DEFORMATIONS OF PRE-SYMPLECTIC STRUCTURES AND THE KOSZUL L_∞-ALGEBRA

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Abstract. We study the deformation theory of pre-symplectic structures, i.e. closed two-forms of fixed rank. The main result is a parametrization of nearby deformations of a given pre-symplectic structure in terms of an L_∞-algebra, which we call the *Koszul L_∞-algebra*. This L_∞-algebra is a cousin of the Koszul dg Lie algebra associated to a Poisson manifold, and its proper geometric understanding relies on Dirac geometry. In addition, we show that a quotient of the Koszul L_∞-algebra is isomorphic to the L_∞-algebra which controls the deformations of the underlying characteristic foliation. Finally, we show that the infinitesimal deformations of pre-symplectic structures and of foliations are both obstructed.

Contents

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Introduction

This paper studies the deformation theory of pre-symplectic structures of a fixed rank. Recall that a 2-form η on a manifold M is called *pre-symplectic* if

i) it is closed, and
ii) the kernel of the vector bundle map $\eta^\sharp : TM \to T^*M$, $v \mapsto \eta(v, \cdot)$ has constant rank.

Pre-symplectic structures arise naturally on certain submanifolds of symplectic manifolds (as in the Dirac theory of constraints in mechanics) and as pullbacks of symplectic forms along submersions. We denote by $\mathsf{Pre\text{-}Sym}^k(M)$ the set of all pre-symplectic structures on M, whose rank equals k. We informally think of $\mathsf{Pre\text{-}Sym}^k(M)$ as some kind of space. Our main goal in this paper is to construct local parametrizations ('charts') for this space.

It is evident that in order to achieve this, we will need to take care simultaneously of the closedness condition and the rank condition. It is not hard to deal with each of these two conditions separately:

i) closedness is the condition of lying in the kernel of a linear differential operator, the de Rham differential $d : \Omega^2(M) \to \Omega^3(M)$,
ii) fibrewise, the rank condition cuts out a (non-linear) fibre-subbundle of $\wedge^2 T^*M \to M$, for which one can construct explicit trivializations.

However, there is a certain tension between the two conditions: The de Rham differential is in no obvious way compatible with the rank condition. Similarly, if one uses in ii) a naive trivialization for the rank condition, one loses control over the closedness condition.

We therefore need a construction which addresses the closedness condition and the rank condition on equal footing. Our solution for this problem originates from *Dirac geometry*, in which one encodes geometric structures on M as subbundles of the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$. The generalized tangent bundle comes with two important structures: 1) a symmetric non-degenerate inner product $\langle \cdot, \cdot \rangle$, encoding the natural pairing between TM and T^*M and 2) a certain bilinear bracket $[\cdot, \cdot]$ on $\Gamma(\mathbb{T}M)$, called the Dorfman bracket. An almost Dirac structure $L \subset \mathbb{T}M$ is a vector subbundle, whose fibres are Lagrangian with respect to $\langle \cdot, \cdot \rangle$. If in addition $\Gamma(L)$ is closed under the Dorfman bracket, L is a *Dirac structure*.

Let us briefly sketch why Dirac geometry is relevant to the deformation theory of pre-symplectic structures: Using auxiliary data, namely the choice of a complement of $\ker(\eta^\sharp)$ in TM, we construct two injections

$$\begin{array}{ccc} \Omega^2(M) & \xhookrightarrow{\Phi_Z} & \\ & & \{\text{almost Dirac structures on } M\}. \\ \Omega^2(M) & \xhookrightarrow{\Phi_0} & \end{array}$$

The transformation $F := \Phi_0^{-1}\Phi_Z$ is well-defined for sufficiently small two-forms and, crucially, it behaves well *both* with respect to the closedness and the rank condition! Indeed, if the restriction of a (sufficiently small) 2-form β to the kernel of η vanishes, then $\eta + F(\beta)$ has the same rank as η, and vice versa (see Theorem. 2.6). Moreover, closedness of $F(\beta)$ is equivalent to $\Phi_Z(\beta)$ being integrable, i.e. Dirac. By the general deformation theory of Dirac structures, which was developed in [12, 5], the integrability of $\Phi_Z(\beta)$ translates into an explicit equation on β of the form

$$d\beta + \frac{1}{2}[\beta,\beta]_Z + \frac{1}{6}[\beta,\beta,\beta]_Z = 0. \tag{1}$$

Here d is the de Rham differential, $[\cdot,\cdot]_Z$ and $[\cdot,\cdot,\cdot]_Z$ is a bilinear, respectively trilinear, map from $\Omega(M)$ to itself. As the notation indicates, $[\cdot,\cdot]_Z$ and $[\cdot,\cdot,\cdot]_Z$ depend on Z, a bivector field on M, which is obtained by partially inverting η (after restriction to the chosen complement of $\ker(\eta^\sharp)$). In a more technical language, d, $[\cdot,\cdot]_Z$ and $[\cdot,\cdot,\cdot]_Z$ equip $\Omega(M)$ with the structure of an L_∞-algebra (after a degree shift), Equation (1) is known as the corresponding *Maurer-Cartan equation*, and its solutions are called *Maurer-Cartan elements*.

To complete our construction, we prove – see Theorem 3.17 – that

$$\Omega_{\mathrm{hor}}(M) := \{\beta \in \Omega(M) \,|\, \beta|_{\ker(\eta^\sharp)} = 0\}$$

is closed under d, $[\cdot,\cdot]_Z$ and $[\cdot,\cdot,\cdot]_Z$, and hence inherits the structure of an L_∞-algebra. The corresponding Maurer-Cartan equation incorporates both the closedness *and* the rank condition which define $\mathsf{Pre\text{-}Sym}^k(M)$. Summing up our discussion, our main result is the construction of the following map (see Theorem 3.19):

Theorem. *There is an injective map*

$$\boxed{\{\textit{small Maurer-Cartan elements of } (\Omega_{\mathrm{hor}}(M), d, [\cdot,\cdot]_Z, [\cdot,\cdot,\cdot]_Z)\} \longrightarrow \mathsf{Pre\text{-}Sym}^k(M)} \tag{2}$$

which bijects onto a ($\mathcal{C}^0$-)neighborhood of η.

Since $[\cdot,\cdot]_Z$ is defined by the same formula as the classical Koszul bracket for a Poisson bivector field, c.f. [9], we refer to $(\Omega_{\mathrm{hor}}(M), d, [\cdot,\cdot]_Z, [\cdot,\cdot,\cdot]_Z)$ as the *Koszul L_∞-algebra of (M,η)*.

The properties of the parametrization (2) can be established without reference to Dirac geometry, and we made an effort to provide two proofs for each statement: a direct, elementary (but sometimes ad hoc) one, and a more conceptual one which relies on Dirac geometry.

The parametrizations of Dirac structures constructed in [12, 5] depend on auxiliary data, and therefore the resulting L_∞-algebras do too. In their recent preprint, cf. [6], M. Gualtieri, M. Matviichuk and G. Scott establish a general framework to control the effects of changing these auxiliary data, and they exhibit explicit canonical L_∞-isomorphisms between the resulting L_∞-algebras. This framework is a natural habitat for the transformation F, which compares the two parametrizations Φ_0 and Φ_Z.

Pre-symplectic structures have a rich geometry. One interesting feature is that each pre-symplectic structure η induces a foliation on the underlying manifold, called the *characteristic foliation* of η, which is given by the kernel of η. This yields a map

$$\rho : \mathsf{Pre\text{-}Sym}^k(M) \to \{\text{foliations on } M\}.$$

We use the local parametrization (2) to construct an algebraic model of this map. To be more precise, we show that a certain quotient of the Koszul L_∞-algebra of (M, η) is isomorphic to the L_∞-algebra whose Maurer-Cartan equation encodes the deformations of the characteristic foliation, see Theorem 4.10.

We finish the paper addressing the obstructedness problem: can every first order deformation be extended to a smooth curve of deformations? We show that the answer to this question is negative, both for deformations of pre-symplectic structures and of foliations. We do so by exhibiting counterexamples on the 4-dimensional torus, using the explicit form of the L_∞-algebras constructed previously.

Let us mention two important issues which we plan to address in a follow-up paper:

- *Geometric vs. algebraic equivalences*: Isotopies act via pullback on the space $\mathsf{Pre\text{-}Sym}^k(M)$ of all pre-symplectic structures of fixed rank. This gives rise to an equivalence relation on $\mathsf{Pre\text{-}Sym}^k(M)$. On the other hand, the Koszul L_∞-algebra of (M, η) comes with the notion of gauge-equivalence on the set of Maurer-Cartan elements. We will show that these two notions of equivalence correspond to each other under the map (2), assuming M is compact. We resolved a problem of this type in our previous work [17].
- *Relation to coisotropic submanifolds*: One of our motivations to develop the deformation theory of pre-symplectic structures is the relationship to coisotropic submanifolds, see [13, 16, 17, 11]. A submanifold M of a symplectic manifold (X, ω) is *coisotropic* if the symplectic orthogonal to TM inside $TX|_M$ is contained in TM. This condition guarantees that $\omega|_M$ is pre-symplectic and, in fact, every pre-symplectic form can be obtained this way. This hints at a tight relationship between the deformation theory of coisotropic submanifolds and the deformation theory of pre-symplectic structures. We will show in forthcoming work that this is indeed the case. On the geometric level, we extend previous work by Ruan, cf. [15]. On the algebraic level, we prove that the Koszul L_∞-algebra of (M, η) is homotopy equivalent – or quasi-isomorphic – to the dg Lie algebra that controls the simultaneous deformations of pairs consisting of a symplectic structure and a coisotropic submanifold, see [2, 5].

Structure of the paper:

Section 1 sets the stage by introducing the relevant deformation problem, and by establishing basic geometric and algebraic facts related to pre-symplectic structures. In Section 1.3 we discuss the toy-example of symplectic structures in the framework which we generalize to arbitrary pre-symplectic structures in the rest of the paper. In Section 1.4 we outline why Dirac geometry provides an effective way to describe the deformation problem.

Section 2 is devoted to the rank condition. We discuss in detail special submanifold charts for the space of skew-symmetric bilinear forms of a fixed rank, explain their Dirac-geometric meaning, and relate them to a more standard construction which makes use of Grassmannians.

Section 3 is the heart of the paper. Here, we construct the Koszul L_∞-algebra associated to a pre-symplectic structure η, and prove that its Maurer-Cartan equation encodes the deformations of η inside the space of pre-symplectic structures of fixed rank. We located the proofs concerning L_∞-algebras in Subsection 3.2, and the discussion of the Dirac geometry underlying our approach in Subsection 3.4. Subsection 3.5 illustrates our approach with three examples.

Section 4 links the deformation theory of pre-symplectic structures to the deformation theory of their characteristic foliations. On the algebraic level, this relationship is quite straightforward: we show that the L_∞-algebra, which encodes the deformations of the foliation underlying η, arises as a quotient of the Koszul L_∞-algebra by an L_∞-ideal.

Section 5 shows that the infinitesimal deformations of pre-symplectic structures as well as foliations are obstructed, using properties of the L_∞-algebras obtained in Section 3 and 4.

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Convention: Let V be a $\mathbb{Z}$-graded vector space, that is we have a decomposition $V = \bigoplus_{k\in\mathbb{Z}} V_k$. For $r \in \mathbb{Z}$, we denote by $V[r]$ the $\mathbb{Z}$-graded vector space whose component in degree k is V_{k+r}.

1. Pre-symplectic structures and their deformations

1.1. **Deformations of pre-symplectic structures.** In this section we set up the deformation problem which we study in this article. Throughout the discussion, M denotes a smooth manifold.

Definition 1.1. *A* 2*-form* η *on* M *is called* pre-symplectic *if*

(1) η *is closed and*
(2) *the vector bundle map* $\eta^\sharp : TM \to T^*M, v \mapsto \iota_v\eta = \eta(v,\cdot)$ *has constant rank.*

In the following, we refer to the rank of $\eta^\sharp$ *as the rank of* η.

Definition 1.2. *A* pre-symplectic *manifold is a pair* (M,η) *consisting of a manifold* M *and a pre-symplectic form* η *on* M.

We denote the space of all pre-symplectic structures on M *by* $\mathsf{Pre\text{-}Sym}(M)$ *and the space of all pre-symplectic structures of rank* k *by* $\mathsf{Pre\text{-}Sym}^k(M)$.

Remark 1.3. Given a pre-symplectic manifold (M,η), the fibrewise kernels of $\eta^\sharp$ assemble into a vector subbundle of TM, which we denote by K:

$$K := \ker(\eta^\sharp).$$

Since η is closed, K is involutive, hence gives rise to a foliation of M. Recall that associated to any foliation, one has the corresponding foliated de Rham complex, which we denote by $\Omega(K) := (\Gamma(\wedge K^*), d_K)$. Restriction of ordinary differential forms on M to sections of K yields a surjective chain map $r : \Omega(M) \to \Omega(K)$. We denote the kernel of r by $\Omega_{\mathrm{hor}}(M)$. It coincides with the multiplicative ideal in $\Omega(M)$ generated by all the section of the annihilator $K^\circ \subset T^*M$ of K. We have the following exact sequence of complexes

$$0 \longrightarrow \Omega_{\mathrm{hor}}(M) \longrightarrow \Omega(M) \xrightarrow{\ r\ } \Omega(K) \longrightarrow 0\ . \tag{3}$$

We denote the cohomology of $\Omega_{\mathrm{hor}}(M)$ by $H_{\mathrm{hor}}(M)$, and the cohomology of $\Omega(K)$ by $H(K)$.

We next compute the formal tangent space to $\mathsf{Pre\text{-}Sym}^k(M)$ at a pre-symplectic form η.

Lemma 1.4. *Let* $(\eta_t)_{t\in[0,\varepsilon)}$ *be a one-parameter family of pre-symplectic forms on* M *of fixed rank* k *with* $\eta_0 = \eta$. *Then* $\frac{d}{dt}|_{t=0}\eta_t$ *is a closed* 2*-form which lies in* $\Omega^2_{\mathrm{hor}}(M)$.

Proof. That $\frac{d}{dt}|_{t=0}\eta_t$ is closed is straight-forward.

Concerning the second claim, let X and Y be two arbitrary smooth sections of K. Since $(K_t = \ker(\eta_t^\sharp))_{t\in[0,\varepsilon)}$ assemble into a smooth vector bundle $\tilde{K}$ over $M \times [0,\varepsilon)$, we can find smooth extensions of X and Y to sections of $\tilde{K}$. Denote these extensions by X_t and Y_t. By definition, we have

$$0 = \eta_t(X_t, Y_t)$$

for all $t \in [0,\varepsilon)$. Differentiation with respect to t yields

$$0 = \left(\frac{d}{dt}|_{t=0}\eta_t\right)(X,Y) + \eta(\frac{d}{dt}|_{t=0}X_t, Y) + \eta(X, \frac{d}{dt}|_{t=0}Y_t) = \left(\frac{d}{dt}|_{t=0}\eta_t\right)(X,Y).$$

□

Remark 1.5. Lemma 1.4 tells us that we have the following identification of the formal tangent space to $\mathsf{Pre\text{-}Sym}^k(M)$ at η:

$$T_\eta\left(\mathsf{Pre\text{-}Sym}^k(M)\right) \cong \{\alpha \in \Omega^2(M) \text{ closed}, r(\alpha) = 0\} = Z^2(\Omega_{\mathrm{hor}}(M)).$$

Definition 1.6. *The group of isotopies* $\mathsf{Diff}_0(M)$ *of* M *acts on* $\mathsf{Pre\text{-}Sym}^k(M)$ *from the right via* $\eta \cdot f := f^*\eta$.

We call two pre-symplectic structures η *and* $\tilde{\eta}$ isotopic *if they lie in the same orbit of this action, and then write* $\eta \sim \tilde{\eta}$. *Given an isotopy* f_t *of* M *and a pre-symplectic structure* η, *we say that the one-parameter family of pre-symplectic structures* $\eta \cdot f_t$ *is generated by* f_t.

Furthermore, we denote the set of orbits by $\mathsf{Pre\text{-}Sym}^k(M)/\mathsf{Diff}_0(M)$.

Here is a slight reformulation of the equivalence relation $\sim$ of pre-symplectic structures given by isotopies:

Proposition 1.7. *Suppose* M *is compact. Two pre-symplectic structures* η *and* $\tilde{\eta}$ *on* M *are isotopic, if and only if there is a smooth one-parameter family of pre-symplectic structures* $(\eta_t)_{t\in[0,1]}$ *joining them, such that the variation* $d\eta_t/dt$ *equals* $d\beta_t$, *with* β_t *a section of* $(\ker(\eta_t))^\circ$.

Proof. Assume that η and $\tilde{\eta}$ are isotopic via f_t. Then the smooth one-parameter family $\eta_t := (f_t)^*\eta$ satisfies the requirements of the proposition since

$$\frac{d}{dt}\eta_t = \mathcal{L}_{X_t}\eta_t = d\iota_{X_t}\eta_t,$$

and $\iota_{X_t}\eta_t$ lies in $(\ker(\eta_t))^\circ$. Here X_t is the time-dependent vector field associated to the isotopy.

One the other hand, if we are given a family η_t and β_t as specified in the proposition, we can apply Moser's trick. In more detail, we make the Ansatz

$$0 = \frac{d}{dt}(g_t^*\eta_t) = g_t^*(d\iota_{X_t}\eta_t + \frac{d}{dt}\eta_t) = g_t^* d(\iota_{X_t}\eta_t + \beta_t),$$

for g_t the isotopy generated by X_t. Now we can find a one-parameter family of vector fields X_t such that

$$\iota_{X_t}\eta_t + \beta_t = 0,$$

since β_t lies in the image of $\eta_t^\sharp$. Observe that the kernel of η_t, $t \in [0,1]$, forms a vector bundle over $M \times [0,1]$ and we can choose a complementary subbundle to it inside the pull back of TM. Requiring that X_t takes values in this subbundle uniquely determines the one-parameter family X_t (this shows in particular that X_t can be chosen in a smooth manner). Since M is compact, $(X_t)_{t\in[0,1]}$ will integrate to an isotopy $(g_t)_{t\in[0,1]}$. Setting $f_t := g_t^{-1}$ yields the desired isotopy satisfying $\eta_t = f_t^*\eta$. □

Definition 1.8. *The* moduli space *of pre-symplectic structures of rank k on M is the set of equivalence classes* $\mathsf{Pre\text{-}Sym}^k(M)/\mathsf{Diff}_0(M)$.

