Portfolio Optimisation with Conditioning Information

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based on joint work with Marc BOISSAUX

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Outline

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1. The portfolio theory of Markowitz
2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights

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2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights
3. Conditioned mean-variance portfolio optimisation with constrained portfolio weights
   - General formulation of the problem
   - Empirical study

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Nevertheless, most people choose game 1, because they are **risk averse**.

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In portfolio theory, expected return is measured by the average of the portfolio return, risk by its dispersion.

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Since portfolio volatility depends on the correlations between the component assets, an investor can reduce portfolio risk simply by holding combinations of instruments that are not perfectly positively correlated.

Efficient portfolio

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An efficient portfolio is defined as the portfolio that maximizes the expected return for a given amount of risk, or the portfolio that minimizes the risk for a given expected return.

Efficient frontier without risk-free assets

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The efficient frontier is the curve that shows all efficient portfolios in a risk-return framework.

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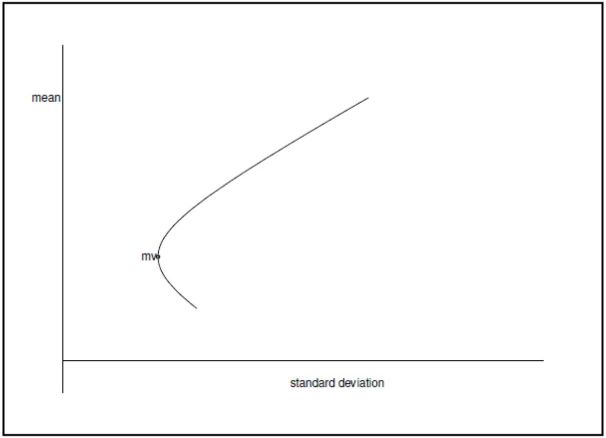

Optimal portfolio in the Markowitz sense

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The theory of Markowitz stipulates that investors try to maximize their utility function u that depends on his risk aversion parameter γ and is given by

$$u(\gamma) = E[P] - \frac{1}{2}\gamma\sigma_P^2,$$

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The optimal portfolio for an investor is the portfolio with maximum utility.

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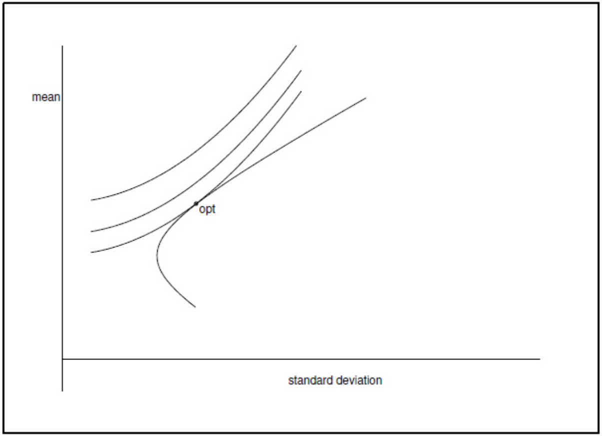

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For a portfolio P, of volatility σ_P, the Sharp ratio SR_P is then defined as

$$SR_P = \frac{E[P] - r_f}{\sigma_P}.$$

Tangency portfolio without risk-free asset

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If an investor wants to invest in a portfolio with maximum Sharpe ratio, getting thus the highest expected return per unit of risk, he chooses the tangency portfolio, which is the most "risk-efficient portfolio".

Graphically, this portfolio is the point where a line through the origin is tangent to the efficient frontier, because this point has the property that is has the highest possible mean-standard deviation ratio.

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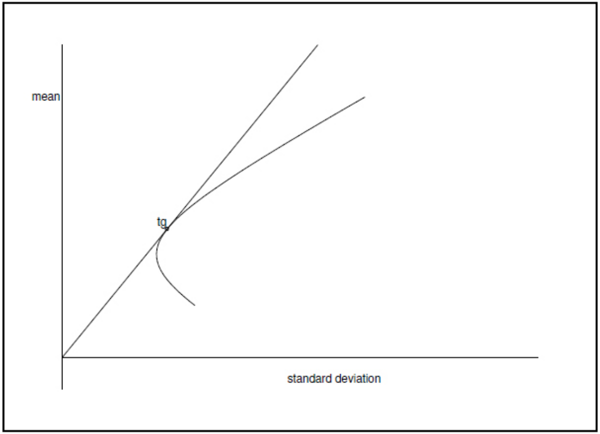

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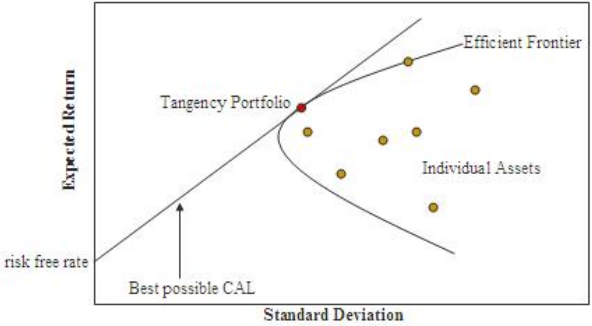
This line, called also the capital allocation line is the efficient frontier in the presence of a risk-free asset.



