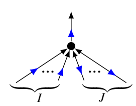
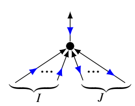
$$\mathcal{A}ss_\infty^{(k+1)} := \mathcal{A}sbig_\infty^{(k+1)}/I_2$$

is a dg free operad generated by the special corollas (essentially, the main point of this discussion is to motivate the claim that the derivation of $\mathcal{A}ss_\infty^{(k+1)}$ given on the generating special corollas by formula (8) is a *differential*). It is called the *multi-oriented operad of strongly homotopy associative algebras*. Let J be the differential closure of the ideal in the free (viewed as a non-differential) operad $\mathcal{A}ss_\infty^{(k+1)}$ generated by the above corollas with $n \geq 3$. The quotient

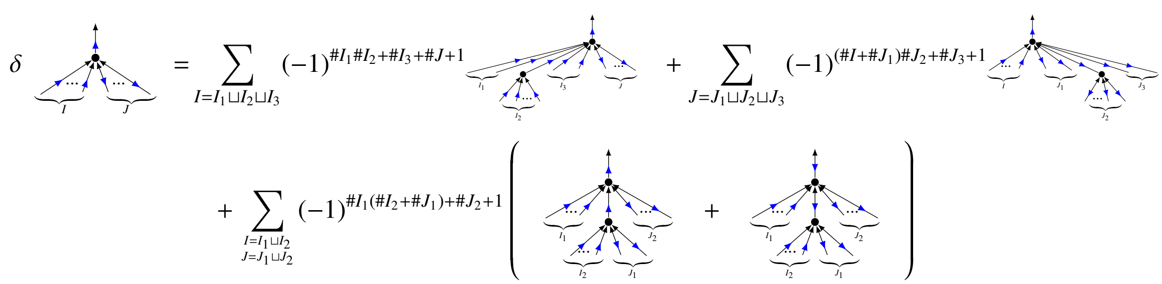
$$\mathcal{A}ss^{(k+1)} := \mathcal{A}ss_\infty^{(k+1)}/J$$

is called a *multi-oriented operad of associative algebras*. We shall see below that this multi-oriented *operad* controls structures which are governed, in some special case, by ordinary *dioperads*. For example, a representation of $\mathcal{A}ss^{(2)}$ in a symplectic vector space with one Lagrangian brane can be identified with an infinitesimal *bi*algebra structure on that brane.

3.1.1. The simplest non-trivial case $k = 1$ in more detail. The dg operad $\mathcal{A}ss_\infty^{(2)}$ is generated by planar corollas of homological degree $2 - \#I - \#J$

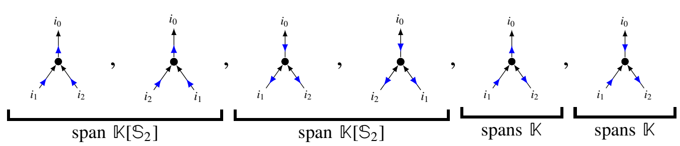
 and  , $\#I + \#J \geq 2.$

where the finite sets of labels I and J are totally ordered (in agreement with the given planar structure of the corollas). The differential is given explicitly by

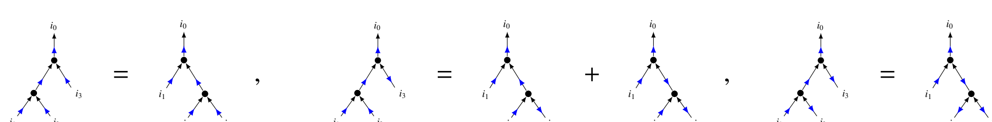


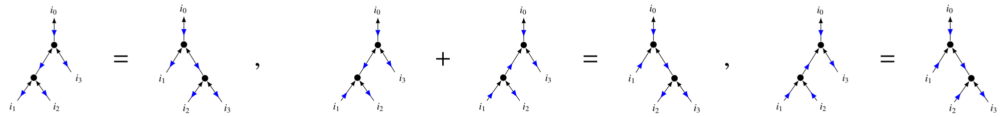
and similarly for the second corolla. Here the summations run over decompositions of the totally ordered sets into disjoint unions of *connected* (with respect to the order) subsets.

The operad $\mathcal{A}ss^{(2)}$ is generated by the following planar corollas (in homological degree zero)



while the relations are given by

(9) 

(10) 

One can describe similarly the operad $\mathcal{A}ss^{(k+1)}$ in terms of generators and relations.

3.1.2. Theorem. *The natural projection $\mathcal{A}ss_\infty^{(k+1)} \longrightarrow \mathcal{A}ss^{(k+1)}$ is a quasi-isomorphism.*

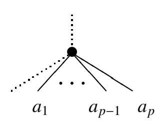
Proof. We have to show that $H^\bullet(\mathcal{A}ss_\infty^{(k+1)}(I,\mathfrak{s})) = \mathcal{A}ss^{(k+1)}(I,\mathfrak{s})$ for any $(k+1)$-oriented set $(I,\mathfrak{s})$. In fact, it is enough to show that the cohomology of the operad $H^\bullet(\mathcal{A}ss_\infty^{(k+1)}(I,\mathfrak{s}))$ is concentrated in degree zero because that would imply the required equality due to the fact that the complex $\mathcal{A}ss_\infty^{(k+1)}(I,\mathfrak{s}))$ is non-positively graded.

We shall prove the claim by induction over $\#I = n+1$, and abbreviate the notation $\mathcal{A}ss_\infty^{(2)}(n) := \mathcal{A}ss_\infty^{(2)}(I,\mathfrak{s})$ and $\mathcal{A}ss^{(2)}(n) := \mathcal{A}ss^{(2)}(I,\mathfrak{s})$. When $n=2$, the equality $H^\bullet(\mathcal{A}ss_\infty^{(2)}(n)) = \mathcal{A}ss^{(2)}(n)$ is obvious. Assume it is true for all multi-oriented sets with $\#I \leq n+1$, and consider the complex $\mathcal{A}ss_\infty^{(k+1)}(n+1)$; we can assume without loss of generality that the input (with respect to the basic colour) legs of any graph from $\mathcal{A}ss_\infty^{(2)}(n+1)$ are labelled from left to right (in accordance with the planar structure) by $1,2,\ldots,n+1$ (while the root vertex by $n+2$).

Consider first a filtration of $\mathcal{A}ss_\infty^{(2)}(n+1)$ by the total number of vertices lying on the path from the root edge to the leg labelled by 1 (and call it a *special* path), and let $Gr(n+1)$ denote the associated graded. Consider next a filtration of $Gr(n+1)$ by the total valency of vertices lying on the special path (and denote the set of such vertices by V_{sp}), and let $(\mathcal{E}_r,\delta_r)$ be the associated spectral sequence (converging to $H^\bullet(\mathcal{A}ss_\infty^{(2)}(n+1))$. The initial page $(\mathcal{E}_0,\delta_0)$ is isomorphic to the direct sum of tensor products of complexes of the form $\mathcal{A}ss_\infty^{(k+1)}(n')$ with all $n' \leq n$ so that by the induction hypothesis we can easily describe the next page of the spectral sequence:

$$\mathcal{E}_1 = H^\bullet(\mathcal{E}_0) \cong \bigoplus_{\text{special paths}} \bigoplus_{\substack{n=\sum_{v\in V_{sp}} n_v \\ n_v \geq 1}} \bigotimes_{v\in V_{sp}} C_v(n_v)$$

where $C_v(n_v)$ is a complex spanned by planar corollas of the form



$, \quad p \geq 1,$

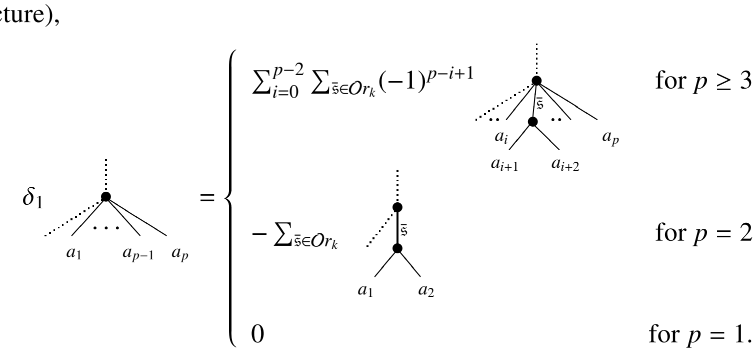
whose dashed legs belong to the given special path (and are equipped with the induced multi-orientations from that special path) while solid legs are decorated by arbitrary elements of the unital extension of the operad $\mathcal{A}ss^{(k+1)}$,

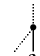
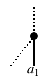
$$a_i \in \mathcal{A}ss_u^{(k+1)}(n_i) := \begin{cases} \mathcal{A}ss^{(k+1)}(n_i) & \text{if } \#n_i \geq 2 \\ \mathbb{K} & \text{if } \#n_i = 1 \\ 0 & \text{if } \#n_i = 0 \end{cases}, \quad i \in [p],$$

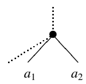
subject to the condition that

$$\sum_{i=1}^{p} n_i = n_v,$$

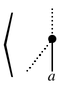
The differential on $C_v(n_v)$ is non-trivial only on the root corolla on which it acts as follows (we suppress some extra orientations in the picture),

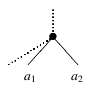
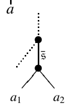
(11) 

Claim. *The cohomology of the complex $C_v(n_v)$ is concentrated in cohomological degree zero.*
Indeed, consider a one-step filtration of $C_v(n_v)$ by the number of three-valent vertices of the form  and the associated two pages spectral sequence. It is easy to see that the complex on the initial page is a direct sum of a trivial complex spanned by graphs of the form  with $a_1 \in \mathcal{A}ss_u^{(k+1)}(n_v)$ and a non-trivial complex which is quasi-isomorphic to the degree shifted (direct summand) subcomplex of $(\mathcal{A}ss_\infty^{(k+1)}(n_v)[1], \delta)$ spanned by graphs with the orientation of the unique root leg fixed by the multi-orientation of the corresponding dashed edge of the given special path (indeed, take a filtration of the latter sub-complex by the valency of the root vertex and use the induction assumption). As $n_v \leq n$ we conclude (again by the induction assumption) that its cohomology is equal to

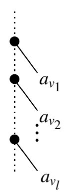
$$A := \text{span}\Big\langle$$  $$\text{mod } \mathcal{A}ss^{(k+1)}\text{-relations},\ a_1 \in \mathcal{A}ss_u^{(k+1)}(n_1),\ a_2 \in \mathcal{A}ss_u^{(k+1)}(n_2),\ n_1 + n_2 = n_v \Big\rangle$$

The induced differential on the next (and final) page of the spectral sequence is an injection

$$d: \quad A \longrightarrow \mathcal{A}ss^{(k+1)}(n_v) = \text{span}\Big\langle$$  $$,\ a \in \mathcal{A}ss_u^{(k+1)}(n_v)\Big\rangle$$

 $$\longrightarrow \quad -\sum_{\bar{\mathfrak{s}} \in Or_k}$$ 

which proves the Claim.
We conclude that the cohomology $H^\bullet(\mathcal{A}ss_\infty^{(k+1)}(n+1))$ is generated by multi-oriented graphs of the form (modulo some relations corresponding to the image of the injection d)



$$\text{where } l := \#V_{sp}, \quad a_{v_i} \in Ass_u^{(k+1)}(n_{v_i}), \quad \sum_{i=1}^{l} n_{v_i} = n$$

which all have cohomological degree zero. Hence $H^\bullet(\mathcal{A}ss_\infty^{(k+1)}(n+1))$ is concentrated in degree zero implying its identification with $\mathcal{A}ss^{(k+1)}(n+1)$. The induction argument and hence the proof the Theorem are completed. □

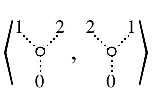
In the next subsection we discuss representations of $\mathcal{A}ss^{(k+1)}$, that is, *associative algebras with k branes*. Rather surprisingly, we recover, in particular, a well-known notion of *infinitesimal bialgebra* as an *associative algebra with one (symplectic Lagrangian) brane*. This interesting fact can be seen already now (i.e. in purely combinatorial way) as follows.

