The final paper appears in: Electron. J. Probab. **23** (2018), no. 20, 1–27.

Evolution systems of measures and semigroup properties on evolving manifolds

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Abstract

An evolving Riemannian manifold $(M, g_t)_{t\in I}$ consists of a smooth d-dimensional manifold M, equipped with a geometric flow g_t of complete Riemannian metrics, parametrized by $I = (-\infty, T)$. Given an additional $C^{1,1}$ family of vector fields $(Z_t)_{t\in I}$ on M. We study the family of operators $L_t = \Delta_t + Z_t$ where Δ_t denotes the Laplacian with respect to the metric g_t. We first give sufficient conditions, in terms of space-time Lyapunov functions, for non-explosion of the diffusion generated by L_t, and for existence of evolution systems of probability measures associated to it. Coupling methods are used to establish uniqueness of the evolution systems under suitable curvature conditions. Adopting such a unique system of probability measures as reference measures, we characterize supercontractivity, hypercontractivity and ultraboundedness of the corresponding time-inhomogeneous semigroup. To this end, gradient estimates and a family of (super-)logarithmic Sobolev inequalities are established.

Keywords: Evolution system of measures; geometric flow; inhomogeneous diffusion; semigroup; supercontractivity; hypercontractivity; ultraboundedness.
AMS MSC 2010: 60J60; 58J65; 53C44.
Submitted to EJP on August 16, 2017, final version accepted on February 2, 2018.
Supersedes arXiv:1708.04951v2.

1 Introduction

Let M be a d-dimensional differentiable manifold equipped with a family of complete Riemannian metrics $(g_t)_{t\in I}$ which is C^1 in t and evolves according to

$$\frac{\partial}{\partial t} g_t = 2h_t, \quad t \in I,$$

where $I = (-\infty, T)$ for some $T \in (-\infty, +\infty]$, and h_t a time-dependent 2-tensor on TM. Denote by ∇^t, Δ_t the Levi-Civita connection, resp. Laplacian on M, both with respect to the metric g_t. For a given $C^{1,1}$ family $(Z_t)_{t\in I}$ of vector fields on M, we study the time-dependent second order differential elliptic operator $L_t = \Delta_t + Z_t$.

In this paper, we develop the basis for a general theory of the following backward Cauchy problem:

$$\begin{Bmatrix} \partial_s u(\cdot, x)(s) = -L_s u(s, \cdot)(x) \\ u(t, x) = \phi(x) \end{Bmatrix}, \quad (s,t) \in \Lambda,\ x \in M, \tag{1.1}$$

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where $\phi \in C^2(M) \cap C_b(M)$ and $\Lambda := \{(s,t) : s \le t \text{ and } s,t \in I\}$.

We investigate this problem from a probabilistic point of view. Let X_t be the diffusion process generated by L_t (called L_t-diffusion) which is assumed to be non-explosive before time T (see [2, 8, 14] for details). As in the time-homogeneous case, we construct L_t-diffusions X_t via horizontal diffusions u_t above X_t.

Let $\mathrm{F}(M)$ be the frame bundle over M and $\mathrm{O}_t(M)$ the orthonormal frame bundle with respect to the metric g_t. We denote by $\pi\colon \mathrm{F}(M) \to M$ the projection from $\mathrm{F}(M)$ onto M. For a frame $u \in \mathrm{O}_t(M)$, denote by $H_Y^t(u)$ the ∇^t-horizontal lift of $Y \in T_{\pi u}M$. This allows one to determine standard-horizontal vector fields H_i^t on $\mathrm{O}_t(M)$, via the formula

$$H_i^t(u) = H_{ue_i}^t(u), \quad i = 1, 2, \ldots, d,$$

where $(e_i)_{i=1}^d$ denotes the canonical orthonormal basis of $\mathbb{R}^d$. Furthermore, we denote by $(V_{\alpha,\beta})_{\alpha,\beta=1}^d$ the standard-vertical fields on $\mathrm{F}(M)$. Then given $s \ge 0$, the diffusion u_t is constructed for $t \ge s$ as the solution to the following Stratonovich SDE:

$$\begin{cases} \mathrm{d}u_t = \sqrt{2}\sum_{i=1}^d H_i^t(u_t) \circ \mathrm{d}B_t^i + H_{Z_t}^t(u_t)\,\mathrm{d}t - \dfrac{1}{2}\sum_{\alpha,\beta=1}^d (\partial_t g_t)(u_t e_\alpha, u_t e_\beta) V_{\alpha,\beta}(u_t)\,\mathrm{d}t, \\ u_s \in \mathrm{O}_s(M),\ \pi u_s = x, \end{cases} \tag{1.2}$$

where B_t is a standard Brownian motion on $\mathbb{R}^d$. The projection $X_t := \pi u_t$ of u_t onto M then gives the wanted L_t-diffusion process on M, see [2]. In the next section we complement existing results on non-explosion of X_t which is a subject already studied in [14].

The backward Cauchy problem (1.1) is the Kolmogorov equation to the following non-autonomous SDE on M:

$$\mathrm{d}X_t = u_t \circ \mathrm{d}B_t + Z_t(X_t)\,\mathrm{d}t, \quad X_s = x. \tag{1.3}$$

Denote by $X_t^{(s,x)}$ the solution to Eq. (1.3) which is assumed to be non-explosive before time T. Then function

$$u(s,x) := \mathbb{E}[\phi(X_t^{(s,x)})]$$

satisfies Eq. (1.1) and gives rise to a family of inhomogeneous Markov evolution operators $(P_{s,t})_{(s,t)\in\Lambda}$ on M:

$$P_{s,t}\phi(x) := \mathbb{E}[\phi(X_t^{(s,x)})] = \mathbb{E}^{(s,x)}[\phi(X_t)].$$

This is completely standard in the case of a fixed metric and a time-independent operator $L_t = L$ where $P_{s,t} = P_{t-s} = \mathrm{e}^{(t-s)L}$ and L^p-spaces are taken with respect to an invariant measure μ, i.e., a Borel probability measure μ on M such that

$$\int_M P_t\phi\,\mathrm{d}\mu = \int_M \phi\,\mathrm{d}\mu, \quad t > 0,\ \phi \in \mathcal{B}_b(M).$$

Under suitable conditions, see [5, 6], existence and uniqueness of the invariant measure can be shown. In this case, P_t extends to a contraction semigroup on $L^p(M,\mu)$ for every $p \in [1,\infty)$, see e.g. [3, 11, 18, 21, 22].

When it comes to the time-inhomogeneous case, the situation turns out to be more involved. For instance, Saloff-Coste and Zúñiga [19, 20] studied the ergodic behavior of time-inhomogeneous Markov chains; more sophisticated and strict conditions are required due to the fact that the generator and the semigroup do not commute and due to the lack of uniqueness of the invariant measure. A first goal will be therefore to construct an evolution system of measures as a family of reference measures which plays a role similar to the invariant measure in the time-homogeneous case.

Let us start by reviewing the notion of an evolution system of measures. A family of Borel probability measures $(\mu_t)_{t\in I}$ on M is called an evolution system of measures (see [9]) if

$$\int_M P_{s,t}\phi \,\mathrm{d}\mu_s = \int_M \phi \,\mathrm{d}\mu_t, \quad \phi \in \mathcal{B}_b(M),\ (s,t)\in\Lambda. \tag{1.4}$$

Recently, Angiuli, Lorenzi, Lunardi et al. investigated evolution system of measures and related topics for non-autonomous parabolic Kolmogorov equations with unbounded coefficients on $\mathbb{R}^d$ (see [1, 12, 15, 16]). For instance, in [12] sufficient conditions for existence and uniqueness of evolution systems of measures are given; in [1], using a unique tight evolution system of measures as reference measures, hypercontractivity and the asymptotic behavior are studied; the asymptotics in time-periodic parabolic problems with unbounded coefficients is addressed in [16]. All this work motivates us to study evolution systems of measures on evolving manifolds and to investigate contractivity properties of the semigroup. Our probabilistic approach simplifies and extends in particular earlier results obtained by analytic methods.

We start by formulating some hypotheses which will be needed later on. Let $\rho_t(x,y)$ be the Riemannian distance from x to y with respect to the metric g_t. Fixing $o\in M$, we write $\rho_t(x) := \rho_t(o,x)$ for simplicity. Let Cut_t be the set of the cut-locus of (M,g_t). Let

$$\mathrm{Cut} := \{(x,t)\colon x\in \mathrm{Cut}_t\}.$$

At different places in the paper, some of the hypotheses listed below will be put in force.

(H1) There exists an increasing function $\varphi\in C^2(\mathbb{R}^+)$ such that

$$\lim_{r\to+\infty}\varphi(r) = +\infty \ \text{ and}$$
$$(L_t+\partial_t)(\varphi\circ\rho_t)(x) \le m(t), \quad (x,t)\in M\times I\setminus \mathrm{Cut},$$

for some continuous function m on I.

(H2) There exists an increasing function $\varphi\in C^2(\mathbb{R}^+)$ such that $\varphi(0)=0$,

$$\lim_{r\to+\infty}\varphi(r) = +\infty \ \text{ and}$$
$$(L_t+\partial_t)(\varphi\circ\rho_t)(x) \le a(t) - c(t)(\varphi\circ\rho_t)(x), \quad (x,t)\in M\times I\setminus \mathrm{Cut},$$

for some non-negative function a and a function c on I such that

$$H(t) := \int_{-\infty}^t \exp\left(-\int_r^t c(u)\,\mathrm{d}u\right) a(r)\,\mathrm{d}r < \infty.$$

(H3) There exists a function k on I such that

$$\mathcal{R}_t^Z := \mathrm{Ric}_t - h_t - \nabla^t Z_t \ge k(t), \quad t\in I.$$

For any $\epsilon>0$, positive function ℓ on I and $t\in I$, set

$$A_1 = 2k-\ell, \quad B_1(t) = 2d + \frac{1}{4}\left(3(d-1)\epsilon^{-1} + 3k_\epsilon(t)\epsilon + 2|Z_t|_t(o)\right)^2 \ell^{-1}(t),$$

where

$$k_\epsilon(t) = \sup\left\{\, |\mathrm{Ric}_t|\,(x)\colon \rho_t(x)\le\epsilon\right\}. \tag{1.5}$$

There exists a positive constant ϵ and a positive function ℓ on I such that

$$H_1(t) := \int_{-\infty}^t \exp\left(-\int_r^t A_1(s)\,\mathrm{d}s\right) B_1(r)\,\mathrm{d}r < +\infty. \tag{1.6}$$

Remark 1.1. In **(H1)** and **(H2)** the Lyapunov function $\varphi \circ \rho_t$ is by definition time-dependent.

(a) From condition (1.6) it can be seen that the function k in **(H3)** must satisfy

$$\int_{-\infty}^{t} \exp\left(-2\int_r^t k(s)\,\mathrm{d}s\right)\mathrm{d}r < \infty \text{ and } \int_{-\infty}^{t} k(s)\,\mathrm{d}s = +\infty, \quad t \in I. \tag{1.7}$$

(b) Hypothesis **(H1)** gives a sufficient condition for non-explosion of L_t-diffusions. Hypothesis **(H2)** ensures existence of an evolution system of measures $(\mu_t)_{t\in I}$, whereas **(H3)** guarantees uniqueness of the evolution system of measures $(\mu_t)_{t\in I}$.

(c) As indicated, the Lyapunov function $\varphi \circ \rho_t$ is time-dependent. Comparatively, in [12] the Euclidean distance is used as reference distance and then a space only Lyapunov condition is sufficient for existence and uniqueness of an evolution system of measures. In [1, 12] the coefficients in the Lyapunov condition are uniformly bounded, and as consequence a time-homogeneous process can be used for comparison with the original process. In our setting, the coefficients in the Lyapunov conditions need to be time-dependent to preserve the information about the varying space.

In general, evolution systems of measures are far from being unique. If there is a unique system it plays an important role. Indeed, it is related to the asymptotic behavior of $P_{s,t}$ as $s \to -\infty$. We shall prove that if Hypothesis **(H3)** holds, then for $x \in M$ and $(s,t) \in \Lambda$,

$$\lim_{s\to-\infty} \|P_{s,t}f(x) - \mu_t(f)\|_{L^2(M,\mu_s)} = 0,$$

where $\mu_t(f)$ denotes the average of f with respect to the measure μ_t.

