This is a preprint version of an article published as *Contributions to General Algebra,* vol. 20, Proceedings of the Salzburg Conference 2011 (AAA81), Verlag Johannes Heyn, Klagenfurt, 2012, pp. 51–84, ISBN: 978-3-7084-0447-9.

PARTIAL ORDERS INDUCED BY QUASILINEAR CLONES

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Abstract. We find sufficient conditions for a subclone $\mathcal{C}$ of Burle's clone and for a subclone of the polynomial clone of a finite semimodule to have the property that the associated $\mathcal{C}$-minor partial order has finite principal ideals. We also prove that for these clones the $\mathcal{C}$-minor partial order is universal for the class of countable partial orders whose principal ideals are finite.

1. Introduction

Every clone $\mathcal{C}$ on a nonempty set A induces a quasi-order, called the $\mathcal{C}$-minor relation, on the set $\mathcal{O}_A$ of all operations on A as follows: an operation f is a $\mathcal{C}$-minor of another operation g if and only if f can be obtained from g by substituting operations from $\mathcal{C}$ for the variables of g. The associated equivalence relation on $\mathcal{O}_A$, called $\mathcal{C}$-equivalence, is a natural extension of Green's $\mathcal{R}$ relation [6, 13] from transformation monoids to operations of higher arity. Early applications of the idea of $\mathcal{C}$-equivalence for some very particular choices of $\mathcal{C}$ can be found in the papers by Harrison [4] and Henno [5]. More recently, the $\mathcal{C}$-minor relation for operations on a 2-element set A has received some attention in the theory of Boolean functions for various essentially unary clones $\mathcal{C}$, see, e.g., [2, 3, 14, 17, 18, 19].

This paper focuses on the partially ordered set $\mathbf{P}_\mathcal{C}$ induced by the $\mathcal{C}$-minor relation on the set of $\mathcal{C}$-equivalence classes of $\mathcal{O}_A$. We will assume that A is finite, so the poset $\mathbf{P}_\mathcal{C}$ is countable for every clone $\mathcal{C}$. In the papers [11, 12] we started a systematic investigation of the clones for which the poset $\mathbf{P}_\mathcal{C}$ is finite. The results suggest that $\mathbf{P}_\mathcal{C}$ is infinite for most clones $\mathcal{C}$. Moreover, it is known that the structure of $\mathbf{P}_\mathcal{C}$ can be complicated; for example, it is proved in [10] that for all but finitely many clones $\mathcal{C}$ on a 2-element set, $\mathbf{P}_\mathcal{C}$ has the property that every countable poset embeds into $\mathbf{P}_\mathcal{C}$. On the other hand, it was observed by Zverovich [19] and the first author [8] that for some clones $\mathcal{C}$, although the poset $\mathbf{P}_\mathcal{C}$ is infinite, it satisfies some finiteness properties like the descending chain condition.

For the clone $\mathcal{C}$ of projections, Couceiro and Pouzet [2] found the exact finiteness strength of $\mathbf{P}_\mathcal{C}$ by showing that $\mathbf{P}_\mathcal{C}$ is universal for the class of countable posets with finite principal (order) ideals; that is, $\mathbf{P}_\mathcal{C}$ has finite principal ideals, and every countable poset with finite principal ideals embeds into $\mathbf{P}_\mathcal{C}$. Our aim in this paper is to extend this result to a broad class of clones $\mathcal{C}$, which includes essentially unary clones, large subclones of Burle's clone, and large subclones of the clone

2010 *Mathematics Subject Classification.* Primary 08A40; secondary 03B50.

Key words and phrases. Clone, minor, semimodule, Burle's clone.

This material is based upon work supported by the Hungarian National Foundation for Scientific Research (OTKA) grants no. K77409 and K83219.

of polynomial operations of a semimodule over a commutative inverse semigroup. The natural context for these results is to look at the $\mathcal{C}$-minor relation on the set of all finitary functions from A to another set U, rather than just on the set of all operations on A. Therefore, our results will be stated and proved in this more general context.

A consequence of our results is that for a finite set A, the family of all clones $\mathcal{C}$ on A for which $\mathbf{P}_{\mathcal{C}}$ has finite principal ideals is closed under taking subclones if and only if $|A| \leq 2$.

2. Preliminaries

2.1. **General notation.** Throughout this paper, we denote the set of natural numbers by $\omega := \{0, 1, 2, \dots\}$ and the set of positive integers by $\omega_+ := \omega \setminus \{0\} = \{1, 2, 3, \dots\}$. For $n \in \omega_+$ we set $[n] := \{0, \dots, n-1\}$ and $\lfloor n \rceil := \{1, \dots, n\}$. Furthermore, if S is a set, we denote by $\mathcal{P}_{\mathrm{f}}(S)$ the set of finite subsets of S.

2.2. **Partially ordered sets.** A *quasi-ordered set* is a pair $(P; \leq)$ where $\leq$ is a *quasi-order,* i.e., a reflexive and transitive binary relation on P. If, in addition, the relation $\leq$ is antisymmetric, then $\leq$ is called a *partial order* and $(P; \leq)$ is called a *partially ordered set* (or a *poset* for short). A quasi-order $\leq$ induces an equivalence relation $\sim$ on P by the rule $x \sim y$ if and only if $x \leq y$ and $y \leq x$. Furthermore, a quasi-order $\leq$ induces a partial order $\preceq$ on the set $P/{\sim}$ of the equivalence classes of $\sim$, which is defined by the rule $x/{\sim} \preceq y/{\sim}$ if and only if $x \leq y$, where $x/{\sim}$ denotes the $\sim$-block of x.

If $(P; \leq)$ is a quasi-ordered set and $x \in P$, then the *principal ideal* generated by x is the set ${\downarrow}x := \{x' \in P : x' \leq x\}$. For two posets $(P; \leq)$ and $(Q; \leq)$ a mapping $h \colon P \to Q$ is called an *embedding* of $(P; \leq)$ into $(Q; \leq)$, if

$$x \leq x' \text{ in } P \quad \text{if and only if} \quad h(x) \leq h(x') \text{ in } Q.$$

Clearly, such a map h is necessarily injective. If h is also surjective, it is called an *isomorphism* of $(P; \leq)$ onto $(Q; \leq)$. We say that $(P; \leq)$ *embeds into* [*is isomorphic to*] $(Q; \leq)$ if there exists an embedding [isomorphism] $h \colon (P; \leq) \to (Q; \leq)$.

Let $\mathcal{K}$ be a class of posets. We say that a poset $(P; \leq)$ is *universal* for $\mathcal{K}$, if $(P; \leq)$ is a member of $\mathcal{K}$ and every member of $\mathcal{K}$ embeds into $(P; \leq)$. We will use the notation FPI for the class of countable posets whose principal ideals are finite.

Lemma 2.1. *A poset $(P; \leq)$ is universal for FPI if and only if $(P; \leq) \in \mathsf{FPI}$ and $(\mathcal{P}_{\mathrm{f}}(\omega); \subseteq)$ embeds into $(P; \leq)$.*

Proof. The necessity of the given condition for $(P; \leq)$ to be universal for FPI is clear, since $(\mathcal{P}_{\mathrm{f}}(\omega); \subseteq) \in \mathsf{FPI}$. To show the sufficiency, assume that $(P; \leq) \in \mathsf{FPI}$ and $(\mathcal{P}_{\mathrm{f}}(\omega); \subseteq)$ embeds into $(P; \leq)$. It is a well-known fact in the theory of ordered sets that $(\mathcal{P}_{\mathrm{f}}(\omega); \subseteq)$ is universal for FPI (see, e.g., [2] for a proof). Therefore, every member of FPI embeds into $(\mathcal{P}_{\mathrm{f}}(\omega); \subseteq)$ and hence into $(P; \leq)$, proving that $(P; \leq)$ is universal for FPI. $\square$

2.3. **Operations and clones.** For arbitrary sets A, U, and for any positive integer n let $\mathcal{F}^{(n)}(A, U)$ denote the set of all n-ary functions $A^n \to U$, and let $\mathcal{F}(A, U) := \bigcup_{n \geq 1} \mathcal{F}^{(n)}(A, U)$. In particular, $\mathcal{O}_A^{(n)} := \mathcal{F}^{(n)}(A, A)$ is the set of all n-ary operations on A, and $\mathcal{O}_A := \mathcal{F}(A, A)$ is the set of all finitary operations on A. For every positive integer t and for every set U the *i-th t-ary projection on U* is the t-ary operation $(u_1, \dots, u_t) \mapsto u_i$, which will be denoted by $\pi_i^{(t)}$ (U will be clear from the context).

A function $f \in \mathcal{F}^{(n)}(A, U)$ *depends on its i-th variable* ($1 \leq i \leq n$) if there exist n-tuples $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in A^n$ with $a_j = b_j$ for all $j \neq i$ such that $f(\mathbf{a}) \neq f(\mathbf{b})$. A variable on which f does not depend is called a *fictitious variable*. We say that f is *essentially unary* if it depends on at most one of its variables. We will use the notation $\mathrm{Im}(f)$ for the range of f.

We will consider $\mathcal{F}(A, U)$ (hence, in particular, $\mathcal{O}_A$) as a multisorted set with sorts $\mathcal{F}^{(n)}(A, U)$. Accordingly, for a subset S of $\mathcal{F}(A, U)$ we will use the notation $S^{(n)}$ for $S \cap \mathcal{F}^{(n)}(A, U)$. Furthermore, for each positive integer t we define the t-th power of S to be $S^t = \bigcup_{n \geq 1} (S^{(n)})^t$, that is, the n-th sort of the t-th power S^t of S is defined to be the t-th power of the n-th sort $S^{(n)}$ of S.

For arbitrary sets A, U and positive integer t there is a natural one-to-one correspondence between $\mathcal{F}(A, U^t)$ and $(\mathcal{F}(A, U))^t$ via the assignment

$$f \mapsto (\pi_1^{(t)} \circ f, \dots, \pi_t^{(t)} \circ f).$$

We will identify $\mathcal{F}(A, U^t)$ and $(\mathcal{F}(A, U))^t$ via this correspondence. In particular, for every set A and for arbitrary positive integers n and t, the set $(\mathcal{O}_A^{(n)})^t$ is identified with the set $\mathcal{F}^{(n)}(A, A^t)$ of all functions $A^n \to A^t$, and hence the set $(\mathcal{O}_A)^t$ is identified with $\mathcal{F}(A, A^t)$. Thus, for arbitrary $k, m, n \geq 1$ and for arbitrary $\mathbf{f} \in (\mathcal{O}_A^{(k)})^m$ and $\mathbf{g} \in (\mathcal{O}_A^{(m)})^n$, the composition $\mathbf{g} \circ \mathbf{f}$ (as functions $\mathbf{f} \colon A^k \to A^m$ and $\mathbf{g} \colon A^m \to A^n$) belongs to $(\mathcal{O}_A^{(k)})^n$. For each n, the identity function in the n-th sort is $\boldsymbol{\pi}^{(n)} := (\pi_1^{(n)}, \dots, \pi_n^{(n)})$.

A subset $\mathcal{C}$ of $\mathcal{O}_A$ is called a *clone on A* if $\mathcal{C}$ contains all projections and is closed under composition; that is, $\boldsymbol{\pi}^{(n)} \in (\mathcal{C}^{(n)})^n$ for all $n \geq 1$, and whenever $\mathbf{f} \in (\mathcal{C}^{(k)})^m$ and $\mathbf{g} \in (\mathcal{C}^{(m)})^n$, then $\mathbf{g} \circ \mathbf{f} \in (\mathcal{C}^{(k)})^n$. The intersection of any family of clones on A is a clone; therefore the set of all clones on A is a complete lattice under inclusion. Hence, for any set F of operations on A, there exists a smallest clone that contains F, which will be denoted by $\langle F \rangle$ and will be referred to as the *clone generated by F*. If F is the set of basic operations of an algebra $\mathbf{A} = (A; F)$, then the members of the clone $\langle F \rangle$ are referred to as the *term operations* $\mathbf{A}$. The *polynomial operations* of $\mathbf{A}$ are the operations in the clone generated by F and all (unary) constant operations on A. For further background information on clones, see, e.g., [7, 16].

2.4. **$\mathcal{C}$-minors.** For arbitrary sets A and U, and for arbitrary clone $\mathcal{C}$ on A, we define the *$\mathcal{C}$-minor relation* $\leq_{\mathcal{C}}$ and the *$\mathcal{C}$-equivalence relation* $\equiv_{\mathcal{C}}$ on $\mathcal{F}(A, U)$ as follows: for $f \in \mathcal{F}^{(m)}(A, U)$ and $g \in \mathcal{F}^{(n)}(A, U)$ we define $f \leq_{\mathcal{C}} g$ to mean that $f = g \circ \mathbf{h}$ for some $\mathbf{h} \in (\mathcal{C}^{(m)})^n$, and $f \equiv_{\mathcal{C}} g$ to mean that $f \leq_{\mathcal{C}} g$ and $g \leq_{\mathcal{C}} f$. In particular, via the identification of $(\mathcal{O}_A)^t$ with $\mathcal{F}(A, A^t)$, this definition yields $\mathcal{C}$-minor and $\mathcal{C}$-equivalence relations $\leq_{\mathcal{C}}$ and $\equiv_{\mathcal{C}}$ on $(\mathcal{O}_A)^t$ for every integer $t \geq 1$.

Proposition 2.2. *For arbitrary sets A and U, and for every clone $\mathcal{C}$ on A,*

(1) *$\leq_{\mathcal{C}}$ is a quasi-order on $\mathcal{F}(A, U)$; and*
(2) *$\equiv_{\mathcal{C}}$ is an equivalence relation on $\mathcal{F}(A, U)$.*

Moreover, for all $\mathbf{g}, \mathbf{g}' \in (\mathcal{O}_A)^m$ and $f \in \mathcal{F}^{(m)}(A, U)$ ($m \geq 1$),

(3) *$\mathbf{g} \leq_{\mathcal{C}} \mathbf{g}'$ implies $f \circ \mathbf{g} \leq_{\mathcal{C}} f \circ \mathbf{g}'$, and*
(4) *$\mathbf{g} \equiv_{\mathcal{C}} \mathbf{g}'$ implies $f \circ \mathbf{g} \equiv_{\mathcal{C}} f \circ \mathbf{g}'$.*

Proof. (2) and (4) follow immediately from (1) and (3), respectively, since $\equiv_{\mathcal{C}}$ is the intersection of $\leq_{\mathcal{C}}$ with its converse.

For (1) we need to show that $\leq_{\mathcal{C}}$ is reflexive and transitive. Let $f \in \mathcal{F}^{(m)}(A,U)$, $f' \in \mathcal{F}^{(m')}(A,U)$, and $f'' \in \mathcal{F}^{(m'')}(A,U)$ be arbitrary elements of $\mathcal{F}(A,U)$. Since $\boldsymbol{\pi}^{(m)} = (\pi_1^{(m)}, \dots, \pi_m^{(m)}) \in (\mathcal{C}^{(m)})^m$ is the identity function $A^m \to A^m$, the equality $f = f \circ \boldsymbol{\pi}^{(m)}$ shows that $\leq_{\mathcal{C}}$ is reflexive. If $f \leq_{\mathcal{C}} f' \leq_{\mathcal{C}} f''$, then by the definition of $\leq_{\mathcal{C}}$, there exist $\mathbf{h}' \in (\mathcal{C}^{(m)})^{m'}$, and $\mathbf{h}'' \in (\mathcal{C}^{(m')})^{m''}$ such that $f = f' \circ \mathbf{h}'$ and $f' = f'' \circ \mathbf{h}''$. Hence $f = f' \circ \mathbf{h}' = (f'' \circ \mathbf{h}'') \circ \mathbf{h}' = f'' \circ (\mathbf{h}'' \circ \mathbf{h}')$ with $\mathbf{h}'' \circ \mathbf{h}' \in (\mathcal{C}^{(m)})^{m''}$, showing that $f \leq_{\mathcal{C}} f''$. Thus $\leq_{\mathcal{C}}$ is transitive.

To prove (3), let $f \in \mathcal{F}^{(m)}(A,U)$, let $\mathbf{g}, \mathbf{g}' \in (\mathcal{O}_A)^m$, that is, $\mathbf{g} \in (\mathcal{O}_A^{(k)})^m$ and $\mathbf{g}' \in (\mathcal{O}_A^{(l)})^m$ for some $k, l \geq 1$, and let us assume that $\mathbf{g} \leq_{\mathcal{C}} \mathbf{g}'$. Thus there exists $\mathbf{h} \in (\mathcal{C}^{(k)})^l$ such that $\mathbf{g} = \mathbf{g}' \circ \mathbf{h}$. Hence $f \circ \mathbf{g} = f \circ (\mathbf{g}' \circ \mathbf{h}) = (f \circ \mathbf{g}') \circ \mathbf{h}$, which shows that $f \circ \mathbf{g} \leq_{\mathcal{C}} f \circ \mathbf{g}'$, and completes the proof. □

The following statement is a straightforward consequence of the definitions.

Proposition 2.3. *For arbitrary sets A and U, and for arbitrary clones $\mathcal{C} \subseteq \mathcal{K}$ on A, the relations $\leq_{\mathcal{C}}$, $\leq_{\mathcal{K}}$ and $\equiv_{\mathcal{C}}$, $\equiv_{\mathcal{K}}$ on $\mathcal{F}(A,U)$ satisfy $\leq_{\mathcal{C}} \subseteq \leq_{\mathcal{K}}$ and $\equiv_{\mathcal{C}} \subseteq \equiv_{\mathcal{K}}$.*

As we discussed in Subsection 2.2 above, for every clone $\mathcal{C}$ on A, the quasi-order $\leq_{\mathcal{C}}$ on $\mathcal{F}(A,U)$ induces a partial order on the quotient set $\mathcal{F}(A,U)/{\equiv_{\mathcal{C}}}$, which we will call the *$\mathcal{C}$-minor partial order*, and will denote by $\preceq_{\mathcal{C}}$. In the next proposition we prove a necessary and sufficient condition for the principal ideals of the partially ordered sets $\bigl(\mathcal{F}(A,U)/{\equiv_{\mathcal{C}}}; \preceq_{\mathcal{C}}\bigr)$ to be finite for all U.

Proposition 2.4. *The following are equivalent for an arbitrary clone $\mathcal{C}$ on a set A:*

(a) *the sets $\mathcal{C}^t/{\equiv_{\mathcal{C}}}$ are finite for all $t \geq 1$;*
(b) *the principal ideals of $\bigl(\mathcal{F}(A,U)/{\equiv_{\mathcal{C}}}; \preceq_{\mathcal{C}}\bigr)$ are finite for all sets U;*
(c) *the principal ideals of $\bigl((\mathcal{O}_A)^t/{\equiv_{\mathcal{C}}}; \preceq_{\mathcal{C}}\bigr)$ are finite for all $t \geq 1$.*

Proof. (a) ⇒ (b). Assume that condition (a) holds for $\mathcal{C}$, and let U and $f \in \mathcal{F}(A,U)$ be arbitrary. Thus $f \in \mathcal{F}^{(k)}(A,U)$ for some $k \geq 1$, and the principal ideal of $\bigl(\mathcal{F}(A,U)/{\equiv_{\mathcal{C}}}; \preceq_{\mathcal{C}}\bigr)$ generated by $f/{\equiv_{\mathcal{C}}}$ is the set

$$I := \{(f \circ \mathbf{g})/{\equiv_{\mathcal{C}}} : \mathbf{g} \in \mathcal{C}^k\}.$$

Proposition 2.2 (4) implies that the assignment $\mathbf{g}/{\equiv_{\mathcal{C}}} \mapsto (f \circ \mathbf{g})/{\equiv_{\mathcal{C}}}$ as $\mathbf{g}$ runs over all elements of $\mathcal{C}^k$ yields a well-defined map of $\mathcal{C}^k/{\equiv_{\mathcal{C}}}$ onto I. The set $\mathcal{C}^k/{\equiv_{\mathcal{C}}}$ is finite by condition (a) (for $t = k$); therefore the ideal I is also finite.