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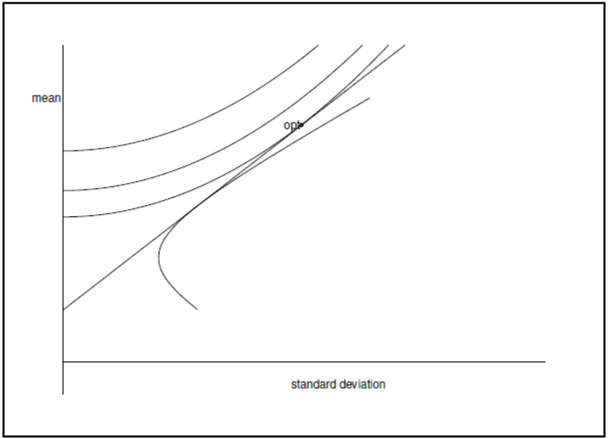

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The Ferson-Siegel paper (JF 2001)

Ferson and Siegel postulate that there exists some relationship $\mu(s)$ between a signal s and each asset return r observed at the end of the investment interval:

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Ferson and Siegel provided an analytical solution for the case where the weights are unconstrained.

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The Ferson-Siegel paper (JF 2001)

Theorem (Ferson and Siegel, 1991)

Given unconditional expected return μ_P, n risky assets, **and no risk-free asset**, *the unique portfolio having minimum unconditional variance is determined by the weights*

$$u'(s) = \frac{e'\Lambda(s)}{e'\Lambda(s)e} + \frac{\mu_P - E\left[\frac{e'\Lambda(s)\mu(s)}{e'\Lambda(s)e}\right]}{E\left[\mu'(s)\left(\Lambda(s) - \frac{\Lambda(s)ee'\Lambda(s)}{e'\Lambda(s)e}\right)\mu(s)\right]}\mu'(s)\left(\Lambda(s) - \frac{\Lambda(s)ee'\Lambda(s)}{e'\Lambda(s)e}\right)$$

where e is an n−vector of ones and $\Lambda(s) = [\mu(s)\mu(s)' + \Sigma_\epsilon^2]^{-1}$; Σ_ϵ^2 is the conditional covariance matrix $E[\epsilon\epsilon'|s]$.

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$$u'(s) = \frac{\mu_P - r_f}{E[(\mu(s) - r_f e)'\Lambda(s)(\mu(s) - r_f e)]}(\mu(s) - r_f e)'\Lambda(s),$$

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For the traditional Markowitz problem, the introduction of portfolio weights constraints of any kind means a closed-form solution is no longer available, and a numerical algorithm has to be used.

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Possible signals

Taken from a continuous scale ranging from purely macroeconomic indices to investor sentiment indicators. Indicators taking into account investor attitude may be based on some model or calculated in an ad-hoc fashion.

Examples include

- short-term treasury bill rates (Fama and Schwert 1977);
- CBOE Market Volatility Index (VIX) (Whaley 1993) or its European equivalents (VDAX etc.);
- risk aversion indices using averaging and normalisation (UBS Investor Sentiment Index 2003) or PCA reduction (Coudert and Gex 2007) of several macroeconomic indicators;
- global risk aversion indices (GRAI) (Kumar and Persaud 2004) based on a measure of rank correlation between current returns and previous risks;
- option-based risk aversion indices (Tarashev et al. 2003);
- sentiment indicators directly obtained from surveys (e.g. University of Michigan Consumer Sentiment Index)

Unconditioned expected return and variance given conditioning information

These are obtained as expectation integrals over the signal domain. If a risk-free asset with return r_t is available,

$$E(P) = E\left[u'(s)(\mu(s) - r_f 1)\right] = E\left[I_1(u, s)\right]$$

and

$$\sigma^2(P) = E\Big[u'(s)\big[(\mu(s) - r_t 1)(\mu(s) - r_t 1)' + \sigma_\epsilon^2\big] u(s)\Big] - \mu_P^2$$

$$= E\left[I_2(u, s)\right] - \mu_P^2$$

for an expected unconditional return of μ_P and a conditional covariance matrix σ_ϵ^2.

Optimal control formulation

$$\begin{aligned}
\text{Minimize} \quad & J_{[s^-,s^+]}(x,u) = \int_{s^-}^{s^+} l_2(u,s)p_s(s)ds \text{ as } s^- \to -\infty, s^+ \to +\infty \\
\text{subject to} \quad & \dot{x}(s) = l_1(u,s)p_s(s)\ \forall s \in [s^-, s^+],\ \text{with} \\
& \lim_{s\to-\infty} x(s) = x_-, \lim_{s\to+\infty} x(s) = x_+, \\
\text{and} \quad & u(s) \in U,\ \forall s \in [s^-, s^+]
\end{aligned}$$

where $U \subseteq \mathbb{R}^n$, $x(s) \in \mathbb{R}^m$ and L as well as f are continuous and differentiable in both x and u.

Since the signal s is not necessarily bounded, the resulting control problem involves expectation integrals with infinite boundaries in the general case.

Necessity and sufficiency results generalised

- The Pontryagin Minimum Principle (PMP) and Mangasarian sufficiency theorem are shown to continue holding if the control problem domain corresponds to the full real axis: the corresponding optimal control problems are well-posed.
- The PMP is then used to show that the given optimal control formulation of the conditioned mean-variance problem generalizes classical (Ferson and Siegel; Markowitz) problem expressions

Aim of the study

Carry out backtests executing constrained-weight conditioned optimization strategies with different settings.