In Sections 3-5 we use Hypothesis **(H3)** as standing assumption. Taking the unique evolution system of measures $(\mu_s)_{s\in I}$ as reference measures, we study contractivity properties of the time-inhomogeneous semigroup $P_{s,t}$. For the sake of brevity, we introduce the following notations:

$$\|P_{s,t}\|_{(p,t)\to(q,s)} := \|P_{s,t}\|_{L^p(M,\,\mu_t)\to L^q(M,\,\mu_s)},$$
$$\|P_{s,t}\|_{(p,t)\to\infty} := \|P_{s,t}f\|_{L^p(M,\,\mu_t)\to L^\infty(M)}.$$

Definition 1.2. *The evolution operator $P_{s,t}$ is called*

(i) *hypercontractive if it maps $L^p(M,\mu_t)$ into $L^q(M,\mu_s)$ for some $1 < p < q < +\infty$ and $(s,t) \in \Lambda$ such that*

$$\|P_{s,t}\|_{(p,t)\to(q,s)} \le 1;$$

(ii) *supercontractive if it maps $L^p(M,\mu_t)$ into $L^q(M,\mu_s)$ for any $1 < p < q < +\infty$ and $(s,t) \in \Lambda$, and if there exists a positive function $C_{p,q} : \Lambda \to (0,+\infty)$ such that*

$$\|P_{s,t}\|_{(p,t)\to(q,s)} \le C_{p,q}(s,t);$$

(iii) *ultrabounded if it maps $L^p(M,\mu_t)$ into $L^\infty(M)$ for every $p > 1$ and $(s,t) \in \Lambda$, and if there exists a function $C_{p,\infty} : \Lambda \to (0,+\infty)$ such that*

$$\|P_{s,t}f\|_{(p,t)\to\infty} \le C_{p,\infty}(s,t). \tag{1.8}$$

Remark 1.3. It is easy to see that due to contractivity of the semigroup, the function $C_{p,q}$ can be chosen such that the following properties are satisfied:

(i) For fixed $s \in I$, the function $C_{p,q}(s, s+\cdot)\colon (0,\infty) \to (0,\infty)$ is a non-increasing function;

(ii) for fixed $t \in I$, the function $C_{p,q}(t-\cdot, t)\colon (0,\infty) \to (0,\infty)$ is a non-increasing function.

Note that the function $C_{p,q}$ takes into account both the position and the length of the interval $[s,t]$.

In what follows, we use the abbreviation

$$\|\cdot\|_{p,\,s} := \|\cdot\|_{L^p(M,\,\mu_s)}.$$

In Section 4, we extend the arguments of [18] to consider hypercontractivity and supercontractivity via logarithmic Sobolev inequalities (in short log-Sobolev inequalities). In fact, under the assumption that $\mathcal{R}_t^Z \geq k(t)$ for $t \in I$, there is a family of log-Sobolev inequalities with respect to $P_{s,t}$:

$$P_{s,t}(f^2 \log f^2) \leq 4 \left(\int_s^t \exp\left(-2 \int_r^t k(u)\,\mathrm{d}u \right) \mathrm{d}r \right) P_{s,t} |\nabla^t f|_t^2 + P_{s,t} f^2 \log P_{s,t} f^2,$$
$$f \in C_b^1(M),\ (s,t) \in \Lambda.$$

Hypercontractivity of $P_{s,t}$ in L^p space, related to the unique evolution system of measures, is then obtained as a consequence of the log-Sobolev inequalities.

In Section 5 we then prove that supercontractivity of the evolution operators $P_{s,t}$ is equivalent to the validity of the following family of super-log-Sobolev inequalities

$$\int_M f^2 \log \frac{|f|}{\|f\|_{2,s}}\,\mathrm{d}\mu_s \leq r \left\| |\nabla^s f|_s \right\|_{2,s}^2 + \beta_s(r)\,\|f\|_{2,s}^2, \quad r > 0,$$

for every $s \in I$, $f \in H^1(M, \mu_s)$ and some positive decreasing function β_s. Note that the function β_s may depend on the current time s which generalizes the notion of super-log-Sobolev inequalities for non-autonomous systems on $\mathbb{R}^d$ in [1]. Moreover, combining the super-log-Sobolev inequalities and dimension-free Harnack inequalities, we prove that the exponential integrability of radial function with respect to $(\mu_t)_{t\in I}$ or $(P_{s,t})_{(s,t)\in\Lambda}$ is equivalent to supercontractivity or ultraboundedness of the corresponding semigroup.

The paper is organized as follows. In Section 2 we first give sufficient conditions for existence and uniqueness of evolution systems of measures. Then in Section 3, by means of Bismut type formulas, gradient estimates in $L^p(M, \mu_s)$ are established for $p \in (1, +\infty]$, which are used in Sections 4-5 to study hypercontractivity, supercontractivity and ultraboundedness for the corresponding semigroup.

2 Diffusion processes and evolution system of measures

2.1 Non-explosion

Recall that $\rho_t(x)$ denotes the distance function $\rho_t(o,x)$ with respect to a fixed reference point $o \in M$. A sufficient condition for non-explosion of L_t-diffusions can be given as follows.

Theorem 2.1. *Suppose that Hypothesis* **(H1)** *holds. Then L_t-diffusion process X_t is non-explosive before time T.*

Proof. Without loss of generality, we suppose that the L_t-process X_t starts from x at time s. For fixed $t^* \in (s, T]$, there exists $c := \sup_{t\in[s,t^*]} m(t) > 0$ such that

$$(L_t + \partial_t)\,\varphi \circ \rho_t(x) \leq c, \quad (t,x) \in [s,t^*] \times M.$$

Then, by the Itô formula for the radial part of X_t (see [14, Theorem 2]), we obtain

$$\begin{aligned}\mathrm{d}\varphi\circ\rho_t(X_t) &\le \sqrt{2}\left\langle u_t^{-1}\nabla^t\varphi\circ\rho_t(X_t),\mathrm{d}B_t\right\rangle+(L_t+\partial_t)\,\varphi\circ\rho_t(X_t)\,\mathrm{d}t\\ &\le \sqrt{2}\left\langle u_t^{-1}\nabla^t\varphi\circ\rho_t(X_t),\mathrm{d}B_t\right\rangle+c\,\mathrm{d}t\end{aligned}$$

up to the lifetime $\zeta\wedge t^*$ where $\zeta:=\lim_{n\to\infty}\zeta_n$ with

$$\zeta_n:=\inf\{t\in(s,T)\colon \rho_t(x,X_t)\ge n\}.$$

In particular, if $X_s=x\in M$, then

$$\varphi(n)\mathbb{P}^x\{\zeta_n\le t\}\le \mathbb{E}^x[\varphi\circ\rho_{t\wedge\zeta_n}(X_{t\wedge\zeta_n})]\le\varphi(\rho_s(x))+ct,\quad t\in[s,t^*].$$

According to Hypothesis **(H1)**, φ is an increasing function such that $\varphi(r)\to+\infty$ as $r\to\infty$. Thus, there exists $m\in\mathbb{N}^+$ such that $\varphi(n)>0$ for all $n\ge m$ and

$$\mathbb{P}^x\{\zeta\le t\}\le\lim_{n\to\infty}\mathbb{P}^x\{\zeta_n\le t\}\le\lim_{n\to\infty}\frac{\varphi(\rho_s(x))+ct}{\varphi(n)}=0,\quad t\in[s,t^*].$$

Therefore we have $\mathbb{P}\{\zeta\ge t^*\}=1$. Since t^* is arbitrary, we obtain

$$\mathbb{P}\{\zeta\ge T\}=1$$

which completes the proof. □

From Theorem 2.1 we get the following corollary which has been proved in [14] in the case of a Lyapunov condition with constant coefficients.

Corollary 2.2. *Let $\psi\in C(\mathbb{R}^+)$ and $h\in C(I)$ be non-negative such that for any $t\in I$,*

$$(L_t+\partial_t)\rho_t(x)\le h(t)\psi(\rho_t(x))\tag{2.1}$$

holds outside $\mathrm{Cut}_t(o)$, the cut-locus of o associated with the metric g_t. If

$$\int_1^\infty\mathrm{d}t\int_1^t\exp\left(-\int_r^t\psi(s)\,\mathrm{d}s\right)\mathrm{d}r=\infty,\tag{2.2}$$

then the L_t-diffusion process is non-explosive.

Proof. Suppose that the process X_t generated by L_t starts from x at time $s\in I$. For fixed $t^*\in(s,T]$, let $c=\sup_{t\in[s,t^*]}h(t)$ and

$$\varphi(s)=\int_1^s\mathrm{d}t\int_1^t\exp\left(-c\int_r^t\psi(u)\,\mathrm{d}u\right)\mathrm{d}r.$$

It is easy to see from condition (2.2) that φ is an increasing function on $\mathbb{R}^+$ with $\varphi(r)\to\infty$ as $r\to\infty$, satisfying

$$(L_t+\partial_t)\varphi(\rho_t(x))\le 1,\quad t\in[s,t^*].$$

This completes the proof. □

2.2 Evolution systems of measures

For $t \in I$ consider the linear second order differential operator L_t given on a smooth function f by

$$L_t f = (\Delta_t + Z_t) f.$$

As indicated, Hypothesis **(H1)** guarantees the existence of a unique Markov semigroup $P_{s,t}$ generated by L_t. Indeed, for fixed $t \in I$ and $f \in C_b(M)$, the function $(s,x) \mapsto P_{s,t} f(x)$ is the unique bounded classical solution in $C_b((-\infty,t] \times M) \cap C^{1,2}((-\infty,t] \times M)$ to the backward Cauchy problem:

$$\begin{cases} \partial_s u(\cdot,x)(s) = -L_s u(s,\cdot)(x), & (s,x) \in (-\infty,t) \times M, \\ u(t,x) = f(x), & x \in M. \end{cases} \tag{2.3}$$

According to the uniqueness of solutions to Eq. (2.3), we obtain

$$P_{s,r}\, P_{r,t} = P_{s,t}, \quad s \le r \le t < T.$$

Moreover, for any $(s,t) \in \Lambda$, $x \in M$ and $f \in C^2(M)$ with $\|L_t f\|_\infty < \infty$, the forward Kolmogorov equation reads as

$$\frac{\partial}{\partial t} P_{s,t} f(x) = P_{s,t} L_t f(x),$$

and for any $f \in \mathcal{B}_b(M)$, $(s,t) \in \Lambda$ and $x \in M$, the backward Kolmogorov equation is given by

$$\frac{\partial}{\partial s} P_{s,t} f(x) = -L_s P_{s,t} f(x).$$

Based on Hypothesis **(H2)** or **(H3)**, one can prove existence and uniqueness of an evolution system.

Theorem 2.3. *Suppose that Hypothesis* **(H2)** *holds, then there exists an evolution system of measures* $(\mu_t)_{t \in I}$ *for* $(P_{s,t})_{(s,t)\in\Lambda}$ *such that*

$$\sup_{s \in (-\infty,t]} \int_M (\varphi \circ \rho_s)(y)\, \mu_s(\mathrm{d}y) \le H(t). \tag{2.4}$$

Suppose that Hypothesis **(H3)** *holds, then there exists a unique evolution system and*

$$\sup_{s \in (-\infty,t]} \int_M \rho_s(y)^2\, \mu_s(\mathrm{d}y) < H_1(t). \tag{2.5}$$

Proof. (a) We first show existence. Given $t \in I$, a family of measures can be constructed as follows (see e.g. [6] for details). For $A \in \mathcal{B}(M)$ and $(s,t) \in \Lambda$, let

$$\mu_{s,t}(A) := \frac{1}{t-s} \int_s^t P_{r,t}(o, A)\, \mathrm{d}r.$$

We claim that under Hypothesis **(H2)**, the family of measures $(\mu_{s,t})_{s \in (-\infty,t]}$ is compact. Suppose that X_t starts from o at time s. Under Hypothesis **(H2)**, applying the Itô formula to the radial process $\rho_t(X_t)$, we get

$$\begin{aligned} \mathrm{d}\varphi(\rho_t(X_t)) &\le \varphi'(\rho_t(X_t)) \left\langle \nabla^t \rho_t(X_t), u_t \mathrm{d}B_t \right\rangle_t + (L_t + \partial_t)\varphi \circ \rho_t(X_t)\, \mathrm{d}t \\ &\le \varphi'(\rho_t(X_t)) \left\langle \nabla^t \rho_t(X_t), u_t \mathrm{d}B_t \right\rangle_t + \big(a(t) - c(t)\, \varphi \circ \rho_t(X_t)\big)\, \mathrm{d}t. \end{aligned}$$