(b) ⇒ (c). This implication is clear from the identification of $(\mathcal{O}_A)^t$ with $\mathcal{F}(A, A^t)$.

(c) ⇒ (a). Now assume that condition (c) holds for $\mathcal{C}$, and let $t \geq 1$ be an arbitrary integer. Since $\boldsymbol{\pi}^{(t)} = (\pi_1^{(t)}, \dots, \pi_t^{(t)})$ is the identity function $A^t \to A^t$, the principal ideal of $\bigl((\mathcal{O}_A)^t/{\equiv_{\mathcal{C}}}; \preceq_{\mathcal{C}}\bigr)$ generated by $\boldsymbol{\pi}^{(t)}/{\equiv_{\mathcal{C}}}$ is the set

$$\{(\boldsymbol{\pi}^{(t)} \circ \mathbf{g})/{\equiv_{\mathcal{C}}} : \mathbf{g} \in \mathcal{C}^t\} = \{\mathbf{g}/{\equiv_{\mathcal{C}}} : \mathbf{g} \in \mathcal{C}^t\} = \mathcal{C}^t/{\equiv_{\mathcal{C}}}.$$

Hence condition (c) implies that $\mathcal{C}^t/{\equiv_{\mathcal{C}}}$ is finite for all $t \geq 1$, which proves (a). □

For further background and results on the $\mathcal{C}$-minor relations, see [8, 10, 11, 12].

2.5. **Polynomial clones of semimodules.** Let $\mathbf{A} = (A; +)$ be a semigroup, that is, $+$ is an associative (not necessarily commutative) operation on A. If $\mathbf{A}$ has a *neutral element*, that is, an element $0 \in A$ such that $a + 0 = a = 0 + a$ for all $a \in A$, then $\mathbf{A}$ is called a monoid. For every element $a \in A$ and positive integer n, the sum $a + \cdots + a$ with n summands is denoted na. An element $a \in A$ is called *idempotent* if $2a = a$, and $\mathbf{A}$ is said to be *idempotent* if every element $a \in A$ is idempotent. An idempotent, commutative semigroup is called a *semilattice*. An *inverse semigroup* is a semigroup $\mathbf{A}$ such that for every $a \in A$ there exists a unique element $-a \in A$ with the properties $a+(-a)+a = a$ and $(-a)+a+(-a) = -a$; $-a$ is called *the inverse of* a. It is easy to see that every group and every semilattice is an inverse semigroup. The next proposition summarizes some basic facts about inverse semigroups that we will need later on (see, e.g., [6, 13]).

Proposition 2.5. *Let* $\mathbf{A} = (A; +)$ *be an inverse semigroup.*

(1) *The set of idempotent elements of* $\mathbf{A}$ *is a (nonempty) subsemilattice of* $\mathbf{A}$.
(2) $\mathbf{A}$ *is a group if and only if* $\mathbf{A}$ *has a unique idempotent element.*
(3) *If* $\mathbf{A}$ *is finite, then there is a positive integer* m *such that* $(m + 1)a = a$ *for all* $a \in A$.

For a finite inverse semigroup $\mathbf{A}$, the least positive integer m such that $(m+1)a = a$ for all $a \in A$ is called *the exponent of* $\mathbf{A}$.

A *semiring* is an algebra $\mathbf{R} = (R; +, \cdot)$ such that $(R; +)$ is a commutative semigroup, $(R; \cdot)$ is a semigroup, and $\cdot$ distributes over $+$. It is easy to check that the set $\mathrm{End}(\mathbf{A})$ of all endomorphisms of a commutative semigroup $\mathbf{A}$ forms a semiring with respect to pointwise addition and composition. A (*left*) *semimodule over a semiring* $\mathbf{R}$ is an algebra ${}_{\mathbf{R}}\mathbf{A} = (A; +, R)$ where $\mathbf{A} = (A; +)$ is a commutative semigroup on which $\mathbf{R}$ acts by endomorphisms; that is, there is a homomorphism $\mathbf{R} \to \mathrm{End}(\mathbf{A})$, $r \mapsto \hat{r}$ of semirings such that the (unary) operation associated to each element $r \in R$ is the endomorphism $\hat{r}$ of $\mathbf{A}$. We will follow the convention of writing r instead of $\hat{r}$.

The definition of a semimodule shows that every semimodule with underlying commutative semigroup $\mathbf{A}$ is a reduct of the semimodule ${}_{\mathbf{E}}\mathbf{A}$ where $\mathbf{E} = \mathrm{End}(\mathbf{A})$ is the endomorphism semiring of $\mathbf{A}$. Therefore, when considering clones of (polynomial) operations of semimodules we may restrict to semimodules of the form ${}_{\mathbf{E}}\mathbf{A}$ with $\mathbf{E} = \mathrm{End}(\mathbf{A})$. Since $\mathbf{E}$ contains the identity endomorphism, it follows easily that every term operation of ${}_{\mathbf{E}}\mathbf{A}$ that depends on the variables $x_1, \dots, x_n$ is of the form $f_1(x_1) + \cdots + f_n(x_n)$ for some $f_1, \dots, f_n \in \mathbf{E}$. Therefore, every polynomial operation of ${}_{\mathbf{E}}\mathbf{A}$ that depends on the variables $x_1, \dots, x_n$ is of the form $f_1(x_1)+\cdots+f_n(x_n)$ or $f_1(x_1)+\cdots+f_n(x_n)+a$ for some $f_1, \dots, f_n \in \mathbf{E}$ and $a \in A$. The clone of all polynomial operations of ${}_{\mathbf{E}}\mathbf{A}$ will be denoted by $\mathrm{PClo}({}_{\mathbf{E}}\mathbf{A})$.

2.6. **Burle's clone.** Let A be a set, $|A| \geq 2$. *Burle's clone on* A, denoted $\mathcal{B}_A$, consists of all operations $f \in \mathcal{O}_A$ such that either f is essentially unary, or f has the form

$$f(x_1, \dots, x_n) = \Psi\big(\psi_1(x_1) + \cdots + \psi_n(x_n)\big) \tag{2.1}$$

where $+$ denotes addition modulo 2, $\psi_1, \dots, \psi_n$ are functions $A \to [2]$, and Ψ is a function $[2] \to A$ (see [1]).

If $|A| = 2$, then $\mathcal{B}_A$ is easily seen to be equal to the clone $\mathrm{PClo}(\mathbf{A})$ of polynomial operations of an(y) abelian group $\mathbf{A} = (A; +)$ on A. Therefore, when considering Burle's clone $\mathcal{B}_A$ we will always assume that $|A| \geq 3$.

Proposition 2.6. *Let* 0 *be a fixed element of* A. *Every operation* $f \in \mathcal{B}_A^{(n)}$ *with* $|\operatorname{Im}(f)| \leq 2$ *can be written in the form* (2.1) *such that* Ψ *is one-to-one and* $\psi_i(0) = 0$ *for all* $1 \leq i \leq n$.

Proof. If $f \in \mathcal{B}_A^{(n)}$ is not essentially unary, then by the definition of Burle's clone, $f(x_1, \dots, x_n) = \Psi'\big(\psi'_1(x_1) + \dots + \psi'_n(x_n)\big)$ for some $\psi'_1, \dots, \psi'_n \colon A \to [2]$ and $\Psi' \colon [2] \to A$. Since $|\operatorname{Im}(f)| \leq 2$, the same conclusion is true even if f is essentially unary. Now, letting $\psi_i(x_i) = \psi'_i(x_i) + \psi'_i(0)$ for all $1 \leq i \leq n$ and $\Psi(y) = \Psi'\big(y + \sum_{i=1}^n \psi'_i(0)\big)$ we get that (2.1) holds with $\psi_i(0) = 0$ for all $1 \leq i \leq n$. Hence, if the function $\Psi \colon [2] \to A$ is one-to-one, we are done. Otherwise, Ψ is a constant function, and hence so is f. In that case we can choose $\psi_1, \dots, \psi_n$ be constant with value 0, and Ψ any one-to-one function that maps 0 to $f(0)$. ☐

Finally, we define some operations in $\mathcal{B}_A$ which will play a role later on in the paper. For this, we choose and fix an element 0 of A. For arbitrary $a \in A \setminus \{0\}$, let $\oplus^a$ denote the binary operation $x \oplus^a y = \Lambda_a\big(\lambda_a(x) + \lambda_a(y)\big)$ in $\mathcal{B}_A$ where the functions $\Lambda_a \colon [2] \to A$ and $\lambda_a \colon A \to [2]$ are defined as follows: $\Lambda_a(0) = 0$, $\Lambda_a(1) = a$, $\lambda_a(a) = 1$, and $\lambda_a(b) = 0$ for all $b \in A \setminus \{a\}$. Since $\lambda_a \circ \Lambda_a$ is the identity function $[2] \to [2]$, we have that

$$(x \oplus^a y) \oplus^a z = \Lambda_a\Big(\lambda_a\big(\Lambda_a(\lambda_a(x) + \lambda_a(y))\big) + \lambda_a(z)\Big) = \Lambda_a\big(\lambda_a(x) + \lambda_a(y) + \lambda_a(z)\big).$$

Similarly, $x \oplus^a (y \oplus^a z) = \Lambda_a\big(\lambda_a(x) + \lambda_a(y) + \lambda_a(z)\big)$; therefore the operation $\oplus^a$ is associative. For every integer $n \geq 2$, we will write the composite operation $(\dots((x_1 \oplus x_2) \oplus^a x_3) \dots) \oplus^a x_n$ without parentheses as $x_1 \oplus^a x_2 \oplus^a \dots \oplus^a x_n$. An easy calculation, similar to the one above, shows that

$$x_1 \oplus^a x_2 \oplus^a \dots \oplus^a x_n = \Lambda_a\big(\lambda_a(x_1) + \lambda_a(x_2) + \dots + \lambda_a(x_n)\big) \tag{2.2}$$

for all $n \geq 2$.

3. The main result

The main result of this paper is the following theorem.

Theorem 3.1. *Let* $\mathcal{C}$ *be a clone on a finite set* A, $|A| \geq 2$, *and let* U *be a finite set such that* $|U| \geq \min(3, |A|)$. *If* $\mathcal{C}$ *satisfies one of the conditions* (A)–(C) *below, then the* $\mathcal{C}$*-minor partial order* $(\mathcal{F}(A, U)/{\equiv_\mathcal{C}}; \preceq_\mathcal{C})$ *is universal for the class* **FPI** *of countable posets whose principal ideals are finite.*

- (A) *There exists a positive integer* m *such that every operation in* $\mathcal{C}$ *depends on at most* m *variables.*
- (B) $\mathcal{C}$ *is a subclone of Burle's clone* $\mathcal{B}_A$ ($|A| \geq 3$) *such that* $\mathcal{C}$ *contains all binary operations* $\oplus^a$ ($a \in A \setminus \{0\}$) *for some fixed element* $0 \in A$.
- (C) *For a commutative inverse semigroup* $\mathbf{A} = (A; +)$ *of exponent* m *and* $\mathbf{E} = \operatorname{End}(\mathbf{A})$, $\mathcal{C}$ *is a subclone of the clone* $\mathrm{PClo}({}_\mathbf{E}\mathbf{A})$ *such that* $\mathcal{C}$ *contains the operation* $x_0 + x_1 + \dots + x_m$.

Proof. Let $\mathcal{C}$ satisfy one of conditions (A)–(C). We will prove in Theorem 4.1 that the sets $\mathcal{C}^t/{\equiv_\mathcal{C}}$ are finite for all $t \geq 1$. Thus by Proposition 2.4, the principal ideals of the $\mathcal{C}$-minor partial order $(\mathcal{F}(A,U)/{\equiv_{\mathcal{C}}};\preceq_{\mathcal{C}})$ are finite. Also, $\mathcal{F}(A,U)$ is countable, because A and U are finite; hence $(\mathcal{F}(A,U)/{\equiv_{\mathcal{C}}};\preceq_{\mathcal{C}}) \in \mathsf{FPI}$. In Theorem 5.1 we will prove that under slightly weaker hypotheses on $\mathcal{C}$ than (A)–(C), the poset $(\mathcal{P}_{\mathrm{f}}(\omega);\subseteq)$ embeds into $(\mathcal{F}(A,U)/{\equiv_{\mathcal{C}}};\preceq_{\mathcal{C}})$. Thus, by Lemma 2.1, $(\mathcal{F}(A,U)/{\equiv_{\mathcal{C}}};\preceq_{\mathcal{C}})$ is universal for FPI, as claimed. □

The special case of part (A) of Theorem 3.1 when $\mathcal{C}$ is the clone of projections (and $U = A = [2]$) was proved in [2] by Couceiro and Pouzet. Two special cases of part (C) of Theorem 3.1 are also worth stating separately, namely when $\mathbf{A}$ is an abelian group or a semilattice. Note that in the case when $\mathbf{A}$ is a semilattice, the exponent of $\mathbf{A}$ is $m = 1$, while in the case when $\mathbf{A}$ is an abelian group of exponent m, then a clone $\mathcal{C}$ contains the operation $x_0 + x_1 + \cdots + x_m$ if and only if it contains the ternary operation $x - y + z$.

Corollary 3.2. *If a clone $\mathcal{C}$ satisfies one of the conditions* $(\mathrm{C})_{\mathrm{gr}}$ *or* $(\mathrm{C})_{\mathrm{sl}}$ *below, then the $\mathcal{C}$-minor partial order $(\mathcal{F}(A,U)/{\equiv_{\mathcal{C}}};\preceq_{\mathcal{C}})$ is universal for FPI for any finite set U with $|U| \geq \min(3,|A|)$.*

- $(\mathrm{C})_{\mathrm{gr}}$ *For a finite abelian group $\mathbf{A} = (A;+)$ $(|A| \geq 2)$ and $\mathbf{E} = \mathrm{End}(\mathbf{A})$, $\mathcal{C}$ is a subclone of the clone $\mathrm{PClo}({}_{\mathbf{E}}\mathbf{A})$ such that $\mathcal{C}$ contains the operation $x-y+z$.*
- $(\mathrm{C})_{\mathrm{sl}}$ *For a finite semilattice $\mathbf{A} = (A;+)$ $(|A| \geq 2)$ and $\mathbf{E} = \mathrm{End}(\mathbf{A})$, $\mathcal{C}$ is a subclone of the clone $\mathrm{PClo}({}_{\mathbf{E}}\mathbf{A})$ such that $\mathcal{C}$ contains the operation $+$.*

We close this section by discussing examples which show that in condition (B) of Theorem 3.1 the assumption "$\mathcal{C}$ contains all binary operations $\oplus^a$ $(a \in A \setminus \{0\})$" cannot be omitted, and similarly, in condition (C) of Theorem 3.1 the assumption "$\mathcal{C}$ contains the operation $x_0+x_1+\cdots+x_m$" cannot be omitted. To this end we will exhibit subclones $\mathcal{C}$ of Burle's clone $\mathcal{B}_A$ and subclones $\mathcal{C}$ of $\mathrm{PClo}({}_{\mathbf{E}}\mathbf{A})$ for certain inverse semigroups $\mathbf{A}$ such that the set $\mathcal{C}/{\equiv_{\mathcal{C}}}$ is infinite. Since $\mathcal{C}/{\equiv_{\mathcal{C}}}$ is a principal ideal of the $\mathcal{C}$-minor partial order $(\mathcal{O}_A/{\equiv_{\mathcal{C}}};\preceq_{\mathcal{C}})$ (see the proof of Proposition 2.4, case $t = 1$ in (c) $\Rightarrow$ (a)), the fact that $\mathcal{C}/{\equiv_{\mathcal{C}}}$ is infinite implies that $(\mathcal{O}_A/{\equiv_{\mathcal{C}}};\preceq_{\mathcal{C}}) = (\mathcal{F}(A,A)/{\equiv_{\mathcal{C}}};\preceq_{\mathcal{C}})$ is not even a member of FPI, let alone universal for FPI.

These examples, along with Theorem 3.1, also show that on a finite set A with more than two elements the family of all clones $\mathcal{C}$ for which the $\mathcal{C}$-minor partial order $(\mathcal{O}_A/{\equiv_{\mathcal{C}}};\preceq_{\mathcal{C}})$ is universal for FPI is not closed under taking subclones.

Example 3.3. Let $0, a, b$ be distinct elements of A $(|A| \geq 3)$, and let $\mathcal{C}$ be the subclone of $\mathcal{B}_A$ generated by the operations f_n $(n \in \omega)$ where $f_n(x_1,\ldots,x_n) = \Lambda_b\big(\lambda_a(x_1) + \cdots + \lambda_a(x_n)\big)$ for $n \geq 1$ and $f_0 = 0$, the unary constant operation with value 0. Since $\Lambda_b \circ \lambda_a \colon A \to A$ is not constant, we get that for $n \geq 1$, f_n depends on all of its variables. However, since $\lambda_a \circ \Lambda_b \colon [2] \to [2]$ is constant 0, it follows that the identities

$$(3.1)\quad f_n\big(x_1,\ldots,x_i,f_m(x_{1+i},\ldots,x_{m+i}),x_{m+i+1},\ldots,x_{m+n-1}\big) \\ = f_{n-1}(x_1,\ldots,x_i,x_{m+i+1},\ldots,x_{m+n-1})$$

hold for all $m, n \geq 1$ and $0 \leq i \leq n-1$. This implies that every operation in $\mathcal{C}$ is either a projection or is of the form $f_n(x_{i_1},\ldots,x_{i_n})$ for some $n \geq 1$ and some variables $x_{i_1},\ldots,x_{i_n}$. Furthermore, we have that whenever $g \leq_{\mathcal{C}} f_n$ holds for some $g \in \mathcal{C}$, then g depends on at most n variables if $n \geq 1$, and g is constant 0 if $n = 0$. Thus $f_m \not\equiv_{\mathcal{C}} f_n$ if $m \neq n$ $(m, n \in \omega)$, so $\mathcal{C}/{\equiv_{\mathcal{C}}}$ is infinite.

Example 3.4. Let $\mathbf{A}$ be a finite inverse semigroup with neutral element 0 such that $\mathbf{A}$ has a nonzero nilpotent endomorphism that fixes 0.[1] Then there exists $r \in \mathrm{End}(\mathbf{A}) = \mathbf{E}$ such that $r \neq 0 = r^2$ and $r(0) = 0$. Let $\mathcal{C}$ be the subclone of $\mathrm{PClo}({}_{\mathbf{E}}\mathbf{A})$ generated by the operations f_n $(n \in \omega)$ where $f_n(x_1, \dots, x_n) = r(x_1) + \dots + r(x_n)$ for $n \geq 1$, and $f_0 = 0$, the unary constant operation with value 0. The assumptions $r \neq 0$ and $r(0) = 0$ imply that for $n \geq 1$, f_n depends on all of its variables, while the assumption $r^2 = 0$ forces that the identities (3.1) hold for all $m, n \geq 1$ and $0 \leq i \leq n-1$. As in Example 3.3, we get that $\mathcal{C}/{\equiv_{\mathcal{C}}}$ is infinite.

4. Finite principal ideals

Our goal in this section is to establish that for the clones $\mathcal{C}$ in Theorem 3.1, the principal ideals of $(\mathcal{F}(A, U)/{\equiv_{\mathcal{C}}}; \preceq_{\mathcal{C}})$ are finite for arbitrary set U. By Proposition 2.4, this will follow if we prove that the sets $\mathcal{C}^t/{\equiv_{\mathcal{C}}}$ are finite for all $t \geq 1$. Thus, our task is reduced to proving the following theorem.