Remark 1.9. Let us determine the formal tangent space to $\mathsf{Pre\text{-}Sym}^k(M)/\mathsf{Diff}_0(M)$ at the equivalence class of η. By Lemma 1.4, the tangent space of $\mathsf{Pre\text{-}Sym}^k(M)$ at η can be identified with the closed 2-forms on M whose restriction to $K = \ker(\eta)$ is zero. On the other hand, the equivalence class of η is infinitesimally modelled by $d\beta$ for $\beta \in \Gamma(K^\circ)$. As the quotient of these two vector spaces, and hence as the candidate for $T_{[\eta]}\left(\mathsf{Pre\text{-}Sym}^k(M)/\mathsf{Diff}_0(M)\right)$, we therefore find $H^2_{\mathrm{hor}}(M)$.

1.2. **Bivector fields induced by pre-symplectic forms.** For later use, we present basic results on the geometry of bivector fields that arise from pre-symplectic forms, after one makes a choice of complement to the kernel. For any bivector field Z, we denote by $\sharp\colon T^*M \to TM$ the map $\xi \mapsto \iota_\xi Z = Z(\xi, \cdot)$. The Koszul bracket of 1-forms associated to Z is

$$[\xi_1, \xi_2]_Z = \iota_{\sharp\xi_1} d\xi_2 - \iota_{\sharp\xi_2} d\xi_1 + d\langle Z, \xi_1 \wedge \xi_2\rangle. \tag{4}$$

In case Z is Poisson, these two pieces of data make T^*M into a Lie algebroid. In general, there is an induced bracket on smooth functions given by $\{f, g\}_Z = (\sharp df)(g)$. In the following, we will make repeated use of the fact that the Koszul bracket has the following derivation property:

$$[\xi_1, f\xi_2]_Z = f[\xi_1, \xi_2]_Z + Z(\xi_1, df)\xi_2$$

where ξ_1, $\xi_2 \in \Omega^1(M)$ and $f \in \mathcal{C}^\infty(M)$.

We start with a lemma about general bivector fields, which in the Poisson case reduces to the fact that $\sharp$ is bracket-preserving:

Lemma 1.10. *For any bivector field Z on M, and for all $\xi_1, \xi_2 \in \Omega^1(M)$ we have*

$$[\sharp\xi_1, \sharp\xi_2] = \sharp[\xi_1, \xi_2]_Z - \frac{1}{2}\iota_{\xi_2}\iota_{\xi_1}[Z, Z]. \tag{5}$$

Proof. By the derivation property of the Koszul bracket, we may assume that ξ_i is exact for $i = 1, 2$. We have $\langle[\sharp df_1, \sharp df_2], df_3\rangle = [\sharp df_1, \sharp df_2](f_3) = \{f_1, \{f_2, f_3\}\} - \{f_2, \{f_1, f_3\}\}$, using in the first equality the definition of the Lie bracket as a commutator. Further $\langle \sharp[df_1, df_2]_Z, df_3\rangle = \langle \sharp d\{f_1, f_2\}, df_3\rangle = \{\{f_1, f_2\}, f_3\}$. Now use the well-known fact that $\frac{1}{2}[Z, Z]$ applied to $df_1 \wedge df_2 \wedge df_3$ equals the Jacobiator $\{\{f_1, f_2\}, f_3\}\} + c.p.$ □

Remark 1.11. Let Z be a constant rank bivector field, and denote by G the image of $\sharp$. Since G° is the kernel of $\sharp$, it is straightforward to check that $[\xi_1, \xi_2]_Z = 0$ for all $\xi_1, \xi_2 \in \Gamma(K^*)$. Lemma 1.10 immediately implies that, for any splitting $TM = K \oplus G$, we have $[Z, Z] \in \Gamma(\wedge^3 G) \oplus \Gamma(\wedge^2 G \otimes K)$.

Let $\eta \in \Omega^2(M)$ is be a pre-symplectic structure on M, and denote its kernel by K. Let us fix a complementary subbundle G, so $TM = K \oplus G$. Define Z to be the bivector field on M determined by $Z^\sharp = -(\eta|_G^\sharp)^{-1}$. Clearly Z is a constant rank bivector field, and the image of $Z^\sharp$ is G.

Together with Remark 1.11, the following Lemma implies that $[Z, Z] \in \Gamma(\wedge^2 G \otimes K)$.

Lemma 1.12. *$[Z, Z]$ has no component in $\wedge^3 G$.*

Proof. Working locally, we may assume that we have a surjective submersion $p\colon M \to M/K$ where M/K is the quotient of M by the foliation integrating K. By pre-symplectic reduction, there is a unique symplectic form Ω on M/K such that

$$p^*\Omega = \eta.$$

Denote by Π the Poisson bivector field on M/K determined by $\Pi^\sharp = -(\Omega^\sharp)^{-1}$. Under the decomposition $TM = K \oplus G$, the only component of η is $\eta|_{\wedge^2 G} \in \Gamma(\wedge^2 G^*)$, whose negative inverse is $Z \in \Gamma(\wedge^2 G)$. By the above equation, Z projects to Π under p. Consequently, the trivector field $[Z, Z]$ projects onto $[\Pi, \Pi] = 0$, finishing the proof. □

We wish to understand the Lie bracket of vector fields $Z^\sharp \xi$ where ξ is a 1-form. Clearly $K^* = G^\circ$ is the kernel of $Z^\sharp$. Hence it is sufficient to assume that ξ be a section of G^*.

Lemma 1.13. *For all $\xi_1, \xi_2 \in \Gamma(G^*)$, Equation (5) expresses $[Z^\sharp \xi_1, Z^\sharp \xi_2]$ in terms of the decomposition $TM = G \oplus K$. In particular,*

$$Z^\sharp[\xi_1, \xi_2]_Z = \mathrm{pr}_G([Z^\sharp \xi_1, Z^\sharp \xi_2]).$$

Proof. The first term on the right-hand side of Equation (5) lies in $G = \mathrm{image}(Z^\sharp)$. Since $[Z, Z] \in \Gamma(\wedge^3 TM)$ has no component in $\wedge^3 G$ by Lemma 1.12, the last term on the right-hand side of Equation (5) lies in K. □

We finish with:

Lemma 1.14. $\Gamma(G^*)$ *is closed under the Koszul bracket.*

Proof. It suffices to show that the Koszul bracket of any elements from a frame for $G^* = K^\circ$ lies again in $\Gamma(K^\circ)$. We use the same notation as in the proof of Lemma 1.12. If we pick a system of coordinates $y_1, \ldots, y_r$ on M/K (we work locally), the 1-forms $d(p^* y_1), \ldots, d(p^* y_r)$ constitute a frame of K°. We have

$$[d(p^* y_i), d(p^* y_j)]_Z = d(\langle Z, d(p^* y_i) \wedge d(p^* y_j) \rangle) = dp^*(\langle \Pi, y_i \wedge y_j \rangle)$$

and this is clearly an element of $\Gamma(K^\circ)$. □

1.3. **The Koszul bracket and deformations of symplectic structures.** In this subsection we describe how the deformations of symplectic structures, once the Poisson geometry point of view is taken, can be described by means of the Koszul bracket. This approach will be generalized to arbitrary pre-symplectic structures in Section 3, relying on some linear algebra developed in Section 2, see in particular Remark 2.5. Of course, the nearby deformations of a symplectic structure ω can also be described as $\omega + \alpha$ for α small 2-forms satisfying $d\alpha = 0$, but this straightforward description does not extend to the pre-symplectic case.

Let π be a Poisson bivector field on M. There is a unique extension of the Koszul bracket $[\cdot, \cdot]_\pi$ – defined on 1-forms by formula 4 – to all differential forms which satisfies

- graded skew-symmetry, i.e.
$$[\alpha, \beta]_\pi = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha]_\pi,$$
- Leibniz rule for d, i.e.
$$d([\alpha, \beta]_\pi) = [d\alpha, \beta]_\pi + (-1)^{|\alpha|-1}[\alpha, d\beta]_\pi,$$
- derivation property with respect to the wedge product $\wedge$:
$$[\alpha, \beta \wedge \gamma]_\pi = [\alpha, \beta]_\pi \wedge \gamma + (-1)^{(|\alpha|-1)|\beta|}\beta \wedge [\alpha, \gamma]_\pi.$$

We recall the following algebraic facts:

Lemma 1.15. *Let π be any Poisson structure on a manifold M.*

(a) *The following is a strict morphism of dg Lie algebras:*

$$I := \wedge \pi^\sharp \colon (\Omega(M), d, [\cdot, \cdot]_\pi) \to (\mathfrak{X}^{\mathrm{multi}}(M), -[\pi, \cdot], [\cdot, \cdot]).$$

(b) *The target dg Lie algebra $\mathfrak{X}^{\mathrm{multi}}(M)$ governs the deformations of the Poisson structure π: Poisson structures nearby π are given by bivector fields $\pi - \tilde{\pi}$, where $\tilde{\pi}$ satisfies the Maurer-Cartan equation*

$$-[\pi, \tilde{\pi}] + \frac{1}{2}[\tilde{\pi}, \tilde{\pi}] = 0.$$

Proof. (a) Recall that the anchor of the cotangent Lie algebroid $\pi^\sharp \colon T^*M \to TM$ is a Lie algebroid morphism. Hence the pullback $\wedge(\pi^\sharp)^* \colon \Omega(M) \to \mathfrak{X}^{\mathrm{multi}}(M)$ relates the Lie algebroid differential d (the de Rham differential) to $d_\pi := [\pi, \cdot]$. Since $(\pi^\sharp)^* = -\pi^\sharp$, it follows that $\wedge\pi^\sharp$ maps d to $-d_\pi$. Further, since $\pi^\sharp$ is a Lie algebroid morphism, $\wedge\pi^\sharp$ preserves (Schouten) brackets.

(b) This follows from $[\pi - \tilde{\pi}, \pi - \tilde{\pi}] = -2[\pi, \tilde{\pi}] + [\tilde{\pi}, \tilde{\pi}]$. □

Suppose that π is invertible, and denote by ω the corresponding symplectic structure, determined by $-\omega^\sharp = (\pi^\sharp)^{-1}$. Denote by $\mathcal{I}_\pi$ the tubular neighborhood of $M \subset \wedge^2 T^*M$ consisting of those bilinear forms β such that $\mathrm{id}_{TM} + \pi^\sharp\beta^\sharp$ is invertible. The following lemma takes the point of view of Poisson geometry to describe the symplectic structures nearby ω, in the sense that instead of deforming ω directly, it deforms π. Diagrammatically:

$$\begin{array}{ccc} \{\text{symplectic forms near } \omega\} & \xleftrightarrow{\text{inversion}} & \{\text{Poisson structures near } \pi\} \\ & & \updownarrow \\ \{\text{small } \beta \text{ s.t. } d\beta + \tfrac{1}{2}[\beta,\beta]_\pi = 0\} & \xleftrightarrow{I} & \{\text{small } \tilde{\pi} \text{ s.t. } -[\pi,\tilde{\pi}] + \tfrac{1}{2}[\tilde{\pi},\tilde{\pi}] = 0\} \end{array}$$

Lemma 1.16. *Let ω be a symplectic structure on M with corresponding Poisson structure π. There is a bijection between*

- *2-forms $\beta \in \Gamma(\mathcal{I}_\pi)$ such that the equation $d\beta + \frac{1}{2}[\beta, \beta]_\pi = 0$ holds.*
- *symplectic forms nearby ω (in the $\mathcal{C}^0$ sense).*

The bijection maps β to the symplectic form with sharp map $\omega^\sharp + \beta^\sharp(\mathrm{id}_{TM} + \pi^\sharp\beta^\sharp)^{-1}$.

Proof. It is well-known that under the correspondence between non-degenerate 2-forms and non-degenerate bivector fields, the closeness of the 2-form ω corresponds to the Poisson condition for π. By Lemma 1.15 (b), the Poisson structures nearby π are given by $\pi - \tilde{\pi}$ where $\tilde{\pi}$ satisfies the Maurer-Cartan equation of the dg Lie algebra $\mathfrak{X}^{\mathrm{multi}}(M)$.

Since π is non-degenerate, the map I is an isomorphism between differential forms and multivector fields. The 2-form $\beta := I^{-1}(\tilde{\pi})$ satisfies the Maurer-Cartan equation $d\beta + \frac{1}{2}[\beta, \beta]_\pi = 0$ by Lemma 1.15 (a). Notice that $\pi - \tilde{\pi} = \pi - I\beta$ has sharp map

$$(\mathrm{id}_{TM} + \pi^\sharp\beta^\sharp)\pi^\sharp \colon T^*M \to TM, \tag{6}$$

so it is non-degenerate iff $\beta \in \Gamma(\mathcal{I}_\pi)$. Hence we obtain a bijection between 2-forms β as in the statement on one side, and symplectic forms corresponding to $\pi - I\beta$ on the other side.

The latter can be described as follows: the sharp map is

$$\begin{aligned} -[(\pi - I\beta)^\sharp]^{-1} &= -(\pi^\sharp)^{-1}(\mathrm{id}_{TM} + \pi^\sharp\beta^\sharp)^{-1} \\ &= \omega^\sharp((\mathrm{id}_{TM} + \pi^\sharp\beta^\sharp) - \pi^\sharp\beta^\sharp)(\mathrm{id}_{TM} + \pi^\sharp\beta^\sharp)^{-1} \\ &= \omega^\sharp + \beta^\sharp(\mathrm{id}_{TM} + \pi^\sharp\beta^\sharp)^{-1}, \end{aligned}$$

where in the first equality we used (6). □

1.4. **The point of view of Dirac geometry.** Let η be a pre-symplectic form on M, with kernel K. The natural way to parametrize deformations of η is by 2-forms α such that $\eta+\alpha$ is again pre-symplectic, but this parametrization has a serious flaw: the space of such α's does not have a natural vector space structure, due to the constant rank condition. Taking the point of view of Dirac geometry, the above approach to parametrize the deformations of η amounts to deforming the Dirac structure graph(η) using $\{0\}\oplus T^*M$ as a complement.[1]

A better way to parametrize the deformations of η in terms of Dirac geometry works as follows: Let us first choose a complement G to K. Then $G\oplus K^*$ is a complement of graph(η): for every $v\in TM$ we have $\iota_v\eta\in K^\circ=G^*$, so requiring that it lies in K^* implies $\iota_v\eta=0$. This means that $v\in K$, so requiring that it lies in G implies $v=0$.
We can now use $G\oplus K^*$ – instead of $\{0\}\oplus T^*M$ – to parametrize deformations of the Dirac structure graph(η). This has the advantage of linearizing the constant rank condition, see Proposition 1.20 below. When η is *symplectic*, the new complement is just TM, hence we are deforming η by viewing it as a Poisson structure, just as in Section 1.3.

We start by giving an alternative characterization of $G\oplus K^*$. Let $\mathfrak{t}_\eta$ be the orthogonal transformation of $TM\oplus T^*M$ given by $(v,\xi)\mapsto(v,\xi+\eta^\sharp(v))$.

Lemma 1.17. *Denote by $Z\in\Gamma(\wedge^2G)$ the bivector field such that $Z^\sharp$ is the inverse of $-(\eta|_G)^\sharp$. Then*

$$G\oplus K^*=\mathfrak{t}_\eta(\mathrm{graph}(Z)).$$

Proof. $\mathfrak{t}_\eta(\mathrm{graph}(Z))=\{(Z^\sharp\xi,\xi|_K):\xi\in T^*M\}=G\oplus K^*$. □

Lagrangian subbundles nearby graph(η) can be written, for some $\bar\beta\in\Gamma(\wedge^2((\mathrm{graph}(\eta))^*)$, as the graph of the map

$$\mathrm{graph}(\eta)\xrightarrow{\bar\beta}(\mathrm{graph}(\eta))^*\cong G\oplus K^*.$$

We denote this graph as $\Phi_{G\oplus K^*}(\bar\beta)$. Moreover, let $\beta\in\Omega^2(M)$ be the 2-form corresponding to $\bar\beta$ under the isomorphism graph$(\eta)\cong TM, v+\iota_v\eta\mapsto v$ and denote by $\Phi_Z(\beta)$ the graph of the map $TM\to T^*M\cong\mathrm{graph}(Z)$ induced by β.

Lemma 1.18. *$\mathfrak{t}_{-\eta}$ maps $\Phi_{G\oplus K^*}(\bar\beta)$ to $\Phi_Z(\beta)$.*

Proof. $\mathfrak{t}_{-\eta}$ preserves the pairing on $TM\oplus T^*M$, clearly maps graph(η) to TM, and maps $G\oplus K^*$ to $\mathfrak{t}_\eta(\mathrm{graph}(Z))$ by Lemma 1.17. Therefore the statement follows by functoriality. □

Since $\mathfrak{t}_{-\eta}$ is actually an automorphism of the standard Courant algebroid $TM\oplus T^*M$, describing the deformations of graph(η) as a Dirac structure using the complement $G\oplus K^*$ is tantamount to describing the deformations of TM using the complement graph(Z). The latter deformation problem is easier to handle, and is the one we address in Subsections 2.2 and 3.4.

Now we explain why the choice of $G\oplus K^*$ is a good one to describe pre-symplectic deformations.

Lemma 1.19. *For any $\bar\beta\in\Gamma(\wedge^2((\mathrm{graph}(\eta))^*)$, the rank of*

$$\Phi_{G\oplus K^*}(\bar\beta)\cap TM \tag{7}$$

equals the rank of

$$\{v\in K:\iota_v\beta\in G^*\}. \tag{8}$$

[1]We refer the reader to Appendix C for the basics of Dirac geometry.