It follows that

$$\mathbb{E}[\varphi(\rho_t(X_{t\wedge\zeta_n}))] - \varphi(0) \le \mathbb{E} \int_s^{t\wedge\zeta_n} \big(a(r) - c(r)\big(\varphi \circ \rho_r(X_r)\big)\big)\, \mathrm{d}r,$$

i.e.,

$$\mathbb{E}\left[\exp\left(\int_s^{t\wedge\zeta_n} c(r)\,\mathrm{d}r\right)\varphi\circ\rho_{t\wedge\zeta_n}(X_{t\wedge\zeta_n})\right] \le \varphi(0) + \mathbb{E}\left[\int_s^{t\wedge\zeta_n}\exp\left(\int_s^r c(u)\,\mathrm{d}u\right)a(r)\,\mathrm{d}r\right].$$

Using the condition $\varphi(0)=0$ and letting $n\to\infty$, we arrive at

$$\begin{aligned}\mathbb{E}\left[\varphi\circ\rho_t(X_t)\right] &\le \left[\int_s^t \exp\left(\int_s^r c(u)\,\mathrm{d}u\right)a(r)\,\mathrm{d}r\right]\exp\left(-\int_s^t c(r)\,\mathrm{d}r\right)\\ &\le \int_s^t \exp\left(-\int_r^t c(u)\,\mathrm{d}u\right)a(r)\,\mathrm{d}r\\ &\le H(t).\end{aligned}$$

Therefore, according to the monotonicity of H, we have

$$\sup_{s\in(-\infty,t]}(P_{s,t}\varphi\circ\rho_t)(o)\le H(t), \tag{2.6}$$

from which it follows that

$$\mu_{s,t}(\varphi\circ\rho_t)=\frac{1}{t-s}\int_s^t (P_{r,t}\varphi\circ\rho_t)(o)\,\mathrm{d}r\le H(t).$$

In addition, since φ is a compact and increasing function such that $\varphi(r)\to+\infty$ as $r\to+\infty$, we know that $(\mu_{s,t})_{s\in(-\infty,t]}$ is a family of compact measures, i.e., for each $n\in\mathbb{Z}$, there exists a sequence (t_{n_k}), $t_{n_k}\to+\infty$ as $k\to+\infty$ such that

$$\mu_{t_{n_k},n}\rightharpoonup^*\mu_n.$$

Let $\mu_s := P^*_{n,s}\mu_n$. It is easy to check that the family μ_s satisfies Eq. (1.4), i.e., for $\phi\in\mathscr{B}_b(M)$,

$$\mu_s(P_{s,t}\phi)=P^*_{n,s}\mu_n(P_{s,t}\phi)=\mu_n(P_{n,t}\phi)=\mu_t(\phi).$$

By this and the bound (2.6), we get the existence of an evolution system $(\mu_s)_{s\in I}$. Moreover, we have the estimate

$$\mu_s(\varphi\circ\rho_s)=\mu_n(P_{n,s}\varphi\circ\rho_s)\le\lim_{t_{n_k}\to\infty}\frac{1}{n-t_{n_k}}\int_{t_{n_k}}^n\sup_{r\in(-\infty,s]}(P_{r,s}\varphi\circ\rho_s)(o)\,\mathrm{d}r\le H(s)$$

which completes the proof of Eq. (2.4).

(b) If Hypothesis **(H3)** holds, we claim that there exists a unique evolution system of probability measures $(\mu_t)_{t\in I}$ such that

$$\sup_{s\in(-\infty,t]}\mu_s(\rho_s^2)<H_1(t).$$

First recall the formula (see [17, Lemma 5 and Remark 6])

$$\partial_t\rho_t(x)=\frac{1}{2}\int\partial_t g_t(\dot r(s),\dot r(s))\,\mathrm{d}s \tag{2.7}$$

where $r\colon[0,\rho_t(x)]\to M$ is a g_t-geodesic connecting o and x. By this formula and the index lemma, we have

$$\begin{aligned}(L_t+\partial_t)\rho_t &= (\Delta_t+Z_t+\partial_t)\rho_t\\ &\le \frac{(d-1)G'(\rho_t)}{G(\rho_t)}+\int_0^{\rho_t}\frac{1}{2}\partial_t g_t(\dot r(s),\dot r(s))\,\mathrm{d}s\\ &\quad+\int_0^{\rho_t}(\nabla^t Z_t)(\dot r(s),\dot r(s))\,\mathrm{d}s+\langle Z_t,\dot r(0)\rangle_t(o)\end{aligned} \tag{2.8}$$

where G is the solution to the equation

$$\begin{cases} G''(s) = \dfrac{-\mathrm{Ric}_t(\dot{r}(s), \dot{r}(s))}{d-1}\, G(s), \\ G(0) = 0,\ G'(0) = 1. \end{cases}$$

Under Hypothesis **(H3)**, by [14, Lemma 9], we have

$$\begin{aligned} (L_t + \partial_t)\rho_t &\le \frac{(d-1)G'(\rho_t)}{G(\rho_t)} - k(t)\rho_t + \int_0^{\rho_t} \mathrm{Ric}_t(\dot{r}(s), \dot{r}(s))\,\mathrm{d}s + \langle Z_t, \dot{r}(0)\rangle_t\,(o) \\ &\le \frac{(d-1)G'(\rho_t)}{G(\rho_t)} - k(t)\rho_t - \int_0^{\rho_t} \frac{(d-1)G''(s)}{G(s)}\,\mathrm{d}s + |Z_t|_t(o) \\ &\le F_t(\rho_t) - k(t)\rho_t + |Z_t|_t(o) \end{aligned}$$

where $F_t(s) = \sqrt{k_\epsilon(t)(d-1)}\coth\left(\sqrt{k_\epsilon(t)/(d-1)}\,(s \wedge \epsilon)\right) + k_\epsilon(t)\,(s\wedge\epsilon)$ and

$$k_\epsilon(t) := \sup\{|\mathrm{Ric}_t| : \rho_t(x) \le \epsilon\}.$$

It is easy to see that $F_t(s)$ is non-increasing in s and $\lim_{r\to 0} rF_t(r) < \infty$. Hence, by means of the positive function ℓ in Hypothesis **(H3)**, we obtain

$$\begin{aligned} (L_t + \partial_t)\rho_t^2 &= 2\rho_t(L_t+\partial_t)\rho_t + 2 \\ &\le 2\rho_t(F_t(\rho_t) - k(t)\rho_t + |Z_t|_t(o)) + 2 \\ &\le 2d + 2\left\{k_\epsilon(t)\epsilon + (d-1)\epsilon^{-1} + \sqrt{(d-1)k_\epsilon(t)} + |Z_t|_t(o)\right\}\rho_t - 2k(t)\rho_t^2 \\ &\le 2d + \left\{3\left(k_\epsilon(t)\epsilon + (d-1)\epsilon^{-1}\right) + 2|Z_t|_t(o)\right\}\rho_t - 2k(t)\rho_t^2 \\ &\le 2d + \frac{\left\{3k_\epsilon(t)\epsilon + 3(d-1)\epsilon^{-1} + 2|Z_t|_t(o)\right\}^2}{4\ell(t)} - (2k(t) - \ell(t))\rho_t^2. \end{aligned}$$

By a similar argument as in part (a), we obtain an evolution system of measures such that

$$\sup_{t\in(-\infty,s]} \mu_t(\rho_t^2) \le H_1(s).$$

We now use a coupling method to prove uniqueness of the evolution system. Let (X_t, Y_t) be a parallel coupling starting from (x,y) at time s. Then, by [7] or [13], we know that if $\mathcal{R}_t^Z \ge k(t)$, $t \in I$, then

$$\mathbb{E}^{(s,(x,y))}[\rho_t(X_t, Y_t)] \le \exp\left(-\int_s^t k(r)\,\mathrm{d}r\right)\rho_s(x,y).$$

Let $(\mu_t)_{t\in I}$ be an evolution system of measures. Then, we have the estimate:

$$\begin{aligned} |P_{s,t}f(o) - \mu_t(f)| &= \left|\int (P_{s,t}f(o) - P_{s,t}f(y))\,\mu_s(\mathrm{d}y)\right| \\ &= \left|\int \mathbb{E}^{(s,(o,y))}\left[\frac{f(X_t) - f(Y_t)}{\rho_t(X_t,Y_t)}\rho_t(X_t,Y_t)\right]\mu_s(\mathrm{d}y)\right| \\ &\le \left\||\nabla^t f|_t\right\|_\infty \int \mathbb{E}^{(s,(o,y))}\left[\rho_t(X_t,Y_t)\right]\,\mu_s(\mathrm{d}y) \\ &\le \exp\left(-\int_s^t k(r)\,\mathrm{d}r\right)\left\||\nabla^t f|_t\right\|_\infty \mu_s(\rho_s) \\ &\le \exp\left(-\int_s^t k(r)\,\mathrm{d}r\right)\left\||\nabla^t f|_t\right\|_\infty \left(\mu_s(\rho_s^2)\right)^{1/2}. \end{aligned} \tag{2.9}$$

In addition, from Eq. (1.7), we know that

$$\exp\left(-\int_{-\infty}^{t} k(r)\,\mathrm{d}r\right)=0 \quad \text{and} \quad \sup_{s\in(-\infty,t]} \mu_s(\rho_s^2)<\infty.$$

Now letting $s\to-\infty$, we conclude that

$$\lim_{s\to-\infty} |P_{s,t}f(o)-\mu_t(f)|=0.$$

If there exists another evolution system of probability measures $(\nu_t)_{t\in I}$, then $\nu_t(f)$ is also the limit of $P_{s,t}f(o)$ as $s\to-\infty$, and hence $\nu_t=\mu_t$. □

Directly from Eq. (2.9) we have the following asymptotic results.

Corollary 2.4. *Suppose that Hypothesis* **(H3)** *holds. Then we have the following convergence result: for any $f\in C^1(M)$ being constant outside a compact set, there exists a function c in $C(I)$ such that*

$$\|P_{s,t}f-\mu_t(f)\|_{2,s}\le c(t)\exp\left(-\int_s^t k(r)\,\mathrm{d}r\right)\big\||\nabla^t f|_t\big\|_\infty, \quad (s,t)\in\Lambda.$$

Proof. Let (X_t,Y_t) be parallel coupling process associated to L_t. For any $f\in C^1(M)$ being constant outside a compact set, we have

$$\begin{aligned}
|P_{s,t}f(x)-\mu_t(f)| &= |P_{s,t}f(x)-P_{s,t}f(o)+P_{s,t}f(o)-\mu_t(f)|\\
&\le \left|\mathbb{E}^{(s,(x,o))}\left[f(X_t)-f(Y_t)\right]\right| + \mathrm{e}^{-\int_s^t k(r)\,\mathrm{d}r}\big\||\nabla^t f|_t\big\|_\infty \left(\mu_s(\rho_s^2)\right)^{1/2}\\
&\le \left|\mathbb{E}^{(s,(x,o))}\left[\frac{f(X_t)-f(Y_t)}{\rho_t(X_t,Y_t)}\rho_t(X_t,Y_t)\right]\right| + \mathrm{e}^{-\int_s^t k(r)\,\mathrm{d}r}\big\||\nabla^t f|_t\big\|_\infty \left(\mu_s(\rho_s^2)\right)^{1/2}\\
&\le \big\||\nabla^t f|_t\big\|_\infty \mathbb{E}^{(s,(x,o))}\left[\rho_t(X_t,Y_t)\right] + \mathrm{e}^{-\int_s^t k(r)\,\mathrm{d}r}\big\||\nabla^t f|_t\big\|_\infty \left(\mu_s(\rho_s^2)\right)^{1/2}\\
&\le \mathrm{e}^{-\int_s^t k(r)\,\mathrm{d}r}\big\||\nabla^t f|_t\big\|_\infty \left(\rho_s(x)+\left(\mu_s(\rho_s^2)\right)^{1/2}\right)
\end{aligned} \tag{2.10}$$

which implies that

$$\|P_{s,t}f-\mu_t(f)\|_{2,s}\le 2\exp\left(-\int_s^t k(r)\,\mathrm{d}r\right)\big\||\nabla^t f|_t\big\|_\infty \left(\mu_s(\rho_s^2)\right)^{1/2}.$$

Now using Theorem 2.3 and

$$\sup_{s\in(-\infty,t]} \mu_s(\rho_s^2)<\infty,$$

we obtain the result directly. □

Corollary 2.5. *Suppose that Hypothesis* **(H3)** *holds and $\sup_{s\in(-\infty,t]}\rho_s(x)<\infty$ for any $x\in M$ and $t\in I$. Then we have the following convergence result: for any $f\in C_b^1(M)$, there exists a function C in $C(I)$ such that*

$$|P_{s,t}f-\mu_t(f)|\le C(t)\exp\left(-\int_s^t k(r)\,\mathrm{d}r\right)\big\||\nabla^t f|_t\big\|_\infty, \quad (s,t)\in\Lambda.$$

Proof. If $\sup_{s\in(-\infty,t]}\rho_s(x)<\infty$ for any $x\in M$ and $t\in I$, then the result can be directly derived from the inequality (2.10). □

Remark 2.6. Actually, our results can be applied to the following forward Cauchy problem via a time reversal: for $s\in[T,+\infty)$,

$$\begin{cases}\partial_t u(\cdot,x)(t)=L_t u(t,\cdot)(x), & (t,x)\in(s,+\infty)\times M;\\ u(s,x)=f(x), & x\in M.\end{cases}$$

2.3 Some examples

We now investigate some non-autonomous systems on evolving manifolds to illustrate the results of Subsection 2.2 above.