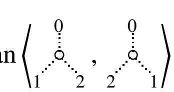
3.2. Infinitesimal bialgebras as 2-oriented associative algebras. Recall that an ordinary (i.e. 1-oriented) dioperad of *infinitesimal associative bialgebras* is, by definition, the quotient of the 1-oriented free dioperad

$$\mathcal{IB} := \mathcal{F}ree_0^{1\text{-or}}\langle B\rangle / R$$

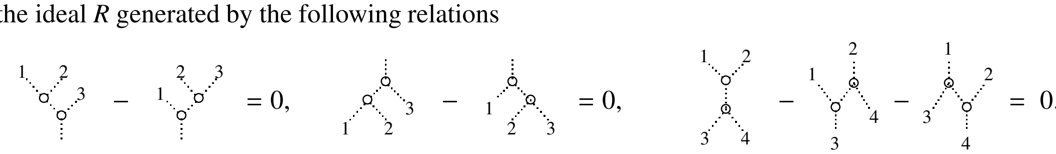
generated by an $\mathbb{S}$-bimodule $B = \{B(m,n)\}$

$$B(m,n) := \begin{cases} \mathbb{K}[\mathbb{S}_2] \otimes \mathbf{1}_1 \equiv \text{span}\langle \ldots \rangle & \text{if } m = 2, n = 1, \\ \mathbf{1}_1 \otimes \mathbb{K}[\mathbb{S}_2] \equiv \text{span}\langle \ldots \rangle & \text{if } m = 1, n = 2, \\ 0 & \text{otherwise} \end{cases}$$





modulo the ideal R generated by the following relations

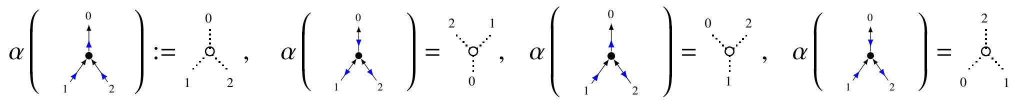


Here all internal edges and legs are assumed to be oriented along the flow running from the bottom of a graph to its top.

3.2.1. Proposition. *There is a (forgetting the basic orientation) morphism of dioperads*

$$\alpha : \mathcal{A}ss^{(2)} \longrightarrow \mathcal{IB}$$

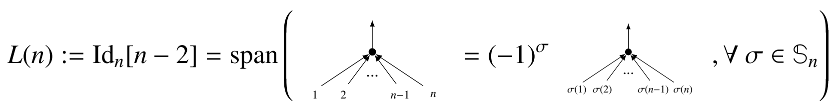
given on the generators as follows:



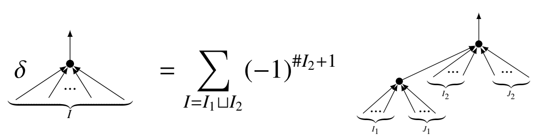
Proof. It is straightforward to check that each of the six relations in (9)-(10) is mapped under α into one of the above three relations for $\mathcal{IB}$. Hence the map is well-defined indeed. □

This proposition indicates that the notion of *representation* of a multi-oriented prop(erad) can not be an immediate generalization of that notion for ordinary coloured prop(erad)s — the extra orientations are not really "colours", and the distinction between operads and dioperads should become non-existent.

3.3. Example: Multi-oriented operad of Lie and $\mathcal{L}ie_\infty$ algebras. Recall that the ordinary operad of strongly homotopy Lie algebras is the free operad $\mathcal{L}ie_\infty := (\mathcal{F}ree^{1\text{-or}}\langle L\rangle, \delta)$ generated by an $\mathbb{S}$-module $L = \{L(n)\}_{\geq 2}$ with



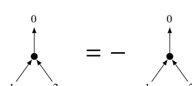
and equipped with the differential given on the generators by



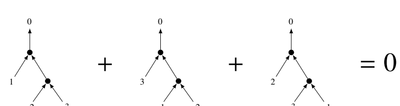
It is essentially a skewsymmetrized version of $\mathcal{A}ss_\infty$ (there is a canonical morphism of operads $\mathcal{L}ie_\infty \to \mathcal{A}ss_\infty$ sending a generator of $\mathcal{L}ie_\infty$ into a skewsymmetrization of the corresponding generator of $\mathcal{A}ss_\infty$). If I is the differential closure of the ideal in $\mathcal{L}ie_\infty$ generated by all corollas with negative cohomological degree, then the quotient

$$\mathcal{L}ie := \mathcal{L}ie_\infty / I$$

is an operad controlling Lie algebras. It is generated by degree zero planar skewsymmetric corollas

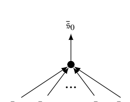


modulo the Jacobi relation,

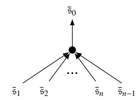
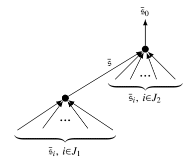


The natural surjection $\mathcal{L}ie_\infty \longrightarrow \mathcal{L}ie$ is a quasi-isomorphism.

An operad of $(k+1)$-oriented strongly homotopy Lie algebras is defined an as obvious skew-symmetrization of the operad $\mathcal{A}ss_\infty^{(k+1)}$ introduced in the previous subsection (so that there is again a canonical morphism of dg operads $\mathcal{L}ie_\infty^{(k+1)} \to \mathcal{A}ss_\infty^{(k+1)}$). More precisely, the prop $\mathcal{L}ie_\infty^{(k+1)}$ is a free $(k+1)$-oriented prop generated by corollas with the same symmetries and degrees as in the case of $\mathcal{L}ie_\infty$, but now with each leg decorated with extra k-orientations,

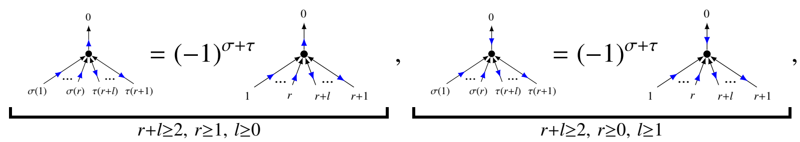


with the condition that there is at least one ingoing edge and outgoing edge with respect to each of the new directions. The differential is given by the same formula as in the case of $\mathcal{H}olieb_{c,d}$ except that now we sum over all possible (and admissible) new orientations attached to the new edge

δ  $= \sum_{\substack{[1,\dots,n]=J_1\sqcup J_2\\ |J_1|\geq 1, |J_2|\geq 1}} \sum_{\bar{\mathfrak{s}}\in Or_k} (-1)^{1+\#J_2+\cdot \mathrm{sgn}(J_1,J_2)}$ 

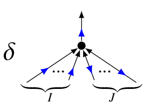
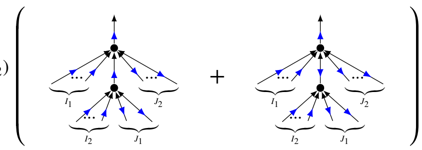
where the first sum run over decompositions of the ordered set $[n]$ into the disjoint union of (not necessarily connected) ordered subsets, and $\mathrm{sign}(I_1, I_2)$ stands for the parity of the permutation $[n] \to I_1 \sqcup I_2$.

In more detail, the operad of 2-oriented strongly homotopy Lie algebras is, by definition, a free 2-oriented operad generated by the following skewsymmetric planar corollas of degree $2 - n$, $n \geq 2$,



for any $\sigma \in \mathbb{S}_r$ and $\tau \in \mathbb{S}_l$. As in the case of $\mathcal{A}ss_\infty^{(k+1)}$ we require that each corolla has at least one ingoing leg and at least one outgoing leg in each extra orientation (in order to avoid curvature terms in representations).

The differential is given by (and it is easy to check that δ is a differential indeed)

δ  $= \sum_{\substack{I=I_1\sqcup I_2\\ J=J_1\sqcup J_2}} (-1)^{\#I_1(\#I_2+\#J_1)+\#J_2+1+\mathrm{sign}(I_1,I_2)+\mathrm{sign}(J_1,J_2)}$ 

and similarly for the second class of corollas. Here the sums run over all admissible decompositions of the ordered sets I and J into the disjoint unions of (not necessarily connected) ordered subsets, and $\mathrm{sign}(I_1, I_2)$ (resp., $\mathrm{sign}(J_1, J_2)$) stands for the parity of the permutation $I \to I_1 \sqcup I_2$ (resp., $J \to J_1 \sqcup J_2$).

If I is the differential closure of the ideal in $\mathcal{L}ie_\infty^{(k+1)}$ generated by all corollas with negative cohomological degree, then the quotient

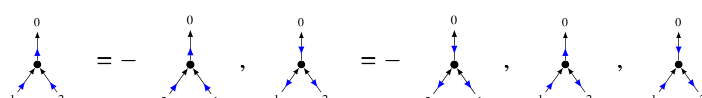
$$\mathcal{L}ie^{(k+1)} := \mathcal{L}ie_\infty^{(k+1)}/I$$

is called an operad of multi-oriented Lie algebras.

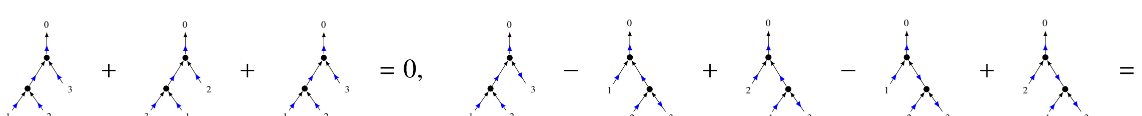
3.3.1. Theorem. *The natural projection $\mathcal{L}ie_\infty^{(k+1)} \longrightarrow \mathcal{L}ie^{(k+1)}$ is a quasi-isomorphism.*

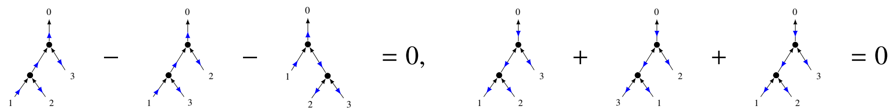
Proof. It is enough to show that $\mathcal{L}ie_\infty^{(k+1)}$ is concentrated in cohomological degree zero, and this can be done by the arity induction in a close analogy to the proof of Theorem **3.1.2**. We omit the details. □

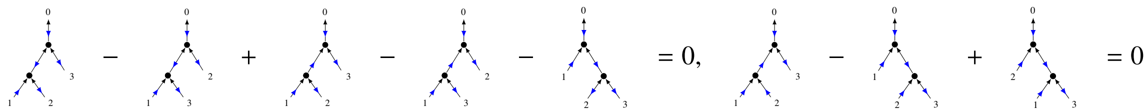
3.4. The operad of 2-oriented Lie algebras versus the ordinary dioperad of Lie bialgebras. The operad $\mathcal{L}ie^{(2)}$ can be explicitly described as follows: it is generated by the following list of degree 0 corollas,



modulo the following relations

(12) 

(13) 

(14) 

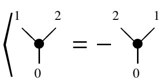
Representations of the operad of 2-oriented Lie algebras in symplectic vector spaces with one Lagrangian brane are studied in the next section where it is shown that they can be identified with famous Manin's triples which give us an alternative (and often very useful) characterization of *Lie bialgebras*. Hence the combinatorics of the latter structures must be hidden in the combinatorics of the former ones, and our next our purpose to make this inter-relation explicit.