Data set

- 11 years of daily data, from January 1999 to February 2010 (2891 samples)
- Risky assets: 10 different EUR-based funds commercialized in Luxembourg chosen across asset categories (equity, fixed income) and across Morningstar style criteria
- Risk-free proxy: EURIBOR with 1 week tenor
- Signals: VDAX, volatility of bond index, PCA-based indices built using both 2 and 4 factors and estimation window sizes of 50, 100 and 200 points, Kumar and Persaud currency-based GRAI obtained using 1 month and 3 month forward rates

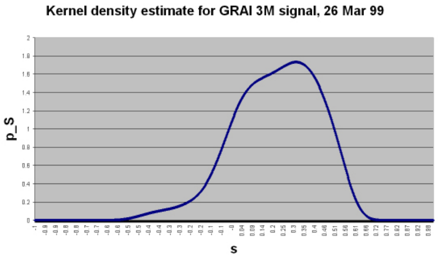
Benchmark problem

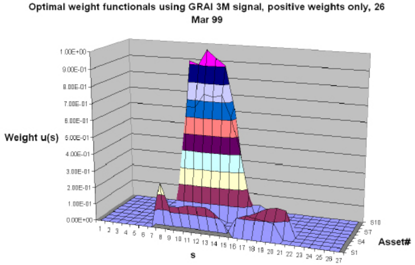
- Take VDAX index as signal, with 60 point estimation window and weights constrained to allow for long investment only
- Rebalance Markowitz-optimal portfolio alongside conditioned optimal portfolio, both with and without the availability of a risk-free proxy asset, over the 11-year period
- Assume lagged relationship $\mu(s)$ between signal and return can be represented by a linear regression
- Use kernel density estimates for signal densities
- Estimate the above using a given rolling window size (15 to 120 points)
- Use direct collocation discretisation method for numerical problem solutions

Benchmark problem (2)

- Vary the parameters to check both for robustness of strategy results and whether results can be further improved while staying with a linear regression model for the relationship between signals and returns
- Obtain efficient frontier for every date and choose portfolio based on quadratic utility functions with risk aversion coefficients between 0 and 10
- Compare sharp ratios (ex ante), additive observed returns (ex post), observed standard deviations (ex post) of both strategies
- Try different window sizes, different signal lags, weight averages over different signal points, different signals

Typical kernel density estimate for signal and resulting optimal weight functionals





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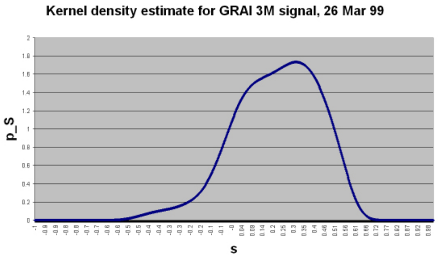
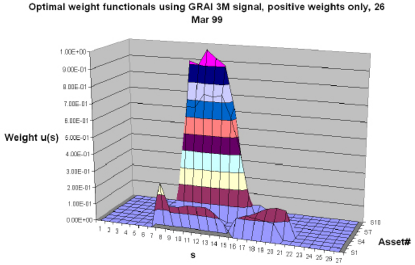
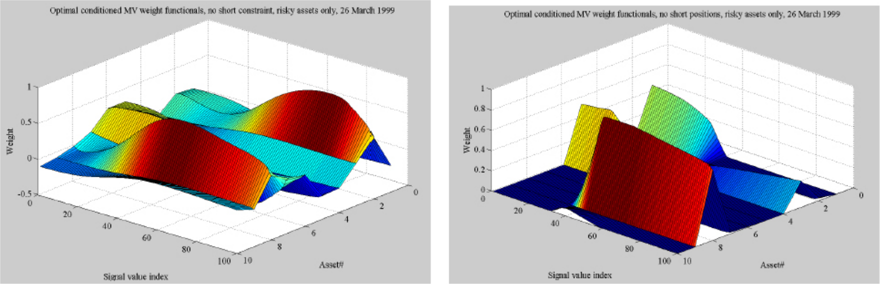




As would be expected, the constrained optimal weights are not simply a truncated version of the unconstrained optimal (Ferson-Siegel) weights.

Weights in the constrained and unconstrained case

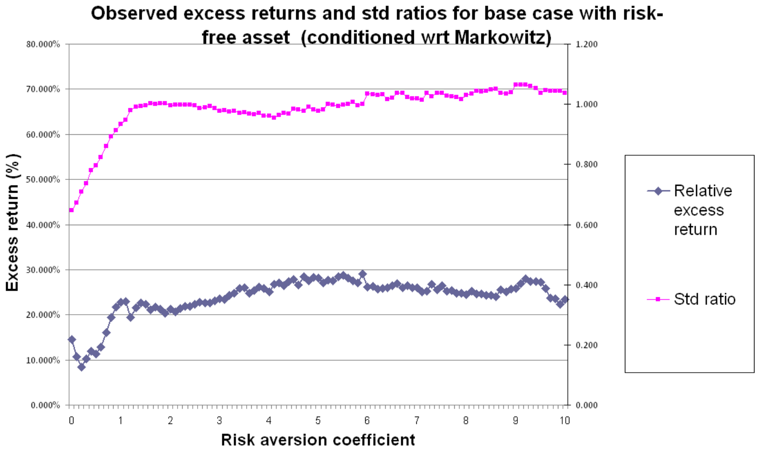


(a) Unconstrained weights. (b) Constrained weights.

FIGURE 4.6: Optimal weight functionals for conditioned problem, no short constraint (left) / weights constrained to be positive(right), 26 March 1999, 60 day estimation windows, VDAX index.