Example 2.7. The manifold M is the Euclidean space $\mathbb{R}^d$ and the geometric flow g_t is given by

$$(g_t)_{ij} = g(t)\delta_{ij}$$

for some positive function $g \in C^1(I)$. Consider the operator $L_t = \Delta_t + Z_t = g(t)^{-1}(\Delta + Z)$. It is easy to see that $k_\varepsilon = 0$ and $|Z_t|_t(o) = g(t)^{-1/2}|Z|(o)$ where k_ε is defined as in (1.5). Moreover, assume that there exists constant C such that $\nabla Z \le C$. Then the curvature satisfies

$$\mathcal{R}_t^Z = -\frac{1}{2}g'(t)\langle\cdot,\cdot\rangle - g^{-2}(t)\langle\nabla_\cdot Z,\cdot\rangle \ge -\frac{1}{2}g'(t) - g^{-2}(t)C =: A(t).$$

Hence, by Theorem 2.3, if we can choose $\ell > 0$ such that

$$\int_t^\infty \exp\left(-\int_r^t (2A-\ell)(s)\,\mathrm{d}s\right)\left(1 + (g(r)\ell(r))^{-1}\right)\mathrm{d}r < \infty,$$

then the non-autonomous system has an unique evolution system of measures.

Example 2.8. The evolving manifold M carries a backward Ricci flow (g_t):

$$\partial_t g_t = 2\mathrm{Ric}_t, \quad t \in (-\infty, T].$$

Consider the operator $L_t = \Delta_t + \nabla^t V$ for some $V \in C_b^2(M)$. Assume that $\mathrm{Hess}_V^t \le -k(t)$, $t \in (-\infty, T]$ and that there exists $o \in M$ such that for small $\varepsilon > 0$,

$$k_\varepsilon(t) \le K \quad \text{and} \quad |Z_t|_t(o) \le C$$

for some constants K and C where k_ε is defined as in (1.5). Hence, by Theorem 2.3, if

$$\int_{-\infty}^t \exp\left(-\int_r^t (2k(s) - \ell(s))\,\mathrm{d}s\right)(1 + \ell(r)^{-1})\,\mathrm{d}r < \infty \tag{2.11}$$

for some positive function ℓ on I, the non-autonomous diffusion system has an unique evolution system of measures. Here, for instance, if $k(t) = |t|^{-\alpha}$ with $0 < \alpha < 1$, choosing $\ell(t) = |t|^{-\alpha}$, it is easy to check that (2.11) holds.

Example 2.9. Suppose that M is a hypersurface parameterized locally by $X = \{x^i\}$ in $\mathbb{R}^d$ and evolving by its backward mean curvature flow, $t \in (-\infty, T]$. Let $\{H_{ij}\}$ be the second fundamental form of M and $H = g^{ij}H_{ij}$ its mean curvature. It is well known that

$$\begin{aligned} &\frac{\partial}{\partial t} g_{ij} = 2HH_{ij}; \\ &\mathrm{Ric}_{ij} = H_{ij}H - g^{mr}H_{ir}H_{mj}. \end{aligned}$$

Consider the process (X_t) generated by $L_t = \Delta_t$. Then

$$(\mathcal{R}_t^Z)_{ij} := (\mathrm{Ric}_t - h_t)_{ij} = -g^{mr}H_{ir}H_{mj}.$$

Assume that $\mathcal{R}_t^Z \ge k(t)$ and that there exists $o \in M$ such that $k_\varepsilon \le K$ for some constant K. Hence, by Theorem 2.3, if

$$\int_{-\infty}^t \exp\left(-\int_r^t (2k(s) - \ell(s))\ \mathrm{d}s\right)(1 + \ell(r)^{-1})\,\mathrm{d}r < \infty$$

for some positive function ℓ on I, the non-autonomous diffusion system has an unique evolution system of measures.

Example 2.10. Consider an evolving manifold (M, g_t) satisfying the following curvature condition:

$$\begin{aligned}&\mathrm{Ric}_t \geq -C_1(t)(1+\rho_t^2);\\&\partial_t\rho_t + Z_t\rho_t \leq C_2(t)(1+\rho_t),\end{aligned}$$

for some non-negative function C_1 and some function C_2. Then

$$\begin{aligned}(L_t+\partial_t)\rho_t &\leq 2\rho_t\Delta_t\rho_t + 2C_2(t)(\rho_t+\rho_t^2)+2\\&\leq 2\rho_t\sqrt{(d-1)C_1(t)\,(1+\rho_t^2)}\,\coth\left(\sqrt{C_1(t)\,(1+\rho_t^2)/(d-1)}\,\rho_t\right)\\&\quad + 2C_2(t)\,(\rho_t+\rho_t^2)+2.\end{aligned}$$

Combining this with the inequality $\coth(s) \leq 1+s^{-1}$, we obtain that for any positive function ℓ on I,

$$\begin{aligned}(L_t+\partial_t)\rho_t^2 &\leq 2\left(\sqrt{(d-1)C_1(t)}+C_2(t)\right)(\rho_t+\rho_t^2)+2d\\&\leq \left(2\sqrt{(d-1)C_1(t)}+2C_2(t)+\ell(t)\right)\rho_t^2\\&\quad+\left(\sqrt{(d-1)C_1(t)}+C_2(t)\right)^2\ell^{-1}(t)+2d\end{aligned}$$

Then by Theorem 2.3, if C_1 and C_2 satisfy

$$\begin{aligned}&\int_{-\infty}^t \exp\left(\int_r^t\left(2\sqrt{(d-1)C_1(s)}+2C_2(s)+\ell(s)\right)\mathrm{d}s\right)\\&\qquad\times\left(\left(\sqrt{(d-1)C_1(r)}+C_2(r)\right)^2\ell^{-1}(r)+2d\right)\mathrm{d}r<\infty\end{aligned}$$

for some positive function ℓ, there exists an evolution system of measures for this system.

3 Gradient estimates

We now turn to gradient estimates for the semigroup. It is well known that the so-called Bismut formula is a powerful tool to derive gradient estimates of semigroups in the fixed metric case (see [4, 10]). Let us first recall a Bismut type formula for $\nabla^s P_{s,t}f$ (see [7, Corollary 3.2]). To this end, define an $\mathbb{R}^d\otimes\mathbb{R}^d$-valued process $(Q_{s,t})_{(s,t)\in\Lambda}$ as the solution to the following ordinary differential equation

$$\frac{\mathrm{d}Q_{s,t}}{\mathrm{d}t} = -\mathcal{R}_t^Z(u_t)Q_{s,t},\quad Q_{s,s}=\mathrm{id},\ (s,t)\in\Lambda, \tag{3.1}$$

where u_t is the horizontal L_t-diffusion process $X_t^{(s,x)}$ with $\pi(u_s)=x$, and $\mathcal{R}_t^Z(u_t)\in\mathbb{R}^d\otimes\mathbb{R}^d$ satisfies

$$\left\langle\mathcal{R}_t^Z(u_t)a,b\right\rangle_{\mathbb{R}^d} = \mathcal{R}_t^Z(u_ta,u_tb),\quad a,b\in\mathbb{R}^d.$$

If $\mathcal{R}_t^Z \geq k(t)$, $t\in I$ then we have

$$\|Q_{r,t}\| \leq \exp\left(-\int_r^t k(s)\,\mathrm{d}s\right),\quad (r,t)\in\Lambda, \tag{3.2}$$

where $\|\cdot\|$ is the operator norm on $\mathbb{R}^d$. The following derivative formula is taken from [7].

Proposition 3.1. *Assume that $\mathcal{R}_t^Z \geq k(t)$ for some continuous function k on I. Let $(s,t) \in \Lambda$. Then for $f \in C^1(M)$ such that f is constant outside a compact set, and for any $h \in C_b^1([s,t])$ satisfying $h(s) = 0$ and $h(t) = 1$, we have*

$$u_s^{-1}\nabla^s P_{s,t} f(x) = \mathbb{E}^{(s,x)}\left[Q_{s,t}^* u_t^{-1} \nabla^t f(X_t)\right] = \frac{1}{\sqrt{2}}\mathbb{E}^{(s,x)}\left[f(X_t)\int_s^t h'(r) Q_{s,r}^* \,\mathrm{d}B_r\right] \tag{3.3}$$

where $Q_{s,t}^*$ is the transpose of $Q_{s,t}$.

The following gradient estimate can be derived from Proposition 3.1.

Theorem 3.2. *Suppose that Hypothesis* **(H3)** *holds. Let $(\mu_t)_{t\in I}$ be the evolution system of measures for $P_{s,t}$. Then,*

(a) *for every $f \in C^1(M)$ such that f is constant outside a compact set and $1 \leq p < \infty$,*

$$\big\||\nabla^s P_{s,t} f|_s\big\|_{p,s} \leq \exp\left(-\int_s^t k(r)\,\mathrm{d}r\right)\big\||\nabla^t f|_t\big\|_{p,t}, \quad (s,t) \in \Lambda; \tag{3.4}$$

(b) *for any $1 < p < \infty$, there exists a positive constant $C_1 = C_1(p)$ such that for every $f \in \mathscr{B}_b(M)$,*

$$\big\||\nabla^s P_{s,t} f|_s\big\|_{p,s} \leq C_1 \left(\max_{r\in[s,(t-1)\vee s]}\int_r^{(r+1)\wedge t}\exp\left(\int_s^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)^{-1}\|f\|_{p,t}$$

for all $(s,t) \in \Lambda$;

(c) *for $f \in \mathscr{B}_b(M)$, there exists a positive constant $C_1 = C_1(p)$ such that for all $(s,t) \in \Lambda$.*

$$\big\||\nabla^s P_{s,t} f|_s\big\|_{\infty} \leq C_1 \left(\max_{r\in[s,(t-1)\vee s]}\int_r^{(r+1)\wedge t}\exp\left(\int_s^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)^{-1}\|f\|_{\infty}.$$

Proof. By the first equality in (3.3) and inequality (3.2), the first assertion in (a) can be derived directly. It is also easy to see that (c) follows from (b). Hence, it suffices to prove (b).