Proof. Applying the transformation $\mathfrak{t}_{-Z} \circ \mathfrak{t}_{-\eta}$ to $\Phi_{G\oplus K^*}(\bar\beta)$, we obtain $\mathfrak{t}_{-Z}(\Phi_Z(\beta)) = \mathrm{graph}(\beta)$ by Lemma 1.18. Applying it to TM we obtain $\{(v + Z^\sharp\iota_v\eta, -\iota_v\eta) \mid v \in V\} = K \oplus G^*$.

Hence applying the transformation to (7) we obtain

$$\mathrm{graph}(\beta) \cap (K \oplus G^*),$$

which is isomorphic to (8). □

Recall that the vector space $\Omega^2_{\mathrm{hor}}(M)$ of horizontal 2-forms was defined in Subsection 1.1 to be the kernel of the restriction map $r^2_\Omega(M) \to \Omega^2(K)$. Since (7) is the kernel of the two-form whose graph is $\Phi_{G\oplus K^*}(\bar\beta)$, we immediately obtain:

Proposition 1.20. *Given $\bar\beta \in \Gamma(\wedge^2((\mathrm{graph}(\eta))^*)$, we consider the two-form on M whose graph is $\Phi_{G\oplus K^*}(\bar\beta)$. Its kernel has rank equal to* $\mathrm{rank}(K)$ *iff β is a horizontal 2 form.*

2. Parametrizing skew-symmetric bilinear forms

In this section, we discuss the rank condition on pre-symplectic structures. Since we postpone the discussion of integrability to Section 3.3, everything boils down to linear algebra. In Section 2.1 we introduce a certain local parametrization of skew-symmetric bilinear forms, which is inspired by Dirac geometry. The link to Dirac geometry is explained in Section 2.2. Another, more standard, parametrization is discussed in Section 2.3 and related to the parametrization from Section 2.1.

2.1. A parametrization inspired by Dirac geometry.

Let V be a finite-dimensional, real vector space. Fix $Z \in \wedge^2 V$ a bivector, which can be encoded by the linear map

$$Z^\sharp \colon V^* \to V, \quad \xi \mapsto \iota_\xi Z = Z(\xi, \cdot).$$

Definition 2.1. *We denote by $\mathcal{I}_Z$ the open neighborhood of $0 \subset \wedge^2 V^*$ consisting of those elements β for which the map $\mathrm{id} + Z^\sharp\beta^\sharp \colon V \to V$ is invertible.*

We consider the map $F \colon \mathcal{I}_Z \to \wedge^2 V^*$ determined by

$$(F(\beta))^\sharp = \beta^\sharp(\mathrm{id} + Z^\sharp\beta^\sharp)^{-1}. \tag{9}$$

This map is clearly non-linear, and it is smooth. A Dirac-geometric interpretation of the definition of F will be given in Section 2.2.

Remark 2.2.
(i) We have $\ker(\beta) = \ker(F(\beta))$. The inclusion "$\subset$" follows directly from Equation (9), using the fact that $\mathrm{id} + Z^\sharp\beta^\sharp$ preserves $\ker(\beta)$. Further, since $\mathrm{id} + Z^\sharp\beta^\sharp$ is an isomorphism, the dimensions of $\ker(\beta)$ and $\ker(F(\beta))$ are the same.
(ii) $F \colon \mathcal{I}_Z \to \wedge^2 V^*$ bijects onto $\mathcal{I}_{-Z}$, with inverse $\alpha \mapsto \alpha^\sharp(\mathrm{id} - Z^\sharp\alpha^\sharp)^{-1}$. Indeed

$$\mathrm{id} - Z^\sharp(F(\beta))^\sharp = \mathrm{id} - Z^\sharp\beta^\sharp(\mathrm{id} + Z^\sharp\beta^\sharp)^{-1} = (\mathrm{id} + Z^\sharp\beta^\sharp)^{-1},$$

showing that

$$(F(\beta))^\sharp(\mathrm{id} - Z^\sharp(F(\beta))^\sharp)^{-1} = \beta^\sharp.$$

F is a diffeomorphism from $\mathcal{I}_Z$ to $\mathcal{I}_{-Z}$, which keeps the origin fixed. We now use this transformation to construct submanifold charts for the space of skew-symmetric bilinear forms on V of some fixed rank.

Definition 2.3. *The* rank *of an element $\eta \in \wedge^2 V^*$ is the rank of the linear map $\eta^\sharp : V \to V^*$. We denote the space of 2-forms on V of rank k by $(\wedge^2 V^*)_k$.*

Assume from now on that $\eta \in \wedge^2 V^*$ is of rank k. We fix a subspace $G \subset V$, which is complementary to the kernel $K = \ker(\eta^\sharp)$. Let $r : \wedge^2 V^* \to \wedge^2 K^*$ be the restriction map; we have the natural identification $\ker(r) \cong \wedge^2 G^* \oplus (G^* \otimes K^*)$. Since the restriction of η to G is non-degenerate, there is a unique element $Z \in \wedge^2 G \subset \wedge^2 V$ determined by the requirement that

$$Z^\sharp : G^* \to G, \quad \xi \mapsto \iota_\xi Z = Z(\xi, \cdot)$$

equals $-(\eta|_G^\sharp)^{-1}$.

Definition 2.4. *The Dirac exponential map* $\exp_\eta$ *of* η *(and for fixed* G*) is the mapping*

$$\exp_\eta \colon \mathcal{I}_Z \to \wedge^2 V^*, \quad \beta \mapsto \eta + F(\beta).$$

Remark 2.5. When η is non-degenerate, the construction of a nearby symplectic form out of a small 2-form β carried out in Subsection 1.3 agrees with $\exp_\eta(\beta)$, the image of β under the Dirac exponential map. This is clear from Lemma 1.16 and Equation (9), and further gives a justification for Definition 2.4.

The following theorem asserts that $\exp_\eta$ is a submanifold chart for $(\wedge^2 V^*)_k \subset \wedge^2 V^*$.

Theorem 2.6.

(i) *Let* $\beta \in \mathcal{I}_Z$*. Then* $\exp_\eta(\beta)$ *lies in* $(\wedge^2 V)_k$ *if, and only if,* β *lies in* $\ker(r) = (K^* \otimes G^*) \oplus \wedge^2 G^*$.

(ii) *Let* $\beta = (\mu, \sigma) \in \mathcal{I}_Z \cap ((K^* \otimes G^*) \oplus \wedge^2 G^*)$*. Then* $\exp_\eta(\beta)$ *is the unique skew-symmetric bilinear form on* V *with the following properties:*
- *its restriction to* G *equals* $(\eta + F(\sigma))|_{\wedge^2 G}$
- *its kernel is the graph of the map* $Z^\sharp \mu^\sharp = -(\eta|_G^\sharp)^{-1} \mu^\sharp : K \to G$.

(iii) *The Dirac exponential map* $\exp_\eta : \mathcal{I}_Z \to \wedge^2 V^*$ *restricts to a diffeomorphism*

$$\mathcal{I}_Z \cap (K^* \otimes G^*) \oplus \wedge^2 G^* \xrightarrow{\cong} \{\eta' \in (\wedge^2 V^*)_k |\ \eta' - \eta \in \mathcal{I}_{-Z}\}$$

onto an open neighborhood of η *in* $(\wedge^2 V^*)_k$.

To prove the theorem we need a technical lemma.

Lemma 2.7. *For any* $\beta \in \mathcal{I}_Z$ *we have*

$$\ker(\exp_\eta(\beta)) = \textit{image of the restriction of } \mathrm{id} + Z^\sharp \beta_G^\sharp \textit{ to } \ker(\beta_K) \subset K.$$

Here we use the notation

$$\beta = \beta_K + \beta_m + \beta_G \in \wedge^2 K^* \oplus (K^* \otimes G^*) \oplus \wedge^2 G^*.$$

Proof. For all $w \in V$ we have

$$\begin{aligned} w \in \ker(\exp_\eta(\beta)) \quad &\Longleftrightarrow \quad \eta^\sharp w = -\beta^\sharp (\mathrm{id} + Z^\sharp \beta^\sharp)^{-1} w \\ &\Longleftrightarrow \quad \eta^\sharp (\mathrm{id} + Z^\sharp \beta^\sharp) v = -\beta^\sharp v, \qquad \text{where } v := (\mathrm{id} + Z^\sharp \beta^\sharp)^{-1} w, \\ &\Longleftrightarrow \quad \eta^\sharp v = -\eta^\sharp Z^\sharp \beta^\sharp v - \beta^\sharp v. \end{aligned} \tag{10}$$

The endomorphism $\eta^\sharp Z^\sharp$ has kernel K and equals $-\mathrm{id}_G$ on G, hence with respect to the decomposition $V = K \oplus G$, the endomorphism $-\eta^\sharp Z^\sharp \beta^\sharp$ reads $\begin{pmatrix} 0 & 0 \\ (\beta_m^\sharp)|_K & \beta_G^\sharp \end{pmatrix}$, and the endomorphism on the right-hand side of Equation (10) reads $-\begin{pmatrix} \beta_K^\sharp & (\beta_m^\sharp)|_G \\ 0 & 0 \end{pmatrix}$ and, in particular, takes values in K^*. In contrast, the endomorphism $\eta^\sharp$ on the left-hand side of Equation (10) takes values in G^*. Hence both sides have to vanish, and Equation (10) is equivalent to the condition $v \in \ker(\beta_K) = \{v \in K : \iota_v \beta_K = 0\}$, where we used $\ker(\eta) = K$. $\square$

Proof of Theorem 2.6. We use the decomposition of β as in Lemma 2.7.

(i) By Lemma 2.7, $\exp_\eta(\beta)$ lies in $(\wedge^2 V)_k$ iff $\beta_K = 0$.

(ii) For any $\beta \in \mathcal{I}_Z$, Lemma 2.7 implies that the second property on the kernel of $\exp_\eta(\beta)$ is satisfied, so we only have to prove the first property. Since $Z^\sharp$ has image G and kernel K, the map $\mathrm{id} + Z^\sharp\beta^\sharp$ sends G isomorphically into itself, and its restriction to G equals $(\mathrm{id}_G + Z^\sharp\beta^\sharp_G)$. Hence

$$(F(\beta))^\sharp|_G = \beta^\sharp(\mathrm{id} + Z^\sharp\beta^\sharp)^{-1}|_G = \beta^\sharp(\mathrm{id}_G + Z^\sharp\beta^\sharp_G)^{-1}.$$

Composing with the projection $V^* = K^* \oplus G^* \to G^*$ we obtain $\beta^\sharp_G(\mathrm{id}_G + Z^\sharp\beta^\sharp_G)^{-1} \colon G \to G^*$, so $F(\beta)(Y_1, Y_2) = F(\beta_G)(Y_1, Y_2)$ for all $Y_1, Y_2 \in G$. This holds for any $\beta \in \mathcal{I}_Z$, in particular also when $\beta_K = 0$.

(iii) One checks that under the canonical isomorphism between $T_\eta((\wedge^2 V^*)_k)$ and $(K^* \otimes G^*) \oplus \wedge^2 G^*$, the differential of

$$\exp_\eta : \mathcal{I}_Z \cap ((K^* \otimes G^*) \oplus \wedge^2 G^*) \to (\wedge^2 V^*)_k$$

at η is the identity. □

Remark 2.8. We notice that the construction of $\exp_\eta$ can be readily extended to the case of vector bundles. In particular, given a pre-symplectic manifold (M, η), the choice of a complementary subbundle G to the kernel K of η yields a fibrewise map

$$\exp_\eta : (K^* \otimes G^*) \oplus (\wedge^2 G^*) \to \wedge^2 T^* M,$$

which maps the zero section to η, and an open neighborhood thereof into the space of 2-forms of rank equal to that of η. As a consequence, we can parametrize deformations of η inside $\mathsf{Pre\text{-}Sym}^k(M)$ by sections $(\mu, \sigma) \in \Gamma(K^* \otimes G^*) \oplus \Gamma(\wedge^2 G^*) \cong \Omega^2_{\mathrm{hor}}(M)$ which are sufficiently close to the zero section, and which satisfy

$$d((\exp_\eta)_*(\mu, \sigma)) = 0,$$

with d the de Rham differential. In Section 3.3 we will show that the latter integrability condition can be rephrased in terms of an $L_\infty[1]$-algebra structure on $\Omega(M)[2]$.

2.2. **Dirac-geometric interpretation of Subsection 2.1.** Using Dirac linear algebra, we explain and re-prove some results obtained in Subsection 2.1. These explanations are natural in view of the Dirac geometric approach to deformations we outlined in Subsection 1.4.

Let V be a finite-dimensional, real vector space. We denote by $\mathbb{V}$ the direct sum $V \oplus V^*$, endowed with the canonical symmetric non-degenerate pairing $\langle \cdot, \cdot \rangle$ (see Appendix C).

We fix a bivector $Z \in \wedge^2 V$. In formula (9), page 10, we defined a map $F \colon \mathcal{I}_Z \to \wedge^2 V^*$ given by $(F(\beta))^\sharp = \beta^\sharp(\mathrm{id} + Z^\sharp\beta^\sharp)^{-1}$, where $\mathcal{I}_Z \subset \wedge^2 V^*$ consists of those elements β for which $\mathrm{id} + Z^\sharp\beta^\sharp \colon V \to V$ is invertible. The following lemma provides a geometric explanation of formula (9). Recall that a linear subspace $L \subset \mathbb{V}$ is called Lagrangian if it is maximally isotropic with respect to $\langle \cdot, \cdot \rangle$.

Lemma 2.9. *Fix $Z \in \wedge^2 V$.*

(i) *Taking graphs with respect to the decompositions $\mathbb{V} = V \oplus V^*$ resp. $\mathbb{V} = V \oplus \mathrm{graph}(Z)$, yields bijections*

$$\begin{aligned} \Phi_0 : \wedge^2 V^* &\xrightarrow{\cong} \{\textit{Lagrangian subspaces of } \mathbb{V} \textit{ transverse to } V^*\} \\ \alpha &\mapsto \{(v, \iota_v \alpha) \mid v \in V\}, \\ \Phi_Z : \wedge^2 V^* &\xrightarrow{\cong} \{\textit{Lagrangian subspaces of } \mathbb{V} \textit{ transverse to } \mathrm{graph}(Z)\} \\ \beta &\mapsto \{(v + Z^\sharp(\iota_v \beta), \iota_v \beta) \mid v \in V\}. \end{aligned}$$

(ii) *Given* $\beta \in \wedge^2 V^*$, *the Lagrangian subspace* $\Phi_Z(\beta)$ *is transverse to* $V^* \subset \mathbb{V}$ *if, and only if,* $(\mathrm{id} + Z^\sharp \beta^\sharp)\colon V \to V$ *is invertible.*

(iii) *The map*

$$F := \Phi_0^{-1} \circ \Phi_Z \colon \mathcal{I}_Z \to \wedge^2 V^*$$

is well-defined and explicitly given by

$$(F(\beta))^\sharp = \beta^\sharp (\mathrm{id} + Z^\sharp \beta^\sharp)^{-1}.$$

Notice that by its very definition, the map F is characterized by the property that

$$\mathrm{graph}(F(\beta)) = \Phi_Z(\beta) \tag{11}$$

for all $\beta \in \mathcal{I}_Z$. In other words, $F(\beta)$ is obtained taking the graph of β w.r.t. the splitting $\mathbb{V} = V \oplus \mathrm{graph}(Z)$.

Proof. (i) According to Remark C.3, any Lagrangian subspace transverse to V^* is the graph of a skew-symmetric linear map $V \to V^*$, and therefore can be written as $\{(v, \iota_v \alpha) \mid v \in V\}$ for some $\alpha \in \wedge^2 V^*$. Similarly, $\mathrm{graph}(Z)$ is transverse to V and the induced isomorphism $\mathrm{graph}(Z) \cong V^*$ is just $(Z^\sharp(\xi), \xi) \mapsto \xi$. Hence any Lagrangian subspace transverse to $\mathrm{graph}(Z)$ can be written as $\{(v, 0) + (Z^\sharp(\iota_v \beta), \iota_v \beta) \mid v \in V\}$ for some $\beta \in \wedge^2 V^*$.

(ii) The expression for $\Phi_Z(\beta)$ in (i) shows that $\Phi_Z(\beta) \cap V^* = \{(0, \iota_v \beta) \mid v \in V, v + Z^\sharp(\iota_v \beta) = 0\}$. This intersection is trivial iff $\ker(\mathrm{id} + Z^\sharp \beta^\sharp) \subset \ker(\beta^\sharp)$. In turn, this condition is equivalent to $(\mathrm{id} + Z^\sharp \beta^\sharp)$ being injective, and thus invertible.

(iii) Finally, if $\mathrm{id} + Z^\sharp \beta^\sharp$ is invertible, $\Phi_Z(\beta)$ is transverse to V^* by item (ii). By item (i) the element $\Phi_0^{-1}(\Phi_Z(\beta))$ is well-defined. In concrete terms, we have to find $\alpha \in \wedge^2 V^*$ such that for all $v \in V$, there is $w \in V$ for which

$$(v + Z^\sharp \beta^\sharp(v), \beta^\sharp(v)) = (w, \alpha^\sharp(w))$$

holds. Equivalently, this means that $\alpha^\sharp(\mathrm{id} + Z^\sharp \beta^\sharp)(v) = \beta^\sharp(v)$ for all $v \in V$. This yields the claimed formula for F. □

The Dirac exponential map gives exactly the 2-form whose graph is the deformation of η we obtained in Subsection 1.4, viewing η as a Dirac structure with complement $G \oplus K^*$. Indeed, using the notation of that subsection, we have

$$\mathrm{graph}(\exp_\eta(\beta)) = \mathfrak{t}_\eta(\Phi_Z(\beta)) = \Phi_{G \oplus K^*}(\bar{\beta}),$$

where the first equality holds by Equation (11) and the second by Lemma 1.18.