Theorem 4.1. *If a clone $\mathcal{C}$ on a finite set A satisfies one of conditions* (A)–(C) *from Theorem 3.1, then the sets $\mathcal{C}^t/{\equiv_{\mathcal{C}}}$ are finite for all $t \geq 1$.*

The next lemma proves the theorem for the case when $\mathcal{C}$ satisfies condition (A).

Lemma 4.2. *If a clone $\mathcal{C}$ on a finite set A satisfies condition* (A) *from Theorem 3.1, then the sets $\mathcal{C}^t/{\equiv_{\mathcal{C}}}$ are finite for all $t \geq 1$.*

Proof. Let $t \geq 1$. Since every operation in $\mathcal{C}$ depends on at most m variables, it follows that every function in $\mathcal{C}^t$ $\bigl(\subseteq \mathcal{F}(A, A^t)\bigr)$ depends on at most mt variables. Therefore, for every $\mathbf{f} \in \mathcal{C}^t$ there exists $\mathbf{f}' \in (\mathcal{C}^{(mt)})^t$ $\bigl(\subseteq \mathcal{F}^{(mt)}(A, A^t)\bigr)$ such that $\mathbf{f}$ and $\mathbf{f}'$ can be obtained from one another by adding or removing fictitious variables. Hence $\mathbf{f} \equiv_{\mathcal{C}} \mathbf{f}'$. This implies that $|\mathcal{C}^t/{\equiv_{\mathcal{C}}}| \leq |(\mathcal{C}^{(mt)})^t/{\equiv_{\mathcal{C}}}| \leq |\mathcal{F}^{(mt)}(A, A^t)/{\equiv_{\mathcal{C}}}|$. Since A is finite, the set $\mathcal{F}^{(mt)}(A, A^t)$ of all functions $A^{mt} \to A^t$ is finite. This proves that $\mathcal{C}^t/{\equiv_{\mathcal{C}}}$ is finite. $\square$

To get the same conclusion for the remaining clones in Theorem 3.1, we will start by setting up a framework in which clones of polynomial operations of semimodules and subclones of Burle's clone can be handled simultaneously.

Definition 4.3. Let A and U be arbitrary sets, and let $f = f(x_1, \dots, x_n)$ be a function in $\mathcal{F}^{(n)}(A, U)$ with essential variables x_j $(j \in J)$. Furthermore, let $\mathbf{B} = (B; +)$ be a commutative monoid such that $B \subseteq U$. We will say that f is *quasilinear with respect to* $\mathbf{B}$, or briefly, $\mathbf{B}$-*quasilinear*, if there exist $b \in B$ and functions $u_j \in \mathcal{F}^{(1)}(A, B)$ $(j \in J)$ such that

$$f(x_1, \dots, x_n) = \sum_{j \in J} u_j(x_j) \qquad \text{for all } x_1, \dots, x_n \in A \tag{4.1}$$

or

$$f(x_1, \dots, x_n) = b + \sum_{j \in J} u_j(x_j) \qquad \text{for all } x_1, \dots, x_n \in A, \tag{4.2}$$

with addition on the right-hand sides of (4.1) and (4.2) performed in $\mathbf{B}$.

We will say that a function in $\mathcal{F}(A, U)$ is *quasilinear* if it is quasilinear with respect to some $\mathbf{B}$, and a subset S of $\mathcal{F}(A, U)$ is quasilinear if every member of

[1] It is not hard to see that such an endomorphism exists if $\mathbf{A}$ is a finite semilattice with neutral element such that $|A| \geq 3$, or if $\mathbf{A}$ is a finite abelian group such that $|A|$ is not square free.

S is. In particular, a clone on A is quasilinear if all operations in the clone are quasilinear.

Let A, U, f, J, and $\mathbf{B}$ be as in Definition 4.3, and assume that f is $\mathbf{B}$-quasilinear. If $\mathbf{B}$ has a neutral element $0_{\mathbf{B}}$, then (4.1) is the special case $b = 0_{\mathbf{B}}$ of (4.2). Moreover, for such a $\mathbf{B}$, the constant map with domain A and range $\{0_{\mathbf{B}}\}$, which will also be denoted by $0_{\mathbf{B}}$, is a member of $\mathcal{F}^{(1)}(A, B)$. Therefore, choosing u_i to be $0_{\mathbf{B}}$ whenever x_i is a fictitious variable of f, we see that there exist $b \in B$ and $u_1, \ldots, u_n \in \mathcal{F}^{(1)}(A, B)$ such that

$$f(x_1, \ldots, x_n) = b + u_1(x_1) + \cdots + u_n(x_n) \qquad \text{for all } x_1, \ldots, x_n \in A, \tag{4.3}$$

where, as before, addition is performed in $\mathbf{B}$.

Next we will introduce notation that will allow us to write $\mathbf{B}$-quasilinear functions f in the form (4.3) even if $\mathbf{B}$ has no neutral element. Let $\circledast$ be an element not in U and let $U^0 := U \cup \{\circledast\}$. As is usual in semigroup theory, if $\mathbf{B}$ has a neutral element, $0_{\mathbf{B}}$, let $\mathbf{B}^0$ denote $\mathbf{B}$ itself. Otherwise, let $\mathbf{B}^0 = (B \cup \{0_{\mathbf{B}}\}; +)$ be the extension of $\mathbf{B}$ by the element $0_{\mathbf{B}} := \circledast$ which acts as a neutral element; that is, $\mathbf{B}$ is a subsemigroup of $\mathbf{B}^0$ and $x + 0_{\mathbf{B}} = x = 0_{\mathbf{B}} + x$ for all $x \in B \cup \{0_{\mathbf{B}}\}$. As before, let $0_{\mathbf{B}}$ denote the constant map with domain A and range $\{0_{\mathbf{B}}\}$, and let $\mathcal{F}^{(1)}(A, B)^0 = \mathcal{F}^{(1)}(A, B) \cup \{0_{\mathbf{B}}\}$ (a set of functions $A \to B \cup \{0_{\mathbf{B}}\}$). Now, the same argument as before shows that every $\mathbf{B}$-quasilinear function $f \in \mathcal{F}(A, U)$ can be written in the form (4.3) for some element $b \in \mathbf{B}^0$ and some functions $u_1, \ldots, u_n \in \mathcal{F}^{(1)}(A, B)^0$ so that addition on the right-hand side is performed in $\mathbf{B}^0$. The expression on the right-hand side of the equality in (4.3) will be referred to as a $\mathbf{B}$-*representation* of f.

We will use the following conventions and notation for sets of quasilinear functions.

Conventions and Notation 4.4. Let A and U be arbitrary sets. Associated to any quasilinear set $S \subseteq \mathcal{F}(A, U)$ of functions, we will fix a set of data witnessing the quasilinearity of S; namely

- a family $\mathbb{B}(S)$ of commutative monoids,
- for each $\mathbf{B} \in \mathbb{B}(S)$,
  - a set $S_{\mathbf{B}}$ of $\mathbf{B}$-quasilinear functions in S,
  - a subset $E_{\mathbf{B}}$ of $\mathcal{F}^{(1)}(A, B)$, and
- along with each $f \in S_{\mathbf{B}}$ a set $\operatorname{repr}_{\mathbf{B}}(f) \neq \emptyset$ of $\mathbf{B}$-representations of f

such that

- $S = \bigcup\{S_{\mathbf{B}} : \mathbf{B} \in \mathbb{B}(S)\}$, and
- for each $f \in S_{\mathbf{B}}$, every $\mathbf{B}$-representation of f in $\operatorname{repr}_{\mathbf{B}}(f)$ has the form $b + u_1(x_1) + \cdots + u_n(x_n)$ as in (4.3) with $b \in \mathbf{B}^0$ and $u_1, \ldots, u_n \in E^0_{\mathbf{B}} := E_{\mathbf{B}} \cup \{0_{\mathbf{B}}\}$.

The set $E^0_{\mathbf{B}} \setminus \{0_{\mathbf{B}}\} = E_{\mathbf{B}} \setminus \{0_{\mathbf{B}}\}$ will be denoted by $E^-_{\mathbf{B}}$.

Example 4.5. Any set S of essentially unary operations on A is $\mathbf{A}$-quasilinear with respect to any fixed commutative monoid $\mathbf{A} = (A; \boxplus)$. To witness the quasilinearity of S we will choose $\mathbb{B}(S) = \{\mathbf{A}\}$, $S_{\mathbf{A}} = S$, and $E_{\mathbf{A}} = \mathcal{O}^{(1)}_A$; moreover, for every operation $f \in S$ we let $\operatorname{repr}_{\mathbf{A}}(f)$ consist of all $\mathbf{A}$-representations of f as in (4.3) with $b = 0_{\mathbf{A}}$ and all, but at most one, $u_i = 0_{\mathbf{A}}$.

Example 4.6. It follows from our discussion in subsection 2.5 that if $\mathbf{A} = (A; +)$ is a commutative semigroup and $\mathbf{E} = \mathrm{End}(\mathbf{A})$, then any set $S \subseteq \mathrm{PClo}({}_{\mathbf{E}}\mathbf{A})$ of polynomial operations of the semimodule ${}_{\mathbf{E}}\mathbf{A}$ is $\mathbf{A}$-quasilinear. To witness the quasilinearity of S we will choose $\mathbb{B}(S) = \{\mathbf{A}\}$, $S_{\mathbf{A}} = S$, and $E_{\mathbf{A}} = \mathbf{E}$; moreover, for every operation $f \in S$ we let $\mathrm{repr}_{\mathbf{A}}(f)$ consist of all $\mathbf{A}$-representations of f as in (4.3) with $u_i \in E^0_{\mathbf{A}}$ for all i. Note that every element of $E^0_{\mathbf{A}}$ is a homomorphism $\mathbf{A} \to \mathbf{A}^0$.

Example 4.7. Let A be a set with $|A| \geq 3$, and let 0 be a fixed element of A. Any subset S of Burle's clone $\mathcal{B}_A$ is quasilinear. In fact, if the range of $f \in S$ has size $|\mathrm{Im}(f)| > 2$, then f is essentially unary, so by Example 4.5, f is $\mathbf{A}$-quasilinear for any (fixed) commutative semigroup $\mathbf{A} = (A; \boxplus)$. ($\mathbf{A}$ can be chosen so that 0 is the neutral element of $\mathbf{A}$.) If the range of $f \in S$ has size $|\mathrm{Im}(f)| \leq 2$, then by Proposition 2.6 f can be written in the form (2.1) such that Ψ is one-to-one and $\psi_i(0) = 0$ for all $1 \leq i \leq n$. Hence there is a unique group operation $+^\Psi$ on the 2-element set $\Psi([2]) \subseteq A$ such that Ψ is an isomorphism $([2]; +) \to \big(\Psi([2]); +^\Psi\big)$. Thus $\Psi(x) +^\Psi \Psi(y) = \Psi(x + y)$ holds for all $x, y \in [2]$. Consequently,

$$\begin{aligned} f(x_1, \dots, x_n) &= \Psi\big(\psi_1(x_1) + \cdots + \psi_n(x_n)\big) \\ &= \Psi\big(\psi_1(x_n)\big) +^\Psi \cdots +^\Psi \Psi\big(\psi_n(x_n)\big), \end{aligned} \tag{4.4}$$

showing that f is $\big(\Psi([2]); +^\Psi\big)$-quasilinear.

Thus, to witness the quasilinearity of $S \subseteq \mathcal{B}_A$, we will fix a commutative semigroup $\mathbf{A} = (A; \boxplus)$ with neutral element 0, and choose

$$\mathbb{B}(S) = \{\mathbf{A}\} \cup \{\big(\Psi([2]); +^\Psi\big) : \Psi \text{ is a one-to-one function } [2] \to A\}.$$

Moreover, for any $\mathbf{B} \in \mathbb{B}(S)$ we let $E_{\mathbf{B}} = E^0_{\mathbf{B}} = \mathcal{F}^{(1)}(A, B)$. If $\mathbf{B} = \mathbf{A}$, then we choose $S_{\mathbf{A}}$ to be the set of all operations $f \in S$ with range of size $|\mathrm{Im}(f)| > 2$, and for each such f we choose $\mathrm{repr}_{\mathbf{A}}(f)$ as in Example 4.5. If $\mathbf{B} = \big(\Psi([2]); +^\Psi\big)$ for a one-to-one $\Psi\colon [2] \to A$, then we let $S_{\mathbf{B}}$ be the set of all operations f of the form (4.4) with $\psi_i(0) = 0$ for all $1 \leq i \leq n$, and for each such f, the possible right-hand sides in (4.4) will form the set $\mathrm{repr}_{\mathbf{B}}(f)$. (It is not hard to see that $\mathrm{repr}_{\mathbf{B}}(f)$ has only one element unless f is a constant operation.)

Lemma 4.8. *Let A be an arbitrary set. If $f_1, \dots, f_t$ are n-ary operations on A such that f_j is $\mathbf{B}_j$-quasilinear for each $1 \leq j \leq t$, then the function $\mathbf{f} = (f_1, \dots, f_t) \in \mathcal{F}^{(n)}\big(A, (A^0)^t\big)$ is $\prod_{j=1}^t \mathbf{B}^0_j$-quasilinear.*

Proof. Indeed, if $b_j + u_{1j}(x_1) + \cdots + u_{nj}(x_n) \in \mathrm{repr}_{\mathbf{B}_j}(f_j)$ with $b_j \in \mathbf{B}^0_j$ and $u_{1j}, \dots, u_{nj} \in E^0_{\mathbf{B}_j}$ for every j $(1 \leq j \leq t)$, then for the tuples $\mathbf{b} = (b_1, \dots, b_t) \in \prod_{j=1}^t \mathbf{B}^0_j$ and $\mathbf{u}_i = (u_{i1}, \dots, u_{it}) \in \prod_{j=1}^t E^0_{\mathbf{B}_j}$ we have that the expression $\mathbf{b} + \mathbf{u}_1(x_1) + \cdots + \mathbf{u}_n(x_n)$ is a $\prod_{j=1}^t \mathbf{B}^0_j$-representation of $\mathbf{f} = (f_1, \dots, f_t) \in \mathcal{F}^{(n)}\big(A, (A^0)^t\big)$. □

Conventions 4.9. Let $S \subseteq \mathcal{O}_A$ be a set of quasilinear operations. For any integer $t \geq 1$, the quasilinearity of the set $S^t \subseteq \mathcal{F}\big(A, (A^0)^t\big)$ will be witnessed by the data suggested by the proof of Lemma 4.8; namely:

$$\mathbb{B}(S^t) = \Big\{\prod_{j=1}^t \mathbf{B}^0_j : \mathbf{B}_j \in \mathbb{B}(S)\Big\},$$

and for each $\mathbf{B} = \prod_{j=1}^{t} \mathbf{B}_j^0 \in \mathbb{B}(S^t)$, we choose $(S^t)_{\mathbf{B}} = \prod_{j=1}^{t} S_{\mathbf{B}_j}$ $(\subseteq S^t)$, $E_{\mathbf{B}} = E_{\mathbf{B}}^0 = \prod_{j=1}^{t} E_{\mathbf{B}_j}^0$, and for every $\mathbf{f} = (f_1, \dots, f_t) \in (S^t)_{\mathbf{B}}$, we let $\mathrm{repr}_{\mathbf{B}}(\mathbf{f})$ consist of all $\mathbf{B}$-representations of $\mathbf{f}$ whose projections onto each coordinate j $(1 \le j \le t)$ belong to $\mathrm{repr}_{\mathbf{B}_j}(f_j)$.

The next lemma provides a sufficient condition for $\mathcal{S}/{\equiv_{\mathcal{C}}}$ to be finite for a set of quasilinear functions $\mathcal{S} \subseteq \mathcal{F}(A, U)$. The statement will primarily be used for the special case when $\mathcal{C}$ is a clone of quasilinear operations and $\mathcal{S} = \mathcal{C}^t$ for some integer $t \ge 1$. However, the statement and the proof are more transparent in the general setting.

Lemma 4.10. *Let $\mathcal{C}$ be a clone on a finite set A, and let U be another finite set. If $\mathcal{S} \subseteq \mathcal{F}(A, U)$ is a set of quasilinear functions, then $\mathcal{S}/{\equiv_{\mathcal{C}}}$ is finite, provided condition $(*)_{p,r}$ below holds for some positive integers p and r:*

$(*)_{p,r}$ *whenever $\mathbf{f}$ is an $(n+r)$-ary function in $\mathcal{S}_{\mathbf{B}}$, for some $\mathbf{B} \in \mathbb{B}(\mathcal{S})$ and $n \ge 0$, such that $\mathbf{f}$ has a $\mathbf{B}$-representation*

$$\mathbf{f}(x_1, \dots, x_n, y_1, \dots, y_r) = \mathbf{b} + \sum_{i=1}^{n} \mathbf{u}_i(x_i) + \sum_{j=1}^{r} \mathbf{v}(y_j), \tag{4.5}$$

with the right-hand side in $\mathrm{repr}_{\mathbf{B}}(\mathbf{f})$ *such that $\mathbf{v} \in E_{\mathbf{B}}^-$, then the $(n + r + p)$-ary function*

$$\mathbf{g}(x_1, \dots, x_n, y_1, \dots, y_{r+p}) = \mathbf{b} + \sum_{i=1}^{n} \mathbf{u}_i(x_i) + \sum_{j=1}^{r+p} \mathbf{v}(y_j) \tag{4.6}$$

is in $\mathcal{S}_{\mathbf{B}}$, the right-hand side of (4.6) *is a $\mathbf{B}$-representation of $\mathbf{g}$ in* $\mathrm{repr}_{\mathbf{B}}(\mathbf{g})$*, and $\mathbf{g} \equiv_{\mathcal{C}} \mathbf{f}$.*

Proof. Assume that there exist integers $p, r \ge 1$ such that condition $(*)_{p,r}$ holds. Our goal is to show that $\mathcal{S}/{\equiv_{\mathcal{C}}}$ is finite. Since U is finite, there are only finitely many commutative monoids $\mathbf{B} = (B; +)$ with $B \subseteq U$. Hence $\mathbb{B}(\mathcal{S})$ is finite. Therefore, since $\mathcal{S} = \bigcup\{\mathcal{S}_{\mathbf{B}} : \mathbf{B} \in \mathbb{B}(\mathcal{S})\}$, it will follow that $\mathcal{S}/{\equiv_{\mathcal{C}}}$ is finite if we show that $\mathcal{S}_{\mathbf{B}}/{\equiv_{\mathcal{C}}}$ is finite for each $\mathbf{B} \in \mathbb{B}(\mathcal{S})$. So, let $\mathbf{B} \in \mathbb{B}(\mathcal{S})$ be arbitrary, and let $p, r \ge 1$ be integers such that $(*)_{p,r}$ holds. To argue that $\mathcal{S}_{\mathbf{B}}/{\equiv_{\mathcal{C}}}$ is finite we will need some notation.

Let $\flat$ be an element not in $E_{\mathbf{B}}^-$. If $\mathbf{h} \in \mathcal{S}_{\mathbf{B}}$ and $\mathbf{h}^* \in \mathrm{repr}_{\mathbf{B}}(\mathbf{h})$, say $\mathbf{h}^*$ is the expression $\mathbf{b} + \sum_{i=1}^{n} \mathbf{w}_i(x_i)$, then we define a function $\mathsf{m}_{\mathbf{h}^*} \colon \{\flat\} \cup E_{\mathbf{B}}^- \to \mathbf{B}^0 \cup \omega$ called *the multiplicity function of* $\mathbf{h}^*$ as follows: $\mathsf{m}_{\mathbf{h}^*}(\flat) = \mathbf{b}$, and for every $\mathbf{w} \in E_{\mathbf{B}}^-$, $\mathsf{m}_{\mathbf{h}^*}(\mathbf{w})$ is the number of summands $\mathbf{w}_i(x_i)$ in $\mathbf{h}^*$ such that $\mathbf{w}_i = \mathbf{w}$.