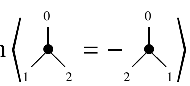
Recall that the ordinary (i.e. 1-oriented) dioperad of Lie bialgebras is the quotient

$$\mathcal{L}ieb_{diop} := \mathcal{F}ree_0^{\text{1-or}} \langle M \rangle / J$$

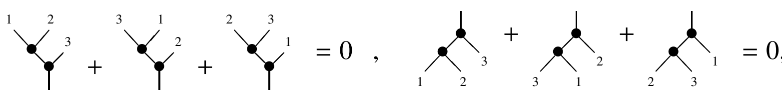
of the free 1-oriented free dioperad generated by an $\mathbb{S}$-bimodule $M = \{M(m,n)\}$ with

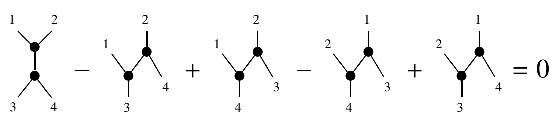
$$M(m,n) := \begin{cases} sgn_2 \otimes \mathbf{1}_1 \equiv \text{span}\langle \cdots \rangle & \text{if } m = 2, n = 1, \\ \mathbf{1}_1 \otimes sgn_2 \equiv \text{span}\langle \cdots \rangle & \text{if } m = 1, n = 2, \\ 0 & \text{otherwise} \end{cases}$$





modulo the ideal J generated by the following relations

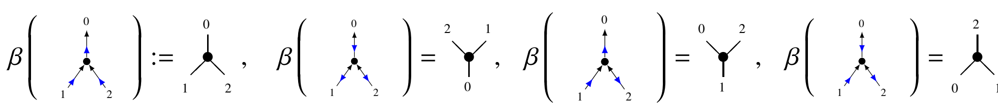




3.4.1. Proposition. *There is a (forgetting the basic orientation) morphism of dioperads*

$$\beta : \mathcal{L}ie^{(2)} \longrightarrow \mathcal{L}ieb_{diop}$$

given on the generators as follows:



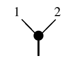
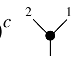
Proof. It is straightforward to check that each of the eight relations in (12)-(14) is mapped under β into one of the above three relations for $\mathcal{L}ieb_{diop}$. Hence the map is well-defined indeed. □

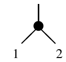
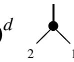
This result gives us a purely combinatorial interpretation of the famous Manin triple construction [D].

3.5. Multi-oriented prop of homotopy Lie bialgebras. Let us recall a graded generalization of the classical prop of Lie bialgebras depending on two integer parameters $c, d \in \mathbb{Z}$. By definition [MW1], $\mathcal{L}ieb_{c,d}$ is a quadratic properad given as the quotient,

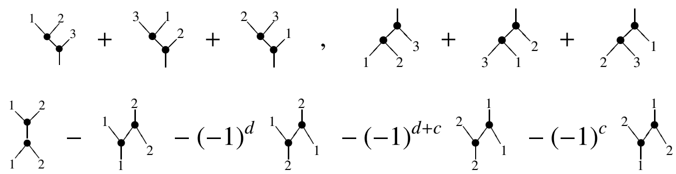
$$\mathcal{L}ieb_{c,d} := \mathcal{F}ree^{1\text{-or}}\langle Q\rangle/\langle\mathcal{R}\rangle,$$

of the free properad generated by an $\mathbb{S}$-bimodule $Q = \{Q(m,n)\}_{m,n\geq 1}$ with all $Q(m,n) = 0$ except

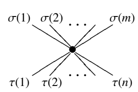
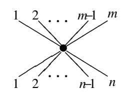
$Q(2,1) := \mathbf{1}_1 \otimes sgn_2^{\otimes c}[c-1] = \text{span}\Big\langle$  $= (-1)^c$  $\Big\rangle$

$Q(1,2) := sgn_2^{\otimes d} \otimes \mathbf{1}_1[d-1] = \text{span}\Big\langle$  $= (-1)^d$  $\Big\rangle$

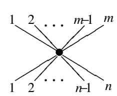
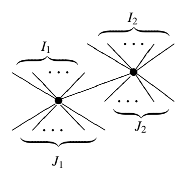
by the ideal generated by the following relations



Its minimal resolution $\mathcal{H}olieb_{c,d}$ is a 1-oriented dg free properad generated by the following (skew)symmetric corollas of degree $1 + c(1-m) + d(1-n)$

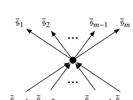
(15)  $= (-1)^{c|\sigma|+d|\tau|}$  $\forall \sigma \in \mathbb{S}_m, \forall \tau \in \mathbb{S}_n$

and has the differential given on the generators by

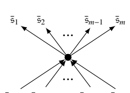
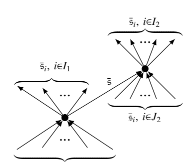
(16) δ  $= \sum\limits_{\substack{[1,\ldots,m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \sum\limits_{\substack{[1,\ldots,n]=J_1\sqcup J_2\\ |J_1|\geq 1, |J_2|\geq 1}} (-1)^{\#I_1\#J_2+\#I_1+\#J_2+c\,\text{sgn}(I_1,I_2)+d\,\text{sgn}(J_1,J_2)}$ 

where $\text{sign}(I_1, I_2)$ (resp., $\text{sign}(J_1, J_2)$) stands for the parity of the permutation $I \to I_1 \sqcup I_2$ (resp., $J \to J_1 \sqcup J_2$). The case $c = d = 0$ corresponds to the ordinary strong homotopy Lie bialgebras, while the case $c = 1$, $d = 0$ to formal Poisson structures on graded vector spaces V viewed as linear manifolds [Me1].

A $(k+1)$-oriented generalization of $\mathcal{H}olieb_{c,d}$ is quite straightforward: the prop $\mathcal{H}olieb_{c,d}^{(k+1)\text{-or}}$ is a free $(k+1)$-oriented prop generated by corollas with the same symmetries and degrees as in the case of $\mathcal{H}olieb_{c,d}$, but now with each leg decorated with k extra orientations,



subject to the condition that there is at least one ingoing edge and outgoing edge in each of the new direction. The differential is given by the same formula as in the case of $\mathcal{H}olieb_{c,d}$ except that now we sum over all possible (and admissible) new orientations attached to the new edge

δ  $= \sum\limits_{\substack{[1,\ldots,m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \sum\limits_{\substack{[m+1,\ldots,m+n]=J_1\sqcup J_2\\ |J_1|\geq 1, |J_2|\geq 1}} \sum\limits_{\bar{\mathfrak{s}}\in Or_k} (-1)^{\#I_1\#J_2+\#I_2+\#J_2+c\cdot\text{sgn}(I_1,I_2)+d\cdot\text{sgn}(J_1,J_2)}$ 

The homotopy theory of such props can be highly non-trivial. As we discuss in more detail below in §5. the automorphism group of $\mathcal{H}olieb_{c,d}^{(c+d-1)\text{-or}}$ contains the Grothendieck-Teichmüller group GRT_1 (for any $c, d \in \mathbb{Z}$ with $c + d \geq 2$) and hence this prop can be a foundation for a rich deformation quantization theory in the geometric dimension $c+d \geq 4$. This fact was one of the our main motivations to introduce and study the multi-oriented props.

Let K be the differential closure of the ideal in $\mathcal{H}olieb_{c,d}^{(k+1)\text{-or}}$ generated by all corollas with total arity ≥ 4 and denote the quotient by

$$\mathcal{L}ieb_{c,d}^{(k+1)\text{-or}} = \mathcal{H}olieb_{c,d}^{(k+1)\text{-or}}/K. \tag{17}$$

This prop gives us a $(k+1)$ oriented version of $\mathcal{L}ieb_{c,d}$. The canonical projection projection

$$p : \mathcal{H}olieb_{c,d}^{(k+1)\text{-or}} \longrightarrow \mathcal{L}ieb_{c,d}^{(k+1)\text{-or}}$$

is *not* a quasi-isomorphism in general. Indeed, it is quite easy to see that the automorphism groups of the props $\mathcal{L}ieb_{c,d}^{(c+d-1)\text{-or}}$ with $c, d \geq 1$ and $c+d \geq 3$ must be almost trivial (i.e. consisting solely of a finite number of rescaling operators on the generators). If the projection p is a quasi-isomorphism in any of these cases, that would contradict the main results in [Z] and [A] which imply a highly non-trivial action of Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$ as (homotopy non-trivial) derivations of the genus completed prop $\widehat{\mathcal{H}olieb}_{c,d}^{(c+d-1)\text{-or}}$ for any[5] $c, d \geq 1$ and $c+d \geq 3$ (see more details on this matter in §5 below). It is likely that the projection (17) is not a quasi-isomorphism for any $k \geq 1$: the proof of the famous Theorem claiming quasi-isomorphism of p in the case $k=0$ (and for any c and d) is based on the Kontsevich idea to use path filtrations, "small" props and "reduced" graphs (see [MaVo] for full details); however that approach does not work in the multidirected case as the process of "reduction" of generic multi-oriented graphs along one of the orientations (say the one which is chosen to define the path filtration) can create wheels in other orientations and hence fails.

The fact that the projection (17) is not a quasi-isomorphism in the case of props $\mathcal{H}olieb_{c,d}^{(c+d-1)\text{-or}}$ with $c, d \geq 1$ and $c+d \geq 4$ is quite useful in the context of deformation quantization in geometric dimension ≥ 4. It can be compared with non-exactness of the wheelification functor in the case of 1-directed props [Me1], a fact which is directly related to the remarkable richness of the set of Kontsevich formality maps.

4. Multidirected endomorphism prop and homotopy algebras with branes

4.1. Tensor algebra of infinite-dimensional vector spaces. By a *countably infinite-dimensional* graded vector space V we understand in this paper any *direct* limit $V := \varinjlim V_p$ of a direct system of finite dimensional vector spaces V_p, $p \geq 1$,

$$V_0 \longrightarrow V_1 \xrightarrow{i_1} V_2 \xrightarrow{i_2} \ldots \xrightarrow{i_{p-1}} V_p \xrightarrow{i_p} V_{p+1} \xrightarrow{i_{p+1}} \ldots. \tag{18}$$

where all arrows i_p are proper injections. For example, $\mathbb{K}^\infty = \varinjlim_n \mathbb{K}^n$ where

$$0 \longrightarrow \mathbb{K} \xrightarrow{i_1} \mathbb{K}^2 \xrightarrow{i_2} \ldots \xrightarrow{i_p} \mathbb{K}^p \xrightarrow{i_{p+1}} \mathbb{K}^{p+1} \xrightarrow{i_{p+2}} \ldots,$$

where $i_p(a_1, \ldots, a_p) = (a_1, \ldots, a_p, 0)$, is an example of a (countably) infinite-dimensional vector space.
Next we define (non-countably) infinite-dimensional vector spaces

$$\mathrm{Hom}(\otimes^r V, \otimes^l V) = \varprojlim_{(p_1,\ldots,p_r)} \left(V_{p_1}^* \otimes \ldots \otimes V_{p_r}^* \otimes V^{\otimes l}\right).$$

which are equipped with the standard projective limit topology. An element $f \in \mathrm{Hom}(\otimes^r V, \otimes^l V)$ is called a *linear map*

$$f : \otimes^r V \longrightarrow \otimes^l V$$

from $\otimes^r V$ to $\otimes^l V$. Such maps can be composed (no divergences) so that one has a well-defined endomorphism prop

$$\mathcal{E}nd_V = \left\{\mathcal{E}nd_V(l,r) := \mathrm{Hom}(\otimes^r V, \otimes^l V)\right\}$$

associated to V and hence talk about representations of ordinary props in V.

[5]In the 1-oriented case ($c+d=2$) this fact was established in [MW1], but in these special cases the automorphism groups of $\mathcal{L}ieb_{c,d}^{(c+d-1)\text{-or}}$ are very rich

Note that Hom(V, V) can contain infinite sums of the form

$$\sum_{n,m=1}^{\infty} a_m^n e^m \otimes e_n, \quad e^m \in V_m^*, e_n \in V_n, a_m^n \in \mathbb{K}, \ \text{ all } a_m^n \neq 0,$$

so that the trace operation on Hom(V, V) (and hence on Hom$(\otimes^k V, \otimes^l V)$) is not well-defined in general so that V can not be used for representations of *wheeled* props.