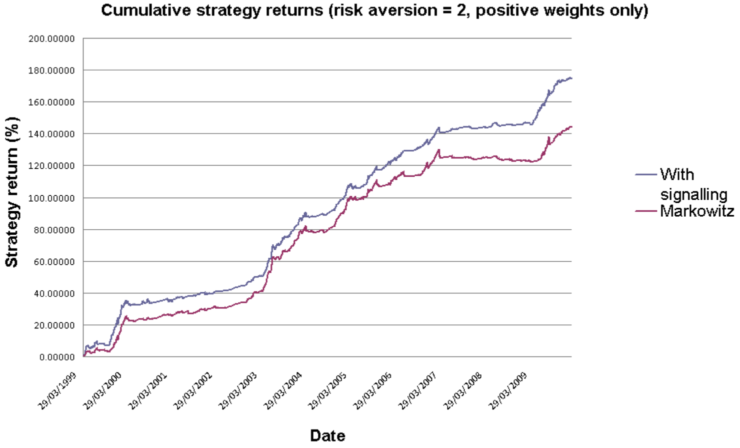
With risk-free asset

Ex post observed relative excess additive returns, standard deviation ratios



With risk-free asset

Time path of additive strategy returns for $\lambda = 2$



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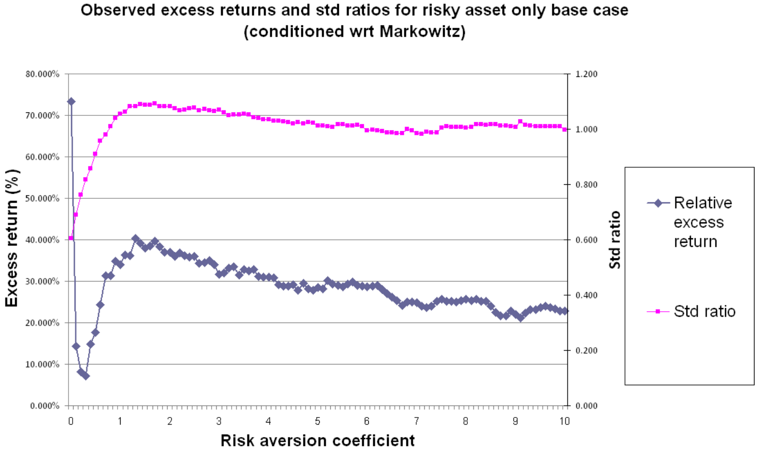

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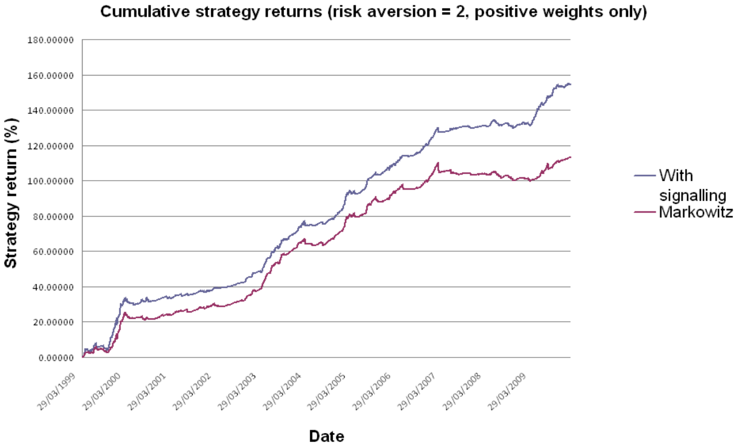
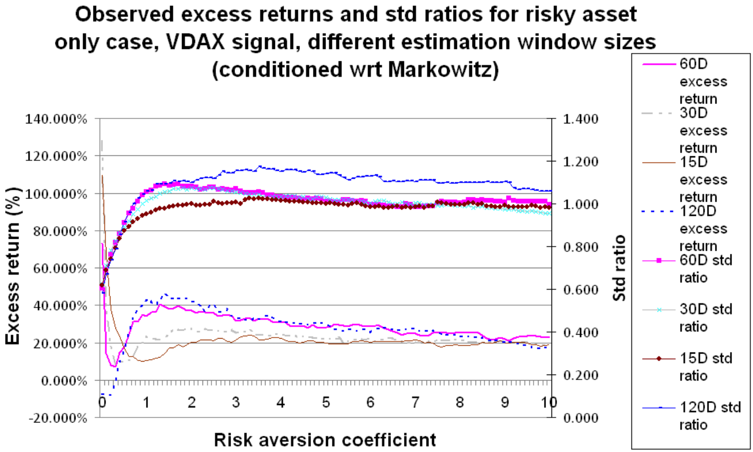
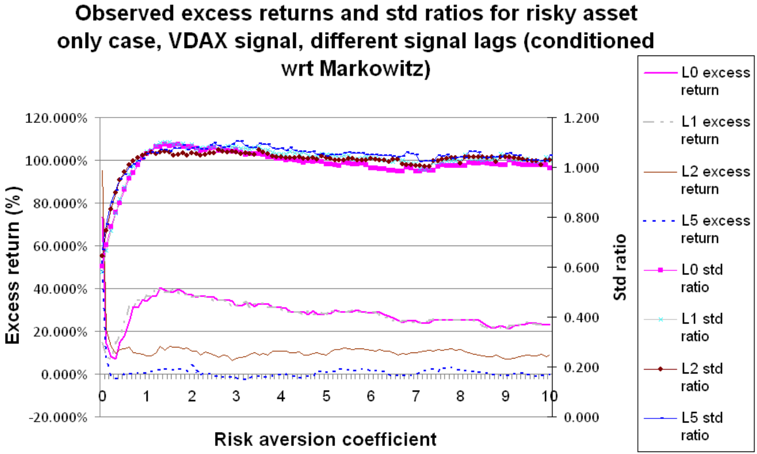