Let M denote the set of all functions $\mathsf{m} \colon \{\flat\} \cup E_{\mathbf{B}}^- \to \mathbf{B}^0 \cup \omega$ satisfying $\mathsf{m}(\flat) \in \mathbf{B}^0$ and $\mathsf{m}(\mathbf{w}) \in \omega$ for all $\mathbf{w} \in E_{\mathbf{B}}^-$. We define the *distance* of two functions $\mathsf{m}, \mathsf{n} \in \mathsf{M}$ by

$$d(\mathsf{m}, \mathsf{n}) = \sum_{\mathbf{w} \in E_{\mathbf{B}}^-} |\mathsf{m}(\mathbf{w}) - \mathsf{n}(\mathbf{w})|.$$

Next we define equivalence relations $\rho_\flat$ and $\rho_{\mathbf{w}}$ ($\mathbf{w} \in E^-_{\mathbf{B}}$) on M as follows: for $\mathsf{m}, \mathsf{n} \in \mathsf{M}$,

$$\begin{aligned} \mathsf{m} \, \rho_\flat \, \mathsf{n} \quad &\iff \quad \mathsf{m}(\flat) = \mathsf{n}(\flat), \text{ and} \\ \mathsf{m} \, \rho_{\mathbf{w}} \, \mathsf{n} \quad &\iff \quad \text{either } \mathsf{m}(\mathbf{w}) = \mathsf{n}(\mathbf{w}) < r, \\ &\qquad\quad \text{or } \mathsf{m}(\mathbf{w}) \equiv \mathsf{n}(\mathbf{w}) \pmod{p} \text{ and } \mathsf{m}(\mathbf{w}), \mathsf{n}(\mathbf{w}) \geq r. \end{aligned}$$

Let $\doteq$ denote the intersection of $\rho_\flat$ and all $\rho_{\mathbf{w}}$ ($\mathbf{w} \in E^-_{\mathbf{B}}$). Clearly, each $\rho_{\mathbf{w}}$ has at most $r+p$ equivalence classes, and $\rho_\flat$ has at most $|B|+1$ equivalence classes. Since A, U are finite, $B \subseteq U$, and hence $E^-_{\mathbf{B}} \subseteq \mathcal{F}(A, B)$ is also finite, it follows that $\doteq$ has only finitely many equivalence classes.

Therefore, to prove that $\mathcal{S}_{\mathbf{B}}/{\equiv_{\mathcal{C}}}$ is finite, it will be sufficient to show that if $\mathbf{f}, \mathbf{h} \in \mathcal{S}_{\mathbf{B}}$ have $\mathbf{B}$-representations $\mathbf{f}^* \in \operatorname{repr}_{\mathbf{B}}(\mathbf{f})$ and $\mathbf{h}^* \in \operatorname{repr}_{\mathbf{B}}(\mathbf{h})$ such that $\mathsf{m}_{\mathbf{f}^*} \doteq \mathsf{m}_{\mathbf{h}^*}$, then $\mathbf{f} \equiv_{\mathcal{C}} \mathbf{h}$. Suppose this implication is false, that is, there exist $\mathbf{f}, \mathbf{h} \in \mathcal{S}_{\mathbf{B}}$ such that for some $\mathbf{B}$-representations $\mathbf{f}^* \in \operatorname{repr}_{\mathbf{B}}(\mathbf{f})$ and $\mathbf{h}^* \in \operatorname{repr}_{\mathbf{B}}(\mathbf{h})$ we have $\mathsf{m}_{\mathbf{f}^*} \doteq \mathsf{m}_{\mathbf{h}^*}$, but $\mathbf{f} \not\equiv_{\mathcal{C}} \mathbf{h}$. Choose and fix $\mathbf{f}$, $\mathbf{h}$ and $\mathbf{f}^*$, $\mathbf{h}^*$ with these properties in such a way that the distance $d(\mathsf{m}_{\mathbf{f}^*}, \mathsf{m}_{\mathbf{h}^*})$ is as small as possible. First we want to argue that $d(\mathsf{m}_{\mathbf{f}^*}, \mathsf{m}_{\mathbf{h}^*}) > 0$. Assume, for a contradiction, that $d(\mathsf{m}_{\mathbf{f}^*}, \mathsf{m}_{\mathbf{h}^*}) = 0$. Then $\mathsf{m}_{\mathbf{f}^*} = \mathsf{m}_{\mathbf{h}^*}$, hence $\mathbf{f}^*$ and $\mathbf{h}^*$ may differ only by renaming variables and adding or removing constant summands $0_{\mathbf{B}}$. It follows that $\mathbf{f}$ and $\mathbf{h}$ may differ only by renaming variables and adding or removing fictitious variables. But then $\mathbf{f} \equiv_{\mathcal{C}} \mathbf{h}$, which contradicts the choice of $\mathbf{f}$ and $\mathbf{h}$. Thus $d(\mathsf{m}_{\mathbf{f}^*}, \mathsf{m}_{\mathbf{h}^*}) > 0$, and hence $\mathsf{m}_{\mathbf{f}^*} \neq \mathsf{m}_{\mathbf{h}^*}$.

Let $\mathbf{v} \in E^-_{\mathbf{B}}$ be such that $\mathsf{m}_{\mathbf{f}^*}(\mathbf{v}) \neq \mathsf{m}_{\mathbf{h}^*}(\mathbf{v})$. The assumption $\mathsf{m}_{\mathbf{f}^*} \doteq \mathsf{m}_{\mathbf{h}^*}$ implies that $\mathsf{m}_{\mathbf{f}^*} \; \rho_{\mathbf{v}} \; \mathsf{m}_{\mathbf{h}^*}$, and the choice of $\mathbf{v}$ excludes the possibility $\mathsf{m}_{\mathbf{f}^*}(\mathbf{v}) = \mathsf{m}_{\mathbf{h}^*}(\mathbf{v}) < r$. Therefore, $\mathsf{m}_{\mathbf{f}^*}(\mathbf{v}) \equiv \mathsf{m}_{\mathbf{h}^*}(\mathbf{v}) \pmod{p}$ and $\mathsf{m}_{\mathbf{f}^*}(\mathbf{v}), \mathsf{m}_{\mathbf{h}^*}(\mathbf{v}) \geq r$.

By switching the roles of $\mathbf{f}$ and $\mathbf{h}$ if necessary we may assume without loss of generality that $\mathsf{m}_{\mathbf{f}^*}(\mathbf{v}) < \mathsf{m}_{\mathbf{h}^*}(\mathbf{v})$, say $\mathsf{m}_{\mathbf{h}^*}(\mathbf{v}) = \mathsf{m}_{\mathbf{f}^*}(\mathbf{v}) + kp$ for some positive integer k. Since $\mathsf{m}_{\mathbf{f}^*}(\mathbf{v}) \geq r$, we get that $\mathbf{f}^*$ has the same form as the expression on the right-hand side of (4.5). Now let $\mathbf{g}^*$ be the expression on the right-hand side of (4.6), and let $\mathbf{g}$ be the function with $\mathbf{B}$-representation $\mathbf{g}^*$. Condition $(*)_{p,r}$ tells us then that $\mathbf{g} \in S_{\mathbf{B}}$, $\mathbf{g}^* \in \operatorname{repr}_{\mathbf{B}}(\mathbf{g})$, and $\mathbf{f} \equiv_{\mathcal{C}} \mathbf{g}$. By the choice of $\mathbf{g}^*$ we have that $\mathsf{m}_{\mathbf{g}^*}(\flat) = \mathsf{m}_{\mathbf{f}^*}(\flat)$, $\mathsf{m}_{\mathbf{g}^*}(\mathbf{v}) = \mathsf{m}_{\mathbf{f}^*}(\mathbf{v}) + p$, and $\mathsf{m}_{\mathbf{g}^*}(\mathbf{w}) = \mathsf{m}_{\mathbf{f}^*}(\mathbf{w})$ for all $\mathbf{w} \in E^-_{\mathbf{B}} \setminus \{\mathbf{v}\}$.

Thus $\mathsf{m}_{\mathbf{g}^*} \doteq \mathsf{m}_{\mathbf{f}^*}$. Hence the functions $\mathbf{g}, \mathbf{h} \in S_{\mathbf{B}}$ with $\mathbf{B}$-representations $\mathbf{g}^* \in \operatorname{repr}_{\mathbf{B}}(\mathbf{g})$ and $\mathbf{h}^* \in \operatorname{repr}_{\mathbf{B}}(\mathbf{h})$ satisfy $\mathsf{m}_{\mathbf{g}^*} \doteq \mathsf{m}_{\mathbf{h}^*}$. Moreover,

$$\begin{aligned} d(\mathsf{m}_{\mathbf{g}^*}, \mathsf{m}_{\mathbf{h}^*}) &= d(\mathsf{m}_{\mathbf{f}^*}, \mathsf{m}_{\mathbf{h}^*}) - |\mathsf{m}_{\mathbf{f}^*}(\mathbf{v}) - \mathsf{m}_{\mathbf{h}^*}(\mathbf{v})| + |\mathsf{m}_{\mathbf{g}^*}(\mathbf{v}) - \mathsf{m}_{\mathbf{h}^*}(\mathbf{v})| \\ &= d(\mathsf{m}_{\mathbf{f}^*}, \mathsf{m}_{\mathbf{h}^*}) - kp + (k-1)p = d(\mathsf{m}_{\mathbf{f}^*}, \mathsf{m}_{\mathbf{h}^*}) - p < d(\mathsf{m}_{\mathbf{f}^*}, \mathsf{m}_{\mathbf{h}^*}). \end{aligned}$$

Since $\mathbf{f}$, $\mathbf{h}$ and $\mathbf{f}^*$, $\mathbf{h}^*$ were chosen with minimum distance $d(\mathsf{m}_{\mathbf{f}^*}, \mathsf{m}_{\mathbf{h}^*})$ such that $\mathsf{m}_{\mathbf{f}^*} \doteq \mathsf{m}_{\mathbf{h}^*}$ and $\mathbf{f} \not\equiv_{\mathcal{C}} \mathbf{h}$, we get that $\mathbf{g} \equiv_{\mathcal{C}} \mathbf{h}$. In view of $\mathbf{f} \equiv_{\mathcal{C}} \mathbf{g}$ this forces that $\mathbf{f} \equiv_{\mathcal{C}} \mathbf{h}$, which contradicts our assumption on $\mathbf{f}$ and $\mathbf{h}$. This completes the proof of the lemma. □

Now we apply Lemma 4.10 to clones $\mathcal{C}$ that satisfy condition (B) or (C) from Theorem 3.1.

Lemma 4.11. *If a clone $\mathcal{C}$ on a finite set A satisfies condition* (C) *from Theorem 3.1, then the sets $\mathcal{C}^t/{\equiv_{\mathcal{C}}}$ are finite for all $t \geq 1$.*

Proof. As in assumption (C), let $\mathcal{C}$ be a subclone of $\mathrm{PClo}({}_{\mathbf{E}}\mathbf{A})$ containing the operation $x_0 + x_1 + \cdots + x_m$, where $\mathbf{A} = (A; +)$ is a finite commutative inverse semigroup of exponent m and $\mathbf{E} = \mathrm{End}(\mathbf{A})$. Let $t \geq 1$ and let $\mathcal{S} = \mathcal{C}^t$. For $\mathbb{B}(\mathcal{C})$, $\mathcal{C}_{\mathbf{A}}$, $E_{\mathbf{A}}$, and $\mathrm{repr}_{\mathbf{A}}(f)$ ($f \in \mathcal{C}$) we will use the choices agreed upon in Example 4.6. Therefore, Convention 4.9 determines the corresponding data for $\mathcal{S} = \mathcal{C}^t$ (in fact, for any subset $\mathcal{S}$ of $\big(\mathrm{PClo}({}_{\mathbf{E}}\mathbf{A})\big)^t$). In particular, $\mathbb{B}(\mathcal{S}) = \{(\mathbf{A}^0)^t\}$, and for $\mathbf{B} = (\mathbf{A}^0)^t$ we have $E_{\mathbf{B}} = E^0_{\mathbf{B}} = (\mathbf{E}^0)^t$. Furthermore, for every $\mathbf{f} \in \mathcal{S}$, the set $\mathrm{repr}_{\mathbf{B}}(\mathbf{f})$ consists of all $\mathbf{B}$-representations $\mathbf{b} + \sum_i \mathbf{w}_i(x_i)$ of $\mathbf{f}$ with $\mathbf{w}_i \in (\mathbf{E}^0)^t$ for all i. Note that since the elements of $\mathbf{E}^0$ are homomorphisms $\mathbf{A} \to \mathbf{A}^0$, the elements of $(\mathbf{E}^0)^t$ are homomorphisms $\mathbf{A} \to (\mathbf{A}^0)^t$.

To show that the set $\mathcal{S}/{\equiv_{\mathcal{C}}}$ is finite we will apply Lemma 4.10. Our goal is to prove that $(*)_{p,r}$ holds for $\mathcal{S}$ with $p = m$ and $r = 1$. So let $\mathbf{B} = (\mathbf{A}^0)^t$ and let $\mathbf{f} \in \mathcal{S}$ be a function as in (4.5) such that $\mathbf{v} \in E^-_{\mathbf{B}}$ and the $\mathbf{B}$-representation on the right-hand side is in $\mathrm{repr}_{\mathbf{B}}(\mathbf{f})$. Then $\mathbf{b} \in \mathbf{B}$ and $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v} \in \big(\mathbf{E}^0\big)^t$. Now let $\mathbf{g}$ be the function in (4.6). Clearly, $\mathbf{g} \in \big(\mathrm{PClo}({}_{\mathbf{E}}\mathbf{A})\big)^t$ and the right-hand side of (4.6) is a $\mathbf{B}$-representation of $\mathbf{g}$ in $\mathrm{repr}_{\mathbf{B}}(\mathbf{g})$. It remains to show that $\mathbf{g} \in \mathcal{S} = \mathcal{C}^t$ and $\mathbf{f} \equiv_{\mathcal{C}} \mathbf{g}$.

Since $\mathbf{A}$ is an inverse semigroup of exponent m, so is $\mathbf{B} = (\mathbf{A}^0)^t$. Therefore,

$$\begin{aligned}\mathbf{g}(x_1, \ldots, x_n, \underbrace{y_1, \ldots, y_1}_{m+1}) &= \mathbf{b} + \sum_{i=1}^{n} \mathbf{u}_i(x_i) + (m+1)\mathbf{v}(y_1)\\ &= \mathbf{b} + \sum_{i=1}^{n} \mathbf{u}_i(x_i) + \mathbf{v}(y_1) = \mathbf{f}(x_1, \ldots, x_n, y_1),\end{aligned}$$

which shows that $\mathbf{f} \leq_{\mathcal{C}} \mathbf{g}$. Using the fact that $\mathbf{v} \in (\mathbf{E}^0)^t$ is a homomorphism $\mathbf{A} \to (\mathbf{A}^0)^t = \mathbf{B}$, we get that

$$\begin{aligned}\mathbf{f}(x_1, \ldots, x_n, y_1 + y_2 + \cdots + y_{m+1}) &= \mathbf{b} + \sum_{i=1}^{n} \mathbf{u}_i(x_i) + \mathbf{v}\Big(\sum_{j=1}^{m+1} y_j\Big)\\ &= \mathbf{b} + \sum_{i=1}^{n} \mathbf{u}_i(x_i) + \sum_{j=1}^{m+1} \mathbf{v}(y_j) = \mathbf{g}(x_1, \ldots, x_n, y_1, \ldots, y_{m+1}).\end{aligned}$$

This shows that $\mathbf{g} \leq_{\mathcal{C}} \mathbf{f}$, since the operation $y_1 + y_2 + \cdots + y_{m+1}$ belongs to $\mathcal{C}$ by assumption. Thus $\mathbf{g} \equiv_{\mathcal{C}} \mathbf{f}$. Moreover, since $\mathcal{C}$ is a clone and $\mathbf{f} \in \mathcal{S} = \mathcal{C}^t$, the relation $\mathbf{g} \leq_{\mathcal{C}} \mathbf{f}$ also implies that $\mathbf{g} \in \mathcal{C}^t = \mathcal{S}$. This completes the proof of $(*)_{m,1}$, and hence shows that $\mathcal{C}^t/{\equiv_{\mathcal{C}}}$ is finite. □

Lemma 4.12. *If a clone $\mathcal{C}$ on a finite set A satisfies condition* (B) *from Theorem 3.1, then the sets $\mathcal{C}^t/{\equiv_{\mathcal{C}}}$ are finite for all $t \geq 1$.*

Proof. For convenience, we will assume that $A = [k]$. As in condition (B), let $k \geq 3$, and let $\mathcal{C}$ be a subclone of Burle's clone $\mathcal{B}_A$ such that $x \oplus^a y$ belongs to $\mathcal{C}$ for all $1 \leq a \leq k-1$. Thus the operation $x_1 \oplus^a x_2 \oplus^a \cdots \oplus^a x_n$ obtained from $x \oplus^a y$ by composition also belongs to $\mathcal{C}$ for all $n \geq 1$ and $1 \leq a \leq k-1$. We start the proof with an auxiliary claim.

Claim 4.12.1. Let $n \geq 1$ and $1 \leq a \leq k-1$. For every function $\psi\colon A \to [2]$ such that $\psi(0) = 0$,

$$\sum_{1 \leq a \leq k-1} \psi(x_1 \oplus^a x_2 \oplus^a \cdots \oplus^a x_n) = \sum_{\ell=1}^{n} \psi(x_\ell) \qquad \text{for all } x_1, \ldots, x_n \in A. \tag{4.7}$$

Proof of Claim 4.12.1. Since $\psi \circ \Lambda_a$ is the function $[2] \to [2]$ given by $0 \mapsto 0$, $1 \mapsto \psi(a)$, we see that $\psi \circ \Lambda_a$ is constant 0 if $\psi(a) = 0$ and $\psi \circ \Lambda_a$ is the identity function if $\psi(a) = 1$. Consequently,

$$\begin{aligned}\sum_{1 \leq a \leq k-1} \psi(x_1 \oplus^a x_2 \oplus^a \cdots \oplus^a x_n) &= \sum_{1 \leq a \leq k-1} \psi\Big(\Lambda_a\Big(\sum_{\ell=1}^{n} \lambda_a(x_\ell)\Big)\Big) \\ &= \sum_{\substack{1 \leq a \leq k-1 \\ \psi(a)=1}} \Big(\sum_{\ell=1}^{n} \lambda_a(x_\ell)\Big) = \sum_{\ell=1}^{n} \Big(\sum_{\substack{1 \leq a \leq k-1 \\ \psi(a)=1}} \lambda_a(x_\ell)\Big).\end{aligned}$$

The proof of (4.7) will be complete if we show that

$$\sum_{\substack{1 \leq a \leq k-1 \\ \psi(a)=1}} \lambda_a(x) = \psi(x) \qquad \text{for all } x \in A. \tag{4.8}$$

Since λ_a $(1 \leq a \leq k-1)$ is the characteristic function of $\{a\}$, the left-hand side of (4.8) is the characteristic function of the set $\{a : 1 \leq a \leq k-1,\ \psi(a) = 1\}$. In view of the fact that $\psi(0) = 0$, ψ is also a characteristic function of this set, which proves the claim. $\diamond$

Now let t be a positive integer and let $\mathcal{S} = \mathcal{C}^t$. For $\mathbb{B}(\mathcal{C})$, $\mathcal{C}_{\mathbf{B}}$, $E_{\mathbf{B}}$, and $\mathrm{repr}_{\mathbf{B}}(f)$ $(\mathbf{B} \in \mathbb{B}(\mathcal{C}),\ f \in \mathcal{C})$ we will use the choices agreed upon in Example 4.7 (for any subset $\mathcal{C}$ of $\mathcal{B}_A$), which can be summarized as follows:

- $\mathbb{B}(\mathcal{C})$ consists of a fixed commutative semigroup $\mathbf{A} = (A; \boxplus)$ with neutral element $0 \in A$, and all 2-element groups $\big(\Psi([2]); +^{\Psi}\big)$ where $\Psi\colon [2] \to A$ is a one-to-one function; hence, each member of $\mathbb{B}(\mathcal{C})$ has a neutral element.
- For each $\mathbf{B} \in \mathbb{B}(\mathcal{C})$, $E_{\mathbf{B}} = E^0_{\mathbf{B}} = \mathcal{F}^{(1)}(A, B)$.
- If $\mathbf{B} = \mathbf{A}$, then $\mathcal{C}_{\mathbf{B}}$ consists of all essentially unary operations in $\mathcal{C}$, and for each $f \in \mathcal{C}_{\mathbf{B}}$, the set $\mathrm{repr}_{\mathbf{B}}(f)$ of $\mathbf{B}$-representations of f consists of all sums with all but at most one summand equal to $0_{\mathbf{A}}$.
- If $\mathbf{B} = \big(\Psi([2]); +^{\Psi}\big)$ for some one-to-one function $\Psi\colon [2] \to A$, then $\mathcal{C}_{\mathbf{B}}$ consists of all operations f which can be written in the form (4.4) with $\psi_i(0) = 0$ for all i; the set all such expressions for f is the set $\mathrm{repr}_{\mathbf{B}}(f)$ of $\mathbf{B}$-representations of f.