For $p \in (1,\infty)$ and $t - s \leq 1$, by using the integration by parts formula, we have

$$\begin{aligned}
|\nabla^s P_{s,t} f|_s^p(x) &= \frac{1}{\sqrt{2}}\Big|\mathbb{E}^{(s,x)}\Big[f(X_t)\int_s^t h'(r) Q_{s,r}^*\,\mathrm{d}B_r\Big]\Big|^p \\
&\leq \frac{1}{\sqrt{2}} P_{s,t}|f|^p(x)\left(\mathbb{E}^{(s,x)}\Big|\int_s^t h'(r) Q_{s,r}^*\,\mathrm{d}B_r\Big|^q\right)^{p/q} \\
&\leq \frac{c_p^p}{\sqrt{2}} P_{s,t}|f|^p(x)\left(\mathbb{E}^{(s,x)}\Big|\int_s^t {h'}^2(r)\|Q_{s,r}\|^2\,\mathrm{d}r\Big|^{q/2}\right)^{p/q} \\
&\leq \frac{c_p^p}{\sqrt{2}} P_{s,t}|f|^p(x)\left(\mathbb{E}^{(s,x)}\Big|\int_s^t {h'}^2(r)\exp\left(-2\int_s^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\Big|^{q/2}\right)^{p/q}
\end{aligned} \tag{3.5}$$

where

$$h(r) = \frac{\int_s^r \exp\left(\int_s^\rho k(u)\,\mathrm{d}u\right)\mathrm{d}\rho}{\int_s^t \exp\left(\int_s^\rho k(u)\,\mathrm{d}u\right)\mathrm{d}\rho}.$$

It then follows that

$$|\nabla^s P_{s,t} f|_s^p(x) \leq \frac{c_p^p}{\sqrt{2}}\, P_{s,t}|f|^p(x)\left(\int_s^t \exp\left(\int_s^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)^{-p}.$$

Integrating both sides of the inequality above with respect to μ_s, we arrive at

$$\mu_s(|\nabla^s P_{s,t} f|_s^p) \leq \frac{c_p^p}{\sqrt{2}}\, \mu_t(|f|^p) \left(\int_s^t \exp\left(\int_s^r k(u)\,\mathrm{d}u \right) \mathrm{d}r \right)^{-p}. \tag{3.6}$$

It leaves us to check the case for $t - s > 1$. For any $r \in [s, t-1]$, combining Eq. (3.4) and Eq. (3.6), we have

$$\begin{aligned}
&|\nabla^s P_{s,r} P_{r,t} f(x)|_s^p \leq \exp\left(-p \int_s^r k(r)\,\mathrm{d}r \right) P_{s,r} |\nabla^r P_{r,t} f|_r^p(x) \\
&\leq \frac{c_p^p}{\sqrt{2}} \exp\left(-p \int_s^r k(r)\,\mathrm{d}r \right) \left(\int_r^{r+1} \exp\left(\int_r^\rho k(u)\,\mathrm{d}u \right) \mathrm{d}\rho \right)^{-p} P_{s,r}(P_{r,r+1}|P_{r+1,t} f|^p)(x) \\
&\leq \frac{c_p^p}{\sqrt{2}} \left(\int_r^{r+1} \exp\left(\int_s^\rho k(u)\,\mathrm{d}u \right) \mathrm{d}\rho \right)^{-p} P_{s,t} |f|^p(x).
\end{aligned}$$

Integrating both sides by μ_s and minimizing the coefficient in r, we obtain the desired conclusion. □

Remark 3.3. In Theorem 3.2 (c), the inequality does not need an evolution system of measures as the reference measures. So the condition for this result can be weaken by only using

$$\mathcal{R}_t^Z \geq k(t)$$

for some function $k \in C(I)$.

4 Log-Sobolev inequality and hypercontractivity

In this section, we prove hypercontractivity for $P_{s,t}$. Let us first introduce the following log-Sobolev inequality, which is essential to the proof of our hypercontractivity theorem.

Proposition 4.1. *If $\mathcal{R}_t^Z \geq k(t)$ for some function $k \in C(I)$ then for any $p \in (1, \infty)$,*

$$\begin{aligned}
P_{s,t}(f^2 \log f^2) \leq 4 \left(\int_s^t \exp\left(-2 \int_r^t k(u)\,\mathrm{d}u \right) \mathrm{d}r \right) P_{s,t} |\nabla^t f|_t^2 \\
+ P_{s,t} f^2 \log P_{s,t} f^2, \quad (s,t) \in \Lambda,
\end{aligned} \tag{4.1}$$

holds for $f \in C_c^1(M)$.

Proof. Without loss of generality, we suppose $f > \delta > 0$. Otherwise, let $f_\delta = (f^2 + \delta)^{1/2}$. Then by letting $\delta \to 0$, we obtain the conclusion.

Consider the process $(P_{r,t} f^2) \log(P_{r,t} f^2)(X_{r \wedge \tau_n})$ where as above

$$\tau_n = \inf\{t \in (s,T] \colon \rho_t(X_t) \geq n\}, \quad n \geq 1. \tag{4.2}$$

Applying Itô's formula, we have

$$\begin{aligned}
\mathrm{d}(P_{r,t} f^2) \log(P_{r,t} f^2)(X_r) &= \mathrm{d}M_r + (L_r + \partial_r)(P_{r,t} f^2 \log P_{r,t} f^2)(X_r)\,\mathrm{d}r \\
&= \mathrm{d}M_r + \left(\frac{1}{P_{r,t} f^2} |\nabla^r P_{r,t} f^2|_r^2 \right) (X_r)\,\mathrm{d}r, \quad s < r < \tau_n \wedge t,
\end{aligned}$$

where M_r is a local martingale. By this and the estimate,

$$\begin{aligned}
|\nabla^r P_{r,t} f^2|_r^2 &\leq \exp\left(-2 \int_r^t k(u)\,\mathrm{d}u \right) \left(P_{r,t} |\nabla^t f^2|_t \right)^2 \\
&\leq 4 \exp\left(-2 \int_r^t k(u)\,\mathrm{d}u \right) (P_{r,t} f^2) P_{r,t} |\nabla^t f|_t^2,
\end{aligned}$$

we obtain

$$\begin{aligned}&\mathrm{d}(P_{r,t}f^2)\log(P_{r,t}f^2)(X_r)\\&\quad\le \mathrm{d}M_r+4\exp\left(-2\int_r^t k(u)\,\mathrm{d}u\right)P_{r,t}|\nabla^t f|_t^2(X_r)\,\mathrm{d}r,\quad s<r<\tau_n\wedge t.\end{aligned}$$

Integrating both sides from s to $t\wedge\tau_n$, we have

$$\begin{aligned}&\mathbb{E}^{(s,x)}\left[f^2\log f^2(X_{t\wedge\tau_n})\right]\\&\le \left(P_{s,t}f^2\log P_{s,t}f^2\right)(x)+\mathbb{E}^{(s,x)}\left[\int_s^{t\wedge\tau_n}4\exp\left(-2\int_r^t k(u)\,\mathrm{d}u\right)P_{r,t}|\nabla^t f|_t^2(X_r)\,\mathrm{d}r\right].\end{aligned}$$

Then again by dominated convergence theorem, letting $n\uparrow+\infty$, we obtain

$$P_{s,t}(f^2\log f^2)\le 4\left(\int_s^t\exp\left(-2\int_r^t k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)P_{s,t}|\nabla^t f|_t^2+P_{s,t}f^2\log P_{s,t}f^2.\qquad\square$$

The log-Sobolev inequality leads to hypercontractivity of $(P_{s,t})$.

Theorem 4.2. *Suppose that Hypothesis* **(H3)** *holds and that* (μ_t) *is the evolution system of measures for* $P_{s,t}$. *Let* $r\le s\le t<T$ *and* $p,q\in(1,\infty)$ *such that*

$$q\le\exp\left(-\int_\cdot^t\left(\int_r^{s_1}\exp\left(-2\int_u^{s_1}k(z)\,\mathrm{d}z\right)\mathrm{d}u\right)^{-1}\mathrm{d}s_1\right)(p-1)+1.$$

Then $P_{s,t}\colon L^p(M,\mu_t)\to L^q(M,\mu_s)$ *satisfies*

$$\|P_{s,t}\|_{(p,t)\to(q,s)}\le 1.$$

Proof. For the sake of conciseness, we assume $f>\delta>0$, otherwise a similar argument as in the proof of Proposition 4.1 can be used. Consider the process

$$(P_{s,t}f)^{q(s)}(X_{s\wedge\tau_n}),\quad s\in[r,t],$$

where

$$q(\cdot)=\exp\left(-\int_\cdot^t\left(\int_r^{s_1}\mathrm{e}^{-2\int_u^{s_1}k(z)\,\mathrm{d}z}\,\mathrm{d}u\right)^{-1}\mathrm{d}s_1\right)(p-1)+1.$$

Using the Itô formula, we have that for $s<\tau_n\wedge t$,

$$\begin{aligned}\mathrm{d}(P_{s,t}f)^{q(s)}(X_s)&=\mathrm{d}M_s+(L_s+\partial_s)(P_{s,t}f)^{q(s)}(X_s)\,\mathrm{d}s\\&=\mathrm{d}M_s+(P_{s,t}f)^{q(s)}\Big(q(s)(q(s)-1)|\nabla^s\log P_{s,t}f|_s^2+q'(s)\log P_{s,t}f\Big)(X_s)\,\mathrm{d}s.\end{aligned}$$

Therefore, for $r\le s\le t<T$,

$$\begin{aligned}&\mathbb{E}^{(r,x)}\left[(P_{s,t}f)^{q(s)}(X_{s\wedge\tau_n})\right]-(P_{r,t}f)^{q(r)}(x)\\&=\int_r^s\bigg(q(u)(q(u)-1)\mathbb{E}^{(r,x)}\left[(P_{u,t}f)^{q(u)-2}|\nabla^u P_{u,t}f|_u^2(X_{u\wedge\tau_n})\right]\\&\qquad+q'(u)\mathbb{E}^{(r,x)}\left[(P_{u,t}f)^{q(u)}\log P_{u,t}f(X_{u\wedge\tau_n})\right]\bigg)\mathrm{d}u.\end{aligned}$$

By using the dominated convergence theorem and letting $n\to+\infty$, we have

$$\begin{aligned}&P_{r,s}(P_{s,t}f)^{q(s)}(x)-(P_{r,t}f)^{q(r)}(x)\\&=\int_r^s\Big[q(u)(1-q(u))P_{r,u}\left((P_{u,t}f)^{q(u)-2}|\nabla^u P_{u,t}f|_u^2\right)(x)\\&\qquad+q'(u)P_{r,u}((P_{u,t}f)^{q(u)}\log P_{u,t}f)(x)\Big]\mathrm{d}u,\end{aligned}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}s}P_{r,s}(P_{s,t}f)^{q(s)} = q'(s)P_{r,s}((P_{s,t}f)^{q(s)}\log P_{s,t}f) \\ + q(s)(q(s)-1)P_{r,s}((P_{s,t}f)^{q(s)-2}|\nabla^s P_{s,t}f|_s^2).$$

Therefore, for $(P_{r,s}(P_{s,t}f)^{q(s)})^{1/q(s)}$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}(P_{r,s}(P_{s,t}f)^{q(s)})^{1/q(s)} \\
&= (P_{r,s}(P_{s,t}f)^{q(s)})^{1/q(s)}\left(-\frac{q'(s)}{q(s)^2}\log P_{r,s}(P_{s,t}f)^{q(s)} + \frac{1}{q(s)}\frac{\partial_s(P_{r,s}(P_{s,t}f)^{q(s)})}{P_{r,s}(P_{s,t}f)^{q(s)}}\right) \\
&= (P_{r,s}(P_{s,t}f)^{q(s)})^{1/q(s)}\Bigg(-\frac{q'(s)}{q(s)^2}\log P_{r,s}(P_{s,t}f)^{q(s)} \\
&\qquad + \frac{q'(s)}{q(s)^2}\frac{P_{r,s}(P_{s,t}f)^{q(s)}\log(P_{s,t}f)^{q(s)}}{P_{r,s}(P_{s,t}f)^{q(s)}} \\
&\qquad + \frac{(q(s)-1)P_{r,s}\big((P_{s,t}f)^{q(s)-2}|\nabla^s P_{s,t}f|_s^2\big)}{P_{r,s}(P_{s,t}f)^{q(s)}}\Bigg) \\
&\le (P_{r,s}(P_{s,t}f)^{q(s)})^{\frac{1-q(s)}{q(s)}}\left[\left(\int_r^s \exp\left(-2\int_u^s k(t)\,\mathrm{d}t\right)\mathrm{d}u\right)q'(s) - q(s) + 1\right] \\
&\quad \times P_{r,s}\left((P_{s,t}f)^{q(s)-2}|\nabla^s P_{s,t}f|_s^2\right)
\end{aligned}$$

where the last inequality comes from the fact that $q' > 0$ along with the log-Sobolev inequality (4.1) with f replaced by $(P_{s,t}f)^{q(s)/2}$. According to the definition of $q(s)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(P_{r,s}(P_{s,t}f)^{q(s)}\right)^{1/q(s)} \le 0.$$

Integrating both sides from r to s, we obtain

$$\left(P_{r,s}(P_{s,t}f)^{q(s)}\right)^{1/q(s)} \le (P_{r,t}f^p)^{1/p}.$$

From this and the fact that $q(s)/p \le 1$, it follows that

$$\mu_r(P_{r,s}(P_{s,t}f)^{q(s)}) \le \mu_r(P_{r,t}f^p)^{q(s)/p} \le (\mu_r(P_{r,t}f^p))^{q(s)/p},$$

which implies

$$\|P_{s,t}f\|_{q(s),s} \le \|f\|_{p,t}.$$

This completes the proof. □

5 Supercontractivity and ultraboundedness

This section is devoted to supercontractivity and ultraboundedness for the semigroup $P_{s,t}$ under Hypothesis **(H3)**.

5.1 Super log-Sobolev inequality and boundedness of semigroup

We present a supercontractivity result first.