Ex post results for different estimation window sizes



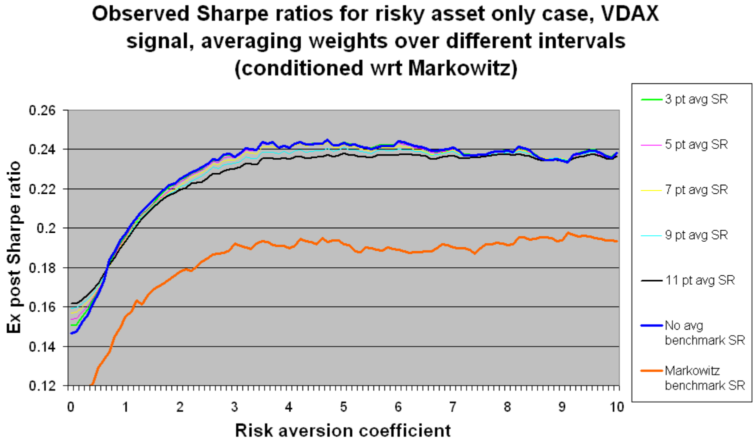
- Excess returns (and standard deviations) larger as window sizes increase
- Trade-off between statistical quality of estimates and impact of conditional nonstationarities

Ex post results for different signal lags



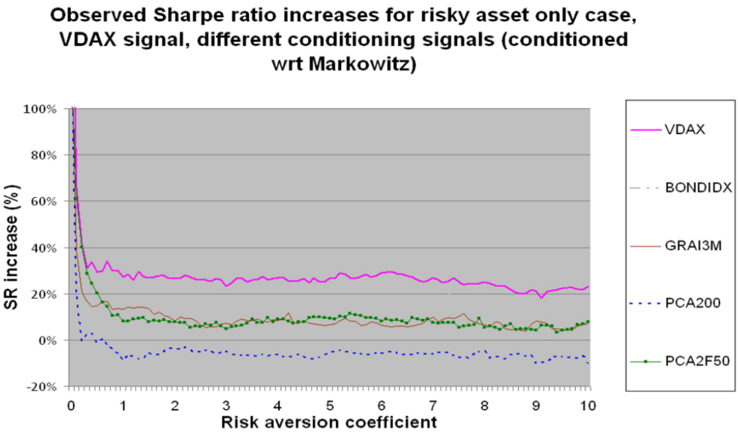
- Excess returns larger and standard deviations smaller as lag size increases
- Trade-off between statistical quality of estimators and easier modelling

Ex post results for weight averages over different number of signal points



- Negligible changes in excess returns, slight changes in standard deviations: little risk attached to signal observations

Ex post results for different signals



- Best results seen for baseline VDAX signal, averaging seems to distract from signal power

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- Model user preferences with respect to the third and fourth moments of returns (skewness (S) and kurtosis (K)) as well as mean (M) and variance (V).
- Can either work as in MV case, replacing expected return or variance by, respectively, skewness or kurtosis (MK efficient frontier)
- or use (polynomial) utility functions to capture investor preferences with respect to more than two moments at the same time (MVK and MVSK optimisation).

MK optimisation as an optimal control problem

$$
\begin{aligned}
&\text{Minimize} && J_{[s^-,s^+]}(x,u) = \int_{s^-}^{s^+} l_4(u,s)p_s(s)ds \text{ as } s^- \to -\infty, s^+ \to +\infty \\
&\text{subject to} && \dot{x}(s) = l_1(u,s)p_s(s)\ \forall s \in [s^-,s^+],\ \text{with} \\
& && \lim_{s\to-\infty} x(s) = x_-,\ \lim_{s\to+\infty} x(s) = x_+, \\
&\text{and} && u(s) \in U,\ \forall s \in [s^-,s^+]
\end{aligned}
$$

where $l_1(u;s)$ and $l_4(u;s)$ are integrands chosen such that the signal domain integral of $l_i p_s(s)$ corresponds in either case to unconditional ith moment metrics of expected portfolio returns, μ_P is the expected unconditional portfolio return and $p_s(s)$ is the signal density function.

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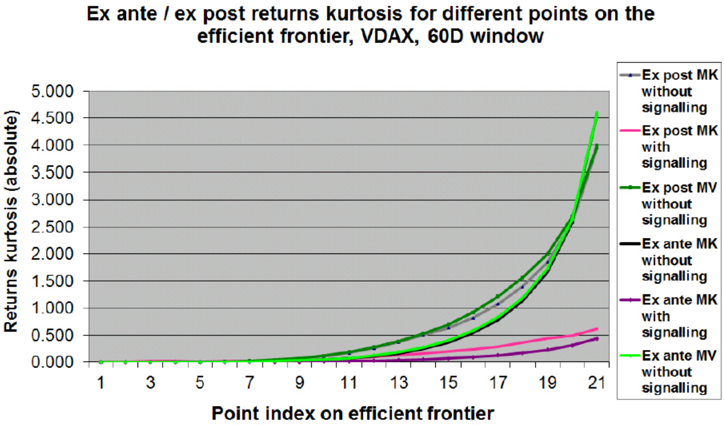
MVK/MVSK optimisation as an optimal control problem

$$
\begin{aligned}
\text{minimise} \quad & J_{I_S}(x(s), u(s)) = -\int_{I_S} \left(a_1 \frac{dx_1}{ds} + a_2 \frac{dx_2}{ds} + a_3 \frac{dx_3}{ds} + a_4 \frac{dx_4}{ds} \right) ds \\
\text{subject to} \quad & \frac{dx_i}{ds} = I_i(u, s) p_S(s), i \in \{1, 2, 3, 4\}, \\
& x_1(s^-) = x_2(s^-) = x_3(s^-) = x_4(s^-) = 0 \\
\text{and} \quad & u(s) \in U \; \forall s \in I_S
\end{aligned}
$$