If one has a collection of k infinite-dimensional vector spaces, $V_i = \varinjlim V_{i,p}$, one can define similarly a k-coloured endomorphism prop $\mathcal{E}nd_{V_1,\dots,V_k}$ based on topological $\mathbb{S}$-modules

(19)

$$\mathcal{E}nd_{V_1,\dots,V_k} = \left\{ \text{Hom}(V_1^{\otimes r_1} \otimes \dots \otimes V_k^{\otimes r_k}, V_1^{\otimes l_1} \otimes \dots \otimes V_k^{\otimes l_k}) := \varprojlim_{(n_1,\dots,n_r)} \left((V_{1,p_1}^*)^{\otimes r_1} \otimes \dots \otimes (V_{k,p_k}^*)^{\otimes r_k} \otimes V_1^{\otimes l_1} \otimes \dots \otimes V_k^{\otimes l_k} \right) \right\}.$$

Its elements give us linear maps $V_1^{\otimes r_1} \otimes \dots \otimes V_k^{\otimes r_k} \longrightarrow V_1^{\otimes l_1} \otimes \dots \otimes V_k^{\otimes l_k}$ and hence can be used to define a representation of a k-coloured prop.

4.2. An infinite-dimensional graded vector space with k branes. Let $V := \varinjlim_p V_p$ be a countably infinite-dimensional graded vector space. It is called a vector space *with k branes* (and is denoted by $(V, W_1, \dots, W_k)$ or simply by $V^{k\text{-br}}$) if the following conditions hold:

(i) V comes equipped with a descending filtration

$$V = F^0 V \supset F^1 V \supset F^2 V \supset \dots \supset F^p V \supset F^{p+1} V \supset \dots$$

such that each quotient vector space $V/F^p V$ is finite-dimensional and equals V_p for any $p \in \mathbb{N}$,

(ii) For any $p \geq 0$ we have k different non trivial direct sum decompositions $V_p = W_{\tau,p}^+ \oplus W_{\tau,p}^-$, $\tau \in [k]$, which are compatible with the given injections $i_p : V_p \to V_{p+1}$,

$$i_p(W_{\tau,p}^\pm) \subset W_{\tau,p+1}^\pm$$

Note that the inclusion $F^{p+1} V \subset F^p V$ induces a projection

$$\pi_{p+1} : V_{p+1} \equiv F^{p+1} V / F^{p+2} V \longrightarrow V_p := F^p V F^{p+1} V$$

so that in this case we have not only direct systems of finite-dimensional vector spaces, but also inverse ones,

$$\dots \xrightarrow{\pi_{p+1}} W_{\tau,p}^\pm \xrightarrow{\pi_p} W_{\tau,p-1}^\pm \xrightarrow{\pi_{p-2}} \dots \longrightarrow 0$$

and hence can consider two limits for branes (and their intersections, see below), the direct and projective ones,

$$W_\tau^\pm := \varinjlim_p W_{\tau,p}^\pm \subset \hat{W}_\tau^\pm := \varprojlim_p W_{\tau,p}^\pm, \quad (W_\tau^\pm)^* = \varprojlim_p (W_{\tau,p}^\pm)^* \supset (\hat{W}_\tau^\pm)^* = \varinjlim_p (W_{\tau,p}^\pm)^*, \quad \forall \tau \in [k].$$

Note that the spaces $W_\tau^\pm$ and $(\hat{W}_\tau^\pm)^*$ are always countably dimensional, while $(W_\tau^\pm)^*$ and $\hat{W}_\tau^\pm$ are, in general, not (but as a compensation they come equipped with a nice topology). Note also that

$$((\hat{W}_\tau^\pm)^*)^* = \hat{W}_\tau^\pm, \quad ((W_\tau^\pm)^*)^* = W_\tau^\pm.$$

To define a suitable multi-oriented endomorphism prop out of an infinite-dimensional vector space with k branes, one has to work with both types of completions simultaneously. This fact motivates the extra filtration condition (i) in the definition of $V^{k\text{-br}}$ above.

4.2.1. Basic example. Let $\{x_1, x_2, \dots,\}$ be a countably infinite set of formal variables of some homological degrees $|x_i| \in \mathbb{Z}$, $i \in \mathbb{N}_{\geq 1}$. The graded vector space

$$V = \text{span}\langle x_1, x_2, \dots \rangle$$

is a typical example of an infinite-dimensional vector space satisfying conditions (i) and (ii) above with

$$F^p V = \text{span}\langle x_i \rangle_{i \geq p+1}, \quad V_p = \text{span}\langle x_1, x_2, \dots, x_p \rangle.$$

Let us choose k injections of countably infinite sets (i.e. k pairs of disjoint countably infinite subsets of $\mathbb{N}_{\geq 1}$)

$$f_i : \mathbb{N}_{\geq 1} \oplus \mathbb{N}_{\geq 1} \longrightarrow \mathbb{N}_{\geq 1}, \quad i \in [k],$$

and define a family of finite-dimensional vector spaces (equipped with the $\mathbb{Z}$-grading induced in the obvious way from the homological grading of the formal variables $(x_1, x_2, \ldots, x_p)$

$$W^+_{i,p} := \mathrm{span}\left\langle \pi_+ \circ f_i^{-1}\{1,2,\ldots,p\}\right\rangle, \qquad W^-_{i,p} := \mathrm{span}\left\langle \pi_- \circ f_i^{-1}\{1,2,\ldots,p\}\right\rangle$$

where $\pi_\pm : \mathbb{N}_{\geq 1} \oplus \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 1}$ is the projection to the first/second summand. The resulting data $(V, W^\pm_1, \ldots, W^\pm_k)$ is an example of an infinite-dimensional graded vector space with k branes.

4.3. Finite-dimensional case. *A finite-dimensional vector space with k branes* is simply a finite-dimensional vector space V equipped with k direct sum decompositions $V = W^+_\tau \oplus W^-_\tau$, $\tau \in [k]$. We shall be most interested in the infinite-dimensional case and hence use all the time use the direct/projective limit notation introduced in the previous section. The finite dimensional version fits into that notation as a special case when $V_p = V$, $W^\pm_{\tau,p} = W^\pm_\tau$ for all p.

4.4. The simplest case of a 2-directed endomorphism prop. Let $V^{1\text{-br}}$ be an infinite-dimensional vector space with one brane, and consider

$$W^+ = \varinjlim_p W^+_p, \quad \hat{W}^- = \varprojlim_p W^-_p, \; (W^+)^* = \varprojlim_p (W^+_p)^*, \quad (\hat{W}^-)^* = \varinjlim_p (W^-_p)^*$$

Note that of these four spaces only W^+ and $(\hat{W}^-)^*$ are always countably dimensional.
Using these data we construct an $\mathcal{S}^{(2)}$-module, that is a functor

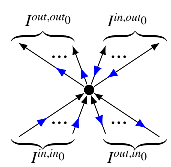
$$\begin{array}{rccl} \mathcal{E}nd_{V^{1\text{-br}}}: & \mathcal{S}^{(2)} & \longrightarrow & \text{Category of graded vector spaces} \\ & (I,\mathfrak{s}) & \longrightarrow & \mathcal{E}nd_{V^{2\text{-br}}}(I,\mathfrak{s}) \end{array}$$

as follows. First, one can identify any 2-oriented set[6]

$$(I, \mathfrak{s}: I \to Or_{1^+}) \equiv \left(I, \mathfrak{s}_{\bar 0}: I \to \{out, in\}, \mathfrak{s}_{\bar 1}: I \to \{out, in\} \text{ such that } \mathfrak{s}_{\bar 0}(i) := \mathfrak{s}_i(\bar 0), \mathfrak{s}_{\bar 1}(i) := \mathfrak{s}_i(\bar 1)\ \forall i \in I\right)$$

with a 2-directed corolla (the subscript 0 indicates the basic orientation)

(20)



where

$$I^{out,out_0} := \mathfrak{s}_{\bar 1}^{-1}(out) \sqcup \mathfrak{s}_{\bar 0}^{-1}(out)), \quad I^{in,out_0} := \mathfrak{s}_{\bar 1}^{-1}(in) \sqcup \mathfrak{s}_{\bar 0}^{-1}(out)), \quad I^{out,in_0} := \mathfrak{s}_{\bar 1}^{-1}(out) \sqcup \mathfrak{s}_{\bar 0}^{-1}(in)), \quad I^{in,in_0} := \mathfrak{s}_{\bar 1}^{-1}(in) \sqcup \mathfrak{s}_{\bar 0}^{-1}(in)),$$

Set

$$\# I^{out,out_0} =: m_1, \quad \# I^{in,out_0} =: m_2, \quad \# I^{in,in_0} =: n_1, \quad \# I^{out,in_0} =: n_2.$$

Next we define[7] (cf. (19))

$$\begin{aligned} \mathcal{E}nd_{V^{1\text{-br}}}(I,\mathfrak{s}) \quad := \quad & \varprojlim_{\substack{p_a \text{ for} \\ a \in I^{in,in_0}_\tau}} \varprojlim_{\substack{p_b \text{ for} \\ b \in I^{in,out_0}_\tau}} \left(\varinjlim_{\substack{p_c \text{ for} \\ c \in I^{out,out_0}_\tau}} \varinjlim_{\substack{p_e \text{ for} \\ e \in I^{out,in_0}_\tau}} \bigotimes_{a \in I^{in,in_0}_\tau} (W^+_{p_a})^* \bigotimes_{b \in I^{in,out_0}_\tau} W^-_{p_b} \bigotimes_{c \in I^{out,out_0}_\tau} W^+_{p_c} \bigotimes_{e \in I^{out,in_0}_\tau} (W^-_{p_e})^* \right) \\ = \quad & \varprojlim_{\substack{p_a \text{ for} \\ a \in I^{in,in_0}_\tau}} \varprojlim_{\substack{p_b \text{ for} \\ b \in I^{in,out_0}_\tau}} \mathrm{Hom}\left(\bigotimes_{a \in I^{in,in_0}_\tau} W^+_{p_a} \bigotimes_{b \in I^{in,out_0}_\tau} (W^-_{p_b})^*\,,\ \otimes^{m_1} W^+ \bigotimes \otimes^{n_2} (\hat{W}^-)^* \right) \\ = \quad & \mathrm{Hom}\left(\otimes^{n_1} W^+ \bigotimes \otimes^{m_2} (\hat{W}^-)^*\,,\ \otimes^{m_1} W^+ \bigotimes \otimes^{n_2} (\hat{W}^-)^* \right) \end{aligned}$$

---

[6]Here we denote the elements of $[1^+]$ by $\bar 0$ and $\bar 1$ so that the value $\mathfrak{s}_i$ of the map $\mathfrak{s}$ on an element $i \in I$ is itself a map of sets $\mathfrak{s}_i : \{\bar 0, \bar 1\} \to \{out, in\}$.

[7]Here we use the facts that for any vector space M and any inverse system of finite-dimensional vector spaces $\{N_i\}$ one has $\varprojlim \mathrm{Hom}(N_i, M) \cong \mathrm{Hom}(\varinjlim N_i, M)$ and $\varprojlim \mathrm{Hom}(M, N_i) \cong \mathrm{Hom}(M, \varprojlim N_i)$, while $\varprojlim (N_i \otimes M) \cong (\varprojlim N_i) \otimes M$ only if M is finite-dimensional. On the other hand, for *any* direct system $\{N_i\}$ the equality $\varinjlim (M \otimes N_i) \cong M \otimes \varinjlim N_i$ holds true for any M, while the equality $\varinjlim \mathrm{Hom}(M, N_i) \cong \mathrm{Hom}(M, \varinjlim N_i)$ is true if and only if M is finite-dimensional.