Now we can give an alternative, more geometric argument for Lemma 2.7. By the proof of Lemma 1.19, the intersection of the above subbundle with TM is $(\mathfrak{t}_\eta \circ \mathfrak{t}_Z)(\mathrm{graph}(\beta) \cap (K \oplus G^*))$, which is precisely the graph of $\mathrm{id} + Z^\sharp \beta_G^\sharp$ over $\ker(\beta_K)$.

2.3. **An alternative parametrization.** Given a vector space V, the local parametrization of $(\wedge^2 V^*)_k$ – the space of skew-symmetric bilinear forms on V of rank k introduced in Subsection 2.1 – is probably not the most obvious one from a geometric point of view. Let us briefly discuss another, more obvious, one.

The underlying idea is that the map

$$\text{2-form } \eta \mapsto \text{ kernel of } \eta^\sharp$$

yields a canonical smooth map from $(\wedge^2 V^*)_k$ to the space of *codimension* k Grassmannians of V. We denote this space by $\mathrm{Gr}^k(V)$ and let

- τ be the tautological vector bundle over $\mathrm{Gr}^k(V)$, i.e. the fibre τ_W over $W \in \mathrm{Gr}^k(V)$ is $W \subset V$ and

- $\wedge^2(\underline{V}/\tau)^*$ be the vector bundle over $\mathrm{Gr}^k(V)$ whose fibre over $W \in \mathrm{Gr}^k(V)$ is given by $\wedge^2(V/W)^*$.

There is a map

$$f : (\wedge^2 V^*)_k \to \wedge^2(\underline{V}/\tau)^*, \quad \eta \mapsto (\ker(\eta^\sharp), \eta|_{V/\ker(\eta^\sharp)}).$$

By dimension reasons, f takes values in the fibre-subbundle of $\wedge^2(\underline{V}/\tau)^*$ consisting of non-degenerate 2-forms. Let us denote this space by $(\wedge^2(\underline{V}/\tau)^*)_{\mathrm{nd}}$.

Lemma 2.10. *The map $f : (\wedge^2 V^*)_k \to (\wedge^2(\underline{V}/\tau)^*)_{nd}$ is a bijection.*

Proof. Given a pair $(K, \alpha) \in (\wedge^2(\underline{V}/\tau)^*)_{\mathrm{nd}}$, simply define $\eta \in \wedge^2 V^*$ by $\eta(v_1, v_2) := \alpha(\pi(v_1), \pi(v_2))$, where $\pi : V \to V/K$ is the quotient map, and check that it has the required properties. This map is inverse to f. □

Explicit formulas for the inverse to f can be constructed as follows: Given $f(\eta) = (K, \alpha) \in (\wedge^2(\underline{V}/\tau)^*)_{\mathrm{nd}}$, fix a complementary subspace G to $K \subset V$. This choice yields an identification of a neighborhood of $K \in \mathrm{Gr}^k(V)$ with $\mathrm{Hom}(K, G)$ and lets us think of α as a non-degenerate 2-form on G. Now suppose (K', α') is a pair sufficiently close to $(K, \alpha) \in (\wedge^2(\underline{V}/\tau)^*)_{\mathrm{nd}}$. Equivalently, we are given a linear map $\phi : K \to G$ close to the zero map, and a 2-form $\sigma := \alpha' - \alpha$ on G close to 0. We define $\eta' \in \wedge^2 V^*$ by

$$\eta'(k_1 + g_1, k_2 + g_2) = (\eta + \sigma)(-\varphi(k_1) + g_1, -\varphi(k_2) + g_2).$$

It is straight-forward to verify that η' coincides with η when both arguments lie in G and that it vanishes on K', which coincides with the graph of φ. This procedure yields a smooth map,

$$\begin{aligned} \mathrm{Hom}(K, G) \times \wedge^2 G^* &\to \wedge^2 V^* \\ (\varphi, \sigma) &\mapsto (\eta + \sigma) \circ (-\varphi + \mathrm{id}_G) \otimes (-\varphi + \mathrm{id}_G), \end{aligned} \tag{12}$$

which maps $(0,0)$ to η and takes values in $(\wedge^2 V^*)_k$ in a neighborhood of $(0,0)$, where it is an inverse to f. Observe that η induces an identification $G \cong G^*, v \mapsto (\iota_v \eta)|_G$, which we can use to think of the above map as a map with domain $(K^* \otimes G^*) \oplus (\wedge^2 G^*)$, the kernel of the restriction map $r : \wedge^2 V^* \to \wedge^2 K^*$. We record this:

Proposition 2.11. *The map in* (12)*, upon applying the identification $G \cong G^*$ induced by η, is a smooth map*

$$\widetilde{\exp}_\eta : (K^* \otimes G^*) \oplus \wedge^2 G^* \to \wedge^2 V^*$$

with the property that $\widetilde{\exp}_\eta(\mu, \sigma)$ is the unique skew-symmetric bilinear form on V such that:

- *its restriction to G equals $\eta|_G + \sigma$ and*
- *its kernel equals the graph of $(\eta|_G^\sharp)^{-1}\mu^\sharp : K \to G$.*

Again, it is clear that the above construction can be applied the vector bundles, i.e. we obtain another local parametrization of a neighborhood of a given pre-symplectic structure η inside $\mathsf{Pre\text{-}Sym}^k(M)$. Observe that the formula for $\widetilde{\exp}_\eta(\mu, \sigma)$ differs from the formula for $\exp_\eta(\mu, \sigma)$ we obtained in Theorem 2.6, Section 2.1. However, $\widetilde{\exp}_\eta$ can be expressed in terms of $\exp_\eta$ and the map $F \colon \mathcal{I}_Z \to \wedge^2 V^*$. Recall that F is a diffeomorphism onto its image (see Lemma 2.9 and Theorem 2.6) and that, by definition, $\exp_\eta(\beta) = \eta + F(\beta)$.

Proposition 2.12. *Given (μ, σ) such that $(\mu, F^{-1}\sigma) \in \mathcal{I}_Z \cap ((K^* \otimes G^*) \oplus \wedge^2 G^*)$ we have*

i) *$F^{-1}\sigma \in \Gamma(\wedge^2 G^*)$,*
ii) *$\widetilde{\exp}_\eta(-\mu, \sigma) = \exp_\eta(\mu, F^{-1}\sigma) = \eta + F(\mu + F^{-1}\sigma)$.*

Proof. i) We saw that $\ker(F^{-1}\sigma)^\sharp = \ker(\sigma^\sharp) \supset K$ in Remark 2.2.
ii) By Theorem. 2.6 (2), $\exp_\eta(\mu, F^{-1}\sigma)$ is determined by the following properties:

– its restriction to G equals $(\eta + F(F^{-1}(\sigma)))|_{\wedge^2 G} = (\eta + \sigma)|_{\wedge^2 G}$
– its kernel is the graph of the map $Z^\sharp \mu^\sharp = -(\eta|_G^\sharp)^{-1}\mu^\sharp : K \to G$.

After applying Proposition 2.11, we are done. □

3. The Koszul L_∞-algebra

In this section we introduce the Koszul L_∞-algebra of a pre-symplectic manifold (M, η). This L_∞-algebra lives on $\Omega_{\mathrm{hor}}(M)$ (with a certain shift in degrees), and the zero set of its Maurer-Cartan equation parametrizes a neighborhood of η in $\mathsf{Pre\text{-}Sym}^k(M)$.

3.1. **An L_∞-algebra associated to a bivector field.** In this subsection, we introduce an L_∞-algebra, which is naturally attached to some bivector field Z on a manifold M. The statements made in this subsection will be proven in Section 3.2 below, and Dirac-geometric interpretations and alternative proofs of some statements will be given in Section 3.4.

There are two two basics operations on differential forms, which one can associate to a multivector field $Y \in \Gamma(\wedge^k TM)$: contraction $\iota_Y : \Omega^\bullet(M) \to \Omega^{\bullet - k}(M)$, and the Lie derivative $\mathcal{L}_Y : \Omega^\bullet(M) \to \Omega^{\bullet - k + 1}(M)$. The precise conventions and basic facts about these operations can be found in Appendix A. Let us just note the crucial relation

$$\mathcal{L}_Y = [\iota_Y, d] = \iota_Y \circ d - (-1)^k d \circ \iota_Y,$$

known as *Cartan's magic formula.*

Definition 3.1. *Let Z be a bivector field on M. The* Koszul bracket *associated to Z is the operation*

$$[\cdot,\cdot]_Z : \Omega^r(M) \times \Omega^s(M) \to \Omega^{r+s-1}(M)$$
$$[\alpha, \beta]_Z := (-1)^{|\alpha|+1}\Big(\mathcal{L}_Z(\alpha \wedge \beta) - \mathcal{L}_Z(\alpha) \wedge \beta - (-1)^{|\alpha|}\alpha \wedge \mathcal{L}_Z(\beta)\Big).$$

When applied to 1-forms α, β, the Koszul bracket can be written as $[\alpha, \beta]_Z = \mathcal{L}_{Z^\sharp \alpha}\beta - \mathcal{L}_{Z^\sharp \beta}\alpha - d\langle Z, \xi_1 \wedge \xi_2\rangle$ and agrees with formula (4) on page 6.

Remark 3.2. The Koszul bracket $[\cdot,\cdot]_Z$ satisfies the following identities for all homogeneous $\alpha, \beta, \gamma \in \Omega(M)$:

- Graded skew-symmetry, i.e. we have
$$[\alpha, \beta]_Z = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha]_Z.$$
- Leibniz rule for d, i.e.
$$d([\alpha, \beta]_Z) = [d\alpha, \beta]_Z + (-1)^{|\alpha|-1}[\alpha, d\beta]_Z.$$
- Derivation property with respect to the wedge product $\wedge$:
$$[\alpha, \beta \wedge \gamma]_Z = [\alpha, \beta]_Z \wedge \gamma + (-1)^{(|\alpha|-1)|\beta|}\beta \wedge [\alpha, \gamma]_Z.$$

After a shift in degree, i.e. when working on the graded vector space $\Omega(M)[1]$ defined by $(\Omega(M)[1])^r = \Omega^{r+1}(M)$, the Koszul bracket therefore satisfies most of the identities required from a differential graded Lie algebra. However, unless we assume that Z is Poisson, i.e. that it commutes with itself under the Schouten-Nijenhuis bracket, the Koszul bracket will fail to satisfy the graded version of the Jacobi identity.

The failure of $[\cdot,\cdot]_Z$ to satisfy the Jacobi-identity is mild. In fact, it can be encoded by a certain trilinear operator on differential forms. As a preparation, we introduce some notation: for a differential form $\alpha \in \Omega^r(M)$, we have

$$\alpha^\sharp : TM \to \wedge^{r-1}T^*M, \quad v \mapsto \iota_v \alpha,$$

and, following [5, §2.3], we extend this definition to a collection of forms $\alpha_1, \dots, \alpha_n$ by setting

$$\begin{aligned} \alpha_1^\sharp \wedge \cdots \wedge \alpha_n^\sharp : \wedge^n TM &\to \wedge^{|\alpha_1|+\cdots+|\alpha_n|-n} T^*M \\ v_1 \wedge \cdots \wedge v_n &\mapsto \sum_{\sigma \in S_n} (-1)^{|\sigma|} \alpha_1^\sharp(v_{\sigma(1)}) \wedge \cdots \wedge \alpha_n^\sharp(v_{\sigma(n)}). \end{aligned}$$

Definition 3.3. *We define the trinary bracket* $[\cdot,\cdot,\cdot]_Z : \Omega^r(M) \times \Omega^s(M) \times \Omega^k(M) \to \Omega^{r+s+k-3}(M)$ *associated to the bivector field* Z *to be*

$$[\alpha, \beta, \gamma]_Z := (\alpha^\sharp \wedge \beta^\sharp \wedge \gamma^\sharp)(\frac{1}{2}[Z, Z]),$$

Remark 3.4. The trinary bracket $[\cdot,\cdot,\cdot]_Z$ is a derivation in each argument, in the sense that

$$[\alpha, \beta, (\gamma \wedge \tilde{\gamma})]_Z = [\alpha, \beta, \gamma]_Z \wedge \tilde{\gamma} + (-1)^{(|\alpha|+|\beta|-1)|\gamma|} \gamma \wedge [\alpha, \beta, \tilde{\gamma}]_Z.$$

The precise compatibility between d, $[\cdot,\cdot]_Z$ and $[\cdot,\cdot,\cdot]_Z$ is the following, and will be proven in Section 3.2 below:

Proposition 3.5. *Let* Z *be a bivector field on* M*. The multilinear maps* $\lambda_1, \lambda_2, \lambda_3$ *on the graded vector space* $\Omega(M)[2]$ *given by*

1. λ_1 *is the de Rham differential* d*, acting on* $\Omega(M)[2]$*,*
2. $\lambda_2(\alpha[2] \odot \beta[2]) = -\Big(\mathcal{L}_Z(\alpha \wedge \beta) - \mathcal{L}_Z(\alpha) \wedge \beta - (-1)^{|\alpha|} \alpha \wedge \mathcal{L}_Z(\beta)\Big)[2] = (-1)^{|\alpha|}([\alpha, \beta]_Z)[2]$,
3. *and*

$$\lambda_3(\alpha[2] \odot \beta[2] \odot \gamma[2]) = (-1)^{|\beta|+1} \Big(\alpha^\sharp \wedge \beta^\sharp \wedge \gamma^\sharp(\frac{1}{2}[Z, Z])\Big)[2]$$

define the structure of an $L_\infty[1]$*-algebra on* $\Omega(M)[2]$*, or – equivalently – the structure of an* L_∞*-algebra on* $\Omega(M)[1]$*, see Appendix B.*

In a nutshell, Proposition 3.5 asserts that λ_1, λ_2 and λ_3 obey a family of quadratic relations. Besides the relations which involve only the de Rham differential d and the Koszul bracket $[\cdot,\cdot]_Z$, there is a relation which asserts that the Jacobiator of $[\cdot,\cdot]_Z$, seen as a chain map from $\Omega(M)[2] \odot \Omega(M)[2] \odot \Omega(M)[2]$ to $\Omega(M)[2]$, is zero-homotopic, with homotopy provided by λ_3. A concise summary on $L_\infty[1]$-algebras can be found in Appendix B.

Proposition 3.6. *Let* Z *be a bivector field on* M*. There is an* $L_\infty[1]$-isomorphism ψ *from the* $L_\infty[1]$*-algebra* $(\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3)$ *of Proposition 3.5 to the* $L_\infty[1]$*-algebra* $(\Omega(M)[2], \lambda_1)$*, i.e. the (twice suspended) de Rham complex of* M*.*

The interested reader can find the proof in Section 3.2 below. Let us just point out that we have a rather explicit knowledge of the $L_\infty[1]$-isomorphism ψ between $(\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3)$ and $(\Omega(M)[2], \lambda_1)$. We will make use of this knowledge below.

Remark 3.7. In case we can find an involutive complement G to $K \subset TM$, the construction from Proposition 3.5 yields the Koszul dg Lie algebra associated to the regular Poisson structure Z on M given by the negative inverse of $\eta|_{\wedge^2 G}$. For Z a Poisson bivector, Proposition 3.6 was established by Fiorenza and Manetti in [4].

We now turn to the geometry encoded by the $L_\infty[1]$-algebra $(\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3)$. To this end, recall that we can naturally associate the following equation to such a structure:

Definition 3.8. *An element* $\beta \in \Omega^2(M)$ *is a* Maurer-Cartan element *of* $(\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3)$ *if it satisfies the* Maurer-Cartan equation

$$d(\beta[2]) + \frac{1}{2}\lambda_2(\beta[2] \odot \beta[2]) + \frac{1}{6}\lambda_3(\beta[2] \odot \beta[2] \odot \beta[2]) = 0.$$

We denote the set of Maurer-Cartan elements by $\mathsf{MC}(Z)$*.*

Recall that in Equation (9), Subsection 2.1 (see also Lemma 2.9), we introduced a map $F\colon \mathcal{I}_Z \to \wedge^2 T^*M$, where $\mathcal{I}_Z \subset \Omega^2(M)$ consists of those 2-forms β for which $\mathrm{id} + Z^\sharp\beta^\sharp$ is invertible.

Corollary 3.9. *There is an open subset $\mathcal{U} \subset \mathcal{I}_Z$, which contains the zero section of $\wedge^2T^*M$, such that a 2-form $\beta \in \Gamma(\mathcal{U})$ is a Maurer-Cartan element of $(\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3)$ if, and only if, the 2-form $F(\beta)$ is closed.*

A proof of Corollary 3.9 is given after Proposition 3.10 below. We will provide a more conceptual proof in Subsection 3.4, which shows that one can in fact take $\mathcal{U} = \mathcal{I}_Z$.

Summing up our findings, we have a smooth, fibrewise mapping

$$F : \mathcal{U} \subset \mathcal{I}_Z \to \wedge^2 T^*M,$$

which has the intriguing property of mapping 2-forms, which are Maurer-Cartan elements of $(\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3)$, to *closed* 2-forms.

We observe that the $L_\infty[1]$-isomorphism from Proposition 3.6

$$\psi : (\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3) \to (\Omega(M)[2], \lambda_1)$$

also induces – modulo convergence issues – a map

$$\psi_* : \Omega^2(M) \to \Omega^2(M), \quad \beta \mapsto \psi_*(\beta) := \sum_{k\geq 1} \frac{1}{k!}\psi_k(\beta^{\odot k}),$$

which has the property that it sends Maurer-Cartan elements of $(\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3)$ to *closed* 2-forms. The following result asserts that for β sufficiently small, the convergence of $\psi_*(\beta)$ is guaranteed, in which case we recover $F(\beta) = \Phi_0^{-1}\Phi_Z(\beta)$ in the limit.