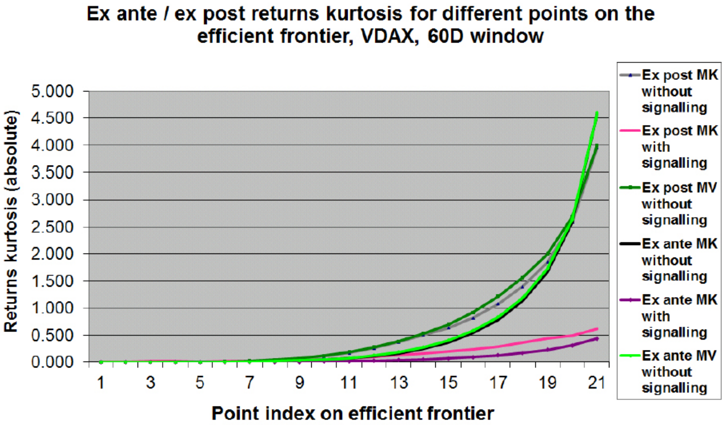
where the $I_i(u, s)$ are integrands chosen such that the $x_i(s^+)$ correspond to unconditional ith moment metrics of expected portfolio returns.

Optimisation involving the third moment always entails a nonconvex cost function: this is problematic from the numerical point of view.

Ex ante and ex post kurtoses for MK optimisation

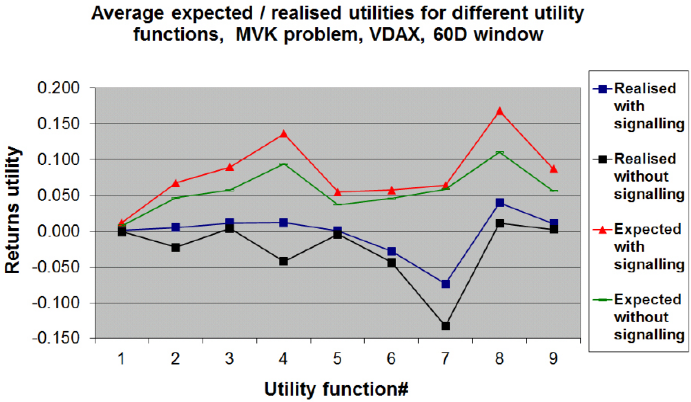


Ex ante and ex post kurtoses for MK optimisation

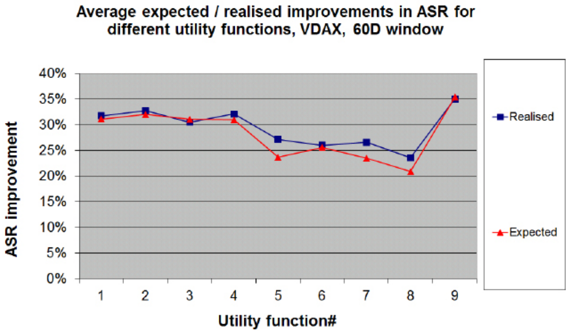


Large reduction in kurtosis both ex ante and ex post seen when conditioning information is used.

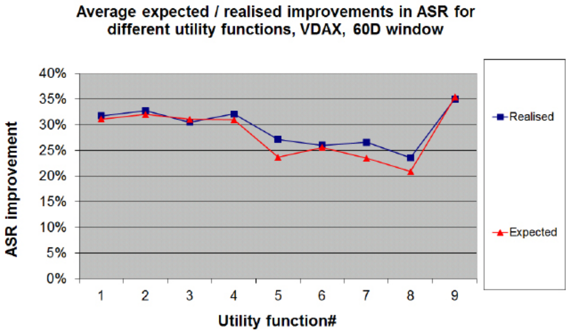
Ex ante and ex post utility values for MVK optimisation



Ex ante and ex post improvements in ASR for MVSK optimisation



Ex ante and ex post improvements in ASR for MVSK optimisation



- The adjusted Sharpe ratio (ASR, Pzier and White (2006)) takes into account third and fourth moments of expected returns.

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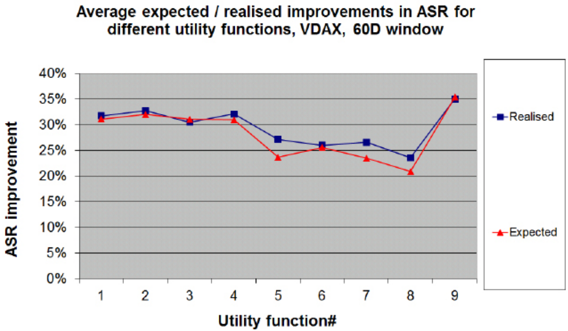
Ex ante and ex post improvements in ASR for MVSK optimisation



- The adjusted Sharpe ratio (ASR, Pzier and White (2006)) takes into account third and fourth moments of expected returns.
- MVSK improvements are consistent with the MVK case: some evidence that skewness preferences may be taken into account in practice.

Summary

- Analysis gives an example of how the optimal control formulation of conditioned problems may be applied to different problem variations not previously accessible.

Summary

- Analysis gives an example of how the optimal control formulation of conditioned problems may be applied to different problem variations not previously accessible.
- Results provide further evidence (in addition to the existing empirical two-moment literature) to suggest that conditioned optimisation increases strategy performance in a universal and robust manner.