Theorem 5.1. *Suppose that Hypothesis* **(H3)** *holds. Let (μ_t) be the evolution system of measures associated with $P_{s,t}$. Then the following properties are equivalent:*

(a) *The semigroup $P_{s,t}$ is supercontractive.*

(b) *The family of super-log-Sobolev inequalities*

$$\int f^2 \log \frac{f^2}{\|f\|_{2,s}^2} \, \mathrm{d}\mu_s \le r \left\| |\nabla^s f|_s \right\|_{2,s}^2 + \beta_s(r) \|f\|_{2,s}^2, \quad r > 0, \tag{5.1}$$

hold for every $f \in H^1(M, \mu_s)$, $s \in I$, *and some positive non-increasing function* $\beta_s \colon (0, +\infty) \to (0, +\infty)$.

First, we give a lemma which makes the proof of this theorem more concise.

Lemma 5.2. *Suppose Hypothesis* **(H2)** *holds. Let* (μ_t) *be an evolution system of measures for* $P_{s,t}$. *If* $f \in C^{1,2}(I \times M) \cap C(I, L^1(M, \mu_r))$ *and there exists some function* $g \in \mathcal{B}_b(M)$ *such that* $|(\partial_r + L_r) f| \le g$ *for all* $r \in I$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}r} \int f(r,x) \, \mu_r(\mathrm{d}x) = \int (\partial_r + L_r) f(r,x) \, \mu_r(\mathrm{d}x) \tag{5.2}$$

for every $r \in I$.

Proof. For $f \in C^{1,2}(I \times M) \cap C(I, L^1(M, \mu_r))$, we have

$$\int f(r,x) \, \mu_r(\mathrm{d}x) = \int P_{s,r} f(r,x) \, \mu_s(\mathrm{d}x), \quad s < r \le T.$$

On the other hand, using Kolmogorov's formula, we have

$$\frac{\mathrm{d}}{\mathrm{d}r} P_{s,r} f(r,x) = P_{s,r}(L_r + \partial_r) f(r,x).$$

We complete the proof by applying the dominated convergence theorem. □

Proof of Theorem 5.1. First we prove "(b) ⇒ (a)". Let $(s,t) \in \Lambda$ and $f \in C_c^\infty(M)$ such that $f > \delta > 0$. By Lemma 5.2, we need to check the following to handle the derivative of $\mu_s (P_{s,t} f)^{q(s)}$ with respect to s:

$$\begin{aligned}
&(L_s + \partial_s)(P_{s,t} f)^{q(s)} \\
&\quad = L_s (P_{s,t} f)^{q(s)} - q(s) (P_{s,t} f)^{q(s)-1} (L_s P_{s,t} f) + q'(s) (P_{s,t} f)^{q(s)} \log P_{s,t} f \\
&\quad = q(s)(q(s)-1) |\nabla^s P_{s,t} f|_s^2 (P_{s,t} f)^{q(s)-2} + q'(s) (P_{s,t} f)^{q(s)} \log P_{s,t} f.
\end{aligned}$$

Under Hypothesis **(H3)**, by Theorem 3.2 (c), there exists a positive constant $c(s,t)$ such that

$$\left\| |\nabla^s P_{s,t} f|_s^2 \right\|_\infty \le c(s,t) \, \|f\|_\infty^2.$$

Moreover, $\|P_{s,t} f\|_\infty \le \|f\|_\infty$ and

$$(P_{s,t} f)^{q(s)} \log^+ (P_{s,t} f) \le (P_{s,t} f)^{q(s)+1} \le \|f\|_\infty^{q(s)+1}.$$

Combining all estimates above, we obtain

$$\|(L_s + \partial_s)(P_{s,t} f)^{q(s)}\|_\infty < \infty.$$

Now using Lemma 5.2, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s} \mu_s ((P_{s,t} f)^{q(s)}) \\
&\quad = \mu_s \Big(L_s (P_{s,t} f)^{q(s)} - q(s) (P_{s,t} f)^{q(s)-1} (L_s P_{s,t} f) + q'(s) (P_{s,t} f)^{q(s)} \log P_{s,t} f \Big) \\
&\quad = q(s)(q(s)-1) \mu_s (|\nabla^s P_{s,t} f|_s^2 (P_{s,t} f)^{q(s)-2}) + q'(s) \mu_s \big((P_{s,t} f)^{q(s)} \log P_{s,t} f \big).
\end{aligned}$$

Furthermore, for $\|P_{s,t}f\|_{q(s),s}$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\|P_{s,t}f\|_{q(s),s} \\
&\quad = \|P_{s,t}f\|_{q(s),s}^{-q(s)+1}(q(s)-1)\mu_s(|\nabla^s P_{s,t}f|_s^2(P_{s,t}f)^{q(s)-2}) \\
&\qquad + \frac{q'(s)}{q(s)}\|P_{s,t}f\|_{q(s),s}^{-q(s)+1}\mu_s((P_{s,t}f)^{q(s)}\log P_{s,t}f) \\
&\qquad - \frac{q'(s)}{q(s)}\|P_{s,t}f\|_{q(s),s}\log\|P_{s,t}f\|_{q(s),s}.
\end{aligned} \tag{5.3}$$

Replacing f in the log-Sobolev inequality (5.1) by $f^{p/2}$, we get

$$\int f^p \log\left(\frac{f^p}{\|f^{p/2}\|_{2,s}^2}\right)\mathrm{d}\mu_s \le r\frac{p^2}{4}\int f^{p-2}|\nabla^s f|_s^2\,\mathrm{d}\mu_s + \beta_s(r)\|f^{p/2}\|_{2,s}^2.$$

Now again replacing f and p by $P_{s,t}f$ and $q(s)$ in the inequality above, respectively, we obtain

$$\begin{aligned}
&\int (P_{s,t}f)^{q(s)}\log(P_{s,t}f)\,\mathrm{d}\mu_s - \|P_{s,t}f\|_{q(s),s}^{q(s)}\log\|P_{s,t}f\|_{q(s),s} \\
&\qquad \le r\frac{q(s)}{4}\int (P_{s,t}f)^{q(s)-2}|\nabla^s P_{s,t}f|_s^2\,\mathrm{d}\mu_s + \frac{\beta_s(r)}{q(s)}\|P_{s,t}f\|_{q(s),s}^{q(s)}.
\end{aligned}$$

Combining this with Eq. (5.3) yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\|P_{s,t}f\|_{q(s),s} \le \frac{\beta_s(r)q'(s)}{q(s)^2}\|P_{s,t}f\|_{q(s),s}, \quad (s,t)\in\Lambda,$$

where

$$q(s) = \mathrm{e}^{4r^{-1}(t-s)}(p-1)+1, \quad q(t)=p.$$

It follows that

$$\|P_{s,t}f\|_{q(s),s} \le \exp\left(\int_s^t \frac{\beta_u(r)q'(u)}{q(u)^2}\,\mathrm{d}u\right)\|f\|_{p,t}. \tag{5.4}$$

If $q(s)=q$, then $r = 4(t-s)\left(\log(q-1)/(p-1)\right)^{-1}$. Taking this r into Eq. (5.4) yields

$$\|P_{s,t}f\|_{q,s} \le \exp\left(\int_s^t \frac{\beta_u(4(t-s)\left(\log(q-1)/(p-1)\right)^{-1})q'(u)}{q(u)^2}\,\mathrm{d}u\right)\|f\|_{p,t}.$$

Next, we prove "(a) $\Rightarrow$ (b)". Suppose that there exists $C_{p,q}(s,t)$ and $1<p<q$ such that

$$\|P_{s,t}\|_{(p,t)\to(q,s)} \le C_{p,q}(s,t).$$

Recall the log-Sobolev inequality with respect to $P_{s,t}$,

$$P_{s,t}(f^2\log f^2) \le 4\left(\int_s^t \mathrm{e}^{-2\int_r^t k(u)\,\mathrm{d}u}\,\mathrm{d}r\right)P_{s,t}|\nabla^t f|_t^2 + P_{s,t}f^2\log(P_{s,t}f^2), \quad f\in C_0^\infty(M). \tag{5.5}$$

From this and the fact that

$$\log^+(P_{s,t}f^2) \le P_{s,t}f^2 \le \|f\|_\infty^2,$$

we are able to integrate both sides of Eq. (5.5) with respect to μ_s,

$$\mu_t(f^2\log f^2) \le 4\left(\int_s^t \mathrm{e}^{-2\int_r^t k(u)\,\mathrm{d}u}\,\mathrm{d}r\right)\mu_t(|\nabla^t f|_t^2) + \mu_s(P_{s,t}f^2\log P_{s,t}f^2). \tag{5.6}$$

Now, we need to deal with the term $\mu_s(P_{s,t}f^2\log P_{s,t}f^2)$. For any $h\in(0,1-\frac{1}{p})$, by the Riesz-Thorin interpolation theorem, we get

$$\|P_{s,t}f\|_{q_h,s}\le C_{p,q}(s,t)^{r_h}\|f\|_{p_h,t},\quad f\in L^p(M,\mu_s),\tag{5.7}$$

where $r_h=\frac{ph}{p-1}\in(0,1)$, $\frac{1}{p_h}=1-r_h+\frac{r_h}{p}$ and $\frac{1}{q_h}=1-r_h+\frac{r_h}{q}$, i.e.,

$$r_h=\frac{ph}{p-1},\quad p_h=\frac{1}{1-h},\quad q_h=\left(1-\frac{p(q-1)}{q(p-1)}h\right)^{-1}.$$

Set $\|f\|_{2,t}=1$. Then from Eq. (5.7), we have

$$\int\left(P_{s,t}|f|^{2(1-h)}\right)^{q_h}\mathrm{d}\mu_s\le C_{p,q}(s,t)^{r_hq_h},$$

which further implies

$$\begin{aligned}&\frac{1}{h}\left[\int\left(P_{s,t}|f|^{2(1-h)}\right)^{q_h}\mathrm{d}\mu_s-\left(\int P_{s,t}|f|^2\,\mathrm{d}\mu_s\right)^{q_h/p_h}\right]\\&\quad=\frac{1}{h}\left(\int\left(P_{s,t}|f|^{2(1-h)}\right)^{q_h}\mathrm{d}\mu_s-1\right)\le\frac{1}{h}\left(C_{p,q}(s,t)^{r_hq_h}-1\right).\end{aligned}$$

As

$$\lim_{h\to0}\frac{1}{h}\left(C_{p,q}(s,t)^{r_hq_h}-1\right)=\frac{p}{p-1}\log C_{p,q}(s,t),$$

by dominated convergence, we obtain

$$\frac{p(q-1)}{q(p-1)}\int P_{s,t}f^2\log P_{s,t}f^2\,\mathrm{d}\mu_s-\int P_{s,t}(f^2\log f^2)\,\mathrm{d}\mu_s\le\frac{p}{p-1}\log C_{p,q}(s,t),$$

or equivalently,

$$\mu_s(P_{s,t}f^2\log P_{s,t}f^2)\le\frac{q(p-1)}{p(q-1)}\,\mu_t(f^2\log f^2)+\frac{q}{q-1}\log C_{p,q}(s,t).$$

Combining this with Eq. (5.6), we arrive at

$$\mu_t(f^2\log f^2)\le\gamma_t(t-s)\,\mu_t(|\nabla^t f|_t^2)+\tilde\beta_t(t-s)\tag{5.8}$$

where $f\in C_0^\infty(M)$, $\|f\|_{2,t}=1$ and

$$\begin{aligned}\gamma_t(t-s)&=\frac{4p(q-1)}{q-p}\int_{t-(t-s)}^t\exp\left(-2\int_r^t k(u)\,\mathrm{d}u\right)\,\mathrm{d}r,\\\tilde\beta_t(t-s)&=\frac{pq}{q-p}\log C_{p,q}(s,t),\end{aligned}\tag{5.9}$$

i.e. $\tilde\beta_t$ is a positive function on $(0,\infty)$ and $2\le p\le q$. We complete the proof by letting $\gamma_t=r$ and then

$$\beta_t(r)=\tilde\beta_t(\gamma_t^{-1}(r)).$$ □

Next, we study the ultraboundedness by using the super-log-Sobolev inequality (5.1).