Proposition 3.10. *For β a 2-form which is sufficiently $\mathcal{C}^0$-small, the power series $\psi_*(\beta)$ converges in the $\mathcal{C}^\infty$-topology to $F(\beta)$.*

As before, we postpone the proof of this result to Subsection 3.2. Let us demonstrate that Proposition 3.10 yields a proof of Corollary 3.9: Suppose β is a 2-form which is sufficiently $\mathcal{C}^0$-small for Proposition 3.10 to apply. We compute

$$dF(\beta) = d\psi_*(\beta) = \sum_{k\geq 1}\frac{1}{k!}d\psi_k(\beta^{\odot k}) = \sum_{k\geq 1}\frac{1}{k!}\psi_{k+1}(\beta^{\odot k}\odot(\lambda_1(\beta) + \frac{1}{2}\lambda_2(\beta\odot\beta) + \frac{1}{6}\lambda_3(\beta\odot\beta\odot\beta)).$$

This shows that if β satisfies the Maurer-Cartan equation, $F(\beta)$ is closed. To obtain the reverse implication, one repeats the same line of arguments to the inverse of ψ.

3.2. **Proofs for Section 3.1.**

In this section we provide proofs of the statements from Section 3.1 which involve L_∞-algebras. We refer to Appendix B for background material.

3.2.1. *Proofs of Proposition 3.5 and Proposition 3.6.* The proofs in this section rely on the calculus of differential operators on graded commutative algebras, and their associated Koszul brackets, see Appendix B. For us, the following example of differential operators is of central importance:

Example 3.11. Let Y be a k-multivector field on M, i.e. a section of $\Gamma(\wedge^k TM)$. The insertion operator

$$\iota_Y : \Omega^\bullet(M) \to \Omega^{\bullet-k}(M)$$

is a differential operator of order $\leq k$ on $\Omega(M)$. Since graded derivations are differential operators of order ≤ 1, and since $[\mathsf{DO}_k(A), \mathsf{DO}_l(A)] \subset \mathsf{DO}_{k+l-1}(A)$, we find that the Lie derivative

$$\mathcal{L}_Y = [\iota_Y, d]$$

is also a differential operator of order $\leq k$ on $\Omega(M)$.

Let A be a graded commutative, unital dg algebra. Given an endomorphism f of A, the *Koszul brackets* of f are a sequence of multilinear operators $\mathcal{K}(f)_n : \odot^n A \to A$ defined iteratively by $\mathcal{K}(f)_1 = f$ and

$$\begin{aligned} \mathcal{K}(f)_n(a_1 \odot \cdots \odot a_n) &= +\mathcal{K}(f)_{n-1}(a_1 \odot \cdots \odot a_{n-2} \odot a_{n-1}a_n) \\ &\quad -\mathcal{K}(f)_{n-1}(a_1 \odot \cdots \odot a_{n-1})a_n \\ &\quad -(-1)^{|a_{n-1}||a_n|}\mathcal{K}(f)_{n-1}(a_1 \odot \cdots \odot a_{n-1} \odot a_n)a_{n-1}, \end{aligned}$$

Using the natural identification $\mathrm{Hom}(\odot A, A) \cong \mathrm{Coder}(\odot A)$, this construction yields a morphism of dg Lie algebras

$$\mathcal{K} : \mathrm{End}_*(A) \to \mathrm{Coder}(\odot A) \tag{13}$$

from the dg Lie algebra of endomorphisms of A which annihilate 1_A, to the dg Lie algebra of coderivations of the (reduced) symmetric coalgebra on A (with the commutator bracket), c.f. [20].

Lemma 3.12. *Let V be a finite-dimensional vector space. Given $Y_1, \dots, Y_n \in V$, we consider $Y := Y_1 \wedge \cdots \wedge Y_n \in \wedge^n V$, and the corresponding differential operator*

$$\iota_Y : \wedge^\bullet V^* \mapsto \wedge^{\bullet - n} V^*$$

on the commutative graded algebra $\wedge V^*$. For $r \leq n$, the Koszul brackets of ι_Y are given by

$$\mathcal{K}(\iota_Y)_r(a_1 \odot \cdots \odot a_r) = \sum_{i_1 + \cdots + i_r = n} \sum_{\sigma \in S(i_1, \dots, i_r)} (-1)^\sharp (\iota_{Y_{\sigma(1)}} \cdots \iota_{Y_{\sigma(i_1)}} a_1) \cdots (\cdots \iota_{Y_{\sigma(n)}} a_r),$$

where we sum over all tuples $(i_1, \dots, i_r) \in \mathbb{Z}^r$ with $i_1 + \cdots + i_r = n$ such that all $i_r \geq 1$, $S(i_1, \dots, i_r)$ is the set of all $(i_1, \dots, i_r)$-unshuffles, and the sign is

$$\sharp = |\sigma| + \sum_{p=2}^{r} i_p(|a_1| + \cdots + |a_{p-1}|).$$

Proof. The proof proceeds by induction on r. For $r = 1$, the claimed formula reads

$$\mathcal{K}(\iota_Y)_1(a) = \iota_{Y_1} \cdots \iota_{Y_n} a,$$

which – by our conventions, see Appendix B – indeed equals $\iota_Y a$.

We now assume that we verified the formula for all $r \geq 1$. Our task is to show that for $r + 1$, the right-hand side of the claimed equality satisfies the same recursion as $\mathcal{K}(\iota_Y)_{r+1}$. This is a straightforward computation. □

Remark 3.13. We will be particularly interested in the special case $n = r$, where the formula specializes to

$$\mathcal{K}(\iota_Y)_n(a_1 \odot \cdots \odot a_n) := (-1)^{(n-1)|a_1| + (n-2)|a_2| + \cdots + |a_{n-1}|}(a_1^\sharp \wedge \cdots \wedge a_n^\sharp)(Y),$$

in the notation of Subsection 3.1.

We now provide a proof of Proposition 3.5, relying on a general construction from [10, 20]. Let Z be an arbitrary bivector field on M. Cartan calculus – which we recap in Appendix A – implies that the operator

$$\Delta_Z := d - t\mathcal{L}_Z - t^2\frac{1}{2}\iota_{[Z,Z]}$$

on $\Omega(M)[[t]]$, t a formal variable of degree 2, squares to zero. In fact, Δ_Z equips $\Omega(M)$ with the structure of a commutative BV_∞-algebra of degree 1 in the sense of [10], which amounts to the fact that d is a derivation, $\mathcal{L}_Z$ is a differential operator of order ≤ 2, and $\iota_{[Z,Z]}$ is a differential operator of order ≤ 3. This BV_∞-algebra structure on $\Omega(M)$ was considered by Fiorenza-Manetti in the case of Z being a Poisson bivector field, cf. [4], and by Dotsenko-Shadrin-Valette in the case of Z being a Jacobi bivector field, cf. [3].

Proof of Proposition 3.5. As noted above, the operator

$$\Delta_Z = d - t\mathcal{L}_Z - t^2\frac{1}{2}\iota_{[Z,Z]}$$

equips $\Omega(M)$ with the structure of a commutative BV_∞-algebra. Proposition B.3 then asserts that the sequence of operations

$$(d, \mathcal{K}(-\mathcal{L}_Z)_2, \mathcal{K}(-\frac{1}{2}\iota_{[Z,Z]})_3)$$

equips $\Omega(M)[2]$ with the structure of an $L_\infty[1]$-algebra. The structure maps λ_1 and λ_2 are easy to match with d and $\mathcal{K}(\mathcal{L}_Z)_2$, respectively. Concerning λ_3, we know by Remark 3.13 that

$$\mathcal{K}(-\frac{1}{2}\iota_{[Z,Z]})_3(\alpha \odot \beta \odot \gamma) = (-1)^{|\beta|+1}(\alpha^\sharp \wedge \beta^\sharp \wedge \gamma^\sharp)(\frac{1}{2}[Z,Z]),$$

which concludes the proof. □

We next turn to Proposition 3.6. We consider now the second Koszul bracket associated to the contraction by the bivector field Z,

$$\mathcal{K}(\iota_Z)_2 : \Omega(M)^{\odot 2} \to \Omega(M)$$

One can extend $\mathcal{K}(\iota_Z)_2$ in a unique way to a coderivation R_Z of $\bigodot(\Omega(M)[2])$, and as such it has degree zero. Since R_Z acts in a pro-nilpotent manner on $\bigodot(\Omega(M)[2])$, it integrates to an automorphism e^{R_Z} of the graded coaugmented coalgebra $\bigodot(\Omega(M)[2])$.

We thank Ruggero Bandiera for helpful conversations which led to the following proof.

Proof of Proposition 3.6. We claim that $\psi := e^{R_Z}$ defines an $L_\infty[1]$-isomorphism

$$\psi : (\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3) \to (\Omega(M)[2], \lambda_1),$$

with inverse given by e^{-R_Z}. Equivalently, we can verify that the coderivation

$$e^{-R_Z} \circ \widehat{\lambda}_1 \circ e^{R_Z}$$

corresponds to $(\lambda_1, \lambda_2, \lambda_3)$, where $\widehat{\lambda}_1$ is the coderivation of $\bigodot(\Omega(M)[2])$ extending λ_1. Notice that

$$e^{-R_Z} \circ \widehat{\lambda}_1 \circ e^{R_Z} = \widehat{\lambda}_1 - [R_Z, \widehat{\lambda}_1] + \frac{1}{2}[R_Z, [R_Z, \widehat{\lambda}_1]] - \frac{1}{3!}[R_Z, [R_Z, [R_Z, \widehat{\lambda}_1]]]] + \cdots,$$

where $[\cdot,\cdot]$ denotes the commutator bracket, i.e. the $(k+1)$-th structure map of $e^{-R_Z} \circ \widehat{\lambda}_1 \circ e^{R_Z}$ can be read off from

$$\frac{1}{k!}(-1)^k \mathrm{ad}(R_Z)^k \widehat{\lambda}_1.$$

We compute the element corresponding to $-[R_Z, \widehat{\lambda}_1]$ under the natural identification $\mathrm{Hom}(\odot A, A) \cong \mathrm{Coder}(\odot A)$. Using the fact that the map (13) respects commutator brackets, and that d is a derivation, while ι_Z is differential operators of order ≤ 2, we obtain

$$-[\mathcal{K}(\iota_Z)_2, \mathcal{K}(d)_1] = -\mathcal{K}([\iota_Z, d])_2 = -\mathcal{K}(\mathcal{L}_Z)_2 = \lambda_2.$$

In the same manner – this time also using that $\mathcal{L}_Z$ is a differential operator of order ≤ 2 – we find

$$\frac{1}{2}[R_Z, [R_Z, \widehat{\lambda}_1]] = \frac{1}{2}[\mathcal{K}(\iota_Z)_2, \mathcal{K}(\mathcal{L}_Z)_2] = \frac{1}{2}\mathcal{K}([\iota_Z, \mathcal{L}_Z])_3 = -\frac{1}{2}\mathcal{K}(\iota_{[Z,Z]})_3 = \lambda_3.$$

Finally, we find

$$\frac{1}{6}[\mathcal{K}(\iota_Z)_2, -\frac{1}{2}\mathcal{K}(\iota_{[Z,Z]})_3] = \frac{1}{12}\mathcal{K}([\iota_Z, -\iota_{[Z,Z]}])_4 = 0,$$

where we made use of the fact that contraction operators commute in the graded sense. □

Remark 3.14. An alternative proof, pointed out to us by V. Dotsenko, proceeds by noticing that the operator

$$\Delta_Z = d - t\mathcal{L}_Z - t^2\frac{1}{2}\iota_{[Z,Z]}$$

is conjugate to d via the automorphism of $\Omega(M)[[t]]$ generated by $t\iota_Z$. In the case of Z being Poisson, this was previously observed by Fiorenza-Manetti in [4], and, in the case of Z being Jacobi, by Dotsenko-Shadrin-Valette in [3].

Combining this observation with the fact that the higher Koszul brackets are compatible with the commutator brackets, see (13), yields Proposition 3.6.

3.2.2. *Proof of Proposition 3.10*. As we just saw, the $L_\infty[1]$-isomorphism $\psi : (\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3) \to (\Omega(M)[2], \lambda_1)$ is given by e^{R_Z}, with the coderivation R_Z determined by

$$R_Z(\alpha[2] \odot \beta[2]) = \Big(\iota_Z(\alpha \wedge \beta) - \iota_Z(\alpha) \wedge \beta - \alpha \wedge \iota_Z(\beta)\Big)[2].$$

Notice that this equals the second Koszul bracket $\mathcal{K}_2(\iota_Z)$ of ι_Z, see Section 3.2.1.

We are interested in $e^{R_Z}(e^{\beta[2]})$ for $\beta \in \Omega^2(M)$. Since e^{R_Z} is an automorphism of the coalgebra $\bigodot(\Omega(M)[2])$ it maps $e^{\beta[2]}$ to an element of the form $e^{\alpha[2]}$. Our aim is to derive a formula for α.

As a preparation, we prove the following

Lemma 3.15. *For β, $\tilde{\beta} \in \Omega^2(M)$, the 2-form $\gamma[2] := R_Z(\beta[2] \odot \tilde{\beta}[2])$ is determined by*

$$\gamma^\sharp = -\left(\beta^\sharp Z^\sharp \tilde{\beta}^\sharp + \tilde{\beta}^\sharp Z^\sharp \beta^\sharp\right).$$

Proof. Let us fix two 2-forms β and $\tilde{\beta}$. We assume without loss of generality that $Z = Z_1 \wedge Z_2$ and we know that $R_Z(\beta[2] \odot \tilde{\beta}[2]) = \mathcal{K}_2(\iota_Z)(\beta \odot \tilde{\beta})[2]$. By Lemma 3.12, we have

$$\gamma = \mathcal{K}(\iota_Z)_2(\beta \odot \tilde{\beta}) = \iota_{Z_1}\beta \wedge \iota_{Z_2}\tilde{\beta} - \iota_{Z_2}\beta \wedge \iota_{Z_1}\tilde{\beta}$$

and therefore

$$\gamma(v_1, v_2) = \beta(Z_1, v_1)\tilde{\beta}(Z_2, v_2) - \beta(Z_1, v_2)\tilde{\beta}(Z_2, v_1) - \beta(Z_2, v_1)\tilde{\beta}(Z_1, v_2) + \beta(Z_2, v_2)\tilde{\beta}(Z_1, v_1).$$

On the other hand, if we evaluate the bilinear skew-symmetric form corresponding to the map $-\beta^\sharp Z^\sharp \tilde{\beta}^\sharp$ on two vectors v_1, $v_2 \in V$, we find

$$-\langle \beta^\sharp Z^\sharp \tilde{\beta}^\sharp(v_1), v_2\rangle = \beta(Z_2, v_2)\tilde{\beta}(Z_1, v_1) - \beta(Z_1, v_2)\tilde{\beta}(Z_2, v_1),$$

and similarly when one switches β and $\tilde{\beta}$. This finally yields

$$\gamma(v_1, v_2) = \langle -\left(\beta^\sharp Z^\sharp \tilde{\beta}^\sharp + \tilde{\beta}^\sharp Z^\sharp \beta^\sharp\right)(v_1), v_2\rangle,$$

which was our claim. □

We introduce a formal parameter t and define $\alpha(t) \in \Omega^2(M)[[t]]$ by

$$e^{tR_Z} e^{\beta} =: e^{\alpha(t)}. \tag{14}$$

We write $\alpha(t) = \sum_{j=0}^{\infty} \alpha_j t^j$.

Lemma 3.16. *The coefficients $\alpha_j \in \Omega^2(M)$ of $\alpha(t)$ are determined by $\alpha_j^\sharp = (-1)^j \beta^\sharp (Z^\sharp \beta^\sharp)^j$.*

Proof. The statement is obviously true for $j = 0$. Now suppose we proved the statement for all $i < j$ already. Differentiating both sides of Equation (14) with respect to t leads to

$$R_Z(e^{tR_Z} e^{\beta}) = R_Z(e^{\alpha(t)}) = \dot{\alpha}(t) \odot e^{\alpha(t)}.$$

After applying the projection $\bigodot \Omega(M)[2] \to \Omega(M)[2]$, one obtains $R_Z(\frac{1}{2}\alpha(t) \odot \alpha(t)) = \dot{\alpha}(t)$. Comparing the coefficients of t^{j-1}, we find the equation

$$\alpha_j = \frac{1}{j} R_Z \Big(\frac{1}{2} \sum_{r+s=j-1} \alpha_r \odot \alpha_s \Big).$$

Applying induction and Lemma 3.15, this yields

$$\begin{aligned} \alpha_j^\sharp &= \frac{1}{j} (-1)^j \sum_{0 \leq i \leq j-1} \beta^\sharp (Z^\sharp \beta^\sharp)^i Z^\sharp \beta^\sharp (Z^\sharp \beta^\sharp)^{j-i-1} \\ &= (-1)^j \beta^\sharp (Z^\sharp \beta^\sharp)^j, \end{aligned}$$

which finishes the proof. □

Proof of Proposition 3.10. Setting $t = 1$ in Equation (14) and applying the projection $\bigodot \Omega(M)[2] \to \Omega(M)[2]$ we find $\alpha = \alpha(1) = \sum_{j \geq 0} \alpha_j$. By Lemma 3.16, we know that

$$(\psi_*(\beta))^\sharp = \alpha^\sharp = \sum_{j \geq 0} \alpha_j = \sum_{j \geq 0} (-1)^j \beta^\sharp (Z^\sharp \beta^\sharp)^j.$$

For β sufficiently small with respect to the $\mathcal{C}^0$-topology, this formal series converges uniformly with respect to the $\mathcal{C}^\infty$-topology to $\beta^\sharp(\mathrm{id} + Z^\sharp \beta^\sharp)^{-1}$. The latter expression coincides with $\Phi_0^{-1} \Phi_Z(\beta)$, compare Lemma 2.9, Subsection 2.1. □

3.3. The Koszul L_∞-algebra of a pre-symplectic manifold.

We return to the pre-symplectic setting, i.e. suppose η is a pre-symplectic structure on M. Let us fix a complementary subbundle G to the kernel $K \subset TM$ of η and let Z be the bivector field on M determined by $Z^\sharp = -(\eta|_G^\sharp)^{-1}$.