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Outline

1. The portfolio theory of Markowitz
2. Conditioned mean-variance portfolio optimisation with unconstrained portfolio weights
3. Conditioned mean-variance portfolio optimisation with constrained portfolio weights
   - General formulation of the problem
   - Empirical study
4. Conditioned portfolio optimisation using higher moments of return
   - Analysis
   - Empirical results
5. Conditioned portfolio optimisation using multiple signals
   - Analysis
   - Empirical results

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Optimal control translation

Two signals $s^{(1)}$ and $s^{(2)}$ with $s = (s^{(1)} s^{(2)})$, investor utility function $U(x) = a_1 x + a_2 x^2$, joint signal density p_s give

$$\begin{aligned}
\text{minimise} \quad & J_{I_S}(x(s), u(s)) = \int_{I_S} \left(a_1 \frac{\partial^2 x_1}{\partial s^{(1)} \partial s^{(2)}} + a_2 \frac{\partial^2 x_2}{\partial s^{(1)} \partial s^{(2)}} \right) ds \\
\text{subject to} \quad & \frac{\partial^2 x_1}{\partial s^{(1)} \partial s^{(2)}} = u'(s)\mu(s)p_s(s), \\
& \frac{\partial^2 x_2}{\partial s^{(1)} \partial s^{(2)}} = \left(\Big(u'(s)\mu(s) \Big)^2 + u'(s)\Sigma_\epsilon^2 u(s) \right) p_s(s), \\
& x_1(s^-) = x_2(s^-) = 0 \\
\text{and} \quad & u(s) \in U \; \forall s \in I_S
\end{aligned}$$

as the resulting mean-variance equivalent optimisation problem formulation.

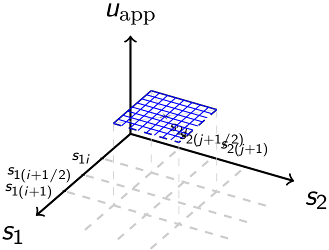
Multidimensional results

- Optimal control problems involving a higher-dimensional objective function integration variable and first-order state PDEs are called *Dieudonné-Rashevsky* problems
- Multidimensional analogues of PMP have been established (Cesari 1969) for problems of the Dieudonné-Rashevsky type
- The problem with cross-derivatives just given represents a form equivalent to Dieudonné-Rashevsky (Udriste 2010)

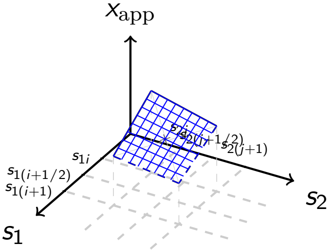
2-D discretisation scheme

- Use a 2-D direct collocation scheme: *direct* means both control and state variables are discretised, *collocation* means PDE and other constraints have to be met exactly at prespecfied (collocation) points on the grid
- Use control values constant on each surface element and state values on vertices to which bilinear interpolation is applied
- Provide analytical expressions for the (sparse) gradient and Hessian matrices to the numerical solver so convergence rate and computational cost remain manageable

2-D discretisation scheme (2)



(a) Control discretisation constant over surface elements.



(b) Bilinear state discretisation.

2-D discretisation scheme convergence result

Theorem

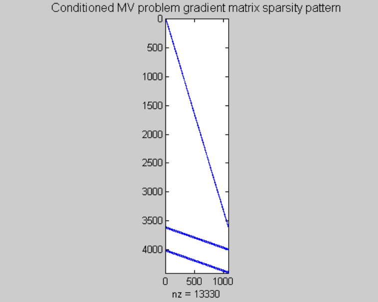
At the collocation points $s_{i+1/2,j+1/2}$, the Pontryagin costate equations are verified to order the chosen grid mesh h:

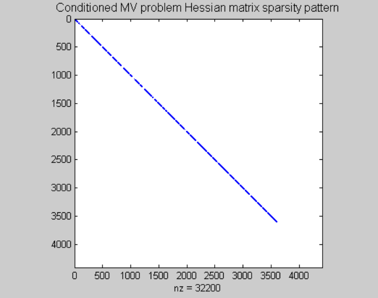
$$\nabla_s \cdot \lambda = -\sum_{\alpha=1}^{2} \lambda^{(\alpha)}_{i+1/2,j+1/2} \frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x} + O(h).$$

Also, for any optimal control interior to the admissible set U, the proposed scheme is consistent with the first-order condition on the Hamiltonian $\mathcal{H}$

$$\frac{\partial \mathcal{H}}{\partial u(s)} = 0 \ \forall s \in I_S.$$

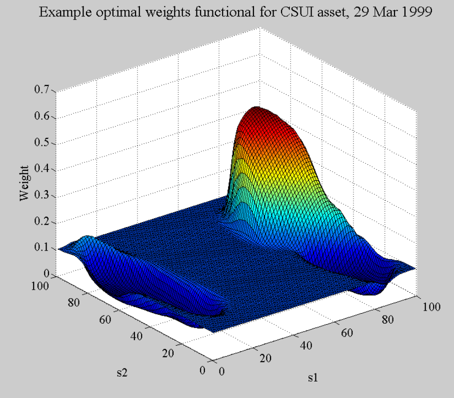
2-D discretisation gradient and Hessian matrix sparsity patterns





- Gradient dimensions for $N \times N$-point grid and n assets are $\left[(N-1)^2 n + 2N^2\right] \times \left[3(N-1)(N-2) + 3(N-2) + 5\right]$
- Hessian dimensions in that case are $\left[(N-1)^2 n + 2N^2\right] \times \left[(N-1)^2 n + 2N^2\right]$