The last two items show that if $\mathbf{B} \in \mathbb{B}(\mathcal{C})$ and $f \in \mathcal{C}_{\mathbf{B}}$, then every $\mathbf{B}$-reperesentation in $\mathrm{repr}_{\mathbf{B}}(f)$ has the form (4.1), i.e., the constant term is $b = 0_{\mathbf{B}}$.

Conventions 4.9 determine the corresponding data for $\mathcal{S} = \mathcal{C}^t$ (for any subset $\mathcal{C}$ of $\mathcal{B}_A$). Namely, $\mathbb{B}(\mathcal{S})$ is the set of all $\prod_{j=1}^{t} \mathbf{B}_j^0$ with $\mathbf{B}_j \in \mathbb{B}(\mathcal{C})$, and for each $\mathbf{B} = \prod_{j=1}^{t} \mathbf{B}_j^0 \in \mathbb{B}(\mathcal{S})$, we have $\mathcal{S}_{\mathbf{B}} = \prod_{j=1}^{t} \mathcal{C}_{\mathbf{B}_j}$, $E_{\mathbf{B}} = E^0_{\mathbf{B}} = \prod_{j=1}^{t} E_{\mathbf{B}_j}$, and for every $\mathbf{f} = (f_1, \ldots, f_t) \in \mathcal{S}_{\mathbf{B}}$, $\mathrm{repr}_{\mathbf{B}}(\mathbf{f})$ consist of all $\mathbf{B}$-representations of $\mathbf{f}$ whose projections onto each coordinate j $(1 \leq j \leq t)$ belong to $\mathrm{repr}_{\mathbf{B}_j}(f_j)$.

As in the proof of Lemma 4.11, we will show the finiteness of $\mathcal{S}/{\equiv_{\mathcal{C}}}$ by applying Lemma 4.10. Our goal is to prove that $(*)_{p,r}$ holds for $\mathcal{S}$ with $p = 2$ and $r = k-1$. So let $\mathbf{B} = \prod_{m=1}^{t} \mathbf{B}_m \in \mathbb{B}(\mathcal{S})$, and let $\mathbf{f} \in \mathcal{S}_{\mathbf{B}}$ be a function as in (4.5) such that $\mathbf{v} \in E_{\mathbf{B}}^{-}$ and the $\mathbf{B}$-representation on the right-hand side is in $\mathrm{repr}_{\mathbf{B}}(\mathbf{f})$. Furthermore, let $\mathbf{g}$ be the function in (4.6). We need to show that (i) $\mathbf{g} \in \mathcal{S}_{\mathbf{B}}$, (ii) the right-hand side of (4.6) is a $\mathbf{B}$-representation of $\mathbf{g}$ in $\mathrm{repr}_{\mathbf{B}}(\mathbf{g})$, and (iii) $\mathbf{g} \equiv_{\mathcal{C}} \mathbf{f}$.

We saw in the proof of Lemma 4.11, that (i)$'$ $\mathbf{g} \in \mathcal{S}$ and (iii) follow if we show that $\mathbf{f} \leq_{\mathcal{C}} \mathbf{g}$ and $\mathbf{g} \leq_{\mathcal{C}} \mathbf{f}$. For this, it will be enough to argue that

$$\begin{aligned}(4.9)\qquad & \mathbf{f}(x_1,\dots,x_n,y_1,\dots,y_{k-1}) \\ &\qquad = \mathbf{g}(x_1,\dots,x_n,y_1,\dots,y_{k-2},y_{k-1},y_{k-1},y_{k-1})\end{aligned}$$

and

$$\begin{aligned}(4.10)\qquad & \mathbf{g}(x_1,\dots,x_n,y_1,\dots,y_{k+1}) \\ &\qquad = \mathbf{f}(x_1,\dots,x_n,y_1 \oplus^1 \cdots \oplus^1 y_{k+1},\dots,y_1 \oplus^{k-1} \cdots \oplus^{k-1} y_{k+1}),\end{aligned}$$

since the operations $x_1 \oplus^a \cdots \oplus^a x_{k+1}$ belong to $\mathcal{C}$ for all $1 \leq a \leq k-1$. To establish (ii) and to strengthen (i)$'$ to (i), we need to prove, in addition, that

$$\begin{aligned}(4.11)\qquad & \mathbf{g} \in \big((\mathcal{B}_A)^t\big)_{\mathbf{B}} \quad \text{and} \\ &\qquad \text{the right-hand side of (4.6) is a } \mathbf{B}\text{-representation of } \mathbf{g} \text{ in } \mathrm{repr}_{\mathbf{B}}(\mathbf{g}).\end{aligned}$$

We will prove (4.9), (4.10), and (4.11) coordinatewise; that is, we will show that for each m ($1 \leq m \leq t$), the analogous equalities

$$\begin{aligned}(4.12)\qquad & f(x_1,\dots,x_n,y_1,\dots,y_{k-1}) \\ &\qquad = g(x_1,\dots,x_n,y_1,\dots,y_{k-2},y_{k-1},y_{k-1},y_{k-1})\end{aligned}$$

and

$$\begin{aligned}(4.13)\qquad & g(x_1,\dots,x_n,y_1,\dots,y_{k+1}) \\ &\qquad = f(x_1,\dots,x_n,y_1 \oplus^1 \cdots \oplus^1 y_{k+1},\dots,y_1 \oplus^{k-1} \cdots \oplus^{k-1} y_{k+1})\end{aligned}$$

hold for the m-th coordinate functions f and g; moreover,

$$\begin{aligned}(4.14)\qquad & g \in (\mathcal{B}_A)_{\mathbf{B}} \quad \text{and} \quad \text{the } m\text{-th coordinate of} \\ &\qquad \text{the right-hand side of (4.6) is a } \mathbf{B}_m\text{-representation of } g \text{ in } \mathrm{repr}_{\mathbf{B}_m}(g).\end{aligned}$$

If $\mathbf{B}_m = \mathbf{A}$, then $f(x_1,\dots,x_n,y_1,\dots,y_{k-1}) = \bigboxplus_{i=1}^{n} u_i(x_i) \boxplus \bigboxplus_{j=1}^{k-1} v(y_j)$ where u_i, v are the m-th coordinate functions of $\mathbf{u}_i$ and $\mathbf{v}$, respectively. Since the expression on the right-hand side is an $\mathbf{A}$-representation of f in $\mathrm{repr}_{\mathbf{A}}(f)$, we get that at most one of $u_1,\dots,u_n$ and the $k-1$ (≥ 2) v's differs from $0_{\mathbf{A}}$. Thus $v = 0_{\mathbf{A}}$ holds in this case. The operation g satisfies an equality similar to f, except that $k-1$ is replaced by $k+1$. Therefore, it is clear that the expression on the right-hand side of this equality belongs to $\mathrm{repr}_{\mathbf{A}}(g)$, proving (4.14). Since $v = 0_{\mathbf{A}}$, neither f nor g depends on any of its variables y_j. Therefore, the equalities (4.12) and (4.13) clearly hold.

Now let us assume that $\mathbf{B}_m = (\Psi([2]); +^{\Psi})$ for some one-to-one function $\Psi\colon [2] \to A$. Then using (4.4) and the injectivity of Ψ we see that

$$(4.15)\qquad f(x_1,\dots,x_n,y_1,\dots,y_{k-1}) = \Psi\Big(\sum_{i=1}^{n} \phi_i(x_i) + \sum_{j=1}^{k-1} \psi(y_j)\Big)$$

and

$$g(x_1,\dots,x_n,y_1,\dots,y_{k+1}) = \Psi\Big(\sum_{i=1}^{n}\phi_i(x_i) + \sum_{j=1}^{k+1}\psi(y_j)\Big) \tag{4.16}$$

where ϕ_i, ψ are functions $A \to [2]$ with $\phi_i(0) = \psi(0) = 0$ for all i. (4.16) shows that (4.14) holds. The equality (4.12) is also clear from (4.16), since $\psi(y_{k-1}) + \psi(y_{k-1}) + \psi(y_{k-1}) = \psi(y_{k-1})$ holds for addition $+$ modulo 2. To prove (4.13) we use (4.15) and (4.16) above together with the equality proved in Claim 4.12.1:

$$\begin{aligned}
&f(x_1,\dots,x_n,y_1 \oplus^1 \cdots \oplus^1 y_{k+1},\dots,y_1 \oplus^{k-1} \cdots \oplus^{k-1} y_{k+1})\\
&\overset{(4.15)}{=} \Psi\Big(\sum_{i=1}^{n}\phi_i(x_i) + \sum_{1\le a\le k-1}\psi(y_1 \oplus^a \cdots \oplus^a y_{k+1})\Big)\\
&\overset{(4.7)}{=} \Psi\Big(\sum_{i=1}^{n}\phi_i(x_i) + \sum_{\ell=1}^{k+1}\psi(y_\ell)\Big)\\
&\overset{(4.16)}{=} g(x_1,\dots,x_n,y_1,\dots,y_{k+1}).
\end{aligned}$$

This completes the proof of $(*)_{2,k-1}$, and hence shows that $\mathcal{C}^t/{\equiv_\mathcal{C}}$ is finite. □

Proof of Theorem 4.1. Combine Lemmas 4.2, 4.11, and 4.12. □

5. Embedding $(\mathcal{P}_\mathrm{f}(\omega);\subseteq)$ into $(\mathcal{F}(A,U)/{\equiv_\mathcal{C}};\preceq_\mathcal{C})$

In this section we will prove that for every clone $\mathcal{C}$ that satisfies one of conditions (A)–(C) from Theorem 3.1, the poset $(\mathcal{P}_\mathrm{f}(\omega);\subseteq)$ embeds into the $\mathcal{C}$-minor partial order $(\mathcal{F}(A,U)/{\equiv_\mathcal{C}};\preceq_\mathcal{C})$ provided $|U| \geq \min(3,|A|)$. In fact, as the theorem below shows, this conclusion is true under somewhat weaker assumptions on $\mathcal{C}$.

Theorem 5.1. *Let $\mathcal{C}$ be a clone on a finite set A, and let U be a set such that $|U| \geq \min(3,|A|)$. If $\mathcal{C}$ satisfies one of the conditions* (A), (B)′, *or* (C)′ *below, then the poset $(\mathcal{P}_\mathrm{f}(\omega);\subseteq)$ embeds into the $\mathcal{C}$-minor partial order $(\mathcal{F}(A,U)/{\equiv_\mathcal{C}};\preceq_\mathcal{C})$.*

- (A) *There exists a positive integer m such that every operation in $\mathcal{C}$ depends on at most m variables.*
- (B)′ *$\mathcal{C}$ is a subclone of Burle's clone $\mathcal{B}_A$ $(|A| \geq 3)$.*
- (C)′ *For a commutative inverse semigroup $\mathbf{A} = (A;+)$ and $\mathbf{E} = \mathrm{End}(\mathbf{A})$, $\mathcal{C}$ is a subclone of the clone $\mathrm{PClo}({}_\mathbf{E}\mathbf{A})$.*

An embedding $(\mathcal{P}_\mathrm{f}(\omega);\subseteq) \to (\mathcal{F}(A,U)/{\equiv_\mathcal{C}};\preceq_\mathcal{C})$ is a mapping $S \mapsto f_S/{\equiv_\mathcal{C}}$ $(S \in \mathcal{P}_\mathrm{f}(\omega))$ such that $S \subseteq T$ if and only if $f_S/{\equiv_\mathcal{C}} \preceq_\mathcal{C} f_T/{\equiv_\mathcal{C}}$ for all $S,T \in \mathcal{P}_\mathrm{f}(\omega)$; equivalently,

$$S \subseteq T \iff f_S \leq_\mathcal{C} f_T \qquad \text{for all } S,T \in \mathcal{P}_\mathrm{f}(N) \tag{5.1}$$

where $N = \omega$.

Definition 5.2. Let N be an arbitrary set, and let $S \mapsto f_S$ be a function $\mathcal{P}_\mathrm{f}(N) \to \mathcal{F}(A,U)$. For a clone $\mathcal{C}$ on A, the family f_S $(S \in \mathcal{P}_\mathrm{f}(N))$ of functions will be called *$\mathcal{C}$-independent* if (5.1) holds, and *strongly $\mathcal{C}$-independent* if the following conditions hold: for arbitrary $S,T \in \mathcal{P}_\mathrm{f}(N)$ and $n \in N$,

(a) $S \subseteq T$ implies that f_S is obtained from f_T by identifying variables, and
(b) $f_{\{n\}} \leq_\mathcal{C} f_T$ implies that $n \in T$.

Lemma 5.3. *Let A and U be sets, and let $\mathcal{C}$ be a clone on A. Every strongly $\mathcal{C}$-independent family of functions in $\mathcal{F}(A,U)$ is $\mathcal{C}$-independent.*

Proof. Let f_S ($S \in \mathcal{P}_{\mathrm{f}}(N)$) be a strongly $\mathcal{C}$-independent family of functions in $\mathcal{F}(A,U)$. Then conditions (a)–(b) from Definition 5.2 hold. We want to prove (5.1). The implication $\Rightarrow$ follows from (a), because the requirement that f_S is obtained from f_T by identifying variables implies that $f_S = f_T \circ \mathbf{h}$ for a tuple $\mathbf{h}$ of projections, so $f_S \leq_{\mathcal{C}} f_T$. To prove the converse implication $\Leftarrow$ let $f_S \leq_{\mathcal{C}} f_T$. We want to show that $S \subseteq T$. Let $n \in S$. Then $\{n\} \subseteq S$, so by condition (a) we have $f_{\{n\}} \leq_{\mathcal{C}} f_S$. Hence, by the transitivity of $\leq_{\mathcal{C}}$, we get that $f_{\{n\}} \leq_{\mathcal{C}} f_T$, which yields by condition (b) that $n \in T$. Thus $S \subseteq T$, as claimed. □

If $\mathcal{C}$ is a clone on A and $B \subseteq A$, let

$$\mathcal{C}_B = \{f|_B : f \in \mathcal{C} \text{ and } f(B,\dots,B) \subseteq B\}.$$

It is easy to see that $\mathcal{C}_B$ is a clone on the set B. Next we show that we can construct strongly $\mathcal{C}$-independent families of functions in $\mathcal{F}(A,U)$, by extension, from strongly $\mathcal{C}_B$-independent families of operations on a common proper subset B of A and U.

Lemma 5.4. *Let A, U, and B be sets such that B is a common proper subset of A and U, and let $\mathcal{C}$ be a clone on A. For every strongly $\mathcal{C}_B$-independent family f_S ($S \in \mathcal{P}_{\mathrm{f}}(N)$) of operations on B there exists a strongly $\mathcal{C}$-independent family $\bar{f}_S$ ($S \in \mathcal{P}_{\mathrm{f}}(N)$) of functions in $\mathcal{F}(A,U)$ such that $\bar{f}_S$ extends f_S for each $S \in \mathcal{P}_{\mathrm{f}}(N)$.*

Proof. Let $0 \in U \setminus B$, and let f_S ($S \in \mathcal{P}_{\mathrm{f}}(N)$) be a strongly $\mathcal{C}_B$-independent family of operations on B. For each $S \in \mathcal{P}_{\mathrm{f}}(N)$ define $\bar{f}_S$ as follows: $\bar{f}_S(\mathbf{x}) = f_S(\mathbf{x})$ if $\mathbf{x}$ is in the domain of f_S, and $\bar{f}(\mathbf{x}) = 0$ otherwise. Clearly, $\bar{f}_S$ extends f_S. We want to argue that $\bar{f}_S$ ($S \in \mathcal{P}_{\mathrm{f}}(N)$) is a strongly $\mathcal{C}$-independent family of functions in $\mathcal{F}(A,U)$.

Suppose that $S \subseteq T$ ($S, T \in \mathcal{P}_{\mathrm{f}}(N)$). Since the family f_S ($S \in \mathcal{P}_{\mathrm{f}}(N)$) is strongly $\mathcal{C}_B$-independent, f_S is obtained from f_T by identifying variables; say $f_S(x_1,\dots,x_k) = f_T(x_{i_1},\dots,x_{i_m})$ where $\{x_{i_1},\dots,x_{i_m}\} = \{x_1,\dots,x_k\}$. Clearly, this identity extends to $\bar{f}_S$ and $\bar{f}_T$, which proves condition (a) from Definition 5.2 for the family $\bar{f}_S$ ($S \in \mathcal{P}_{\mathrm{f}}(N)$).

Now assume that $\bar{f}_{\{n\}} \leq_{\mathcal{C}} \bar{f}_T$ ($n \in N$, $T \in \mathcal{P}_{\mathrm{f}}(N)$), say $\bar{f}_{\{n\}}$ is k-ary and $\bar{f}_T$ is m-ary. Then there exists $\mathbf{h} \in (\mathcal{C}^{(k)})^m$ such that $\bar{f}_{\{n\}} = \bar{f}_T \circ \mathbf{h}$; that is, $\bar{f}_{\{n\}}(\mathbf{x}) = \bar{f}_T\big(\mathbf{h}(\mathbf{x})\big)$ for all $\mathbf{x} \in A^k$. If $\mathbf{x} \in B^k$, then $\bar{f}_{\{n\}}(\mathbf{x}) = f_{\{n\}}(\mathbf{x}) \in B$, so $\bar{f}_T\big(\mathbf{h}(\mathbf{x})\big) \in B$, which implies that $\mathbf{h}(\mathbf{x}) \in B^m$. Thus each coordinate function h_i of $\mathbf{h}$ satisfies $h_i(B,\dots,B) \subseteq B$, and hence can be restricted to B, so $\mathbf{h}|_B \in (\mathcal{C}_B^{(k)})^m$. By definition, all functions $\bar{f}_S$ can also be restricted to B. Therefore, the equality $\bar{f}_{\{n\}} = \bar{f}_T \circ \mathbf{h}$ yields that $(\bar{f}_{\{n\}})|_B = (\bar{f}_T)|_B \circ \mathbf{h}|_B$, that is, $f_{\{n\}} = f_T \circ \mathbf{h}|_B$. Hence $f_{\{n\}} \leq_{\mathcal{C}_B} f_T$. Since the family f_S ($S \in \mathcal{P}_{\mathrm{f}}(N)$) is strongly $\mathcal{C}_B$-independent, we get that $n \in T$. This proves condition (b) from Definition 5.2 for the family $\bar{f}_S$ ($S \in \mathcal{P}_{\mathrm{f}}(N)$), establishing its strong $\mathcal{C}$-independence. □

Our main tool in proving Theorem 5.1 will be the following corollary of Lemmas 5.3 and 5.4.