Thus an element $f \in \mathcal{E}nd_{V^{1\text{-br}}}(I, \mathfrak{s})$ gives us a well-defined map

$$f : \otimes^{n_1} W^+ \bigotimes \otimes^{m_2} (\hat{W}^-)^* \longrightarrow \otimes^{m_1} W^+ \bigotimes \otimes^{n_2} (\hat{W}^-)^*$$

between countably dimensional vector spaces and hence such elements can be composed along the "blue direction". What about the basic direction? Note that we can try rearranging tensor factors in $\mathcal{E}nd_{V^{1\text{-br}}}(I, \mathfrak{s})$ as follows

$$\begin{aligned}
\mathcal{E}nd_{V^{1\text{-br}}}(I, \mathfrak{s}) &:= \varprojlim_{p_a} \varprojlim_{p_b} \left(\varinjlim_{p_c} \varinjlim_{p_e} \bigotimes_{a \in I_\tau^{in,in_0}} (W^+_{p_a})^* \bigotimes_{e \in I_\tau^{out,in_0}} (W^-_{p_e})^* \bigotimes_{c \in I_\tau^{out,out_0}} W^+_{p_c} \bigotimes_{b \in I_\tau^{in,out_0}} W^-_{p_b} \right) \\
&= \varprojlim_{p_a} \varprojlim_{p_b} \left(\varinjlim_{p_c} \varinjlim_{p_e} \mathrm{Hom} \left(\bigotimes_{a \in I_\tau^{in,in_0}} W^+_{p_a} \bigotimes_{e \in I_\tau^{out,in_0}} W^-_{p_e} , \bigotimes_{c \in I_\tau^{out,out_0}} W^+_{p_c} \bigotimes_{b \in I_\tau^{in,out_0}} W^-_{p_b} \right) \right) \\
&= \mathrm{Hom} \left(\varinjlim_{p_a} \varprojlim_{p_e} \bigotimes_{a \in I_\tau^{in,in_0}} W^+_{p_a} \bigotimes_{e \in I_\tau^{out,in_0}} W^-_{p_e} , \varprojlim_{p_b} \varinjlim_{p_c} \bigotimes_{c \in I_\tau^{out,out_0}} W^+_{p_c} \bigotimes_{b \in I_\tau^{in,out_0}} W^-_{p_b} \right)
\end{aligned}$$

However, in general,

$$\otimes^{n_1} W^+ \bigotimes \hat{\otimes}^{n_2} \hat{W}^- := \varinjlim_{p_a} \varprojlim_{p_e} \bigotimes_{a \in I_\tau^{in,in_0}} W^+_{p_a} \bigotimes_{e \in I_\tau^{out,in_0}} W^-_{p_e} \neq \varprojlim_{p_e} \varinjlim_{p_a} \bigotimes_{a \in I_\tau^{in,in_0}} W^+_{p_a} \bigotimes_{e \in I_\tau^{out,in_0}} W^-_{p_e} =: \otimes^{m_1} W^+ \widehat{\bigotimes} \hat{\otimes}^{m_2} \hat{W}^-$$

with the l.h.s. being a (proper, in general!) subspace of the r.h.s. Hence elements of

$$\mathcal{E}nd_{V^{1\text{-br}}}(I, \mathfrak{s}) \cong \mathrm{Hom}\left(\otimes^{n_1} W^+ \bigotimes \hat{\otimes}^{n_2} \hat{W}^- , \otimes^{m_1} W^+ \widehat{\bigotimes} \hat{\otimes}^{m_2} \hat{W}^- \right)$$

can not, in general, be composed along graphs of the type shown in (7). (Nevertheless the latest formula shows that any element of $\mathcal{E}nd_{V^{1\text{-br}}}(I, \mathfrak{s})$ can nevertheless be understood as some linear map along the basic direction.)

We conclude that the $\mathcal{S}^{(2)}$-module $\mathcal{E}nd_{V^{1\text{-br}}}$ admits nice compositions μ_Γ along any graphs *not containing closed paths of directed edges in blue color* as in (6) (with the "associativity" axioms are obviously satisfied) and hence gives us an example of 2-oriented prop. We call it the *endomorphism prop* of $V^{1\text{-br}}$.

Note that if $V^{1\text{-br}}$ is finite-dimensional (or at least if W^- is finite-dimensional), then

$$\mathcal{E}nd_{V^{1\text{-br}}}(I, \mathfrak{s}) \cong \mathrm{Hom}\left(\otimes^{n_1} W^+ \bigotimes \otimes^{n_2} W^- , \otimes^{m_1} W^+ \bigotimes \otimes^{m_2} W^-\right) \cong \mathrm{Hom}\left(\otimes^{n_1} W^+ \bigotimes \otimes^{m_2} (\hat{W}^-)^* , \otimes^{m_1} W^+ \bigotimes \otimes^{n_2} (\hat{W}^-)^*\right)$$

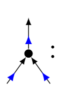
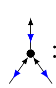
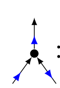
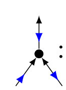
and the compositions μ_Γ (in the definition of a multi-directed prop) make sense for any graphs $\Gamma \in G^{0\uparrow k+1}$.

4.4.1. Definition. Let $\mathcal{P}^{2\text{-or}}$ be a 2-oriented prop(erad). A morphism of 2-oriented prop(erad)s

$$\rho : \mathcal{P}^{2\text{-or}} \longrightarrow \mathcal{E}nd_{V^{1\text{-br}}}$$

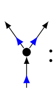
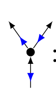
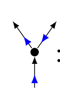
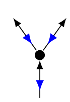
is called a *representation* of $\mathcal{P}^{2\text{-or}}$ in the vector space V with one brane.

4.4.2. Example. A representation of a 2-oriented operad $\mathcal{A}ss^{(2)}$ (resp., $\mathcal{L}ie^{(2)}$) in $V^{1\text{-br}}$ is given by a collection of linear maps

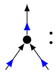
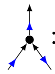
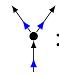
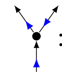
 $: W^+ \otimes W^+ \to W^+,$  $: (\hat{W}^-)^* \to (\hat{W}^-)^* \otimes (\hat{W}^-)^*,$  $: W^+ \to W^+ \otimes (W^-)^*,$  $: W \otimes (\hat{W}^-)^* \to (\hat{W}^-)^*$

such that their compositions satisfy relations (9)-(10) (respectively, (12)-(14)).

To illustrate divergency phenomenon let us consider generic representations of a prop in which compositions along graphs with *non-trivial genus* make sense. For example, a representation of, say, the prop of 2-oriented Lie bialgebras $\mathcal{L}ieb^{2\text{-or}}$ in $V^{1\text{-br}}$ is given by maps as above plus the following ones

 $: W^+ \to W^+ \otimes W^+,$  $: (\hat{W}^-)^* \to W^+ \otimes (\hat{W}^-)^*$  $: W^+ \otimes (\hat{W}^-)^* \to W^+,$  $: (\hat{W}^-)^* \otimes (\hat{W}^-)^* \to (\hat{W}^-)^*$

satisfying certain quadratic relations. If $\{x_{a^+}\}_{a^+\in N_{\geq 1}}$ is a countably infinite basis of W^+, $\{x_{a^-}\}_{a^-\in N_{\geq 1}}$ is a countably infinite basis of W^-, and $\{y^{a_+}\}_{a_+\in N_{\geq 1}}$ and $\{y^{a_-}\}_{a_-\in N_{\geq 1}}$ the associated countably infinite dual bases for $(\hat{W}^+)^*$ and $(\hat{W}^-)^*$ then the corresponding maps, say the following ones

 $: W^+ \otimes W^+ \xrightarrow{\mu_1} W^+$,  $: W^+ \xrightarrow{\mu_2} W^+ \otimes (\hat{W}^-)^*$,  $: W^+ \xrightarrow{\mu_3} W^+ \otimes W^+$,  $: W^+ \otimes (\hat{W}^-)^* \xrightarrow{\mu_4} W^+$,

can be represented by the following infinite, in general, sums

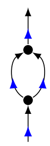
$$\mu_1 = \sum_{a^+,b^+,c^+\in\mathbb{N}_{\geq 1}} \Phi^{c^+}_{a^+b^+} y^{a^+}\otimes y^{b^+}\otimes x_{c^+}, \quad \mu_2 = \sum_{a^+,b_-,c^+\in\mathbb{N}_{\geq 1}} \Phi^{a^+}_{c^+b_-} x^{c^+}\otimes x_{a^+}\otimes y^{b_-}, \qquad \Phi^{c^+}_{a^+b^+}, \Phi^{a^+}_{c^+b_-}\in\mathbb{K}$$

$$\mu_3 = \sum_{a_+,b_+,c_+\in\mathbb{N}_{\geq 1}} \Psi^{a_+b_+}_{c_+} y^{c^+}\otimes x_{a^+}\otimes x_{b^+} \quad \mu_4 = \sum_{a^+,b_-,c^+\in\mathbb{N}_{\geq 1}} \Psi^{a^+b_-}_{c^+} y^{c^+}\otimes x_{b^-}\otimes x_{a^+}, \qquad \Psi^{a_+b_+}_{c_+}, \Psi^{a^+b_-}_{c^+}\in\mathbb{K}.$$

where the coefficients satisfy the conditions:

- for fixed a_+, b_- only finitely many $\Phi^{c^+}_{a^+b^+}\neq 0$;for fixed c_+ only finitely many $\Phi^{a^+}_{c^+b_-}\neq 0$;
- for fixed c_+ only finitely many $\Psi^{a_+b_+}_{c_+}\neq 0$; for fixed a_+, b_- only finitely many $\Psi^{c^+}_{a^+b_-}\neq 0$.

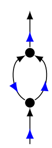
Then the element

 $\in \mathcal{L}ieb^{\text{2-or}}$

gets represented in $V^{\text{1-br}}$ as a linear map

$$\sum_{c^+,d^+\in\mathbb{N}_{\geq 1}} \left(\underbrace{\sum_{a^+,b^+\in\mathbb{N}_{\geq 1}} \Phi^{c^+}_{a^+b^+}\Psi^{a_+b_+}_{d_+}}_{\text{only finitely many terms non-zero}} \right) y^{d^+}\otimes x_{c^+} : W^+ \to W^+$$

which is always well defined, while the element

 $\in \mathcal{L}ieb^{1\uparrow 2}$

gets represented in $V^{\text{1-br}}$ as a formal sum of linear maps

$$\sum_{c^+,d^+\in\mathbb{N}_{\geq 1}} \left(\underbrace{\sum_{a^+,b^+\in\mathbb{N}_{\geq 1}} \Phi^{c^+}_{a^+b_-}\Psi^{a^+b_-}_{d_+}}_{\text{infinitely many terms can be non-zero ingeneral}} \right) y^{d^+}\otimes x_{c^+} : W^+ \to W^+$$

which in general diverges. Such an element can be represented in general only in the case $\dim V < \infty$.

4.4.3. Symplectic vector space with Lagrangian branes. Let $(V, \omega : \wedge^2 V \to \mathbb{K})$ be a finite-dimensional vector space equipped with a symplectic form; in general $\dim V = 2n$ for some $n\in\mathbb{N}_{\geq 1}$. A subspace $W\subset V$ is called isotropic if

$$W \subset W^\perp := \{v\in V \mid \omega(v,w) = 0\ \forall w\in W\}.$$

Such a subspace is called *Lagrangian* if $\dim W = n$. It is well-known that a Lagrangian subspace $W^+\subset V$ always has a complement $W^-\subset V$ which is also Lagrangian. Moreover, in this case the symplectic form induces a canonical isomorphism

$$\omega : (W^-)^* \longrightarrow W^+.$$

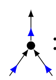
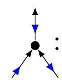
The data (V, W^+, W^-) is called a *finite-dimensional symplectic vector space with one Lagrangian brane*. Similarly one defines *finite-dimensional symplectic vector space with k Lagrangian branes*, $(V, W^+_\tau, W^-_\tau)_{\tau\in[k]}$. We generalize this notion to infinite dimensions as follows.