Theorem 5.3. *Suppose that Hypothesis* **(H3)** *holds. Let (μ_t) be an evolution system of measures associated with $P_{s,t}$.*

(i) *If the function k in Hypothesis* **(H3)** *is almost surely non-negative and $P_{s,t}$ satisfies*

$$\|P_{s,t}\|_{(2,t)\to\infty}\le C_{2,\infty}(s,t),$$

then Eq. (5.1) *holds for $\beta_s(r)=2\log C_{2,\infty}(s,s+\frac{r}{8})$.*

(ii) *Conversely, assume Eq.* (5.1) *holds for some positive non-increasing function* β : $(0,+\infty)\to(0,+\infty)$*, which is independent of* s*. If there exists a function* $r\in C([2,\infty))$ *such that*

$$t_0 := \int_2^\infty \frac{r(p)}{p-1}\,\mathrm{d}p < \infty,$$

then for $t-s\ge t_0$*, we have*

$$\|P_{s,t}\|_{(2,t)\to\infty} \le \exp\left(\int_2^\infty \frac{\beta(r(p))}{p^2}\,\mathrm{d}p\right).$$

Proof. Letting $p=2$ and $q\to+\infty$ in Eq. (5.8), we know from Eq. (5.9) that for $f\in C_0^\infty(M)$ with $\|f\|_{2,t}=1$,

$$\begin{aligned}\mu_t(f^2\log f^2) &\le 8\left(\int_s^t \mathrm{e}^{-2\int_r^t k(u)\,\mathrm{d}u}\,\mathrm{d}r\right)\mu_t(|\nabla^t f|_t^2) + 2\log C_{2,\infty}(s,t)\\ &\le 8(t-s)\mu_t(|\nabla^t f|_t^2) + 2\log C_{2,\infty}(s,s+(t-s)).\end{aligned}$$

Letting $r=8(t-s)$, we obtain (i) directly.

Given $(s,t)\in\Lambda$. Let q and N be two functions in $C^1((-\infty,t])$ such that $q'<0$, which will be given later. It follows from Eq. (5.3) that for $f\in C_c^\infty(M)$ such that $f>0$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{-N(s)}\,\|P_{s,t}f\|_{q(s),s}\\ &\quad= \frac{q'(s)}{q(s)}\mathrm{e}^{-N(s)}\|P_{s,t}f\|_{q(s),s}^{-q(s)+1}\Bigg[\mu_s\left((P_{s,t}f)^{q(s)}\log P_{s,t}f\right) - \|P_{s,t}f\|_{q(s),s}^{q(s)}\log\|P_{s,t}f\|_{q(s),s}\\ &\qquad + \frac{q(s)(q(s)-1)}{q'(s)}\mu_s\left(|\nabla^s P_{s,t}f|_s^2(P_{s,t}f)^{q(s)-2}\right) - N'(s)\frac{q(s)}{q'(s)}\|P_{s,t}f\|_{q(s),s}^{q(s)}\Bigg].\end{aligned}\tag{5.10}$$

From this, and applying the super-log-Sobolev inequality (5.1) to $(P_{s,t}f)^{q(s)/2}$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-N(s)}\|P_{s,t}f\|_{q(s),s}\\ &\quad\ge \mathrm{e}^{-N(s)}\|P_{s,t}f\|_{q(s),s}^{-q(s)+1}\frac{q'(s)}{q(s)}\Bigg[\left(\frac{q(s)(q(s)-1)}{q'(s)} + r\frac{q(s)}{4}\right)\mu_s\left(|\nabla^s P_{s,t}f|_s^2(P_{s,t}f)^{q(s)-2}\right)\\ &\qquad + \left(\frac{1}{q(s)}\beta(r) - N'(s)\frac{q(s)}{q'(s)}\right)\|P_{s,t}f\|_{q(s),s}^{q(s)}\Bigg].\end{aligned}\tag{5.11}$$

Let $q(s)$ and $N(s)$ be the solutions to the following equations respectively:

$$q'(s) = \frac{-4(q(s)-1)}{r\circ q(s)},\quad q(t)=2;$$

$$N'(s) = \frac{q'(s)\beta(r\circ q(s))}{q(s)^2},\quad N(t)=0.$$

It is easy to see that $q'<0$. Then from this and (5.11), we conclude that:

$$\mathrm{e}^{-N(s)}\|P_{s,t}f\|_{q(s),s} \le \|f\|_{2,t}.\tag{5.12}$$

Defining

$$t_0 := \int_2^\infty \frac{r(p)}{4(p-1)}\,\mathrm{d}p < \infty,$$

we claim that $q(s)\to+\infty$ as $s\to t-t_0$, and then $N(s)\to\int_2^\infty \frac{\beta(r(p))}{p^2}\,\mathrm{d}p$. Indeed, $q(t-t_0)=\infty$ follows from the fact that

$$\int_2^{q(t-t_0)} \frac{r(s)}{4(s-1)}\,\mathrm{d}s = \int_t^{t-t_0}\frac{r\circ q(s)\,q'(s)}{4(q(s)-1)}\,\mathrm{d}s = t_0 = \int_2^\infty \frac{r(s)}{4(s-1)}\,\mathrm{d}s,$$

i.e. $q(t-t_0)=\infty$. By this and Eq. (5.12), we have

$$\|P_{t-t_0,t}\|_{(2,t)\to\infty} \le \exp\left(\int_2^\infty \frac{\beta(r(p))}{p^2}\,\mathrm{d}p\right). \qquad \square$$

5.2 Dimension-free Harnack inequality and boundedness of semigroup

Next, we will use integrability of the Gaussian function $\mathrm{e}^{\lambda\rho_t^2}$ (for $\lambda>0$ and $t\in I$) with respect to the families of measures $(\mu_s)_{s\in I}$ or $(P_{s,t})_{(s,t)\in\Lambda}$ to give another criterion which is equivalent to supercontractivity or ultraboundedness. To this end, we need the following preliminary result which is a dimension-free Harnack-type estimate for $P_{s,t}$ (see [7]).

Lemma 5.4. *Assume that $\mathcal{R}_t^Z \ge k(t)$ holds for $t\in I$. For every $f\in C_b(M)$, $p>1$, $(s,t)\in\Lambda$ and $x,y\in M$, we have the inequality:*

$$|P_{s,t}f|^p(x) \le \left(P_{s,t}|f|^p\right)(y)\exp\left(\frac{p}{4(p-1)}\left(\int_s^t \exp\left(2\int_s^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)^{-1}\rho_s^2(x,y)\right). \tag{5.13}$$

The main result of this section is the following:

Theorem 5.5. *Suppose that Hypothesis* **(H3)** *holds. Let (μ_s) be the evolution system of measures. Then*

(i) *$P_{s,t}$ is supercontractive with respect to (μ_s) if and only if $\mu_t\left(\exp(\lambda\rho_t^2)\right)<\infty$ for any $\lambda>0$ and $t\in I$;*

(ii) *$P_{s,t}$ is ultrabounded with respect to (μ_s) if and only if $\left\|P_{s,t}\exp(\lambda\rho_t^2)\right\|_\infty<\infty$ for any $\lambda>0$ and $(s,t)\in\Lambda$.*

Proof. (a) From Hypothesis **(H3)** we know that $\mathcal{R}_t^Z\ge k(t)$ for $t\in I$. It follows by the Harnack inequality (5.13) that for $(s,t)\in\Lambda$, $p>1$ and $f\in C_b(M)$,

$$|P_{s,t}f|^p(x)\le P_{s,t}|f|^p(y)\exp\left(\frac{p}{4(p-1)}\left(\int_s^t\exp\left(2\int_s^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)^{-1}\rho_s(x,y)^2\right).$$

If $\mu_t(|f|^p)=1$, then

$$\begin{aligned}1&\ge |P_{s,t}f(x)|^p\int\exp\left(-\frac{p}{4(p-1)}\left(\int_s^t\exp\left(2\int_s^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)^{-1}\rho_s(x,y)^2\right)\mu_s(\mathrm{d}y)\\&\ge |P_{s,t}f(x)|^p\mu_s(B_s(o,R))\exp\left(-\frac{p(\rho_s(x)+R)^2}{4(p-1)}\left(\int_s^t\exp\left(2\int_s^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)^{-1}\right),\end{aligned} \tag{5.14}$$

where $B_s(o,R):=\{y\in M:\rho_s(y)\le R\}$. Since (μ_s) is compact, there exists $R>0$, which may depend on s, such that

$$\mu_s(B_s(o,R))=\mu_s(\{x:\rho_s(x)\le R\})\ge 1-\frac{\mu_s(\rho_s^2)}{R^2}\ge 1-\frac{H_2(s)}{R^2}\ge 2^{-p}.$$

By this and Eq. (5.14), we arrive at

$$1\ge|P_{s,t}f(x)|^p2^{-p}\exp\left(-\left(\int_s^t\exp\left(2\int_s^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)^{-1}\frac{p(\rho_s^2(x)+R^2)}{4(p-1)}\right)$$

which further implies

$$|P_{s,t}f(x)| \le 2\exp\left(\left(\int_s^t \exp\left(2\int_s^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)^{-1}\frac{\rho_s^2(x)+R^2}{4(p-1)}\right), \quad s<t. \tag{5.15}$$

Therefore, we have

$$\|P_{s,t}f\|_{q,s} \le \left\{\mu_s\left(\exp\left(q(c_1+c_2\rho_s^2)\right)\right)\right\}^{1/q}$$

for some positive constants c_1, c_2 depending on s,t. Hence, if $\mu_s(\exp(\lambda\rho_s^2))<\infty$ for any $\lambda>0$ and $s\in I$, then $P_{s,t}$ is supercontractive, i.e.

$$\|P_{s,t}\|_{(p,t)\to(q,s)}<\infty$$

for any $1<p<q<\infty$.

Conversely, if the semigroup $P_{s,t}$ is supercontractive, then by Theorem 5.1, we know that the family of super-log-Sobolev inequalities (5.1) holds. Now our first step is to prove $\mu_s(\mathrm{e}^{\lambda\rho_s})<\infty$ for any $s\in I$ and $\lambda>0$. Let $\rho_s^n=\rho_s\wedge n$ and $h_{s,n}(\lambda)=\mu_s(\exp(\lambda\rho_s^n))$. Taking $\exp(\frac{\lambda}{2}\rho_s^n)$ into the super-log-Sobolev inequality (5.1) above, we have

$$\lambda h'_{s,n}(\lambda)-h_{s,n}(\lambda)\log h_{s,n}(\lambda)\le h_{s,n}(\lambda)\lambda^2\left(\frac{r}{4}+\frac{\beta_s(r)}{\lambda^2}\right).$$

This implies

$$\left(\frac{1}{\lambda}\log h_{s,n}(\lambda)\right)'=\frac{\lambda h'_{s,n}(\lambda)-h_{s,n}(\lambda)\log h_{s,n}(\lambda)}{\lambda^2 h_{s,n}(\lambda)}\le\frac{r}{4}+\frac{\beta_s(r)}{\lambda^2}. \tag{5.16}$$

Integrating both sides of Eq. (5.16) from λ to 2λ, we obtain

$$h_{s,n}(2\lambda)\le h_{s,n}^2(\lambda)\exp\left(\frac{r}{2}\lambda^2+\beta_s(r)\right). \tag{5.17}$$

By this and the fact that there exists a constant M_s such that

$$\mu_s(\{\lambda\rho_s\ge M_s\})\le\frac{1}{4}\exp\left(-\left(\frac{r}{2}\lambda^2+\beta_s(r)\right)\right),$$

it follows that:

$$\begin{aligned}
h_{s,n}(\lambda)&=\int_{\{\lambda\rho_s\ge M_s\}}\mathrm{e}^{\lambda\rho_s^n}\,\mathrm{d}\mu_s+\int_{\{\lambda\rho_s<M_s\}}\mathrm{e}^{\lambda\rho_s^n}\,\mathrm{d}\mu_s\\
&\le\mu_s(\{\lambda\rho_s\ge M_s\})^{1/2}\,\mu_s(\mathrm{e}^{2\lambda\rho_s^n})^{1/2}+\mathrm{e}^{M_s}\\
&\le\left(\frac{1}{4}\exp\left(-\left(\frac{r}{2}\lambda^2+\beta_s(r)\right)\right)\right)^{1/2}\exp\left(\frac{r}{4}\lambda^2+\frac{1}{2}\beta_s(r)\right)h_{s,n}(\lambda)+\mathrm{e}^{M_s}\\
&\le\frac{1}{2}h_{s,n}(\lambda)+\mathrm{e}^{M_s},
\end{aligned}$$

which implies $h_{s,n}(\lambda)\le 2\mathrm{e}^{M_s}$ for $s\in I$. As M_s is independent of n, letting n go to infinity, we obtain

$$\mu_s(\mathrm{e}^{\lambda\rho_s})<\infty,\quad \text{for } s\in I.$$

Our second step is to prove $\mu_s(\mathrm{e}^{\lambda\rho_s^2})<\infty$ for all $s\in I$ and $\lambda>0$. Let $h_s(\lambda):=\lim_{n\to\infty}h_{s,n}(\lambda)$. Integrating both sides of Eq. (5.16) from 1 to λ and letting $n\to\infty$, we obtain

$$h_s(\lambda)\le\exp\left(\lambda c_0(s)+\frac{r}{4}(\lambda^2-\lambda)+\beta_s(r)(1-\lambda)\right) \tag{5.18}$$

where $c_0(s) := \log \mu_s(\exp(\rho_s))$. Now, we observe that for any positive constant ϵ,

$$\int_1^\infty h_s(\lambda)\, \mathrm{e}^{-(\frac{r}{4}+\epsilon)\lambda^2}\,\mathrm{d}\lambda = \int_M \mathrm{d}\mu_s \int_1^\infty \mathrm{e}^{\lambda\rho_s}\,\mathrm{e}^{-(\frac{r}{4}+\epsilon)\lambda^2}\,\mathrm{d}\lambda < \infty.$$