Corollary 5.5. *Let $\mathcal{C}$ be a clone on a set A, let $B \subseteq A$, and let U be a set such that $|U| \geq \min(|B|+1, |A|)$. If, for some countably infinite set N, there exists a strongly $\mathcal{C}_B$-independent family f_S ($S \in \mathcal{P}_{\mathrm{f}}(N)$) of operations on B, then $(\mathcal{P}_{\mathrm{f}}(\omega); \subseteq)$ embeds into $(\mathcal{F}(A,U)/{\equiv_{\mathcal{C}}}; \preceq_{\mathcal{C}})$.*

Proof. The assumption $|U| \geq \min(|B|+1, |A|)$ implies that U has a subset C with $|C| = |B|$ such that $C \neq U$ if $B \neq A$. Since every bijection $\varphi\colon U \to V$ induces an isomorphism $(\mathcal{F}(A,U)/{\equiv_\mathcal{C}}; \preceq_\mathcal{C}) \to (\mathcal{F}(A,V)/{\equiv_\mathcal{C}}; \preceq_\mathcal{C})$ via the map $g/{\equiv} \mapsto (\varphi \circ g)/{\equiv}$, we get that by applying an appropriate bijection that maps C to B, we may assume for the proof of Corollary 5.5 that $C = B$, that is, $B \subseteq A \cap U$, and $B \neq U$ if $B \neq A$.

If $B = A$, then $\mathcal{C}_B = \mathcal{C}$, and the given strongly $\mathcal{C}_B$-independent family f_S ($S \in \mathcal{P}_\mathrm{f}(N)$) of operations on B becomes, by increasing the codomain to U, a strongly $\mathcal{C}$-independent family $\bar{f}_S$ ($S \in \mathcal{P}_\mathrm{f}(N)$) of functions in $\mathcal{F}(A,U)$. If $B \neq A$, then $B \neq U$, and Lemma 5.4 yields a strongly $\mathcal{C}$-independent family $\bar{f}_S$ ($S \in \mathcal{P}_\mathrm{f}(N)$) of functions in $\mathcal{F}(A,U)$. In either case, we get from Lemma 5.3 that $S \subseteq T \iff \bar{f}_S \leq_\mathcal{C} \bar{f}_T$ holds for all $S, T \in \mathcal{P}_\mathrm{f}(N)$. Thus the mapping $S \mapsto \bar{f}_S/{\equiv_\mathcal{C}}$ is an embedding of $(\mathcal{P}_\mathrm{f}(N); \subseteq)$ into $(\mathcal{F}(A,U)/{\equiv_\mathcal{C}}; \preceq_\mathcal{C})$. Since N is countably infinite, there exists a bijection $\omega \to N$, which induces an isomorphism $(\mathcal{P}_\mathrm{f}(\omega); \subseteq) \to (\mathcal{P}_\mathrm{f}(N); \subseteq)$. Thus $(\mathcal{P}_\mathrm{f}(\omega); \subseteq)$ embeds into $(\mathcal{F}(A,U)/{\equiv_\mathcal{C}}; \preceq_\mathcal{C})$. □

For the proof of Theorem 5.1 we will use the special case $|B| = 2$ of Corollary 5.5. The main step is to exhibit a strongly $\mathcal{D}$-independent family of operations on B for every subclone $\mathcal{D}$ of $\mathrm{PClo}({}_{\mathrm{End}(\mathbf{B})}\mathbf{B}) = \mathrm{PClo}(\mathbf{B})$ where $\mathbf{B}$ is a 2-element group or semilattice.

We will assume without loss of generality that $B = [2]$, and will use the following notation. For a finite set $S \in \mathcal{P}_\mathrm{f}(\omega)$, $\Sigma S := \sum_{i \in S} i$ is the sum of the elements of S. For $n \in \omega$, denote $S^{<n} := \{i \in S : i < n\}$. For $S \subseteq \omega_+$, set

$$D_S := \bigcup_{n \in S} (\{n\} \times \lfloor n \rceil) = \{(n,i) : n \in S,\ 1 \leq i \leq n\}.$$

(Recall the notation $\lfloor n \rceil := \{1, \dots, n\}$.) For every nonempty $S \in \mathcal{P}_\mathrm{f}(\omega_+)$, let us fix a bijection $\beta_S\colon D_S \to \lfloor \Sigma S \rceil$. Our arguments do not depend on the choice of β_S, but for convenience we will choose β_S to be the mapping $(n,i) \mapsto \Sigma S^{<n} + i$; that is, β_S is the unique bijection $D_S \to \lfloor \Sigma S \rceil$ that is an order isomorphism between the ordered sets $(D_S; \sqsubseteq)$ and $(\lfloor \Sigma \rceil; \leq)$ where $\leq$ is the natural order on any subset of ω_+, and $\sqsubseteq$ is the lexicographic order on D_S ($\subseteq \omega_+ \times \omega_+$) with respect to $\leq$.

Let $S \in \mathcal{P}_\mathrm{f}(\omega_+)$ be nonempty. For an $(\Sigma S + 1)$-tuple $\mathbf{u} := (u_1, \dots, u_{\Sigma S+1})$ (of variables or elements of $[2]$) and for an element $n \in S$, we will refer to the n-tuple $\mathbf{u}_{(S,n)} := (u_{\beta_S(n,1)}, u_{\beta_S(n,2)}, \dots, u_{\beta_S(n,n)})$ as the (S,n)-*block* of $\mathbf{u}$. An S-*block* of $\mathbf{u}$ is an (S,n)-block of $\mathbf{u}$ for some $n \in S$. Note that the last entry $u_{\Sigma S+1}$ does not contribute to any S-block of $\mathbf{u}$. Denote $B_{S,n} := \{\beta_S(n,i) : 1 \leq i \leq n\}$.

Define the $(\Sigma S + 1)$-tuples $\boldsymbol{\eta}^S_{n,i}, \boldsymbol{\mu}^S_{n,i}, \boldsymbol{\iota}^S_n \in [2]^{\Sigma S+1}$ by

$$\boldsymbol{\eta}^S_{n,i}(j) := \begin{cases} 0, & \text{if } j \in B_{S,n} \setminus \{\beta_S(n,i)\}, \\ 1, & \text{otherwise,} \end{cases} \qquad \boldsymbol{\mu}^S_{n,i}(j) := \begin{cases} 0, & \text{if } j = \beta_S(n,i), \\ 1, & \text{otherwise,} \end{cases}$$

$$\boldsymbol{\iota}^S_n(j) := \begin{cases} 1, & \text{if } j \in B_{S,n}, \\ 0, & \text{otherwise,} \end{cases}$$

for all $j \in \lfloor \Sigma S + 1 \rceil$. For β_S chosen above, we can write the tuples $\boldsymbol{\eta}^S_{n,i}$, $\boldsymbol{\mu}^S_{n,i}$, and $\boldsymbol{\iota}^S_n$ as follows, indicating the various S-blocks and also positions $\beta_S(n,i)$ and $\Sigma S+1$:

$$\begin{aligned}
\boldsymbol{\eta}^S_{n,i} &= (1\cdots1 \ \ \cdots\ \ 1\cdots1\ \ \overset{\beta_S(n,i)}{\overset{\downarrow}{0\cdots010\cdots0}}\ \ 1\cdots1\ \ \cdots\ \ 1\cdots1\ \ \overset{\Sigma S+1}{\overset{\downarrow}{1}}),\\
\boldsymbol{\mu}^S_{n,i} &= (1\cdots1 \ \ \cdots\ \ 1\cdots1\ \ 1\cdots101\cdots1\ \ 1\cdots1\ \ \cdots\ \ 1\cdots1\ \ 1),\\
\boldsymbol{\iota}^S_n &= (\underbrace{\underbrace{0\cdots0}\ \ \cdots\ \ \underbrace{0\cdots0}\ \ \underbrace{1\cdots111\cdots1}_{(S,n)\text{-block}}\ \ \underbrace{0\cdots0}\ \ \cdots\ \ \underbrace{0\cdots0}}_{S\text{-blocks}}\ \ 0).
\end{aligned}$$

We will denote the all-0 and all-1 $(\Sigma S + 1)$-tuples by $\mathbf{0}^S := (0,\dots,0)$ and $\mathbf{1}^S := (1,\dots,1)$. Furthermore, we will use the following notation:

$$\begin{aligned}
E_{S,n} &:= \{\boldsymbol{\eta}^S_{n,i}, \boldsymbol{\mu}^S_{n,i} : i \in \lfloor n \rceil\} \qquad (n \in S),\\
E_S &:= \bigcup_{n\in S} E_{S,n}.
\end{aligned}$$

Now we define operations $f_S \in \mathcal{O}_{[2]}$ for each finite subset S of ω_+. For $S \neq \emptyset$, let $f_S \colon [2]^{\Sigma S+1} \to [2]$ be the characteristic function of the set E_S, that is,

$$f_S(\mathbf{a}) = \begin{cases} 1, & \text{if } \mathbf{a} \in E_S,\\ 0, & \text{if } \mathbf{a} \in [2]^{\Sigma S+1} \setminus E_S. \end{cases}$$

(In other words, $f_S(\mathbf{a}) = 1$ if and only if there exists an S-block of $\mathbf{a}$ such that exactly one or all but one entries of that S-block are 0, and all remaining entries of $\mathbf{a}$ are 1.) For $S = \emptyset$, let $f_\emptyset \colon [2] \to [2]$ be the unary constant operation 0. The operations f_S $(S \in \mathcal{P}_{\mathrm{f}}(\omega_+))$ are essentially the Boolean functions constructed by Couceiro and Pouzet in [2], which in turn were based on functions defined by Pippenger [14].

Lemma 5.6. *Let $S, T \in \mathcal{P}_{\mathrm{f}}(\omega_+)$ and $n \in \omega_+$. If $S \subseteq T$, then f_S is obtained from f_T by identifying variables.*

Proof. Let $S, T \in \mathcal{P}_{\mathrm{f}}(\omega_+)$. If $S = \emptyset$ and T is arbitrary, then it is easy to verify that the identity $f_\emptyset(x) = f_T(x,\dots,x)$ holds. Assume now that $S \neq \emptyset$. For any $(\Sigma S+1)$-tuple $\mathbf{u} = (u_1,\dots,u_{\Sigma S+1})$ (of variables or elements of [2]) let $\tilde{\mathbf{u}} = (\tilde{u}_1,\dots,\tilde{u}_{\Sigma T+1})$ be the $(\Sigma T + 1)$-tuple defined as follows:

$$\tilde{u}_j := \begin{cases} u_{\beta_S(n,i)}, & \text{if } j = \beta_T(n,i) \text{ for some } n \in S,\ i \in \lfloor n \rceil,\\ u_{\Sigma S+1}, & \text{otherwise.} \end{cases}$$

Denoting the list of variables of f_S by $\mathbf{x} = (x_1,\dots,x_{\Sigma S+1})$ we claim that the identity $f_S(\mathbf{x}) = f_T(\tilde{\mathbf{x}})$ holds whenever $S \subseteq T$. This follows by observing that for every $\mathbf{a} \in A^{\Sigma S+1}$, $\tilde{\mathbf{a}}$ is the unique $(\Sigma T+1)$-tuple $\mathbf{b}$ for which we have that $\mathbf{b}_{(T,n)} = \mathbf{a}_{(S,n)}$ for all $n \in S$, and the remaining entries of $\mathbf{b}$, i.e., those outside of the (T,n)-blocks for $n \in S$, are equal to $a_{\Sigma S+1}$. This completes the proof. □

Next we want to prove that, for some infinite set $N \subseteq \omega_+$, f_S $(S \in \mathcal{P}_{\mathrm{f}}(N))$ is a strongly $\mathcal{D}$-independent family of operations on [2], provided $\mathcal{D}$ is a subclone of $\mathrm{PClo}(\mathbf{B})$ for a semilattice or group $\mathbf{B} = ([2];+)$ with neutral element 0.

Lemma 5.7. *Let* $\mathbf{B} = ([2]; +)$ *be the unique semilattice with neutral element* 0*, and let* $\mathcal{D}$ *be a subclone of* $\mathrm{PClo}(\mathbf{B})$*. For* $N := \{n \in \omega : n \geq 4\}$*,* f_S $(S \in \mathcal{P}_\mathrm{f}(N))$ *is a strongly* $\mathcal{D}$*-independent family of operations on* $[2]$*.*

Proof. In view of Lemma 5.6 and Proposition 2.3, we only need to verify that condition (b) in Definition 5.2 holds for the clone $\mathcal{C} := \mathrm{PClo}(\mathbf{B})$ and the family f_S $(S \in \mathcal{P}_\mathrm{f}(N))$.

Let $T \in \mathcal{P}_\mathrm{f}(N)$, $n \in N$, and assume that $f_{\{n\}} \leq_\mathcal{C} f_T$. Thus $n \geq 4$. We want to show that $n \in T$. The assumption $f_{\{n\}} \leq_\mathcal{C} f_T$ implies that there exists a map $\mathbf{g} \in (\mathcal{C}^{(n+1)})^{\Sigma T+1}$ such that $f_{\{n\}} = f_T \circ \mathbf{g}$.

Claim 5.7.1. $\mathbf{g}$ has the following properties:

(1) $\mathbf{g}$ is a homomorphism from $\mathbf{B}^{n+1}$ to $\mathbf{B}^{\Sigma T+1}$, i.e.,

$$\mathbf{g}(\mathbf{u} + \mathbf{v}) = \mathbf{g}(\mathbf{u}) + \mathbf{g}(\mathbf{v}) \qquad \text{for all } \mathbf{u}, \mathbf{v} \in [2]^{n+1}. \tag{5.2}$$

(2) If $i \in \lfloor n \rceil$, then

$$\mathbf{g}(\boldsymbol{\mu}_{n,i}^{\{n\}}) = \sum_{j \in \lfloor n \rceil \setminus \{i\}} \mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}}). \tag{5.3}$$

(3) $\mathbf{g}$ maps the set $E_{\{n\}}$ into E_T, and its complement $[2]^{n+1} \setminus E_{\{n\}}$ into the complement $[2]^{\Sigma T+1} \setminus E_T$ of E_T.

Proof of Claim 5.7.1. $\mathrm{PClo}(\mathbf{B})$ is generated by $+$ and the unary constant operations 0, 1. Therefore (1) follows from the fact that $+$ is a homomorphism $\mathbf{B}^2 \to \mathbf{B}$, and 0, 1 are homomorphisms $\mathbf{B} \to \mathbf{B}$.

(2) follows from (1) and the fact that $\boldsymbol{\mu}_{n,i}^{\{n\}} = \sum_{j \in \lfloor n \rceil \setminus \{i\}} \boldsymbol{\eta}_{n,j}^{\{n\}}$ for all $i \in \lfloor n \rceil$.

Finally, to prove (3), let $\mathbf{a} \in [2]^{n+1}$, and use the equality $f_{\{n\}} = f_T \circ \mathbf{g}$. If $\mathbf{a} \in E_{\{n\}}$, then $1 = f_{\{n\}}(\mathbf{a}) = f_T\big(\mathbf{g}(\mathbf{a})\big)$, so $\mathbf{g}(\mathbf{a}) \in E_T$. Similarly, if $\mathbf{a} \notin E_{\{n\}}$, then $0 = f_{\{n\}}(\mathbf{a}) = f_T\big(\mathbf{g}(\mathbf{a})\big)$, so $\mathbf{g}(\mathbf{a}) \notin E_T$. ◇

Claim 5.7.2. For all $i, j \in \lfloor n \rceil$ with $i \neq j$, we have that $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) \neq \mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}})$ and $\mathbf{g}(\boldsymbol{\mu}_{n,i}^{\{n\}}) \neq \mathbf{g}(\boldsymbol{\mu}_{n,j}^{\{n\}})$.

Proof of Claim 5.7.2. Suppose, on the contrary, that $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) = \mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}})$ for some $i \neq j$. Then, by (5.2) and by the idempotence of $+$, we get that

$$\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}} + \boldsymbol{\eta}_{n,j}^{\{n\}}) = \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) + \mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}}) = \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}).$$

This contradicts Claim 5.7.1 (3), because $\boldsymbol{\eta}_{n,i}^{\{n\}} + \boldsymbol{\eta}_{n,j}^{\{n\}} \in [2]^{n+1} \setminus E_{\{n\}}$ and $\boldsymbol{\eta}_{n,i}^{\{n\}} \in E_{\{n\}}$. A contradiction can be derived in a similar way, if we suppose that $\mathbf{g}(\boldsymbol{\mu}_{n,i}^{\{n\}}) = \mathbf{g}(\boldsymbol{\mu}_{n,j}^{\{n\}})$ for some $i \neq j$. ◇

Claim 5.7.3. There exists an element $t \in T$ and a map $\sigma\colon \lfloor n \rceil \to \lfloor t \rceil$ such that $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) = \boldsymbol{\eta}_{t,\sigma(i)}^{T}$ for all $i \in \lfloor n \rceil$.

Proof of Claim 5.7.3. First we will argue that if $i \in \lfloor n \rceil$, then $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) \neq \boldsymbol{\mu}_{p,q}^{T}$ for all $p \in T$, $q \in \lfloor p \rceil$. Suppose, on the contrary, that there is $i \in \lfloor n \rceil$ such that

$\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}) = \boldsymbol{\mu}^T_{p,q}$ for some $p \in T$, $q \in \lfloor p \rceil$. Then, by (5.3), we get for all $\ell \in \lfloor n \rceil \setminus \{i\}$ that

$$\mathbf{g}(\boldsymbol{\mu}^{\{n\}}_{n,\ell}) = \sum_{j \in \lfloor n \rceil \setminus \{\ell\}} \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,j}) = \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}) + \sum_{j \in \lfloor n \rceil \setminus \{\ell, i\}} \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,j})$$
$$= \boldsymbol{\mu}^T_{p,q} + \sum_{j \in \lfloor n \rceil \setminus \{\ell, i\}} \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,j}) \geq \boldsymbol{\mu}^T_{p,q},$$

where $\leq$ denotes the natural ordering of the semilattice $\mathbf{B}^{\Sigma T+1}$ induced by the ordering $0 < 1$ of $\mathbf{B}$; that is, $\leq$ is the coordinatewise ordering of the set $[2]^{\Sigma T+1}$ induced by the ordering $0 < 1$ of $[2]$. Since the only tuples in $[2]^{\Sigma T+1}$ that are greater than or equal to $\boldsymbol{\mu}^T_{p,q}$ by $\leq$ are $\boldsymbol{\mu}^T_{p,q} \in E_T$ and $\mathbf{1}^T \in [2]^{\Sigma T+1} \setminus E_T$, it follows from Claim 5.7.1 (3) that $\mathbf{g}(\boldsymbol{\mu}^{\{n\}}_{n,\ell}) = \boldsymbol{\mu}^T_{p,q}$ for all $\ell \in \lfloor n \rceil \setminus \{i\}$. This contradicts Claim 5.7.2, because $n \in N$ implies that $n \geq 4$.

Thus, we have that for each $i \in \lfloor n \rceil$, $\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}) \neq \boldsymbol{\mu}^T_{p,q}$ for all $p \in T$, $q \in \lfloor p \rceil$. On the other hand, since $\boldsymbol{\eta}^{\{n\}}_{n,i} \in E_{\{n\}}$, we know from Claim 5.7.1 (3) that $\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}) \in E_T = \{\boldsymbol{\eta}^T_{p,q},\ \boldsymbol{\mu}^T_{p,q} : p \in T,\ q \in \lfloor p \rceil\}$. Hence, each $\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i})$ ($i \in \lfloor n \rceil$) is an $\boldsymbol{\eta}$-tuple from E_T. To complete the proof, it remains to show that all these $\boldsymbol{\eta}$-tuples have the same first subscripts.