Let $(V, W_1, \ldots, W_k)$ be a countably infinite dimensional vector space with k branes such that for each p and each $\tau \in [k]$ the vector space $V_p = W^+_{\tau,p} \oplus W^-_{\tau,p}$ is a finite dimensional symplectic vector space with k Lagrangian branes. Then the symplectic forms induce a non-degenerate pairing

$$\omega : (\hat{W}^-_\tau)^* \longrightarrow W^+_\tau$$

which is an isomorphism for each $\tau \in [k]$. The resulting datum $(V, W_1, \ldots, W_k, \omega)$ is called an *infinite-dimensional symplectic vector space with k Lagrangian branes* and often is denoted by $V^{k\text{-br}}_{symp}$.

If we consider now a generic representation ρ of, say, $\mathcal{A}ss^{(2)}$ or $\mathcal{L}ie^{(2)}$ in $V^{1\text{-br}}_{symp}$, then, due to the canonical isomorphism $(\hat{W}^-_\tau)^* = W^+_\tau$, we see that multioriented generators which differ only in the basic orientation stand for linear maps of the same type, for example

 $: W^+ \otimes W^+ \to W^+,$  $: W^+ \otimes W^+ \to W^+,$

and hence it makes sense to identify them. We call the representation ρ in $V^{1\text{-br}}_{symp}$ *reduced symplectic Lagrangian* if ρ takes identical values on all generating corollas of $\mathcal{P}^{2\text{-or}}$ which become identical (as k-oriented graphs) once the basic directions on edges are forgotten. Then we can reformulate observations **3.2.1** and **3.4.1** as follows.

4.4.4. Proposition. *(i) There is a one-to-one correspondence between reduced symplectic Lagrangian representations of $\mathcal{A}ss^{(2)}$ in $V^{1\text{-}br}_{symp}$ and infinitesimal bialgebra structures in the Lagrangian subspace W^+.*

(ii) There is a one-to-one correspondence between reduced symplectic Lagrangian representations of $\mathcal{L}ie^{(2)}$ in $V^{1\text{-}br}_{symp}$ and Lie bialgebra structures in the Lagrangian subspace W^+.

4.5. Remark. In principle one can use symplectic structures on V of homological degree $q \neq 0$ so that the induced isomorphism takes the form $\omega : (\hat{W}^-_\tau)^* \longrightarrow W^+_\tau[q]$ but then the basic direction can not be forgotten completely in representations as it stands now for a degree shift of linear maps.

4.6. Endomorphism multidirected prop of a graded vector space with k branes. Given any multi-oriented set $(I, \mathfrak{s} : I \to Or_{k^+})$. Recall that for any fixed $i \in I$ there is an associated map

$$\mathfrak{s}_i : [k^+] \xrightarrow{\mathfrak{s}(i)} \{out, in\}.$$

while for any fixed $\tau \in [k^+]$ there is a map

$$\begin{array}{rccc} \mathfrak{s}_\tau : & I & \longrightarrow & \{out, in\} \\ & i & \longrightarrow & \mathfrak{s}_\tau(i) := \mathfrak{s}_i(\tau). \end{array}$$

The latter map can be used to decompose I into two disjoint subsets

$$I = \mathfrak{s}_\tau^{-1}(out) \sqcup \mathfrak{s}_\tau^{-1}(in)$$

The basic direction $\tau = 0$ plays a special role. For any $\tau \neq 0$, i.e. for any $\tau \in [k]$ we can further decompose the set I into four disjoint subsets

$$I = (\mathfrak{s}_\tau^{-1}(out) \sqcup \mathfrak{s}_\tau^{-1}(in)) \cap (\mathfrak{s}_0^{-1}(out) \sqcup \mathfrak{s}_0^{-1}(in)) := I_\tau^{out,out_0} \sqcup I_\tau^{out,in_0} \sqcup I_\tau^{in,out_0} \sqcup I_\tau^{in,in_0}$$

where

$$I_\tau^{out,out_0} := \mathfrak{s}_\tau^{-1}(out) \cap \mathfrak{s}_0^{-1}(out), \quad I_\tau^{out,in_0} := \mathfrak{s}_\tau^{-1}(out) \cap \mathfrak{s}_0^{-1}(in), \quad I_\tau^{in,out_0} := \mathfrak{s}_\tau^{-1}(in) \cap \mathfrak{s}_0^{-1}(out), \quad I_\tau^{in,in_0} := \mathfrak{s}_\tau^{-1}(in) \cap \mathfrak{s}_0^{-1}(in),$$

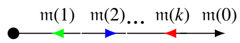
Given a graded vector space with k branes, $V^{k\text{-br}} = (V = \varinjlim V_p, W_1, \ldots, W_k)$, consider a collection of linear subspaces for each $p \in \mathbb{N}$,

$$W_p^{\mathfrak{m}} := W_{1,p}^{\mathfrak{m}(1)} \cap W_{2,p}^{\mathfrak{m}(2)} \cap \ldots \cap W_{k,p}^{\mathfrak{m}(k)},$$

one for each multidirection $\mathfrak{m} : [k^+] \to \{out, in\}$ from $\mathcal{O}r_{k^+}$, where we set for each $\tau \in [k]$,

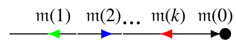
$$W_{\tau,p}^{\mathfrak{m}(\tau)} := \begin{cases} W_{\tau,p}^{+} & \text{if } \mathfrak{m}(0) = \mathfrak{m}(\tau) = out \\ (W_{\tau,p}^{+})^* & \text{if } \mathfrak{m}(0) = \mathfrak{m}(\tau) = in \\ W_{\tau,p}^{-} & \text{if } \mathfrak{m}(0) = out,\ \mathfrak{m}(\tau) = in \\ (W_{\tau,p}^{-})^* & \text{if } \mathfrak{m}(0) = in,\ \mathfrak{m}(\tau) = out \end{cases}$$

Note that $(W_{\tau,p}^{\mathfrak{m}})^* = W_{\tau,p}^{\mathfrak{m}^{opp}}$. For example,

for $\mathfrak{m} =$ 

one has $W_p^{\mathfrak{m}} = W_{1,p}^{-} \cap W_{2,p}^{+} \cap \ldots \cap W_{k,p}^{-}$

while

for $\mathfrak{m}^{opp} =$ 

one has $W_p^{\mathfrak{m}^{opp}} = (W_{1,p}^{-})^* \cap (W_{2,p}^{+})^* \cap \ldots \cap (W_{k,p}^{-})^*$.

We define a countably dimensional vector space

$$W^{\mathfrak{m}} := \varinjlim_{p} W_p^{\mathfrak{m}},$$

Define an $\mathcal{S}^{(k+1)}$-module $\mathcal{E}nd_{V^{k\text{-br}}}$, that is, a functor

$$\begin{array}{rccc} \mathcal{E}nd_{V^{k\text{-br}}} : & \mathcal{S}^{(k+1)} & \longrightarrow & \text{Category of graded vector spaces} \\ & (I, \mathfrak{s}) & \longrightarrow & \mathcal{E}nd_{V^{k\text{-br}}}(I, \mathfrak{s}) \end{array},$$

by setting

$$\mathcal{E}nd_{V^{k\text{-br}}}(I, \mathfrak{s}) := \bigcap_{\tau \in [k]} \mathrm{Hom}_\tau(\mathfrak{s}, I),$$

where (cf. (19))

$$\begin{aligned} \mathrm{Hom}_\tau(\mathfrak{s}, I) &:= \varprojlim_{\substack{p_i \text{ for} \\ i \in I_\tau^{in,out_0} \cup I_\tau^{in,in_0}}} \left(\varinjlim_{\substack{p_j \text{ for} \\ i \in I_\tau^{out,out_0} \cup I_\tau^{out,in_0}}} \bigotimes_{i \in I} W_{p_i}^{\mathfrak{s}_i} \right) \\ &= \varprojlim_{p_i} \left(\bigotimes_{i \in i \in I_\tau^{in,out_0} \cup I_\tau^{in,in_0}} W_{p_i}^{\mathfrak{s}_i} \otimes \bigotimes_{j \in I_\tau^{out,out_0} \cup I_\tau^{out,in_0}} W^{\mathfrak{s}_j} \right) \\ &= \mathrm{Hom}\left(\bigotimes_{i \in I_\tau^{in,out_0} \cup I_\tau^{in,in_0}} \varinjlim_{p_i} W_{p_i}^{\mathfrak{s}_i^{opp}}, \bigotimes_{j \in I_\tau^{out,out_0} \cup I_\tau^{out,in_0}} W^{\mathfrak{s}_j} \right) \\ &= \mathrm{Hom}\left(\bigotimes_{i \in i \in I_\tau^{in,out_0} \cup I_\tau^{in,in_0}} W^{\mathfrak{s}_i^{opp}}, \bigotimes_{j \in I_\tau^{out,out_0} \cup I_\tau^{out,in_0}} W^{\mathfrak{s}_j} \right). \end{aligned}$$

Thus a single element $f \in \mathcal{E}nd_{V^{k\text{-br}}}(I, \mathfrak{s})$ has k incarnations as a linear map, one for each "coloured direction" $\tau \in [k]$. Note that all the k spaces $\mathrm{Hom}_\tau(\mathfrak{s}, I)$, $\tau \in [k]$, belong to one and the same vector space

$$\mathrm{Hom}(\mathfrak{s}, I) := \varprojlim_{p_i} \bigotimes_{i \in I} W_{p_i}^{\mathfrak{s}_i} \tag{21}$$

so that it makes sense to talk about their intersection. If V is finite-dimensional, then, of course, $\mathrm{Hom}_\tau(\mathfrak{s}, I) = \mathrm{Hom}(\mathfrak{s}, I)$ for any $\tau \in [k]$.

Therefore elements of $\mathcal{E}nd_{V^{k\text{-br}}}$ can be composed (when it makes sense) along each of the "coloured" direction, but in general they can *not* be composed along the basic direction (i.e. compositions of the type (7) have no sense in general).

Let $\mathcal{P}^{(k+1)\text{-or}}$ be a $(k+1)$-oriented prop(erad). A morphism of $(k+1)$-oriented prop(erad)s

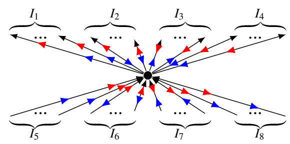
$$\rho : \mathcal{P}^{(k+1)\text{-or}} \longrightarrow \mathcal{E}nd_{V^{k\text{-br}}}$$

is called a *representation* of $\mathcal{P}^{(k+1)\text{-or}}$ in a vector space with one k branes $V^{k\text{-br}}$. If $V^{k\text{-br}}$ happens to be a symplectic vector space with k Lagrangian branes, that a representation ρ is called *reduced symplectic Lagrangian* if ρ takes identical values on all those generating corollas of $\mathcal{P}^{(k+1)\text{-or}}$ which become identical (as k-oriented graphs) once the *basic* directions on edges are forgotten.