Typical optimal weight functional

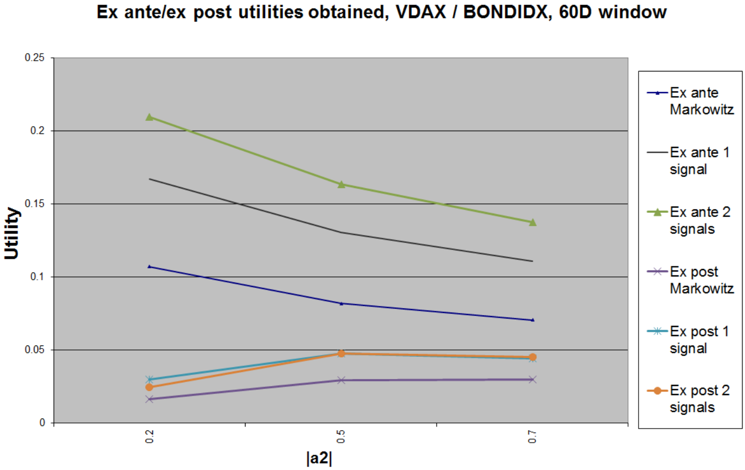


Optimal weights are found as vector functions of the two signals

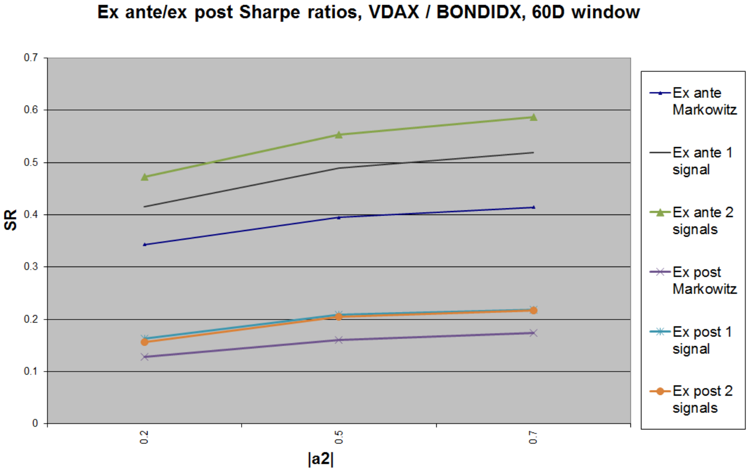
2-signal backtest

- Simultaneously use VDAX (pure equity risk) and BONDIDX (volatility of Barclays Aggregate Euro Bond Index, pure interest rate risk) as signals
- Obtain optimal portfolio weights for daily rebalancing by optimising unconditional expected utilities for quadratic investor utility functions $U(x) = a_1 x + a_2 x^2$ and three different levels of risk aversion: $a_2 = -0.2$, $a_2 = -0.5$ and $a_2 = -0.7$.
- Compare utilities and Sharpe ratios (ex ante and ex post), maximum drawdowns / drawdown durations (MD/MDD) and observed returns time paths for Markowitz, 1 signal and 2 signal strategies

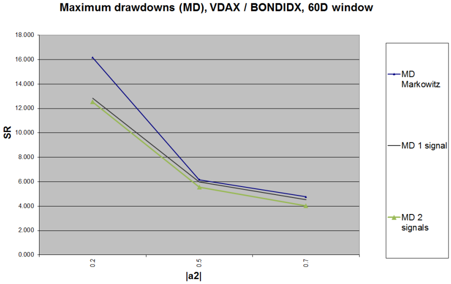
Backtest average utility values

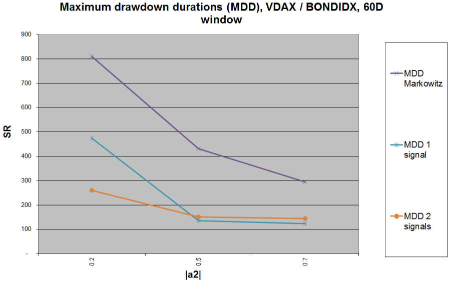


Backtest average Sharpe ratios

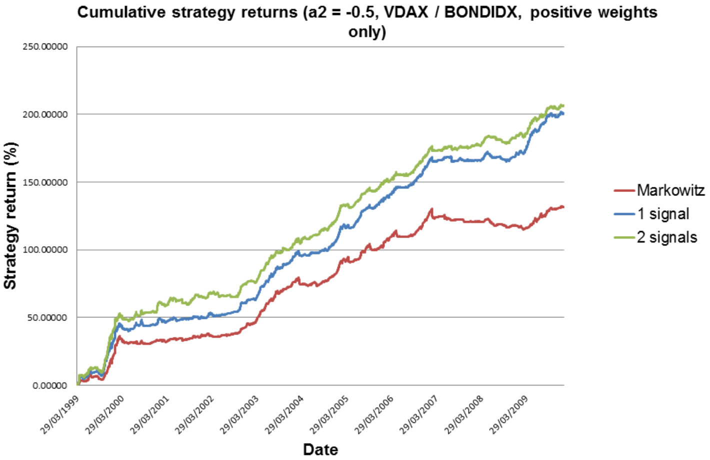


Backtest average maximum drawdown (durations)





Backtest cumulative return time paths, $a_2 = -0.5$



Summary

- Improvement with a second signal is substantial ex ante, but very marginal ex post: estimation risk larger than for a single signal
- The suggested numerical solution scheme can be generalised to even more signals, but a curse of dimensionality applies:
  - computational cost: will diminish in impact over time
  - statistical (kernel density estimate): fundamentally prevents the use of more than three signals unless simplifications are made.
- Marginal ex post improvements, however, suggest an averaging effect (as seen for single PCA indices in earlier single signal study) takes place for more signals, such that this limitation is not seen as that restrictive