Suppose then, on the contrary, that there exist $i, i' \in \lfloor n \rceil$, $p, p' \in T$, $q \in \lfloor p \rceil$, $q' \in \lfloor p' \rceil$ such that $p \neq p'$, $\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}) = \boldsymbol{\eta}^T_{p,q}$, $\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i'}) = \boldsymbol{\eta}^T_{p',q'}$; then necessarily $i \neq i'$, and $\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}) + \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i'}) = \boldsymbol{\eta}^T_{p,q} + \boldsymbol{\eta}^T_{p',q'} = \mathbf{1}^T$. By (5.3), we have that for all $\ell \in \lfloor n \rceil \setminus \{i, i'\}$,

$$\mathbf{g}(\boldsymbol{\mu}^{\{n\}}_{n,\ell}) = \sum_{j \in \lfloor n \rceil \setminus \{\ell\}} \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,j}) = \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}) + \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i'}) + \sum_{j \in \lfloor n \rceil \setminus \{i, i', \ell\}} \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,j})$$
$$= \mathbf{1}^T + \sum_{j \in \lfloor n \rceil \setminus \{i, i', \ell\}} \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,j}) = \mathbf{1}^T.$$

This equality contradicts Claim 5.7.1 (3), because $\boldsymbol{\mu}^{\{n\}}_{n,\ell} \in E_{\{n\}}$, but $\mathbf{1}^T \notin E_T$.
$\diamond$

By Claim 5.7.2, the mapping $\sigma \colon \lfloor n \rceil \to \lfloor t \rceil$ given by Claim 5.7.3 is injective; hence $t \geq n$. Let $\mathbf{d} := \mathbf{g}(\boldsymbol{\mu}^{\{n\}}_{n,i})$. Since $\boldsymbol{\mu}^{\{n\}}_{n,i} \in E_{\{n\}}$, Claim 5.7.1 (3) implies that $\mathbf{d} \in E_T$. By (5.3), we have that

$$\mathbf{d} = \mathbf{g}(\boldsymbol{\mu}^{\{n\}}_{n,i}) = \sum_{j \in \lfloor n \rceil \setminus \{i\}} \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,j}) = \sum_{j \in \lfloor n \rceil \setminus \{i\}} \boldsymbol{\eta}^T_{t,\sigma(j)},$$

and since σ is injective, exactly $n-1$ entries in the (T,t)-block of $\mathbf{d}$ are equal to 1, and the entries outside the (T,t)-block are 1. In order for $\mathbf{d}$ to be in E_T, it is necessary that either 1 or $t-1$ entries in the (T,t)-block of $\mathbf{d}$ are equal to 1. Thus, either $n = 2$ or $n = t$. The former case is not possible, because the assumption that $n \in N$ forces $n \geq 4$. Hence we conclude that $n = t \in T$. $\square$

Lemma 5.8. *Let* $\mathbf{B} = ([2]; +)$ *be the unique group on* $[2]$ *with neutral element* 0*, and let* $\mathcal{D}$ *be a subclone of* $\mathrm{PClo}(\mathbf{B})$*. For* $N := \{n \in \omega : n \geq 6,\ n \text{ even}\}$*,* f_S $(S \in \mathcal{P}_{\mathrm{f}}(N))$ *is a strongly* $\mathcal{D}$*-independent family of operations on* $[2]$*.*

Proof. In view of Lemma 5.6 and Proposition 2.3, we only need to verify that condition (b) in Definition 5.2 holds for the clone $\mathcal{C} := \mathrm{PClo}(\mathbf{B})$ and the family f_S $(S \in \mathcal{P}_\mathrm{f}(N))$.

Let $T \in \mathcal{P}_\mathrm{f}(N)$, $n \in N$, and assume that $f_{\{n\}} \leq_\mathcal{C} f_T$. Thus n is even and $n \geq 6$. We want to show that $n \in T$. The assumption $f_{\{n\}} \leq_\mathcal{C} f_T$ implies that there exists a map $\mathbf{g} \in (\mathcal{C}^{(n+1)})^{\Sigma T+1}$ such that $f_{\{n\}} = f_T \circ \mathbf{g}$.

Claim 5.8.1. $\mathbf{g}$ has the following properties:

(1) For the ternary group $\mathbf{T} = ([2]; x+y+z)$, $\mathbf{g}$ is a homomorphism from $\mathbf{T}^{n+1}$ to $\mathbf{T}^{\Sigma T+1}$, and hence for every odd natural number $2k+1$,

$$\mathbf{g}\Big(\sum_{i=1}^{2k+1} \mathbf{u}_i\Big) = \sum_{i=1}^{2k+1} \mathbf{g}(\mathbf{u}_i) \qquad \text{for all } \mathbf{u}_1, \dots, \mathbf{u}_{2k+1} \in [2]^{n+1}. \tag{5.4}$$

(2) If $i \in \lfloor n \rceil$, then

$$\mathbf{g}(\boldsymbol{\mu}_{n,i}^{\{n\}}) = \sum_{j \in \lfloor n \rceil \setminus \{i\}} \mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}}) \qquad \text{and} \qquad \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) = \sum_{j \in \lfloor n \rceil \setminus \{i\}} \mathbf{g}(\boldsymbol{\mu}_{n,j}^{\{n\}}). \tag{5.5}$$

(3) $\mathbf{g}$ maps the set $E_{\{n\}}$ into E_T, and its complement $[2]^{n+1} \setminus E_{\{n\}}$ into the complement $[2]^{\Sigma T+1} \setminus E_T$ of E_T.

Proof of Claim 5.8.1. $\mathrm{PClo}(\mathbf{B})$ is generated by $+$ and the unary constant operation 1. Since $+$ is a homomorphism $\mathbf{T}^2 \to \mathbf{T}$, and 1 is a homomorphisms $\mathbf{T} \to \mathbf{T}$, it follows that $\mathbf{g}$ is a homomorphism $\mathbf{T}^{n+1} \to \mathbf{T}^{\Sigma T+1}$. This means that $\mathbf{g}(\mathbf{u}+\mathbf{v}+\mathbf{w}) = \mathbf{g}(\mathbf{u}) + \mathbf{g}(\mathbf{v}) + \mathbf{g}(\mathbf{w})$ holds for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in [2]^{n+1}$. Repeated application of this equality yields (5.4).

(2) is true, because the fact that $n \in N$ is even implies that we have $\boldsymbol{\mu}_{n,i}^{\{n\}} = \sum_{j \in \lfloor n \rceil \setminus \{i\}} \boldsymbol{\eta}_{n,j}^{\{n\}}$ and $\boldsymbol{\eta}_{n,i}^{\{n\}} = \sum_{j \in \lfloor n \rceil \setminus \{i\}} \boldsymbol{\mu}_{n,j}^{\{n\}}$ for all $i \in \lfloor n \rceil$; moreover, since the number of summands, $n-1$, is odd, (5.4) applies.

(3) can be proved the same way as the analogous statement was proved in Claim 5.7.1. $\diamond$

Claim 5.8.2. For all $i, j \in \lfloor n \rceil$ with $i \neq j$, we have that $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) \neq \mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}})$ and $\mathbf{g}(\boldsymbol{\mu}_{n,i}^{\{n\}}) \neq \mathbf{g}(\boldsymbol{\mu}_{n,j}^{\{n\}})$.

Proof of Claim 5.8.2. Suppose, on the contrary, that $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) = \mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}})$ for some $i \neq j$, and let $k \in \lfloor n \rceil \setminus \{i, j\}$. By (5.4) and by the fact that $\mathbf{B}$ is a Boolean group, we get that

$$\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}} + \boldsymbol{\eta}_{n,j}^{\{n\}} + \boldsymbol{\eta}_{n,k}^{\{n\}}) = \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) + \mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}}) + \mathbf{g}(\boldsymbol{\eta}_{n,k}^{\{n\}}) = \mathbf{g}(\boldsymbol{\eta}_{n,k}^{\{n\}}).$$

This contradicts Claim 5.8.1 (3), because $\boldsymbol{\eta}_{n,i}^{\{n\}} + \boldsymbol{\eta}_{n,j}^{\{n\}} + \boldsymbol{\eta}_{n,k}^{\{n\}} \in [2]^{n+1} \setminus E_{\{n\}}$ and $\boldsymbol{\eta}_{n,i}^{\{n\}} \in E_{\{n\}}$. Replacing $\boldsymbol{\eta}$'s by $\boldsymbol{\mu}$'s throughout, we can get a contradiction in a similar way, if we suppose that $\mathbf{g}(\boldsymbol{\mu}_{n,i}^{\{n\}}) = \mathbf{g}(\boldsymbol{\mu}_{n,j}^{\{n\}})$ for some $i \neq j$. $\diamond$

Claim 5.8.3. There do not exist any $i, j \in \lfloor n \rceil$, $p \in T$, $q \in \lfloor p \rceil$ such that $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) = \boldsymbol{\eta}_{p,q}^T$ and $\mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}}) = \boldsymbol{\mu}_{p,q}^T$, or $\mathbf{g}(\boldsymbol{\mu}_{n,i}^{\{n\}}) = \boldsymbol{\eta}_{p,q}^T$ and $\mathbf{g}(\boldsymbol{\mu}_{n,j}^{\{n\}}) = \boldsymbol{\mu}_{p,q}^T$.

Proof of Claim 5.8.3. Suppose, on the contrary, that there exist i, j, p, q such that $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) = \boldsymbol{\eta}_{p,q}^{T}$ and $\mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}}) = \boldsymbol{\mu}_{p,q}^{T}$. Hence $i \neq j$ and

$$\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) + \mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}}) = \boldsymbol{\eta}_{p,q}^{T} + \boldsymbol{\mu}_{p,q}^{T} = \boldsymbol{\iota}_{p}^{T}. \tag{5.6}$$

Let $\ell \in \lfloor n \rceil \setminus \{i, j\}$. Combining (5.4) and (5.6) we get that

$$\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}} + \boldsymbol{\eta}_{n,j}^{\{n\}} + \boldsymbol{\eta}_{n,\ell}^{\{n\}}) = \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) + \mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}}) + \mathbf{g}(\boldsymbol{\eta}_{n,\ell}^{\{n\}}) = \boldsymbol{\iota}_{p}^{T} + \mathbf{g}(\boldsymbol{\eta}_{n,\ell}^{\{n\}}).$$

Here $\boldsymbol{\eta}_{n,i}^{\{n\}} + \boldsymbol{\eta}_{n,j}^{\{n\}} + \boldsymbol{\eta}_{n,\ell}^{\{n\}} \in [2]^{n+1} \setminus E_{\{n\}}$, therefore Claim 5.8.1 (3) shows that $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}} + \boldsymbol{\eta}_{n,j}^{\{n\}} + \boldsymbol{\eta}_{n,\ell}^{\{n\}}) \in [2]^{\Sigma T+1} \setminus E_T$, and hence $\boldsymbol{\iota}_{p}^{T} + \mathbf{g}(\boldsymbol{\eta}_{n,\ell}^{\{n\}}) \in [2]^{\Sigma T+1} \setminus E_T$. For the same reason, $\mathbf{g}(\boldsymbol{\eta}_{n,\ell}^{\{n\}}) \in E_T = \{\boldsymbol{\eta}_{r,s}^{T},\ \boldsymbol{\mu}_{r,s}^{T} : r \in T,\ s \in \lfloor r \rceil\}$, because $\boldsymbol{\eta}_{n,\ell}^{\{n\}} \in E_{\{n\}}$. Since $\boldsymbol{\iota}_{p}^{T} + \boldsymbol{\eta}_{p,s}^{T} = \boldsymbol{\mu}_{p,s}^{T} \in E_T$ and $\boldsymbol{\iota}_{p}^{T} + \boldsymbol{\mu}_{p,s}^{T} = \boldsymbol{\eta}_{p,s}^{T} \in E_T$ hold for all $s \in \lfloor p \rceil$, we conclude that $\mathbf{g}(\boldsymbol{\eta}_{n,\ell}^{\{n\}})$ equals $\boldsymbol{\eta}_{r,s}^{T}$ or $\boldsymbol{\mu}_{r,s}^{T}$ for some $r \neq p$. Thus, the (T, p)-block of $\mathbf{g}(\boldsymbol{\eta}_{n,\ell}^{\{n\}})$ is $(1, \dots, 1)$.

Now fix $k \in \lfloor n \rceil \setminus \{i, j\}$, and consider

$$\mathbf{w} := \sum_{\ell \in \lfloor n \rceil \setminus \{k\}} \mathbf{g}(\boldsymbol{\eta}_{n,\ell}^{\{n\}}).$$

By Claim 5.8.1, $\mathbf{w} = \mathbf{g}(\boldsymbol{\mu}_{n,k}^{\{n\}}) \in E_T$. However, in the sum on the right hand side, $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) + \mathbf{g}(\boldsymbol{\eta}_{n,j}^{\{n\}}) = \boldsymbol{\iota}_{p}^{T}$, and, as we established in the preceding paragraph, the (T, p)-block of the remaining summands is $(1, \dots, 1)$. The number of summands is odd, therefore the (T, p)-block of $\mathbf{w}$ equals $(0, \dots, 0)$, which contradicts $\mathbf{w} \in E_T$.

The claim about $\mathbf{g}(\boldsymbol{\mu}_{n,i}^{\{n\}})$ and $\mathbf{g}(\boldsymbol{\mu}_{n,j}^{\{n\}})$ is proved similarly, by switching the roles of the $\boldsymbol{\eta}$'s and the $\boldsymbol{\mu}$'s. $\diamond$

Let us define the map $\rho \colon \lfloor n \rceil \to D_T$ by the rule $\rho(i) = (p, q)$ if and only if $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) \in \{\boldsymbol{\eta}_{p,q}^{T}, \boldsymbol{\mu}_{p,q}^{T}\}$. By Claims 5.8.2 and 5.8.3, ρ is injective.

Denote $\mathbf{d} := \sum_{i=1}^{n} \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}})$. It follows from (5.5) that

$$\mathbf{d} = \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) + \mathbf{g}(\boldsymbol{\mu}_{n,i}^{\{n\}}) \qquad \text{for all } i \in \lfloor n \rceil. \tag{5.7}$$

By Claim 5.8.1 (3), the summands $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}})$ and $\mathbf{g}(\boldsymbol{\mu}_{n,i}^{\{n\}})$ belong to E_T, because $\boldsymbol{\eta}_{n,i}^{\{n\}}, \boldsymbol{\mu}_{n,i}^{\{n\}} \in E_{\{n\}}$. Thus, for each i, (5.7) is a decomposition $\mathbf{d} = \mathbf{u} + \mathbf{v}$ of $\mathbf{d}$ with summands $\mathbf{u}, \mathbf{v} \in E_T$. The condition $\mathbf{u}, \mathbf{v} \in E_T$ yields that there exist $\kappa, \lambda \in T$, $\alpha \in \lfloor \kappa \rceil$, $\beta \in \lfloor \lambda \rceil$ such that $\mathbf{u} \in \{\boldsymbol{\eta}_{\kappa,\alpha}^{T}, \boldsymbol{\mu}_{\kappa,\alpha}^{T}\}$, $\mathbf{v} \in \{\boldsymbol{\eta}_{\lambda,\beta}^{T}, \boldsymbol{\mu}_{\lambda,\beta}^{T}\}$.

Claim 5.8.4. If $\mathbf{u} + \mathbf{v} = \mathbf{d}$ holds for $\mathbf{u} \in \{\boldsymbol{\eta}_{\kappa,\alpha}^{T}, \boldsymbol{\mu}_{\kappa,\alpha}^{T}\}$ and $\mathbf{v} \in \{\boldsymbol{\eta}_{\lambda,\beta}^{T}, \boldsymbol{\mu}_{\lambda,\beta}^{T}\}$, then $\kappa = \lambda$ and $\alpha = \beta$.

Proof of Claim 5.8.4. Let $\mathbf{u}$, $\mathbf{v}$ satisfy the assumptions, and suppose first that $\kappa \neq \lambda$. Assuming, without loss of generality, that $\kappa < \lambda$, and denoting the entries in the (T, κ)-block of $\mathbf{u}$ by x, y, and the entries in the (T, λ)-block of $\mathbf{v}$ by χ, ψ, we can write $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{d}$ as follows:

$$\begin{array}{rl} & \qquad\qquad \beta_T(\kappa,\alpha) \qquad\qquad\qquad\qquad \beta_T(\lambda,\beta) \\ & \qquad\qquad\quad \downarrow \qquad\qquad\qquad\qquad\qquad\quad \downarrow \\ \mathbf{u} = & (1\cdots 1 \;\; x\cdots xyx\cdots x \;\; 1\cdots 1 \;\; 1\cdots 111\cdots 1 \;\; 1\cdots 1 \;\; 1), \\ \mathbf{v} = & (1\cdots 1 \;\; 1\cdots 111\cdots 1 \;\; 1\cdots 1 \;\; \chi\cdots\chi\psi\chi\cdots\chi \;\; 1\cdots 1 \;\; 1), \\ \mathbf{d} = & (0\cdots 0 \;\; \underbrace{y\cdots yxy\cdots y}_{(T,\kappa)\text{-block}} \;\; 0\cdots 0 \;\; \underbrace{\psi\cdots\psi\chi\psi\cdots\psi}_{(T,\lambda)\text{-block}} \;\; 0\cdots 0 \;\; 0). \end{array}$$

The equality $\mathbf{d} = \mathbf{u} + \mathbf{v}$ holds, because $\{x, y\} = \{\chi, \psi\} = [2]$ and $+$ is performed modulo 2. This shows that

- the (T, κ)-block of $\mathbf{d}$ contains either a single occurrence of 0 and $\kappa - 1$ occurrences of 1, or a single occurrence of 1 and $\kappa - 1$ occurrences of 0,
- the (T, λ)-block of $\mathbf{d}$ contains either a single occurrence of 0 and $\lambda - 1$ occurrences of 1, or a single occurrence of 1 and $\lambda - 1$ occurrences of 0, and
- the remaining entries of $\mathbf{d}$, i.e., those belonging neither to the (T, κ)-block nor to the (T, λ)-block, are all 0.

It is easy to verify that the only possible way of decomposing $\mathbf{d}$ into a sum of two elements of E_T is $\mathbf{u} + \mathbf{v}$. Thus, $\{\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}), \mathbf{g}(\boldsymbol{\mu}^{\{n\}}_{n,i})\} = \{\mathbf{u}, \mathbf{v}\}$ for all $i \in \lfloor n \rceil$. Since $n \geq 6$, there exist distinct $i, j \in \lfloor n \rceil$ such that $\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}) = \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,j})$, which contradicts Claim 5.8.2.