On the other hand, it is easy to see that for $\epsilon > 0$,

$$\begin{aligned}
&\int_M \mathrm{d}\mu_s \int_1^\infty \mathrm{e}^{\lambda\rho_s}\,\mathrm{e}^{-(\frac{r}{4}+\epsilon)\lambda^2}\,\mathrm{d}\lambda \\
&\quad = \int_M \exp\left(\rho_s^2/(r+4\epsilon)\right)\mathrm{d}\mu_s \int_1^\infty \exp\left(-\left(\frac{1}{2}\sqrt{r+4\epsilon}\lambda - \rho_s/\sqrt{r+4\epsilon}\right)^2\right)\mathrm{d}\lambda \\
&\quad \ge \frac{2}{\sqrt{r+4\epsilon}} \int_M \exp\left(\rho_s^2/(r+4\epsilon)\right)\mathrm{d}\mu_s \int_{\sqrt{r+4\epsilon}/2}^\infty \exp(-t^2)\,\mathrm{d}t.
\end{aligned}$$

By the arbitrariness of r, we obtain that there exists a number N_s such that for any $\lambda > 0$,

$$\int \mathrm{e}^{\lambda\rho_s^2}\,\mathrm{d}\mu_s < N_s, \quad s \in I,$$

which completes the proof of (i).

(b) If $\|P_{s,t}\exp(\lambda\rho_t^2)\|_\infty < \infty$ for any $\lambda > 0$ and $(s,t) \in \Lambda$, then we know from Eq. (5.15) that for any $p > 1$ and $f \in C_b(M)$ satisfying $f > 0$ and $\|f\|_{p,t} = 1$,

$$\left|P_{(s+t)/2,t} f(x)\right| \le 2\exp\left(\left(\int_{(s+t)/2}^t \exp\left(2\int_{(s+t)/2}^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)^{-1} \frac{\rho_{(s+t)/2}^2(x) + R^2}{4(p-1)}\right)$$

which implies that there exist constants c_1 and c_2 such that

$$\begin{aligned}
&\|P_{s,t}f\|_\infty \\
&\le 2\left\| P_{s,(t+s)/2}\exp\left(2\left(c_1 + c_2\rho_{(s+t)/2}^2(x)\right)\left(\int_{\frac{s+t}{2}}^t \exp\left(2\int_{\frac{s+t}{2}}^r k(u)\,\mathrm{d}u\right)\mathrm{d}r\right)^{-1}\right)\right\|_\infty < \infty.
\end{aligned}$$

On the other hand, if $\|P_{s,t}\|_{(p,t)\to\infty} < \infty$ for all $p > 1$, then

$$\left\|P_{s,t}\exp(\lambda\rho_t^2)\right\|_\infty \le \|P_{s,t}\|_{(p,t)\to\infty}\left\|\exp(\lambda\rho_t^2)\right\|_{p,t} < \infty$$

provided $\mu_t(\exp(\lambda\rho_t^2))$ is bounded for all $t \in I$. Hence, it suffices to prove that

$$\mu_t(\exp(\lambda\rho_t^2)) < \infty.$$

Since $P_{s,t}$ is ultrabounded, $P_{s,t}$ is supercontractive. Using Theorem 5.5 (i), we get $\mu_t(\exp(\lambda\rho_t^2)) < \infty$. This completes the proof. □

5.3 Other criteria on supercontractivity and ultraboundedness

It is straightforward to check that Hypothesis **(H3)** implies Hypothesis **(H2)** for $\varphi(r) = r^2$, $r > 0$. As far as supercontractivity and ultraboundedness of $P_{s,t}$ is concerned, we have the following results in terms of other types of space-time Lyapunov conditions.

Theorem 5.6. *Let $\gamma \in C((0,\infty))$ be a positive increasing function such that*

$$\lim_{r\to+\infty}\frac{\gamma(r)}{r} = +\infty.$$

(i) *If*

$$(L_t + \partial_t)\rho_t^2(x) \le c - \gamma(\rho_t^2(x)) \tag{5.19}$$

holds for $t \in I$, $c > 0$ *and* $x \notin \mathrm{Cut}_t(o)$*, then* $P_{s,t}$ *has an evolution system of measures* (μ_s) *and* $P_{s,t}$ *is supercontractive with respect to* (μ_s)*.*

(ii) *If* (5.19) *holds for* γ *such that* $g_\lambda(r) = r\gamma(\lambda \log r)$ *is convex on* $(0, \infty)$ *and such that for any* $\lambda > 0$*,*

$$\int_0^\infty \frac{\mathrm{d}r}{r\gamma(\lambda \log r)} < \infty,$$

then $P_{s,t}$ *has an evolution system of measures* (μ_s) *and* $P_{s,t}$ *is ultrabounded with respect to* (μ_s)*.*

(iii) *If* (5.19) *holds for* $\gamma(r) = \alpha r^\delta$*, where* $\alpha > 0$ *and* $\delta > 1$*, then* $P_{s,t}$ *is ultrabounded with respect to* (μ_s) *and*

$$\|P_{s,t}\|_{(2,t)\to\infty} \le \exp\left(c(t-s)^{-\delta/(\delta-1)}\right)$$

holds for some constant $c > 0$ *and all* $(s,t) \in \Lambda$*.*

Proof. It is an immediate consequence of Theorem 2.1 that under condition (5.19) the process $X_\cdot$ is non-explosive up to time T. The idea of following proof is similar to [22, Corollary 5.7.6]. We include a proof for convenience.

(a) Let X_t be a diffusion processes generated by L_t. Then by Itô's formula,

$$\begin{aligned}&\mathrm{d}\exp(\lambda\rho_t^2(X_t))\\ &= 2\lambda\rho_t(X_t)\exp(\lambda\rho_t^2(X_t))\,\mathrm{d}b_t + \mathbf{1}_{\{X_t\notin \mathrm{Cut}_t(o)\}}(L_t+\partial_t)\exp(\lambda\rho_t^2(X_t))\,\mathrm{d}t - \mathrm{d}\ell_t,\end{aligned} \tag{5.20}$$

where b_t is a one-dimensional Brownian motion and ℓ_t an increasing process supported on $\{t \ge s : X_t \in \mathrm{Cut}_t(o)\}$. By Eq. (5.19), it follows that

$$\begin{aligned}(L_t+\partial_t)\exp(\lambda\rho_t^2) &= \lambda\exp(\lambda\rho_t^2)(L_t+\partial_t)\rho_t^2 + 4\lambda^2\rho_t^2\exp(\lambda\rho_t^2)\\ &\le \exp(\lambda\rho_t^2)(c-\gamma(\rho_t^2)) + 4\lambda^2\rho_t^2\exp(\lambda\rho_t^2)\end{aligned} \tag{5.21}$$

holds outside $\mathrm{Cut}_t(o)$. If $\limsup_{r\to\infty}\frac{\gamma(r)}{r} = +\infty$, then there exist $c_1, c_2 > 0$ such that for each $t \in I$,

$$(L_t+\partial_t)\left(\exp(\lambda\rho_t^2)-1\right) \le c_1 - c_2\left(\exp(\lambda\rho_t^2)-1\right)$$

holds outside $\mathrm{Cut}_t(o)$. According to Theorem 2.3, there exists an evolution system of measures (μ_s) such that $\sup_{s\in I}\mu_s(\mathrm{e}^{\lambda\rho_s^2}) < \infty$. We then obtain the first conclusion by Theorem 5.5 (i).

(b) We use Theorem 5.5 (ii) to give the proof. So it suffices for us to check $\|P_{s,t}\exp(\lambda\rho_t^2)\|_\infty < \infty$. By Eq. (5.21), we have

$$\begin{aligned}(L_t+\partial_t)\exp(\lambda\rho_t^2) &\le \lambda\exp(\lambda\rho_t^2)(c-\gamma(\rho_t^2)) + 4\lambda^2\rho_t^2\exp\left(\lambda\rho_t^2\right)\\ &\le \lambda\exp(\lambda\rho_t^2)(c_1 - \gamma(\rho_t^2)/2),\end{aligned} \tag{5.22}$$

where $c_1 = 0 \vee \sup\left(c - \frac{1}{2}\gamma(r) + 4\lambda r\right) < \infty$. According to Eq. (5.22), there exists a positive constant $C(\lambda)$ such that

$$(L_t+\partial_t)\exp(\lambda\rho_t^2) \le C(\lambda), \tag{5.23}$$

where $\lambda > 0$. For fixed $x \in M$, let

$$\theta_s(t) := \mathbb{E}^{(s,x)}\left[\exp(\lambda\rho_t^2(X_t))\right], \quad t \geq s.$$

We need to show that $\theta_s(t)$ is uniformly bounded. Since the set $\{t \in [s,T) \colon X_t \in \mathrm{Cut}_t(o)\}$ is of measure zero, it follows from Eq. (5.20) and Eq. (5.23) that

$$\mathbb{E}^{(s,x)}\left[\lambda\rho_t^2(X_{t\wedge\tau_n})\right] \leq \exp\left(\lambda\rho_s^2(x)\right) + C(\lambda)\,\mathbb{E}^{(s,x)}[t\wedge\tau_n - s],$$

where $\tau_n := \inf\{t \in [s,T) : \rho_t(X_t) \geq n\}$. Since $\tau_n \uparrow T$ as $n \to \infty$, we further conclude that

$$\theta_s(t) \leq \exp(\lambda\rho_s(x)^2) + C(\lambda)(t-s)$$

for any $\lambda > 0$ and $(s,t) \in \Lambda$. In particular, here θ_s is continuous and

$$M_t := 2\sqrt{2}\lambda\int_s^t \rho_r(X_r)\exp\left(\lambda\rho_r^2(X_r)\right)\mathrm{d}b_r$$

is a square integrable martingale. By Fubini's theorem, along with Eq. (5.20) and Eq. (5.22), we have

$$\begin{aligned}\frac{\theta_s(t+r)-\theta_s(t)}{r} &\leq \frac{\lambda c_1}{r}\int_t^{t+r}\theta_s(u)\,\mathrm{d}u - \frac{\lambda}{2r}\int_t^{t+r}\mathbb{E}^{(s,x)}\left[\exp\left(\lambda\rho_u^2(X_u)\right)\gamma(\rho_u^2(X_u))\right]\,\mathrm{d}u\\ &\leq \frac{\lambda c_1}{r}\int_t^{t+r}\theta_s(u)\,\mathrm{d}u - \frac{\lambda}{2r}\int_t^{t+r}\theta_s(u)\gamma(\lambda^{-1}\log\theta_s(u))\,\mathrm{d}u\end{aligned}$$

where the second inequality comes from the fact that for $\lambda > 0$, the function $r \mapsto r\gamma(\lambda\log r)$ is convex for $r \geq 1$. Therefore,

$$\theta_s'(t) \leq \lambda c_1\theta_s(t) - \frac{\lambda}{2}\theta_s(t)\gamma(\lambda^{-1}\log\theta_s(t)), \quad t \in [s,T).$$

Then, by a similar discussion as in the fixed metric case (see the proof of [22, Corollary 5.7.6]), we obtain

$$\theta_s(t) \leq G^{-1}(\lambda(t-s)/4) \vee c_2 < \infty, \tag{5.24}$$

for some positive constant c_2 where

$$G(r) := \int_r^\infty \frac{\mathrm{d}s}{s\gamma(\lambda^{-1}\log s)}, \quad r > 1.$$

In particular, for $\gamma(r) = \alpha r^\delta$ for $\delta > 1$ and $\alpha > 0$, we have

$$G(r) = \frac{\lambda^\delta}{\alpha(\delta-1)}(\log r)^{1-\delta}.$$

Combining this with Eq. (5.24), we complete the proof of (ii) and (iii). □

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Acknowledgments. This work has been supported by the Fonds National de la Recherche Luxembourg (FNR) under the OPEN scheme (project GEOMREV O14/7628746). The first named author acknowledges support by NSFC (Grant No. 11501508) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ16A010009).