Theorem 3.17. *The $L_\infty[1]$-algebra structure on $\Omega(M)[2]$ associated to the bivector field Z, see Proposition 3.5, maps $\Omega_{\mathrm{hor}}(M)[2]$ to itself. The subcomplex $\Omega_{\mathrm{hor}}(M)[2] \subset \Omega(M)[2]$ therefore inherits the structure of an $L_\infty[1]$-algebra, which we call the* Koszul $L_\infty[1]$-algebra *of (M, η). Moreover, we call the corresponding L_∞-algebra structure on $\Omega_{\mathrm{hor}}(M)[1]$, see Appendix B, the* Koszul L_∞-algebra *of (M, η).*

Proof. We have to show that $\Omega_{\mathrm{hor}}(M)[2]$ is preserved by the operations λ_1, λ_2 and λ_3 as defined in Proposition 3.5. Notice that these structure maps operate in a local manner on $\Omega(M)$.

We already know that $\Omega_{\mathrm{hor}}(M)$ is a subcomplex of $\Omega(M)$, so the claim is clear for λ_1.

For the binary map λ_2, we notice that it suffices to show that $\Gamma(K^\circ)$ is closed under the Koszul bracket; the reasons being that $\Omega_{\mathrm{hor}}(M)$ is the ideal generated by $\Gamma(K^\circ)$ and that the Koszul bracket is a derivation in each argument. We already showed in Lemma 1.14, Section 1.2, that $\Gamma(K^\circ)$ is closed under the Koszul bracket.

To see that λ_3 maps $\Omega_{\mathrm{hor}}(M)[2]^{\odot 3}$ to $\Omega_{\mathrm{hor}}(M)[2]$, recall that $\Omega_{\mathrm{hor}}(M) = \Gamma(\wedge^{\geq 1} G^* \otimes \wedge K^*)$ and that the component of $[Z, Z]$ in $\Gamma(\wedge^3 G)$ vanishes by Lemma 1.12 in Section 1.2. □

Definition 3.18. *We denote by* $\mathsf{MC}(\eta)$ *the set of Maurer-Cartan elements of the Koszul* $L_\infty[1]$*-algebra of* (M,η)*.*

Recall from Section 2.1 that $\mathcal{I}_Z$ denotes the subset of those elements β of $\wedge^2 T^*M$, for which $\mathrm{id} + Z^\sharp\beta^\sharp$ is invertible. Recall further that we constructed a map

$$\exp_\eta : \mathcal{I}_Z \to \wedge^2 T^*M, \quad \beta \mapsto \eta + F(\beta), \tag{15}$$

which is fibre-preserving, and maps $\mathcal{I}_Z$ diffeomorphically onto $\{\eta' \in \Omega^2(M)|\ \eta' - \eta \in \mathcal{I}_{-Z}\}$, sending the zero section to η. Finally, we saw that this map restricts to a diffeomorphism from $\mathcal{I}_Z \cap ((K^* \otimes G^*) \oplus \wedge^2 G^*)$ to the 2-forms η' of rank k such that $\eta' - \eta \in \mathcal{I}_{-Z}$.

Drawing from the results established up to this point – namely Theorem 2.6 and Corollary 3.9 – we are now ready to prove our main result:

Theorem 3.19. *Let* (M,η) *be a pre-symplectic manifold. The choice of a complement* G *to the kernel of* η *determines a bivector field* Z *by requiring* $Z^\sharp = -(\eta^\sharp|_G)^{-1}$*. Suppose* β *is a 2-form on* M*, which lies in* $\mathcal{I}_Z$*. The following statements are equivalent:*

1. *β is a Maurer-Cartan element of the Koszul $L_\infty[1]$-algebra $\Omega_{\mathrm{hor}}(M)[2]$ of (M,η), which was introduced in Theorem 3.17.*
2. *The image of β under the map $\exp_\eta$, which is recalled in (15), is a pre-symplectic structure of the same rank as η.*

Proof. We already know that β being horizontal is equivalent to $\exp_\eta(\beta)$ being of the same rank as η, see Theorem 2.6 (iii) in Section 2.1. Clearly β is Maurer-Cartan in $\Omega_{\mathrm{hor}}(M)[2]$ if, and only if, it is Maurer-Cartan in $\Omega(M)[2]$. By Corollary 3.9, the latter is equivalent to $F(\beta)$ being closed. In turn this is equivalent to $\exp_\eta(\beta)$ being closed, since these two forms differ by the closed 2-form η. □

Rephrasing the above result, the fibrewise map

$$\exp_\eta : \mathcal{I}_Z \cap ((K^* \otimes G^*) \oplus \wedge^2 G^*) \to (\wedge^2 T^*M)_k$$

restricts, on the level of sections, to an injective map

$$\boxed{\exp_\eta : \Gamma(\mathcal{I}_Z) \cap \mathsf{MC}(\eta) \to \mathsf{Pre\text{-}Sym}^k(M)}$$

with image an open neighborhood of η in $\mathsf{Pre\text{-}Sym}^k(M)$ (equipped with the $\mathcal{C}^0$-topology). That is, for $\beta \in \Omega^2_{\mathrm{hor}}(M)$ sufficiently $\mathcal{C}^0$-small, being Maurer-Cartan is equivalent to $\exp_\eta(\beta)$ being a pre-symplectic structure of rank equal to the rank of η.

3.3.1. *Quotient $L_\infty[1]$-algebras.* Theorem 3.17 asserts that the multiplicative ideal of horizontal forms $\Omega_{\mathrm{hor}}(M)$ is closed with respect to the $L_\infty[1]$-algebra structure maps $\lambda_1, \lambda_2, \lambda_3$ from Proposition 3.5. We now refine this result. For $k \geq 0$, we denote by $F^k\Omega(M)$ the k'th power of $\Omega_{\mathrm{hor}}(M) \subset \Omega(M)$. Given the choice of a subbundle $G \subset TM$ which is a complement to the kernel K of the pre-symplectic form η, we have the identification

$$F^k\Omega(M) = \Gamma(\wedge^\bullet K^* \otimes \wedge^{\geq k} G^*).$$

This gives us a filtration

$$F^0\Omega(M) = \Omega(M) \supset F^1\Omega(M) = \Omega_{\mathrm{hor}}(M) \supset F^2\Omega(M) \supset \ldots \supset \{0\}.$$

Lemma 3.20. *The* $L_\infty[1]$*-algebra structure maps* $\lambda_1, \lambda_2, \lambda_3$ *on* $\Omega(M)[2]$ *associated to the bivector field* Z*, see Proposition 3.5, satisfy for all* $k \geq 0$*:*

i) $\lambda_1(F^k\Omega(M)[2]) \subset F^k\Omega(M)[2]$,
ii) $\lambda_2(F^k\Omega(M)[2], F^l\Omega(M)[2]) \subset F^{k+l-1}\Omega(M)[2]$,
iii) $\lambda_3(F^k\Omega(M)[2], F^l\Omega(M)[2], F^m\Omega(M)[2]) \subset F^{k+l+m-2}\Omega(M)[2]$.

Proof. We use the notation $\Omega^{j,k} := \Gamma(\wedge^j K^* \otimes \wedge^k G^*)$ for brevity and also suppress the shift in degrees.

i) For $\alpha \in \Gamma(G^*) = \Gamma(K^\circ)$, the involutivity of K and the usual formula for the de Rham differential imply that $d\alpha|_{\wedge^2 K} = 0$, i.e. $d\alpha \in \Omega^{1,1} + \Omega^{0,2}$. Since d is a (degree one) derivation of the wedge product, we obtain $d(\Omega^{0,k}) \subset \Omega^{1,k} + \Omega^{0,k+1}$. For $\alpha \in \Gamma(K^*)$ in general we can only state that $d\alpha \in \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}$, so that $d(\Omega^{j,0}) \subset \Omega^{j+1,0} + \Omega^{j,1} + \Omega^{j-1,2}$. The statement follows from this.

ii) We have
$$\lambda_2(\Gamma(G^*), \Gamma(G^*)) \subset \Gamma(G^*)$$
$$\lambda_2(\Gamma(K^*), \Gamma(K^*)) \equiv 0$$
$$\lambda_2(\Gamma(K^*), \Gamma(G^*)) \subset \Gamma(K^*) + \Gamma(G^*)$$
The first statement is established in Lemma 1.14, Section 1.2. The second is a consequence of the formula (4), page 6. The third statement is clear. Since λ_2 is a graded bi-derivation with respect to the wedge product, item ii) follows.

iii) By Lemma 1.12 we know that $[Z, Z]$ has no component in $\wedge^3 G$. The formula for λ_3 in Proposition 3.5 implies the statement.

□

Remark 3.21. When G is involutive, Lemma 3.20 can be improved to the following statement: $\lambda_1(\Omega^{j,k}) \subset \Omega^{j+1,k} + \Omega^{j,k+1}$ and $\lambda_2(\Omega^{\bullet,k}, \Omega^{\bullet,l}) \subset \Omega^{\bullet,k+l-1}$ (recall that λ_3 vanishes). This follows from the proof of Lemma 3.20, together with the following observations which hold when G is involutive. First: $\alpha \in \Gamma(K^*)$ satisfies $d\alpha \in \Omega^{2,0} + \Omega^{1,1}$. Second: using this and formula (4), page 6, one has $\lambda_2(\Gamma(K^*), \Gamma(G^*)) \subset \Gamma(K^*)$.

Returning to the general case, Lemma 3.20 allows us to refine Theorem 3.17 as follows:

Corollary 3.22.

1) $(F^1\Omega(M))[2] = \Omega_{\mathrm{hor}}(M)[2]$ *is an* $L_\infty[1]$*-subalgebra of* $\Omega(M)[2]$.
2) $(F^k\Omega(M))[2]$ *is an* $L_\infty[1]$*-ideal of* $\Omega_{\mathrm{hor}}(M)[2]$ *for all* $k \geq 2$. *Hence we get a sequence of* $L_\infty[1]$*-algebras and strict morphisms between them*
$$\left(\Omega_{\mathrm{hor}}(M)\big/_{F^2\Omega(M)}\right)[2] \longleftarrow \left(\Omega_{\mathrm{hor}}(M)\big/_{F^3\Omega(M)}\right)[2] \longleftarrow \cdots \longleftarrow \Omega_{\mathrm{hor}}(M)[2].$$

Proof. Both statements are immediate consequences of Lemma 3.20. □

In Section 4, we identify the $L_\infty[1]$-algebra $(\Omega_{\mathrm{hor}}(M)/F^2\Omega(M))[2]$ with the $L_\infty[1]$-algebra that controls the deformations of the characteristic foliation $K = \ker(\eta)$ of the pre-symplectic manifold (M, η).

3.3.2. Relation to the alternative parametrization. The construction of the Koszul $L_\infty[1]$-algebra associated to a pre-symplectic form in Subsection 3.3 relies on the linear algebra that we considered in Section 2.1. We briefly discuss how the alternative parametrization from Section 2.3 fits into our treatment.

To this end, we first introduce a fibrewise-linear endomorphism ξ_G of bundles of graded algebras
$$\xi_G : \wedge T^*M \cong \wedge K^* \otimes \wedge G^* \xrightarrow{\mathrm{pr}} \wedge G^* \hookrightarrow \wedge T^*M.$$
Observe that it commutes with the endomorphism ι_Z, since the latter is linear over $\wedge K^*$. From this it follows that
$$\mathcal{K}(\iota_Z)_2 \circ \xi_G \otimes \xi_G = \xi_G \circ \mathcal{K}(\iota_Z)_2 \;:\; \Omega(M) \odot \Omega(M) \to \Omega(M).$$

We denote by $\widetilde{R}_Z$ the coderivation of $\bigodot\Omega(M)[2]$ which extends $\xi_G\circ\mathcal{K}(\iota_Z)_2$ and by $e^{\widetilde{R}_Z}$ the corresponding automorphism of $\bigodot\Omega(M)[2]$. Observe that since the coderivation $\widetilde{R}_Z$ maps $\bigodot\Omega_{\mathrm{hor}}(M)[2]\subset\bigodot\Omega(M)[2]$ to itself, the automorphism $e^{\widetilde{R}_Z}$ restricts to $\bigodot\Omega_{\mathrm{hor}}(M)[2]$. Finally, let τ_K be the automorphism of $\Omega_{\mathrm{hor}}(M)[2]$, which maps $\alpha\in\Gamma(\wedge^k K^*\otimes\wedge^l G^*)$ to $(-1)^k\alpha$, and lift it to $\bigodot\Omega_{\mathrm{hor}}(M)[2]$ by

$$\tau_K(\alpha_1[2]\odot\cdots\odot\alpha_r[2]):=(\tau_K(\alpha_1)[2]\odot\cdots\odot\tau_K(\alpha_r)[2]).$$

Lemma 3.23. *Let $\beta=(\mu,\sigma)\in(K^*\otimes G^*)\oplus\wedge^2G^*$ be sufficiently small. The pushforward of $e^{\beta[2]}$ along*

$$e^{R_Z}e^{-\widetilde{R}_Z}\tau_K:\bigodot\Omega_{\mathrm{hor}}(M)[2]\to\bigodot\Omega(M)[2]$$

yields $e^{\gamma[2]}$ with

$$\gamma=\widetilde{\exp}_\eta(\mu,\sigma)-\eta.$$

Here $\widetilde{\exp}_\eta$ is the alternative exponential map from Section 2.3.

Proof. It follows from Proposition 3.10 that for β sufficiently small, the power series

$$e^{R_Z}e^{-\widetilde{R}_Z}\tau_K(e^{\beta[2]})$$

converges in the $\mathcal{C}^\infty$-topology to $F(-\mu,F^{-1}(\sigma))$, which is $\widetilde{\exp}_\eta(\mu,\sigma)-\mu$, according to Proposition 2.12 from Section 2.3. □

The meaning of Lemma 3.23 is as follows:

- We can use the automorphism $e^{-\widetilde{R}_Z}\tau_K$ of the coalgebra $\bigodot\Omega_{\mathrm{hor}}(M)[2]$, to twist the Koszul $L_\infty[1]$-algebra structure there, i.e. we define a new $L_\infty[1]$-algebra on $\Omega_{\mathrm{hor}}(M)[2]$ by the requirement that $e^{-\widetilde{R}_Z}\tau_K$ becomes an isomorphism of $L_\infty[1]$-algebras from the new $L_\infty[1]$-algebra structure on $\Omega_{\mathrm{hor}}(M)[2]$ to $(\Omega_{\mathrm{hor}}(M)[2],\lambda_1,\lambda_2,\lambda_3)$.
- The Maurer-Cartan elements of this new $L_\infty[1]$-algebra structure are still in bijection to pre-symplectic structures on M of the same rank as η, and sufficiently close to η. Lemma 3.23 asserts that this bijection is now given by the alternative exponential map $\widetilde{\exp}_\eta$.

3.4. **Dirac-geometric interpretation of Subsection 3.1.** Using Dirac geometry, we explain and re-prove some of the central results obtained in Subsection 3.1, "*An L_∞-algebra associated to a bivector field*". Recall that for any manifold M, the vector bundle $\mathbb{T}M=TM\oplus T^*M$ comes equipped with a non-degenerate pairing $\langle\cdot,\cdot\rangle$ and the Dorfman bracket $[\![\cdot,\cdot]\!]$, which allows one to define Dirac structures as suitable subbundles of $\mathbb{T}M$. We recall the basic notions from Dirac geometry in Appendix C.

Corollary 3.9 relates the transformation F to the $L_\infty[1]$-algebra $(\Omega(M)[2],\lambda_1,\lambda_2,\lambda_3)$ associated to a bivector field Z. In view of graph$(F(\beta))=\Phi_Z(\beta)$ (see Equation (11)), it states that the 2-form β is a Maurer-Cartan element iff $\Phi_Z(\beta)$ is a Dirac structure. The latter is the graph of β w.r.t the splitting $\mathbb{T}M=TM\oplus\mathrm{graph}(Z)$, suggesting the strategy we use in this subsection, namely: consider deformations of the Dirac structure TM.

3.4.1. *Revisiting Proposition 3.5.* The $L_\infty[1]$-algebra $(\Omega(M)[2],\lambda_1,\lambda_2,\lambda_3)$ was exhibited in Proposition 3.5. It can be recovered using Dirac geometry – or more precisely, the deformation theory of Dirac structures – as a special case of the construction from [5, Section 2.2].

Proposition 3.24. *Let L be a Dirac structure and R a complementary almost Dirac structure, i.e. we have a vector bundle decomposition[2] $L\oplus R=\mathbb{T}M$. Then $\Gamma(\wedge L^*)[2]$ has an induced $L_\infty[1]$-algebra structure, whose only non-trivial multibrackets are μ_1,μ_2,μ_3 given as follows:*

[2]More generally, Proposition 3.24 holds replacing $\mathbb{T}M$ by any Courant algebroid.

(1) *μ_1 is the differential d_L associated to the Lie algebroid L,*
(2)
$$\mu_2(\alpha[2] \odot \beta[2]) = -(-1)^{|\alpha|}[\alpha, \beta]_{L^*}[2],$$
where $[\alpha, \beta]_{L^*}$ denotes the (extension of) the bracket of the almost Lie algebroid $R \cong L^*$,
(3)
$$\mu_3(\alpha[2] \odot \beta[2] \odot \gamma[2]) = (-1)^{|\beta|}(\alpha^\sharp \wedge \beta^\sharp \wedge \gamma^\sharp)\psi[2]$$
where $\psi \in \Gamma(\wedge^3 L)$ is given by
$$\Gamma(\wedge^3 L^*) \to \mathcal{C}^\infty(M) \ , \ \eta_1 \wedge \eta_2 \wedge \eta_3 \mapsto \langle \mathrm{pr}_L([\![\eta_1, \eta_2]\!]), \eta_3 \rangle,$$
where we made use of the identification $R \cong L^*$.