4.7. Example: 3-oriented endomorphism prop. Let $V^{2\text{-or}} = (V, W_1^+, W_2^+)$ be a countably dimensional graded vector space with 2 branes. We would like to describe in more details the structure of the associated endomorphism prop which is a functor

$$\mathcal{E}nd_{V^{2\text{-or}}} : \qquad \mathcal{S}^{(3)} \qquad \longrightarrow \quad \text{Category of dg vector spaces}$$

$$(I, \mathfrak{s}) = $$



$$\longrightarrow \qquad \mathcal{E}nd_{V^{2\text{-or}}}(I, \mathfrak{s})$$

The multi-orientation $\mathfrak{s}$ defines (and can be reconstructed from) the decomposition $I = I_1 \sqcup I_2 \sqcup \ldots \sqcup I_8$ as explained in the picture. If

$$\# I_i = m_i \ \text{ for } \ i \in \{1, 2, 3, 4\}\,, \ \# I_i = n_{i-4} \ \text{ for } \ i \in \{5, 6, 7, 8\}.$$

then, by definition, $\mathcal{E}nd_{V^{2\text{-or}}}(I, \mathfrak{s})$ is the intersection in (21) of two graded vector spaces

$$\mathrm{Hom}(I, \mathfrak{s}) := \mathrm{Hom}\left(\otimes^{n_1} W^{++} \otimes^{n_3} W^{+-} \otimes^{m_2} (\hat{W}^{-+})^* \otimes^{m_4} (\hat{W}^{--})^*\,,\ \otimes^{m_1} W^{++} \otimes^{m_3} W^{+-} \otimes^{n_2} (\hat{W}^{-+})^* \otimes^{n_4} (\hat{W}^{--})^*\right)$$

and

$$\mathrm{Hom}(I, \mathfrak{s}) := \mathrm{Hom}\left(\otimes^{n_1} W^{++} \otimes^{n_2} W^{-+} \otimes^{m_3} (\hat{W}^{+-})^* \otimes^{m_4} (\hat{W}^{--})^*\,,\ \otimes^{m_1} W^{++} \otimes^{m_2} W^{-+} \otimes^{n_3} (\hat{W}^{+-})^* \otimes^{n_4} (\hat{W}^{--})^*\right)$$

where we set

$$W^{++} := \lim_{\substack{\longrightarrow \\ p}} W^+_{1,p} \cap W^+_{2,p}, \quad W^{-+} := \lim_{\substack{\longrightarrow \\ p}} W^-_{1,p} \cap W^+_{2,p}, \quad W^{+-} := \lim_{\substack{\longrightarrow \\ p}} W^+_{1,p} \cap W^-_{2,p}, \quad W^{--} := \lim_{\substack{\longrightarrow \\ p}} W^-_{1,p} \cap W^-_{2,p},$$

$$\hat{W}^{++} := \lim_{\substack{\longrightarrow \\ p}} W^-_{1,p} \cap W^-_{2,p}, \quad \hat{W}^{-+} := \lim_{\substack{\longleftarrow \\ p}} W^-_{1,p} \cap W^+_{2,p}, \quad \hat{W}^{--} := \lim_{\substack{\longrightarrow \\ p}} W^-_{1,p} \cap W^-_{2,p}, \quad \hat{W}^{+-} := \lim_{\substack{\longleftarrow \\ p}} W^+_{1,p} \cap W^-_{2,p},$$

All the tensor factors shown in the above formulae for $\mathrm{Hom}(I, \mathfrak{s})$ and $\mathrm{Hom}(I, \mathfrak{s})$ are countably dimensional vector spaces. Let $\{x_{A_{++}}\}$, $\{x_{A_{+-}}\}$, $\{x_{A_{-+}}\}$, $\{x_{A_{--}}\}$ be bases for the (direct limit) vector spaces W^{++}, W^{+-}, W^{-+} and W^{--}, while $y^{A^{++}}$, $y^{A^{+-}}$, $y^{A^{-+}}$ and $y^{A^{--}}$ be the associated dual bases for (also direct limit) vector spaces $(\hat{W}^{++})^*$, $(\hat{W}^{+-})^*$, $(\hat{W}^{+-})^*$ and $(\hat{W}^{--})^*$. Then the "big" vector space (21) consists of all formal power series of the form

$$\sum_{A_{\bullet\bullet}, B^{\bullet\bullet}} F^{A_{++}A_{+-}A_{-+}A_{--}}_{B^{++}B^{+-}B^{-+}B^{--}}\ x_{A_{-+}} \otimes x_{A_{-+}} \otimes x_{A_{-+}} \otimes x_{A_{-+}} \otimes y^{B^{++}} \otimes y^{B^{+-}} \otimes y^{B^{-+}} \otimes y^{B^{--}}, \qquad F^{A_{++}A_{+-}A_{-+}A_{--}}_{B^{++}B^{+-}B^{-+}B^{--}} \in \mathbb{K},$$

its subspace $\mathrm{Hom}(I, \mathfrak{s})$ is spanned by those formal series whose coefficients satisfy the condition

- for any fixed values of indices A_{++}, A_{+-}, B^{-+} and B^{--} only *finitely many* $F^{A_{++}A_{+-}A_{-+}A_{--}}_{B^{++}B^{+-}B^{-+}B^{--}} \neq 0$,

while the subspace $\mathrm{Hom}(I, \mathfrak{s})$ is characterized by

- for any fixed values of indices A_{++}, A_{-+}, B^{+-} and B^{--} only *finitely many* $F^{A_{++}A_{+-}A_{-+}A_{--}}_{B^{++}B^{+-}B^{-+}B^{--}} \neq 0$,

This gives us a "down to earth" characterization of the endomorphism prop $\mathcal{E}nd_{V^{2\text{-br}}} \cong \{\mathrm{Hom}(I, \mathfrak{s}) \cap \mathrm{Hom}(I, \mathfrak{s})\}$.

5. Action of the Grothendieck-Tiechmüller group on some multi-oriented props

5.1. An operad of multi-oriented graphs. For any $l \geq -1$ and $k \geq 0$ let $G^{l+1\uparrow k+1}_{n,p}$ be a set of $(l+1)$-oriented $(k+1)$-directed (see §**2.3**) graphs Γ with n vertices and p edges such that some bijections $V(\Gamma) \to [n]$ and $E(\Gamma) \to [p]$ are fixed, i.e. every vertex and every edge of Γ has a numerical label. There is a natural right action of the group $\mathbb{S}_n \times \mathbb{S}_p$ on the set $G^{l+1\uparrow k+1}_{n,p}$ with $\mathbb{S}_n$ acting by relabeling the vertices and $\mathbb{S}_p$ by relabeling the edges. For each fixed integer d, consider a collection of $\mathbb{S}_n$-modules $\mathcal{G}ra_d^{l+1\uparrow k+1} = \{\mathcal{G}ra_d^{l+1\uparrow k+1}(n)\}_{n\geq 1}$, where

$$\mathcal{G}ra_d^{l+1\uparrow k+1}(n) := \prod_{p\geq 0} \mathbb{K}\langle G^{l+1\uparrow k+1}_{n,p}\rangle \otimes_{\mathbb{S}_p} sgn_p^{\otimes|d-1|}[p(d-1)].$$

where sgn_p is the 1-dimensional sign representation of $\mathbb{S}_p$. It has an (ordinary, i.e 1-oriented!) operad structure with the composition rule,

$$\begin{array}{rccc} \circ_i: & \mathcal{G}ra_d^{l+1\uparrow k+1}(n)\times \mathcal{G}ra_d^{l+1\uparrow k+1}(m) & \longrightarrow & \mathcal{G}ra_d^{l+1\uparrow k+1}(n+m-1), \ \ \forall\, i\in[n] \\ & (\Gamma_1,\Gamma_2) & \longrightarrow & \Gamma_1\circ_i\Gamma_2, \end{array}$$

given by substituting the graph Γ_2 into the i-labeled vertex v_i of Γ_1 and taking the sum over re-attachments of dangling edges (attached before to v_i) to vertices of Γ_2 in all possible ways. If $l=k$ we abbreviate $\mathcal{G}ra_d^{(k+1)\text{-or}} = \mathcal{G}ra_d^{k+1\uparrow k+1}$ and call it the *operad of* $(k+1)$*-oriented graphs*.

Note also that for $l>l'$ the operad $\mathcal{G}ra_d^{l+1\uparrow k+1}$ is a suboperad of $\mathcal{G}ra_d^{l'+1\uparrow k+1}$.

There is a canonical injection

$$\mathcal{G}ra_d^{l+1\uparrow k+1} \longrightarrow \mathcal{G}ra_d^{l+1\uparrow k+2}$$

sending a $(k+1)$-directed graph Γ into a sum of $(k+2)$-directed graphs obtained from Γ by attaching the new $(k+2)$nd direction to each edge in all (i.e. two) possible ways.

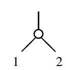
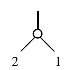
Let $\mathcal{L}ie_d$ be a (degree shifted) ordinary operad of Lie algebras whose representations are graded Lie algebras equipped with the Lie bracket in degree $1-d$, and consider the standard (cf. [MV]) deformation complex of the trivial morphism of operads,

$$\mathsf{fGC}_d^{l+1\uparrow k+1} := \mathsf{Def}\left(\mathcal{L}ie_d \xrightarrow{0} \mathcal{G}ra_d^{l+1\uparrow k+1}\right) \simeq \prod_{n\geq 1} \mathcal{G}ra_d^{l+1\uparrow k+1}(n)^{\mathbb{S}_n}[d(1-n)] \quad \forall\, k\geq 0,\ l\in\{-1,0,1,\dots,k\}. \tag{22}$$

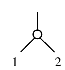
This is a Lie algebra. Moreover, it admits a non-trivial Maurer-Cartan element γ_0 which corresponds to a morphism

$$\gamma_0 : \mathcal{L}ie_d \longrightarrow \mathcal{G}ra_d^{l+1\uparrow k+1}$$

given explicitly on the generator (of homological degree $1-d$)

 $= (-1)^d$  $\in \mathcal{L}ie_d(2)$

by the following explicit formula (cf. [W1, W2])

(23) $\gamma_0\Big($  $\Big) = \sum_{\mathfrak{a}\in Or_k} \left((1)\xrightarrow{\mathfrak{a}}(2) + (-1)^d\, (2)\xrightarrow{\mathfrak{a}}(1) \right) =: \bullet\!\!\to\!\!\bullet$

where the summation runs over all possible ways to attach extra k directions to the 1-oriented edge. Note that elements of $\mathsf{fGC}_d^{l+1\uparrow k+1}$ can be identified with graphs from $\mathcal{G}ra_d^{l+1\uparrow k+1}$ whose vertices' labels are symmetrized (for d even) or skew-symmetrized (for d odd) so that in pictures we can forget about labels of vertices and denote them by unlabelled black bullets as in the formula above. Note also that graphs from $\mathcal{G}ra_d^{l+1\uparrow k+1}$ come equipped with a *orientation* which is a choice of ordering of edges (for d even) or a choice of ordering of vertices (for d odd) up to an even permutation in both cases. Thus every graph $\Gamma\in\mathsf{fGC}_d^{l+1\uparrow k+1}$ has at most two different orientations, or and or^{opp}, and one has the standard relation, $(\Gamma, or) = -(\Gamma, or^{opp})$; as usual, the data (Γ, or) is abbreviated to Γ (with some choice of orientation implicitly assumed). Note that the homological degree of graph Γ from $\mathsf{fGC}_d^{l+1\uparrow k+1}$ is given by

$$|\Gamma| = d(\#V(\Gamma)-1) + (1-d)\#E(\Gamma).$$

We show in [Me3] some other explicit Maurer-Cartan elements in the Lie algebra $\mathsf{fGC}_d^{l+1\uparrow k+1}$ given by transcendental formulae; in this paper we need only γ_0.

The above Maurer-Cartan (23) makes $(\mathsf{fGC}_d^{l+1\uparrow k+1}, [\ ,\ ])$ into a *differential* Lie algebra with the differential

$$\delta_0 := [\bullet\!\!\to\!\!\bullet,\ ]. \tag{24}$$

This dg Lie algebra contains a dg subalgebra $\mathsf{fcGC}_d^{l+1\uparrow k+1}$ spanned by *connected* graphs which in turn contains a dg Lie algebra $\mathsf{GC}_d^{l+1\uparrow k+1}$ spanned by connected graphs with at least bivalent vertices. It was proven in [W1, W2] (for the case $k=0$ and $l=-1,0$ but the arguments works in greater generality) that the latter two subalgebras are quasi-isomorphic,

$$H^\bullet(\mathsf{fcGC}_d^{l+1\uparrow k+1},\delta_0) = H^\bullet(\mathsf{GC}_d^{l+1\uparrow k+1},\delta_0)$$

It was proven in [W1, W2] (in the cases $k=0$ and $l\in\{-1,0\}$ but the arguments work in greater generality) that

$$H^\bullet(\mathsf{fGC}_d^{l+1\uparrow k+1},\delta_0) = \odot^{\bullet\geq 1}\left(\mathsf{GC}_d^{l+1\uparrow k+1}[-d]\right)[d]$$

so that there is no loss of information when working with $\mathsf{GC}_d^{l+1\uparrow k+1}$ instead of the full graph complex $\mathsf{fGC}_d^{l+1\uparrow k+1}$. There is a remarkable isomorphism of Lie algebras [W1],

$$H^0(\mathsf{GC}_2^{0\uparrow 1}, \delta_0) = \mathfrak{grt}_1,$$

where $\mathfrak{grt}_1$ is the Lie algebra of the Grothendieck-Teichmüller group GRT_1 introduced by Drinfeld in the context of deformation quantization of Lie bialgebras. Nowadays, this group plays an important role in many areas of mathematics.