This proves that $\kappa = \lambda$. Now suppose that $\alpha \neq \beta$. Assuming, without loss of generality, that $\alpha < \beta$, and denoting the entries in the (T, κ)-blocks of $\mathbf{u}$ and $\mathbf{v}$ by x, y and χ, ψ, respectively, we can write $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{d}$ as follows:

$$\begin{array}{rl} & \qquad\qquad \beta_T(\kappa,\alpha) \qquad\quad \beta_T(\kappa,\beta) \\ & \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \\ \mathbf{u} = & (1\cdots 1 \;\; x\cdots x\, y\, x\cdots x\, x\, x\cdots x \;\; 1\cdots 1 \;\; 1), \\ \mathbf{v} = & (1\cdots 1 \;\; \chi\cdots\chi\,\chi\,\chi\cdots\chi\,\psi\,\chi\cdots\chi \;\; 1\cdots 1 \;\; 1), \\ \mathbf{d} = & (0\cdots 0 \;\; \underbrace{X\cdots XYX\cdots XYX\cdots X}_{(T,\kappa)\text{-block}} \;\; 0\cdots 0 \;\; 0). \end{array}$$

To check that $\mathbf{d}$ has the given form, notice that $\{x, y\} = \{\chi, \psi\} = [2]$, therefore $x \neq y$ and $\chi \neq \psi$, so with addition $+$ modulo 2 we get that $x + \chi \neq y + \chi = x + \psi$. Hence, with the choice $X := x + \chi$, $Y := y + \chi = x + \psi$, the equality $\mathbf{d} = \mathbf{u} + \mathbf{v}$ holds. This shows that

- the (T, κ)-block of $\mathbf{d}$ contains either two occurrences of 0 and $\kappa - 2$ occurrences of 1, or two occurrences of 1 and $\kappa - 2$ occurrences of 0, and
- the entries outside the (T, κ)-block are all 0.

In the former case, the only possible decompositions of $\mathbf{d}$ into a sum of two elements of E_T are $\boldsymbol{\eta}^T_{\kappa,\alpha} + \boldsymbol{\mu}^T_{\kappa,\beta}$ and $\boldsymbol{\mu}^T_{\kappa,\alpha} + \boldsymbol{\eta}^T_{\kappa,\beta}$; that is, for all $i \in \lfloor n \rceil$, $\{\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}), \mathbf{g}(\boldsymbol{\mu}^{\{n\}}_{n,i})\} = \{\boldsymbol{\eta}^T_{\kappa,\alpha}, \boldsymbol{\mu}^T_{\kappa,\beta}\}$ or $\{\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}), \mathbf{g}(\boldsymbol{\mu}^{\{n\}}_{n,i})\} = \{\boldsymbol{\mu}^T_{\kappa,\alpha}, \boldsymbol{\eta}^T_{\kappa,\beta}\}$. In the latter case, the only possible decompositions of $\mathbf{d}$ into a sum of two elements of E_T are $\boldsymbol{\eta}^T_{\kappa,\alpha} + \boldsymbol{\eta}^T_{\kappa,\beta}$ and $\boldsymbol{\mu}^T_{\kappa,\alpha} + \boldsymbol{\mu}^T_{\kappa,\beta}$; that is, for all $i \in \lfloor n \rceil$, $\{\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}), \mathbf{g}(\boldsymbol{\mu}^{\{n\}}_{n,i})\} = \{\boldsymbol{\eta}^T_{\kappa,\alpha}, \boldsymbol{\eta}^T_{\kappa,\beta}\}$ or $\{\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}), \mathbf{g}(\boldsymbol{\mu}^{\{n\}}_{n,i})\} = \{\boldsymbol{\mu}^T_{\kappa,\alpha}, \boldsymbol{\mu}^T_{\kappa,\beta}\}$. Since $n \geq 6$, the pigeonhole principle yields in both cases that there exist distinct $i, j \in \lfloor n \rceil$ such that $\mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,i}) = \mathbf{g}(\boldsymbol{\eta}^{\{n\}}_{n,j})$, which contradicts Claim 5.8.2. $\diamond$

Claim 5.8.5. For the tuple $\mathbf{d}$ defined above, we have that

$$\text{either} \quad \mathbf{d} = \mathbf{0}^T \qquad \text{or} \qquad \mathbf{d} = \boldsymbol{\iota}_\kappa^T \quad \text{for some } \kappa \in T. \tag{5.8}$$

Proof of Claim 5.8.5. (5.7) and the discussion following it shows that $\mathbf{d} = \mathbf{u} + \mathbf{v}$ for some $\mathbf{u}, \mathbf{v} \in E_T$. Therefore, by Claim 5.8.4, there exist $\kappa \in T$ and $\alpha \in \lfloor \mu \rceil$ such that $\mathbf{u}, \mathbf{v} \in \{\boldsymbol{\eta}_{\kappa,\alpha}^T, \boldsymbol{\mu}_{\kappa,\alpha}^T\}$. Using the fact that $\boldsymbol{\eta}_{\kappa,\alpha}^T + \boldsymbol{\mu}_{\kappa,\alpha}^T = \boldsymbol{\iota}_\kappa^T$, we get that $\mathbf{d} = \mathbf{0}^T$ if $\mathbf{u} = \mathbf{v}$, and $\mathbf{d} = \boldsymbol{\iota}_\kappa^T$ if $\mathbf{u} \neq \mathbf{v}$. ◇

Recall that ρ is the injective map $\lfloor n \rceil \to D_T$ that assigns to each $i \in \lfloor n \rceil$ the pair $\rho(i) = (p, q)$ such that $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) \in \{\boldsymbol{\eta}_{p,q}^T, \boldsymbol{\mu}_{p,q}^T\}$.

Claim 5.8.6. For each $m \in T$, either $D_{\{m\}} \subseteq \operatorname{Im} \rho$ or $D_{\{m\}} \cap \operatorname{Im} \rho = \emptyset$.

Proof of Claim 5.8.6. Suppose, on the contrary, that there is $m \in T$ such that $\emptyset \neq D_{\{m\}} \cap \operatorname{Im} \rho \subsetneq D_{\{m\}}$. Consider the (T, m)-block of $\mathbf{d} = \sum_{i=1}^n \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}})$. The (T, m)-block of each summand $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) \in \{\boldsymbol{\eta}_{\rho(i)}^T, \boldsymbol{\mu}_{\rho(i)}^T\}$ is

(i) the constant tuple $(1, \dots, 1)$, if $\rho(i) = (p, q)$ with $p \neq m$, and
(ii) an almost constant tuple of the form $(0, \dots, 0, 1, 0, \dots, 0) + (c, \dots, c)$ ($c \in [2]$) with the sole 1 in the q-th position, if $\rho(i) = (m, q)$.

Our assumption $\emptyset \neq D_{\{m\}} \cap \operatorname{Im} \rho \subsetneq D_{\{m\}}$ implies that, as i runs over the elements of $\lfloor n \rceil$, each one of the two cases (i)–(ii) occurs at least once. Furthermore, the injectivity of ρ implies that for distinct i's for which case (ii) applies, the 1's will occur in different positions. It follows that the (T, m)-block of $\mathbf{d}$ contains both 0 and 1 as an entry, which contradicts Claim 5.8.5. ◇

Claim 5.8.7. There is exactly one $m \in T$ such that $D_{\{m\}} \subseteq \operatorname{Im} \rho$.

Proof of Claim 5.8.7. Since $\operatorname{Im} \rho \neq \emptyset$, we get from Claim 5.8.6 that there exists at least one $m \in T$ such that $D_{\{m\}} \subseteq \operatorname{Im} \rho$. Suppose, on the contrary, that there are at least two such elements of T, and fix m to be one of them. Then we obtain from $m \in T \subseteq N$ and $D_{\{m\}} \subsetneq \operatorname{Im} \rho$ that m is even and $6 \leq m = |D_{\{m\}}| < |\operatorname{Im} \rho| = n$.

Let $R := \rho^{-1}(D_{\{m\}})$. We know that ρ is injective and that for each $i \in R$, $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) \in \{\boldsymbol{\eta}_{\rho(i)}^T, \boldsymbol{\mu}_{\rho(i)}^T\} = \{\boldsymbol{\eta}_{m,q}^T, \boldsymbol{\mu}_{m,q}^T\}$ for some $q \in \lfloor m \rceil$. Hence, $|R| = m$ is even and $\{\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) : i \in R\}$ is a transversal for $\big\{\{\boldsymbol{\eta}_{m,q}^T, \boldsymbol{\mu}_{m,q}^T\} : q \in \lfloor m \rceil\big\}$. We have $\boldsymbol{\mu}_{m,q}^T = \boldsymbol{\eta}_{m,q}^T + \boldsymbol{\iota}_m^T$ for each $q \in \lfloor m \rceil$, and since m is even, we also have that $\sum_{q \in \lfloor m \rceil} \boldsymbol{\eta}_{m,q}^T = \boldsymbol{\iota}_m^T$. Therefore, if the number of $\boldsymbol{\mu}$'s among the tuples $\mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}})$ ($i \in R$) is k, then

$$\sum_{i \in R} \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) = \boldsymbol{\iota}_m^T + k\boldsymbol{\iota}_m^T = \mathbf{0}^T \quad \text{or} \quad \boldsymbol{\iota}_m^T,$$

depending on the parity of k.

Now let us fix an element $r \in R$. Then $\rho(r) \in D_{\{m\}}$, say $\rho(r) = (m, r')$, so $\mathbf{g}(\boldsymbol{\eta}_{n,r}^{\{n\}}) \in \{\boldsymbol{\eta}_{m,r'}^T, \boldsymbol{\mu}_{m,r'}^T\}$. Thus, the last displayed equality, along with the equality $\boldsymbol{\mu}_{m,r'}^T = \boldsymbol{\eta}_{m,r'}^T + \boldsymbol{\iota}_m^T$, implies that

$$\begin{aligned} \sum_{i \in R \setminus \{r\}} \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) &= \mathbf{g}(\boldsymbol{\eta}_{n,r}^{\{n\}}) + \sum_{i \in R} \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) \\ &\in \{\boldsymbol{\eta}_{m,r'}^T + \mathbf{0}^T, \boldsymbol{\mu}_{m,r'}^T + \mathbf{0}^T, \boldsymbol{\eta}_{m,r'}^T + \boldsymbol{\iota}_m^T, \boldsymbol{\mu}_{m,r'}^T + \boldsymbol{\iota}_m^T\} = \{\boldsymbol{\eta}_{m,r'}^T, \boldsymbol{\mu}_{m,r'}^T\}. \end{aligned}$$

Since $|R\setminus\{r\}| = |R|-1 = m-1$ is odd, we can apply Claim 5.8.1 (1) to conclude that

$$\mathbf{g}\Big(\sum_{i\in R\setminus\{r\}} \boldsymbol{\eta}_{n,i}^{\{n\}}\Big) = \sum_{i\in R\setminus\{r\}} \mathbf{g}(\boldsymbol{\eta}_{n,i}^{\{n\}}) \in \{\boldsymbol{\eta}_{m,r'}^{T}, \boldsymbol{\mu}_{m,r'}^{T}\} \subseteq E_T.$$

On the other hand, the fact that $5 \leq |R\setminus\{r\}| = m-1 \leq n-2$ shows that

$$\sum_{i\in R\setminus\{r\}} \boldsymbol{\eta}_{n,i}^{\{n\}} \in [2]^n \setminus E_{\{n\}}.$$

This contradicts Claim 5.8.1 (3), and hence finishes the proof. $\diamond$

Let m be the element of T given by Claim 5.8.7. Then actually $\operatorname{Im}\rho = D_{\{m\}}$, and we conclude that ρ is a bijection from the n-element set $\lfloor n\rceil$ onto the m-element set $D_{\{m\}}$. Hence, $n = m \in T$. $\square$

We are now ready to prove the main result of this section.

Proof of Theorem 5.1. The theorem will follow from the special case $|B| = 2$ of Corollary 5.5 and from Lemmas 5.7 and 5.8, if we show that for each clone $\mathcal{C}$ on A satisfying one of the conditions (A), (B)$'$, (C)$'$, there exists a two-element subset $B \subseteq A$ such that $\mathcal{C}_B$ is a subclone of $\mathrm{PClo}(\mathbf{B})$, where $\mathbf{B} = (B;+)$ is a semilattice or a group.

If $\mathcal{C}$ satisfies condition (A), then for any choice of a two-element subset B of A, the operations in $\mathcal{C}_B$ depend on at most m variables. Post's description [15] of all clones on a two-element set shows that in this case the members of $\mathcal{C}_B$ depend, in fact, on at most one variable. Hence $\mathcal{C}_B$ is a subclone of $\mathrm{PClo}(\mathbf{B})$, where $\mathbf{B} = (B;+)$ is a two-element group.

If $\mathcal{C}$ satisfies condition (B)$'$, then for any choice of a two-element subset B of A, $\mathcal{C}_B$ is a subclone of $\mathrm{PClo}(\mathbf{B})$, where $\mathbf{B} = (B;+)$ is a two-element group.

Assume therefore that $\mathcal{C}$ satisfies condition (C)$'$. Let m denote the exponent of $\mathbf{A}$, and let $\mathbf{S} = (S;+)$ be the semilattice of all idempotent elements of $\mathbf{A}$ (see Proposition 2.5). First we will consider the case when $|S| > 1$. Let $\leq$ denote the natural order on $\mathbf{S}$ defined by $x \leq y$ iff $x + y = y$, and choose $\mathbf{B} = (B;+)$ to be a two-element subsemilattice of $\mathbf{S}$ with $B = \{o, e\}$ such that $o < e$ and there is no $s \in S$ such that $o < s < e$.

Our goal is to show that $\mathcal{C}_B$ is a subclone of $\mathrm{PClo}(\mathbf{B})$. We may assume without loss of generality that $\mathcal{C} = \mathrm{PClo}({}_{\mathbf{E}}\mathbf{A})$ ($\mathbf{E} = \mathrm{End}(\mathbf{A})$). Let $f(x_1,\dots,x_n) = a + \sum_{i=1}^n u_i(x_i)$ ($a \in A^0$, $u_1,\dots,u_n \in \mathbf{E}^0$) be an n-ary operation in $\mathcal{C}$ such that $f(B,\dots,B) \subseteq B$, and recall that $\mathbf{E}^0$ is a set of endomorphisms of $\mathbf{A}^0$. Let $c = ma + o$, and for each i, let v_i be the function $A \to A$ defined by $v_i(x) = mu_i(x) + o$. Since $\mathbf{E}^0$ is closed under $+$, each $v_i \in \mathbf{E}^0$. Thus the operation $g(x_1,\dots,x_n) = c + \sum_{i=1}^n v_i(x_i)$ is a member of $\mathcal{C}$ and satisfies $g(\mathbf{x}) = mf(\mathbf{x}) + o$ for all $\mathbf{x} \in A^n$. Since $mb + o = b$ for each $b \in B$, we get that $g(B,\dots,B) \subseteq B$ and $f|_B = g|_B$. Moreover, for all $b_1,\dots,b_n \in B$, the definitions of c and v_i ($1 \leq i \leq n$) imply that $c \in S$ with $c \geq o$ and $v_i(b_i) \in S$ with $v_i(b_i) \geq o$. On the other hand, the condition $c + \sum_{i=1}^n v_i(b_i) = g(b_1,\dots,b_n) \in B$ implies that $c \leq e$ and $v_i(b_i) \leq e$. Thus it follows from our choice of B that $c \in B$ and $v_i(b_i) \in B$ for all i. Thus $f|_B(x_1,\dots,x_n) = g|_B(x_1,\dots,x_n) = c + \sum_{i=1}^n v_i|_B(x_i)$ where $c \in B$ and $v_1|_B,\dots,v_n|_B \in \mathrm{End}(\mathbf{B})$. Since the only elements of $\mathrm{End}(\mathbf{B})$ are the identity endomorphism and the two constant endomorphisms, it follows that $f|_B \in \mathrm{PClo}(\mathbf{B})$, as claimed.

It remains to consider the case when $|S| = 1$. Thus $\mathbf{A}$ is an abelian group (see Proposition 2.5). Let 0 denote the neutral element of $\mathbf{A}$, let $b \in A$ be an element of prime order p, and let $B = \{0, b\}$. Let $f(x_1, \ldots, x_n) = a + \sum_{i=1}^{n} u_i(x_i)$ ($a \in A^0 = A$, $u_1, \ldots, u_n \in \mathbf{E}^0 = \mathbf{E}$) be an n-ary operation in $\mathcal{C}$ such that $f(B, \ldots, B) \subseteq B$. Since $u_i(0) = 0$ for each i, we get that $a = f(0, \ldots, 0) \in B$ and $a + u_i(b) = f(0, \ldots, 0, b, 0, \ldots, 0) \in B$. Hence, for each i, there are only two possibilities for the function $u_i|_B \colon B \to A$, one being constant 0, and the other one

(1) $u_i|_B(x) = x$ for all $x \in B$ if $a = 0$, and
(2) $u_i|_B(x) = -x$ for all $x \in B$ if $a = b$.

If $p = 2$, then $\mathbf{B} = (B; +)$ is a subgroup of $\mathbf{A}$, and these considerations show that u_i is either the constant 0 or the identity endomorphism of $\mathbf{B}$. Hence it follows that $f|_B \in \mathrm{PClo}(\mathbf{B})$. If $p > 2$, then for $i \neq j$, $u_i|_B$ and $u_j|_B$ cannot simultaneously be nonconstant. Otherwise, if, say, $u_1|_B$ and $u_2|_B$ are both nonconstant, then $a + u_1(b) + u_2(b) = f(b, b, 0, \ldots, 0) \in B$, and conditions (1)–(2) imply that $2b = 0 + b + b \in B$ if $a = 0$, and $-b = b + (-b) + (-b) \in B$ if $a = b$, which is impossible for $p > 2$. Therefore $f|_B$ depends on at most one of its variables. This implies that $\mathcal{C}_B$ is a subclone of $\mathrm{PClo}(\mathbf{B})$ for any two-element group $\mathbf{B} = (B; \boxplus)$ on B. □

6. $\mathcal{C}$-minors of Boolean functions

We conclude this paper by bringing together results pertaining to the $\mathcal{C}$-minor partial orders on the set $\mathcal{O}_{[2]}$ of Boolean functions. The two semilattice operations on [2] will be denoted by $\wedge$ and $\vee$, and addition modulo 2 by $+$. For posets, lexicographic product and disjoint union will be denoted by $\times_{\mathrm{lex}}$ and $\uplus$, respectively. For the two-element antichain we use the notation $\overline{2}$.

Theorem 6.1. *For a clone $\mathcal{C}$ on* [2]*, the $\mathcal{C}$-minor partial order $(\mathcal{O}_{[2]}/{\equiv_{\mathcal{C}}}; \preceq_{\mathcal{C}})$ is*

(i) *finite, if $\mathcal{C}$ contains the discriminator function*

$$t(x, y, z) = \begin{cases} x, & \text{if } x \neq y, \\ z, & \text{otherwise,} \end{cases} \qquad (x, y, z \in [2]);$$

(ii) *isomorphic to*

$$(\omega; \leq) \times_{\mathrm{lex}} \overline{2}, \quad (\omega; \leq) \uplus (\omega; \leq) \quad \text{or} \quad (\omega; \leq) \uplus (\omega; \leq) \uplus (\omega; \leq) \uplus (\omega; \leq),$$

if $\mathcal{C}$ is one of the clones $\langle \wedge, \vee, 0, 1 \rangle$, $\langle \wedge, \vee, 0 \rangle$, $\langle \wedge, \vee, 1 \rangle$, or $\langle \wedge, \vee \rangle$;

(iii) *universal for the class* **FPI** *of countable posets whose principal ideals are finite, if $\mathcal{C}$ is a subclone of $\langle +, 0, 1 \rangle$, $\langle \wedge, 0, 1 \rangle$ or $\langle \vee, 0, 1 \rangle$; and*

(iv) *universal for the class of all countable posets, otherwise.*

Proof. Statement (i) is established in [11, Corollary 4.2]. Statement (ii) follows from the results obtained in [8] and [9]; see also the discussion in [10] for further explanation. Statement (iii) can be derived from Theorem 3.1 of the current paper as follows. By Post's description [15] of all clones on [2], every subclone $\mathcal{C}$ of $\langle +, 0, 1 \rangle$, $\langle \wedge, 0, 1 \rangle$ or $\langle \vee, 0, 1 \rangle$ either satisfies condition (A) of Theorem 3.1 (with $m = 1$) or condition (C) of Theorem 3.1. Finally, statement (iv) is proved in [10, Theorem 15]. □

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