Proof. The proof is a minor adaptation of the first part of the proof of [5, Lemma 2.6], setting $\varphi = 0$ there. We recall briefly the idea of the proof. By [14] there is a natural description of the Courant algebroid structure on $\mathbb{T}M$ in terms of graded geometry. One can use it to apply Voronov's Higher Derived Brackets construction (see [20, 21] and Theorem B.1) and obtain an $L_\infty[1]$-algebra structure on $\Gamma(\wedge L^*)[2]$. The multibrackets obtained are the ones in the statement of the lemma, as one can check using [14] and via computations in local coordinates. □

Alternative proof of Proposition 3.5. Let Z be a bivector field. Apply Proposition 3.24, choosing $L = TM$ and $R = \mathrm{graph}(Z)$. In that case d_L is the de Rham differential, and the bracket on K is given by the formula for the Koszul bracket. One checks that ψ is the trivector field $-\frac{1}{2}[Z, Z]$ using Lemma 1.10. Hence the $L_\infty[1]$-brackets on $\Omega(M)[2]$ given by Proposition 3.24 are $\mu_1 = \lambda_1$, $\mu_2 = -\lambda_2$ and $\mu_3 = \lambda_3$. Applying the automorphism $-\mathrm{id}$ to $\Omega(M)[2]$ yields Proposition 3.5. □

3.4.2. *Revisiting Corollary 3.9.* We now turn to Maurer-Cartan elements. Every bivector field Z gives rise to an almost Dirac structure, by taking its graph. In Lemma 2.9, we found a parametrization of all almost Dirac structures that are transverse to $\mathrm{graph}(Z)$ in terms of 2-forms β on M. As in Subsection 2.1, we denote this parametrization by Φ_Z:
$$\beta \mapsto \Phi_Z(\beta) := \{(v + Z^\sharp(\iota_v\beta), \iota_v\beta) \,|\, v \in TM\}.$$

Proposition 3.25. *For any 2-form $\beta \in \Omega^2(M)$ we have: $-\beta$ is a Maurer-Cartan element of $(\Omega(M)[2], \mu_1, \mu_2, \mu_3)$ if, and only if, the almost Dirac structure $\Phi_Z(\beta)$ is a Dirac structure.*

The above proposition can be regarded as an extension of the work by Liu-Weinstein-Xu recalled in Remark C.5.

Proof. The second part of [5, Lemma 2.6] states the following, given a Dirac structure L, and R a complementary almost Dirac structure: an element $\Phi \in \Gamma(\wedge^2 L^*)[2]$ is a Maurer-Cartan element of the $L_\infty[1]$-algebra structure given in Proposition 3.24 iff the graph
$$\Gamma_\Phi := \{(X - \iota_X\Phi) : X \in L\} \subset L \oplus R$$
is a Dirac structure. The above inclusion makes use of the identification $R \cong L^*$.

We apply this to the Dirac structure $L = TM$ and to the almost Dirac structure $R = \mathrm{graph}(Z)$. Notice that, for any $\beta \in \Omega^2(M) = \Gamma(\wedge^2 T^*M)$, we have $\Gamma_{-\beta} = \{(v + Z^\sharp(\iota_v\beta), +\iota_v\beta) \mid v \in TM\} = \Phi_Z(\beta)$. □

Corollary 3.9 states that for $\beta \in \Omega^2(M)$ taking values in some sufficiently small neighborhood $\mathcal{U}$ of the zero section in $\wedge^2 T^*M$ – in particular we require that β takes values in $\mathcal{I}_Z$, i.e. $\mathrm{id} + Z^\sharp\beta^\sharp$ is invertible – is a Maurer-Cartan element of $(\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3)$ iff $F(\beta)$ is closed. Using Proposition 3.25, we now provide an alternative proof of this result, which also shows that one can choose $\mathcal{U}$ to equal $\mathcal{I}_Z$.

Alternative proof of Corollary 3.9. For $\beta \in \Gamma(\mathcal{I}_Z)$ to be a Maurer-Cartan element of the $L_\infty[1]$-algebra $(\Omega(M)[2], \lambda_1, \lambda_2, \lambda_3)$ is equivalent to $\Phi_Z(\beta)$ being a Dirac structure, by Proposition 3.25. Further, by Section 2.1, $\Phi_Z(\beta)$ is transverse to $TM \subset \mathbb{T}M$, i.e. it can be written as the graph of a 2-form, which we denoted $F(\beta)$. Now use the fact that the graph of a 2-form is a Dirac structure if, and only if, the 2-form is closed. □

We summarize these insights in the following table, which places concepts from geometry next to their corresponding concepts from algebra:

geometry	*algebra*
almost Dirac structures transversal to the graph of Z	$\Omega^2(M)$
$\cup$	$\cup$
almost Dirac structures transversal to the graph of Z and to TM	$\Gamma(\mathcal{I}_Z)$
$\cup$	$\cup$
Dirac structures transversal to the graph of Z and TM	$\mathsf{MC}(\eta) \cap \Gamma(\mathcal{I}_Z)$
$\wr\Vert\ F$	$\downarrow \psi_*$
graphs of closed 2-forms	closed 2-forms

3.5. **Examples.** We present two examples for Corollary 3.9 and one for Theorem 3.19.

Example 3.26 (An example with quadratic Maurer-Cartan equation)**.** On the open subset $(\mathbb{R} \setminus \{-1\}) \times \mathbb{R}^2$ of $\mathbb{R}^3$ consider the Poisson bivector field $Z = \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$. The closed 2-form $\alpha := (xdy + ydx) \wedge dz$ lies in $\mathcal{I}_{-Z}$. We check that $\beta := F^{-1}(\alpha)$ satisfies

$$d\beta + \frac{1}{2}[\beta, \beta]_Z = 0,$$

i.e. that $\beta[2]$ is a Maurer-Cartan element of the $L_\infty[1]$-algebra of Proposition 3.5, as predicted by Corollary 3.9. We have $\beta^\sharp = \alpha^\sharp(\mathrm{id} - Z^\sharp\alpha^\sharp)^{-1}$ by Remark 2.2, whence

$$\beta = \frac{1}{1+x}\alpha = \frac{x}{1+x}dy \wedge dz + \frac{y}{1+x}dx \wedge dz.$$

We are done by computing

$$\begin{aligned} d\beta &= -\frac{x}{(1+x)^2}dx \wedge dy \wedge dz, \\ [\beta, \beta]_Z &= 2\mathcal{L}_Z\beta \wedge \beta = 2\frac{x}{(1+x)^2}dx \wedge dy \wedge dz. \end{aligned}$$

Example 3.27 (An example with cubic Maurer-Cartan equation)**.** Let a be a smooth function on the real line. On the open subset U of $\mathbb{R}^4$ consisting of points (x, y, z, w) so that $1 + a(y) \neq 0$, let $Z = \frac{\partial}{\partial y} \wedge (\frac{\partial}{\partial z} - a(y)\frac{\partial}{\partial x})$. From now on we write a instead of $a(y)$ for the ease of notation. Notice that Z is a Poisson structure iff G is involutive, which happens exactly when the derivative a' vanishes (i.e. a is locally constant).

The closed 2-form $\alpha := dx \wedge dy + dz \wedge dw$ lies in $\mathcal{I}_{-Z}$. We check that $\beta := F^{-1}(\alpha)$ satisfies

$$d\beta + \frac{1}{2}[\beta, \beta]_Z - \frac{1}{6}(\beta^\sharp \wedge \beta^\sharp \wedge \beta^\sharp)(\frac{1}{2}[Z, Z]) = 0, \tag{16}$$

i.e. that $\beta[2]$ is a Maurer-Cartan element of the $L_\infty[1]$-algebra of Proposition 3.5, as prediced by Corollary 3.9.

We know that $\beta^\sharp = \alpha^\sharp(\mathrm{id} - Z^\sharp\alpha^\sharp)^{-1}$ by Remark 2.2. In matrix form w.r.t. the frames $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial w}$ and dx, dy, dz, dw we have

$$Z^\sharp\alpha^\sharp = \begin{pmatrix} -a & 0 & 0 & 0 \\ 0 & -a & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (\mathrm{id} - Z^\sharp\alpha^\sharp)^{-1} = \frac{1}{1+a}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1+a & 0 \\ 0 & 0 & 0 & 1+a \end{pmatrix}$$

and therefore

$$\beta = \frac{1}{1+a}(dx \wedge dy + dx \wedge dw) + dz \wedge dw.$$

We compute

$$d\beta = \frac{a'}{(1+a)^2} dx \wedge dy \wedge dw. \tag{17}$$

Next we compute $[\beta, \beta]_Z$. We have $\beta \wedge \beta = \frac{2}{1+a} dx \wedge dy \wedge dz \wedge dw$, so

$$\mathcal{L}_Z(\beta \wedge \beta) = -d\Big(\iota_{\frac{\partial}{\partial y}}\iota_{\frac{\partial}{\partial z} - a\frac{\partial}{\partial x}}(\beta \wedge \beta)\Big) = 2\frac{a'}{(1+a)^2}(dx \wedge dy \wedge dw + dy \wedge dz \wedge dw).$$

Further $\mathcal{L}_Z\beta = \iota_Z d\beta - d\iota_Z\beta = \frac{a'}{(1+a)^2}(dy - adw)$, so that

$$-2\mathcal{L}_Z\beta \wedge \beta = 2\frac{a'}{(1+a)^2}(dx \wedge dy \wedge dw - dy \wedge dz \wedge dw).$$

Therefore there is a cancellation and

$$[\beta, \beta]_Z = -\mathcal{L}_Z(\beta \wedge \beta) + 2\mathcal{L}_Z\beta \wedge \beta = -4\frac{a'}{(1+a)^2}(dx \wedge dy \wedge dw). \tag{18}$$

Finally, using $[Z, Z] = -2a'\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$ we see that

$$(\beta^\sharp \wedge \beta^\sharp \wedge \beta^\sharp)(\frac{1}{2}[Z, Z]) = -6\frac{a'}{(1+a)^2} dx \wedge dy \wedge dw. \tag{19}$$

Using Equations (17), (18), (19), we see that the left-hand side of Equation (16) reads

$$[1 + \frac{1}{2}\cdot(-4) - \frac{1}{6}\cdot(-6)]\frac{a'}{(1+a)^2} dx \wedge dy \wedge dw = 0.$$

Example 3.28 (The four-dimensional torus)**.** Consider the four-dimensional torus $M = (S^1)^{\times 4}$, equipped with angular coordinates θ_1, θ_2, θ_3 and θ_4. The 2-form $\eta := d\theta_3 \wedge d\theta_4$ is pre-symplectic, with kernel $K \subset TM$ spanned by the global vector fields $\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}$. The complementary subbundle G, defined as the span of $\frac{\partial}{\partial\theta_3}, \frac{\partial}{\partial\theta_4}$, is clearly involutive. The Poisson bivector field corresponding to η then reads $\pi := \frac{\partial}{\partial\theta_3} \wedge \frac{\partial}{\partial\theta_4}$.

A 2-form β on M, when written in the global frame of TM, reads

$$\beta = \sum_{i<j} f_{ij} d\theta_i d\theta_j, \quad f_{ij} \in \mathcal{C}^\infty(M).$$

The form β is horizontal iff the component f_{12} is identically zero. Our aim is to explicitly work out $F(\beta)$ and the Maurer-Cartan equation

$$d\beta + \frac{1}{2}[\beta, \beta]_\pi = 0.$$

Let us first deal with the linear algebra, i.e. let us work out i) when $\mathrm{id} + \pi^\sharp\beta^\sharp$ is invertible, and ii) how its inverse looks like. Working with respect to the frames of TM and T^*M induced by the coordinates $(\theta_i)_{i=1,\dots,4}$, we find the following matrices

$$\pi^\sharp = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right), \quad \beta^\sharp = \left(\begin{array}{cc|cc} 0 & -f_{12} & -f_{13} & -f_{14} \\ f_{12} & 0 & -f_{23} & -f_{24} \\ \hline f_{13} & f_{23} & 0 & -f_{34} \\ f_{14} & f_{24} & f_{34} & 0 \end{array}\right)$$

and therefore

$$\mathrm{id} + \pi^\sharp\beta^\sharp = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -f_{14} & -f_{24} & (1-f_{34}) & 0 \\ f_{13} & f_{23} & 0 & (1-f_{34}) \end{array}\right).$$

and it is clearly invertible iff $f_{34} \neq 1$. Its inverse is

$$\frac{1}{(1-f_{34})}\left(\begin{array}{cc|cc} 1-f_{34} & 0 & 0 & 0 \\ 0 & 1-f_{34} & 0 & 0 \\ \hline f_{14} & f_{24} & 1 & 0 \\ -f_{13} & -f_{23} & 0 & 1 \end{array}\right).$$

From this one finds the following formula for $F(\beta)^\sharp = \beta^\sharp(\mathrm{id} + \pi^\sharp\beta^\sharp)^{-1}$:

$$F(\beta)^\sharp = \frac{1}{(1-f_{34})}\left(\begin{array}{cc|cc} 0 & -(f_{12}(1-f_{34}) + f_{13}f_{24} - f_{14}f_{23}) & -f_{13} & -f_{14} \\ f_{12}(1-f_{34}) + f_{13}f_{24} - f_{14}f_{23} & 0 & -f_{23} & -f_{24} \\ \hline f_{13} & f_{23} & 0 & -f_{34} \\ f_{14} & f_{24} & f_{34} & 0 \end{array}\right).$$

Adding $\eta^\sharp$, we find

$$\eta^\sharp + F(\beta)^\sharp = \frac{1}{(1-f_{34})}\left(\begin{array}{cc|cc} 0 & -(f_{12}(1-f_{34}) + f_{13}f_{24} - f_{14}f_{23}) & -f_{13} & -f_{14} \\ f_{12}(1-f_{34}) + f_{13}f_{24} - f_{14}f_{23} & 0 & -f_{23} & -f_{24} \\ \hline f_{13} & f_{23} & 0 & -1 \\ f_{14} & f_{24} & 1 & 0 \end{array}\right).$$

It is straightforward to check that the kernel of this matrix is non-trivial iff $f_{12} = 0$, i.e. iff β is horizontal, and that it then coincides with the span of

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline -f_{14} & -f_{24} \\ f_{13} & f_{23} \end{array}\right). \tag{20}$$

Moreover, when $\eta + F(\beta)$ is restricted to G, it coincides with $(\eta + F(\sigma))|_G$. All this is in agreement with Theorem 2.6.

Let us turn to the Maurer-Cartan equation for β (without assuming it to be horizontal). First, the above formula for $F(\beta)^\sharp$ can be rewritten as

$$F(\beta) = \frac{1}{(1-f_{34})}\left(\beta + (-f_{12}f_{34} + f_{13}f_{24} - f_{14}f_{23})d\theta_1 d\theta_2\right).$$

This 2-form is closed iff

$$\begin{aligned} 0 \overset{!}{=}\ & (1-f_{34})\left(d\beta + d(-f_{12}f_{34} + f_{13}f_{24} - f_{14}f_{23})d\theta_1 d\theta_2\right) \\ & + df_{34}\left(\beta + (-f_{12}f_{34} + f_{13}f_{24} - f_{14}f_{23})d\theta_1 d\theta_2\right). \end{aligned} \tag{21}$$

We next want to relate this equation to the Maurer-Cartan equation. Notice that $[d\theta_i, d\theta_j]_\pi = 0$, since the Poisson structure π is constant in the frame. For $f \in \mathcal{C}^\infty(M)$, we find

$$\begin{aligned}
[d\theta_i, f]_\pi &= \mathcal{L}_\pi(d\theta_i f) = [\iota_\pi, d](d\theta_i f) \\
&= -\iota_\pi(d\theta_i \wedge df) = -\iota_{\frac{\partial}{\partial\theta_3}}\iota_{\frac{\partial}{\partial\theta_4}}(d\theta_i df) \\
&= \delta_{i3}\frac{\partial f}{\partial\theta_4} - \delta_{i4}\frac{\partial f}{\partial\theta_3}.
\end{aligned}$$

Using this, we arrive at

$$\frac{1}{2}[\beta,\beta]_\pi = \sum_{j=1}^{4}\sum_{r<s}(f_{j3}\frac{\partial f_{rs}}{\partial\theta_4} - f_{j4}\frac{\partial f_{rs}}{\partial\theta_3})d\theta_j d\theta_r d\theta_s,$$

where we use the convention that for $j > i$, $f_{ji} := -f_{ij}$ and $f_{jj} = 0$.

Out of β, we construct two 3-forms, namely the right-hand side of Equation (21) and the Maurer-Cartan series for β:

$$\begin{aligned}
A &:= (1 - f_{34})\left(d\beta + d(-f_{12}f_{34} + f_{13}f_{24} - f_{14}f_{23})d\theta_1 d\theta_2\right) \\
&\quad + df_{34}\left(\beta + (-f_{12}f_{34} + f_{13}f_{24} - f_{14}f_{23})d\theta_1 d\theta_2\right) \\
B &:= d\beta + \frac{1}{2}[\beta,\beta]_\pi.
\end{aligned}$$

We want to show that A vanishes iff B does. Straightforward computations show that the $d\theta_1 d\theta_3 d\theta_4$-components A_{134} and B_{134} of A and B, respectively, are equal, and that the same holds for the $d\theta_2 d\theta_3 d\theta_4$-components A_{234} and B_{234}. To be more precise, one finds:

$$\begin{aligned}
A_{134} = B_{134} &= \frac{\partial f_{34}}{\partial\theta_1} - (1 - f_{34})\left(\frac{\partial f_{14}}{\partial\theta_3} - \frac{\partial f_{13}}{\partial\theta_4}\right) - f_{14}\frac{\partial f_{34}}{\partial\theta_3} + f_{13}\frac{\partial f_{34}}{\partial\theta_4}, \\
A_{234} = B_{234} &= \frac{\partial f_{34}}{\partial\theta_2} - (1 - f_{34})\left(\frac{\partial f_{24}}{\partial\theta_3} - \frac{\partial f_{23}}{\partial\theta_4}\right) - f_{24}\frac{\partial f_{34}}{\partial\theta_3} + f_{23}\frac{\partial f_{34}}{\partial\theta_4}.
\end{aligned}$$

This establishes our claim that $A = 0$ iff $B = 0$ for half of the components. We assume from now on that these components actually vanish and turn to the remaining components.