The multidirected graph complexes have been introduced and studied in [Z]; more precisely, Marko Živković studied fully oriented graph complexes which are dual to the complexes $\left(\mathsf{GC}_d^{k+1\uparrow k+1}, \delta_0\right)$, $k \geq 0$. We often abbreviate $\mathsf{GC}_d^{(k+1)\text{-or}} := \mathsf{GC}_d^{k+1\uparrow k+1}$ for $k \geq 0$ and $\mathsf{GC}_d^{0\text{-or}} := \mathsf{GC}_d^{0\uparrow 1}$

Note that for $l' < l$ the Lie algebra $\mathsf{GC}_d^{l+1\uparrow k+1}$ is a Lie subalgebra of $\mathsf{GC}_d^{l'+1\uparrow k+1}$.

5.2. Cohomology of (partially) oriented multi-directed graph complexes. For any $k \geq 0$ and any $-1 \leq l \leq k$ there is an obvious map of graph complexes

$$i : \mathsf{GC}_d^{l+1\uparrow k+1} \longrightarrow \mathsf{GC}_d^{l+1\uparrow k+2}$$

which sends an $(l+1)$-oriented graph with $k+1$ directions to an $(l+1)$-oriented graph with $k+2$ directions by taking a sum over all possible ways to attach a new $(k+2)$-nd direction to each $(k+1)$-directed edge.

5.2.1. Theorem [W1]. *The injection* $i : \mathsf{GC}_d^{l+1\uparrow k+1} \longrightarrow \mathsf{GC}_d^{l+1\uparrow k+2}$ *is a quasi-isomorphism of dg Lie algebras.*

This theorem was proved in [W1] by Thomas Willwacher for the case $k = 0$, $l \in \{-1, 0\}$, but the argument works in greater generality. This result implies

$$H^\bullet(\mathsf{GC}_d^{l+1\uparrow k+1}, \delta_0) = H^\bullet(\mathsf{GC}_d^{(l+1)\text{-or}}, \delta_0) \quad \forall k \geq 0, \ \ -1 \leq l \leq k.$$

Put another way, multidirections which are not *oriented* can be forgotten, they do not give us something really new.

Thomas Willwacher also proved the following

5.2.2. Theorem [W2]. $H^\bullet(\mathsf{GC}_d^{0\text{-or}}, \delta_0) = H^\bullet(\mathsf{GC}_{d+1}^{1\text{-or}}; \delta_0)$ *for any* $d \in \mathbb{Z}$.

In particular, one has an isomorphism

$$H^0(\mathsf{GC}_3^{1\text{-or}}, \delta_0) = H^0(\mathsf{GC}_2^{0\text{-or}}, \delta_0) = \mathfrak{grt}_1$$

which plays an important role in the homotopy theory of (involutive) Lie bialgebras [MW1].

This Theorem has been recently generalized to k-directed oriented graphs by Marko Živković.

5.2.3. Theorem [Z]. $H^\bullet(\mathsf{GC}_d^{(k+1)\text{-or}}, \delta_0) = H^\bullet(\mathsf{GC}_{d+1}^{(k+2)\text{-or}}, \delta_0)$ *for any* $d \in \mathbb{Z}$ *and any* $k \geq 0$.

Theorems **5.2.1** and **5.2.3** imply the equalities

$$H^\bullet(\mathsf{GC}_d^{l+1\uparrow k+1}, \delta_0) = H^\bullet(\mathsf{GC}_{d+1}^{l+2\uparrow k+2}, \delta_0) = \quad \forall d \in \mathbb{Z}, \ \ k \geq 0, \ \ -1 \leq l \leq k. \tag{25}$$

In particular we have isomorphisms of Lie algebras,

$$H^0(\mathsf{GC}_{d+2}^{d\text{-or}}, \delta_0) = H^0(\mathsf{GC}_2^{0\uparrow 1}, \delta_0) = \mathfrak{grt}_1, \tag{26}$$

for any $d \geq 0$. For $d = 2$ and $d = 3$ the algebro-geometric meanings of the associated graph complex incarnations of the Grothendieck-Teichmüller group GRT_1 are clear: the $d = 2$ case corresponds to the action of GRT_1 (through cocycles representative in $\mathsf{GC}_2^{0\uparrow 1}$) on universal Kontsevich formality maps associated with deformation quantizations of Poisson structures (given explicitly with the help of suitable configuration spaces in the *two* dimensional upper half-plane [Ko]), while the case $d = 3$ corresponds to the action of GRT_1 (through cocycles representatives in $\mathsf{GC}_3^{1\text{-or}}$) on universal formality maps associated with deformation quantizations of Lie bialgebras (see [MW3] where compactified configuration spaces in *three* dimensions have been used).

The above results tell us that the Grothendieck-Teichmüller group survives in any geometric dimension ≥ 4 but now in the multi-oriented graph complex incarnation. What can the associated to $\mathfrak{grt}_1$ degree zero cocycles in $\mathsf{GC}_{d+2}^{d\text{-or}}$ act on? It is an attempt to answer this question which motivated much of the present work. In the first approximation the answer to that question is that it acts on the multi-oriented props $\mathcal{H}olieb_{c,d}^{(c+d-1)\text{-or}}$ (more precisely, on their genus completed versions $\widehat{\mathcal{H}olieb}_{c,d}^{(c+d-1)\text{-or}}$, and it is not hard to see how. Recall the main result of [MW1] which says that there is a morphism of dg Lie algebras

$$F:\ \mathsf{GC}^{1\text{-or}}_{c+d+1}\to \mathrm{Der}(\widehat{\mathcal{H}olieb}_{c,d})$$

where $\widehat{\mathcal{H}olieb}_{c,d}$ is the genus completion of $\mathcal{H}olieb_{c,d}$ and $\mathrm{Der}(\widehat{\mathcal{H}olieb}_{c,d})$ is the Lie algebra of continuous derivations of $\widehat{\mathcal{H}olieb}_{c,d}$ (see [MW1] for some small subtleties in its definition). This map is a quasi-isomorphism (up to two rescaling classes), and it can be given by a simple formula: for any $\Gamma\in\mathsf{GC}^{1\text{-or}}_{c+d+1}$ one has

$$F(\Gamma)=\sum_{m,n\geq 1}\ \sum_{\substack{s:[n]\to V(\Gamma)\\ \hat{s}:[m]\to V(\Gamma)}} \begin{array}{c} {\scriptstyle 1\ \ 2\ \ \ \ \ \ \ \ m}\\ \diagdown\diagdown\cdots\diagup\\ \Gamma\\ \diagup\diagup\cdots\diagdown\\ {\scriptstyle 1\ \ 2\ \ \ \ \ \ \ \ n}\end{array}$$

where the second sum in taken over all ways, s and $\hat{s}$, of attaching the in- and outgoing legs to the graph Γ, and then setting to zero every graph containing a vertex with valency ≤ 2 or with no input legs or no output legs (there is an implicit rule of signs in-built into this formula). In a complete analogy one can define an action of the dg Lie algebra $\mathsf{GC}^{(k+1)\text{-or}}_{c+d+1}$ as derivations on the multi-oriented dg prop $\widehat{\mathcal{H}olieb}_{c,d}^{(k+1)\text{-or}}$, that is, a morphism of dg Lie algebras

$$F:\ \mathsf{GC}^{(k+1)\text{-or}}_{c+d+1}\to \mathrm{Der}(\widehat{\mathcal{H}olieb}_{c,d}^{(k+1)\text{-or}}).$$

It was proven by Assar Andersen in [A] that this map is a quasi-isomorphism (up to a finite number of classes corresponding to the rescaling automorphisms). This result together with equality (26) imply a highly non-trivial action of GRT_1 on the infinite family of the multi-oriented props $\widehat{\mathcal{H}olieb}_{c,d}^{(c+d-1)\text{-or}}$, $c+d\geq 4$.

References

[A] A. Andersson, *Deformation theory of multi-oriented props of homotopy Lie bialgebras*, to appear.

[D] V. G. Drinfeld, *Quantum groups*. In: Proceedings of the International Congress of Mathematicians (Berkeley, Calif., 1986).

[EK] P. Etingof and D. Kazhdan. *Quantization of Lie bialgebras, I.* Selecta Math. (N.S.) **2** (1996), 1-41.

[Ko] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), 157-216.

[LV] J.-L. Loday and B. Vallette, *Algebraic Operads*, Number 346 in Grundlehren der mathematischen Wissenschaften. Springer, Berlin, 2012.

[M] M. Markl, *Operads and props*. In: "Handbook of Algebra" vol. 5, 87140, Elsevier 2008.

[MMS] M. Markl, S. Merkulov and S. Shadrin, *Wheeled props and the master equation*, J. Pure and Appl. Algebra **213** (2009), 496-535.

[MSS] M. Markl, S. Shnider and J.D. Stasheff, Operads in Algebra, Topology and Physics, AMS, Providence, 2002.

[MaVo] M. Markl and A.A. Voronov, *PROPped up graph cohomology*, in: Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, Progr. Math., 270, Birkhäuser Boston, Inc., Boston, MA (2009) pp. 249–281.

[Me1] S.A. Merkulov, *Graph complexes with loops and wheels*. In: "Algebra, Arithmetic and Geometry - Manin Festschrift" (eds. Yu. Tschinkel and Yu. Zarhin), Progress in Mathematics, Birkhaüser (2008).

[Me2] S. Merkulov, *Formality Theorem for Quantizations of Lie Bialgebras*. Lett. Math. Physics **106**(2) (2016) 169-195.

[Me3] S. Merkulov, *Deformation quantization of homotopy algebras with branes*, to appear.

[MV] S. Merkulov and B. Vallette, *Deformation theory of representations of prop(erad)s I & II*, Journal für die reine und angewandte Mathematik (Qrelle) **634**, 51-106, & **636** (2009) 123-174.

[MW1] S. Merkulov and T. Willwacher, *Deformation theory of Lie bialgebra properads*, arXiv:1512.05252 (2015) 19pp. To appear in the memorial volume "Hitchin 70".

[MW2] S. Merkulov and T. Willwacher, *Classification of universal formality maps for quantizations of Lie bialgebras*, preprint arXiv:1605.01282 (2016), 30 pp.

[MW3] S. Merkulov and T. Willwacher, *An explicit two step quantization of Poisson structures and Lie bialgebras*, preprint arXiv:1612.00368 (2016), 57 pp.

[W1] T. Willwacher, *M. Kontsevich's graph complex and the Grothendieck-Teichmueller Lie algebra*, Invent. Math. 200 (2015) 671-760.

[W2] T. Willwacher, *Oriented graph complexes*, Comm. Math. Phys. **334** , no. 3 (2015) 1649–1666.

[V] B. Vallette, *A Koszul duality for props*, Trans. Amer. Math. Soc., **359** (2007), 4865–4943.

[Z] M. Živković, *Multi-oriented graph complexes and quasi-isomorphisms between them*, preprint arXiv:1703.09605 (2017)

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