Lemma 3.29. *The equations*

$$\begin{aligned}
A_{123} &= (1 - f_{34})B_{123} + f_{23}A_{134} - f_{13}A_{234}, \\
A_{124} &= (1 - f_{34})B_{124} + f_{24}A_{134} - f_{14}A_{234}
\end{aligned}$$

hold.

Proof. Working out the definitions, we have

$$\begin{aligned}
A_{123} &= (1 - f_{34})\left((d\beta)_{123} + \frac{\partial}{\partial\theta_3}(-f_{12}f_{34} + f_{13}f_{24} - f_{14}f_{23})\right) \\
&\quad + \frac{\partial f_{34}}{\partial\theta_1}f_{23} - \frac{\partial f_{34}}{\partial\theta_2}f_{13} + \frac{\partial f_{34}}{\partial\theta_3}\left(f_{12}(1 - f_{34}) + f_{13}f_{24} - f_{14}f_{23}\right)
\end{aligned}$$

We now use the expressions for A_{134} and A_{234} to replace the terms $\frac{\partial f_{34}}{\partial \theta_1}$ and $\frac{\partial f_{34}}{\partial \theta_2}$. After rearranging of terms, we arrive at

$$\begin{aligned} A_{123} &= f_{23}A_{134} - f_{13}A_{234} + (1-f_{34})(d\beta)_{123} \\ &\quad +(1-f_{34})\Bigg(\frac{\partial}{\partial\theta_3}(-f_{12}f_{34}+f_{13}f_{24}-f_{14}f_{23}) + \\ &\qquad + \left(\frac{\partial f_{14}}{\partial\theta_3}-\frac{\partial f_{13}}{\partial\theta_4}\right)f_{23} - \left(\frac{\partial f_{24}}{\partial\theta_3}-\frac{\partial f_{23}}{\partial\theta_4}\right)f_{13} + \frac{\partial f_{34}}{\partial\theta_3}f_{12}\Bigg) \\ &= f_{23}A_{134} - f_{13}A_{234} + (1-f_{34})(d\beta)_{123} \\ &\quad +(1-f_{34})\left(-\frac{\partial f_{12}}{\partial\theta_3}f_{34} + \frac{\partial f_{13}}{\partial\theta_3}f_{24} - \frac{\partial f_{13}}{\partial\theta_4}f_{23} - \frac{\partial f_{23}}{\partial\theta_3}f_{14} + \frac{\partial f_{23}}{\partial\theta_4}f_{13}\right) \\ &= f_{23}A_{134} - f_{12}A_{234} + (1-f_{34})B_{123}. \end{aligned}$$

The second claimed equality is obtained in a similar fashion. □

We established that, under the assumption $f_{34} \neq 1$, the following equivalence holds:

$$\boxed{d(F(\beta)) = \frac{1}{(1-f_{34})^2}A \text{ vanishes if, and only if } d\beta + \frac{1}{2}[\beta,\beta]_\pi = B \text{ vanishes.}}$$

4. The characteristic foliation

Recall that for any pre-symplectic structure η on M, the kernel $K = \ker(\eta)$ is a constant rank, involutive distribution. Thus we have a map from the space of pre-symplectic structures on M of some given rank k to the space of codimension k foliations, denoted by

$$\rho : \mathsf{Pre\text{-}Sym}^k(M) \to \mathsf{Fol}^k(M).$$

Our aim is to understand this map from an algebraic perspective.

4.1. The tangent map.

First we consider the infinitesimal version of ρ. In Section 1.1 we established that the formal tangent space to $\mathsf{Pre\text{-}Sym}^k(M)$ can be identified with the space of closed 2-forms on M, which vanish when restricted to the foliation. On the other hand, the formal tangent space to the space of foliations at K can be identified with closed elements of degree 1 in the Bott-complex, which we recall below.

Definition 4.1. *Let K be a foliation of M. Denote the normal bundle TM/K by ν_K. It comes equipped with a natural flat, partial connection, called the Bott-connection, given by*

$$\begin{aligned} \Gamma(K)\times\Gamma(\nu_K) &\to \Gamma(\nu_K), \\ (X,Y) &\mapsto [X,\tilde{Y}] \bmod K, \end{aligned}$$

where $\tilde{Y}$ is a lift of Y to a vector field on M.

The Bott-complex *of K is the graded vector space $\Omega(K,\nu_K) := \Gamma(\wedge K^*\otimes\nu_K)$, equipped with the derivative corresponding to the Bott-connection.*

Remark 4.2. Let us sketch why the formal tangent space to K inside $\mathsf{Fol}^k(M)$ can be identified with the closed elements of $\Omega^1(K,\nu_K)$. First, to deform the foliation K means in particular to deform the fibres $K_x \subset T_xM$ in a smooth manner. Said another way, a one-parameter family of foliations is in particular a one-parameter family of section of the bundle $\mathrm{Gr}^k(TM)$, the bundle of codimension k Grassmannians of TM. It is well-known that the tangent space $T_{K_x}(\mathrm{Gr}^k(T_xM))$ is canonical isomorphic to $\mathrm{Hom}(K_x, T_xM/K_x)$. To be more explicit, suppose we are given a one-parameter family K_t of foliations with $K_0 = K$. We want to compute the image of some element $v \in K$ under the linear map $\psi : K \to \nu_K$ corresponding to the tangent vector of K_t at $t = 0$. One first picks any extension v_t of v to a family on K_t. With this one has

$$\psi(v) = \left(\frac{d}{dt}|_{t=0} v_t\right) \mod K.$$

We conclude that we can identify $T_K \mathsf{Fol}^k(M)$ with a certain subspace of $\Omega^1(K, \nu_K)$.

To see that the formal tangent space to $K \subset \mathsf{Fol}^k(M)$ coincides with the space of *closed* elements of $\Omega^1(K, \nu_K)$, one argues as follows: Suppose K_t is a one-parameter family of foliations with $K_0 = K$, whose tangent vector at $t = 0$ corresponds to $\psi : K \to \nu_K$. Given two sections X, Y of K, we can extend them to sections X_t and Y_t of K_t. We compute

$$\begin{aligned} \psi([X, Y]) &= \frac{d}{dt}|_{t=0}[X_t, Y_t] \mod K \\ &= \frac{d}{dt}|_{t=0}([X + (X_t - X), Y + (Y_t - Y)]) \mod K \\ &= \frac{d}{dt}|_{t=0}([X, Y_t] + [X_t, Y]) \mod K \\ &= ([X, \widetilde{\psi(Y)}] + [\widetilde{\psi(X)}, Y]) \mod K. \end{aligned}$$

Hence ψ is closed with respect to the differential on $\Omega(K, \nu_K)$.

The preceding discussion suggests that for any given pre-symplectic form η on M, there should be a chain map

$$\Omega_{\mathrm{hor}}(M) \to \Omega(K, \nu_K)$$

which corresponds to the map

$$T_\eta \rho : T_\eta \mathsf{Pre\text{-}Sym}^k(M) \to T_K \mathsf{Fol}^k(M).$$

Our next aim is to construct this chain map.

Lemma 4.3. (1) *Let K° be the dual bundle to ν_K, equipped with the dual (partial) connection. The map $\eta^\sharp : TM \to T^*M$ restricts to a vector bundle isomorphism $\nu_K \cong K^\circ$, which is compatible with the flat connections.*

(2) *The following map is compatible with the differentials:*

$$\begin{aligned} \varphi : \Omega^k_{\mathrm{hor}}(M) &\to \Omega^{k-1}(K, K^\circ) \\ \beta &\mapsto \varphi(\beta)(X_1, \ldots, X_{k-1}) := \beta(X_1, \ldots, X_{k-1}, \cdot). \end{aligned}$$

Proof. Concerning item (1), the only (slightly) non-obvious fact is the compatibility with the connections. For all X, $Y \in \Gamma(\nu_K)$ and $W \in \Gamma(K)$ we need to verify

$$\langle \nabla_W \eta^\sharp(X), Y\rangle \stackrel{!}{=} \langle \eta^\sharp(\nabla_W X), Y\rangle = \eta(\nabla_W X, Y).$$

By definition of the connection on K°, the left-hand side can be written as

$$W(\eta(X, Y)) - \eta(X, \nabla_W Y).$$

In total, we see that compatibility of $\eta^\sharp$ with the connections is equivalent to the equation

$$W(\eta(X, Y)) = \eta([W, \tilde{X}], Y) + \eta(X, [W, \tilde{Y}]),$$

where $\tilde{X}$ and $\tilde{Y}$ are extensions of X and Y to vector fields. This equation is satisfied because it is precisely given by the non-vanishing terms on the left-hand side of $d\eta(W, \tilde{X}, \tilde{Y}) = 0$.

For item (2), we have to check for all $X_1, \ldots, X_k \in \Gamma(K)$ and $Y \in \Gamma(\nu_K)$, that the equality

$$\langle \varphi(d\beta)(X_1, \ldots, X_k), Y\rangle = \langle d_\nabla(\varphi(\beta))(X_1, \ldots, X_k), Y\rangle$$

holds. By definition, the left-hand side is equal to $d\beta(X_1,\dots,X_k,\tilde{Y})$, where $\tilde{Y}$ is an extension of Y to a vector field, while the right-hand side is equal to

$$\sum_{i=1}^{k}(-1)^{i-1}\langle\nabla_{X_i}(\beta(X_1,\dots,\hat{X}_i,\dots,X_k,\cdot)),\tilde{Y}\rangle \\ +\sum_{i<j}(-1)^{i+j}\beta([X_i,X_j],\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,\tilde{Y}),$$

where the first line equals

$$\sum_{i=1}^{k}(-1)^{i-1}X_i(\beta(X_1,\dots,\hat{X}_i,\dots,X_k,\tilde{Y}))+\sum_{i=1}^{k}(-1)^{i}\beta(X_1,\dots,\hat{X}_i,\dots,X_k,[X_i,\tilde{Y}]).$$

Using that β vanishes when restricted to K, we see that, in total, the above expression sums up to $d\beta(X_1,\dots,X_k,\tilde{Y})$ as well. □

Remark 4.4. The lemma above yields a chain map

$$\Omega^\bullet_{\mathrm{hor}}(M)\xrightarrow{\varphi}\Omega^{\bullet-1}(K,K^\circ)\xrightarrow{(\eta^\sharp)^{-1}}\Omega^{\bullet-1}(K,\nu_K). \tag{22}$$

Proposition 4.5. *The linear map*

$$g:\{\alpha\in\Omega^2_{\mathrm{hor}}(M),\, d\alpha=0\}\to\{\beta\in\Omega^1(K,\nu_K),\, d_\nabla\beta=0\}$$

obtained by restricting (22) *coincides with* minus *the formal tangent map of*

$$\rho:\mathsf{Pre\text{-}Sym}^k(M)\to\mathsf{Fol}^k(M)$$

at $\eta\in\mathsf{Pre\text{-}Sym}^k(M)$.

Proof. Fix a pre-symplectic form η on M with underlying foliation K. Let η_t be a one-parameter family of pre-symplectic forms on M with $\eta_0=\eta$ and of constant rank. Denote the corresponding one-parameter family of foliations by K_t. The claim of the proposition amounts to the equality

$$\frac{d}{dt}|_{t=0}K_t=-(\eta^\sharp)^{-1}\left(\frac{d}{dt}|_{t=0}\eta_t^\sharp\right).$$

To check this, consider $v\in K$ and recall that the image of v under the map $\frac{d}{dt}|_{t=0}K_t$ is given by $\frac{d}{dt}|_{t=0}v_t \bmod K$. By applying the isomorphism $\nu_K\cong K^\circ$ given by $\eta^\sharp$, we obtain

$$\left(\eta^\sharp\circ\frac{d}{dt}|_{t=0}K_t\right)(v)=\eta^\sharp\left(\frac{d}{dt}|_{t=0}v_t\right).$$

Moreover, since $\eta_t\in K_t$ we have $\eta_t^\sharp(v_t)=0$ and, after differentiation,

$$\eta^\sharp\left(\frac{d}{dt}|_{t=0}v_t\right)=-\left(\frac{d}{dt}|_{t=0}\eta_t^\sharp\right)(v).$$

This concludes the proof. □

4.2. **Deformation theory of foliations.** We first review the $L_\infty[1]$-algebra governing deformations of foliations. We follow the exposition of Xiang Ji [8, Theorem 4.20] (who works in the wider setting of deformations of Lie subalgebroids). For foliations, these (and more) results were first obtained by Huebschmann [7] and Vitagliano [19] (see [19, Section 8] and [8, Rem. 4.23] for a comparison of results).

Proposition 4.6. *Let K be an involutive distribution on a manifold M, and let G be a complement. There is an $L_\infty[1]$-algebra structure on $\Gamma(\wedge K^* \otimes G)[1]$, whose only non-vanishing brackets are $l_1, -l_2, l_3$, with the property that the graph of $\phi \in \Gamma(K^* \otimes G)[1]$ is involutive iff ϕ is a Maurer-Cartan element.*

We remark that l_1 is the differential associated to the flat K-connection[3] on G which, under the identification $G \cong TM/K$, corresponds to the Bott connection from Section 4.1. The formulae for l_1, l_2, l_3 are spelled out in the following remark.

Remark 4.7. Ji [8, Section 4] derives this $L_\infty[1]$-multibrackets l_1, l_2, l_3 as follows: he views $TM = G \oplus K$ as a vector bundle over K (with fibres isomorphic to those of G), and applies Voronov's derived bracket construction (see Appendix B) to:

- the graded Lie algebra V of vector fields on $T[1]M$,
- the abelian Lie subalgebra $\mathfrak{a}$ of vector fields which are vertical and fibrewise constant with respect to the projection $T[1]M \to K[1]$,
- the Lie subalgebra $\mathfrak{h}$ of vector fields that are tangent to the base manifold $K[1]$,
- the homological vector field X on $T[1]M$ encoding the de Rham differential on M (notice that X is tangent to $K[1]$ because K is an involutive distribution).

This construction delivers the multibrackets l_1, l_2, l_3 in the formulae below. To relate these formulas to the formulas obtained in [8, Remark 4.16], one has to apply the isomorphism[4] of $\Gamma(\wedge K^* \otimes G)$ which acts on each homogeneous component $\Gamma(\wedge^k K^* \otimes G)$ via multiplications by $(-1)^{\frac{k(k+1)}{2}}$.

For all $\xi \in \Gamma(\wedge^k K^* \otimes G)[1], \psi \in \Gamma(\wedge^l K^* \otimes G)[1], \phi \in \Gamma(\wedge^m K^* \otimes G)[1]$ we have:

[3]Explicitly, this is the flat K-connection of G given by $\nabla_X Y = \mathrm{pr}_G[X, Y]$ for all $X \in \Gamma(K), Y \in \Gamma(G)$.

[4]The necessity to apply this isomorphism is related to the following piece of linear algebra. Let V be a vector space, and consider $V[1]$ (a graded manifold with linear coordinates of degree 1). The functions on $V[1]$ coincide with the exterior algebra $\wedge V^*$, and the vector fields on $V[1]$ coincide with $\wedge V^* \otimes V$ (in particular, degree -1 vector fields on $V[1]$ are just elements of V). Now take a k-multilinear skew-symmetric map on V with values in V, i.e. an element $\alpha \in \wedge^k V^* \otimes V$. This element can also be viewed canonically as a (degree $k-1$) vector field on $V[1]$, which we denote by X. Consider the k-multilinear skew-symmetric map on V with values in V given by

$$v_1, \ldots, v_k \mapsto [[\ldots[X, \iota_{v_1}], \ldots], \iota_{v_k}],$$

where ι_{v_i} is the (degree -1) vector field on $V[1]$ given by $v_i \in V$, and the bracket is the graded Lie bracket of vector fields on $V[1]$. Our point is that this map *differs from α by a factor of* $(-1)^{\frac{k(k+1)}{2}}$, as one can easily check in coordinates, and this factor is omitted in [8].

$$\begin{aligned}
l_1(\xi)(X_1,\dots,X_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\mathrm{pr}_G\Big[X_i\,,\,\xi(X_1,\dots,\hat{X}_i,\dots,X_{k+1})\Big] \\
&+ \sum_{i<j}(-1)^{i+j}\xi\Big([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+1}\Big) \\
l_2(\xi,\psi)(X_1,\dots,X_{k+l}) &= (-1)^k\sum_{\tau\in S_{k,j}}(-1)^\tau\mathrm{pr}_G\Big[\xi(X_{\tau(1)},\dots,X_{\tau(k)})\,,\,\psi(X_{\tau(k+1)},\dots,X_{\tau(k+l)})\Big] \\
&- (-1)^{k(l+1)}\sum_{\tau\in S_{l,1,k-1}}(-1)^\tau\xi\Big(\mathrm{pr}_K\Big[\psi(X_{\tau(1)},\dots,X_{\tau(l)})\,,\,X_{\tau(l+1)}\Big],\dots \\
&\qquad\qquad \dots,X_{\tau(l+2)},\dots,X_{\tau(l+k)}\Big) \\
&+ (-1)^k(\xi\leftrightarrow\psi,k\leftrightarrow l)
\end{aligned}$$

$$l_3(\xi,\psi,\phi)(X_1,\dots,X_{k+l+m-1}) = (-1)^{m+k(l+m)}\cdot$$

$$\sum_{\tau\in S_{l,m,k-1}}(-1)^\tau\xi\Big(\mathrm{pr}_K\Big[\psi(X_{\tau(1)},\dots,X_{\tau(l)})\,,\,\phi(X_{\tau(l+1)},\dots,X_{\tau(l+m)})\Big],\dots,X_{\tau(l+m+k-1)}\Big)\pm\circlearrowleft$$

Here $X_i\in\Gamma(K)$, pr_G is the projection $TM=G\oplus K\to G$, and similarly for pr_K. $S_{i,j,k}$ denotes the set of permutations τ of $i+j+k$ elements such that the order is preserved within each block: $\tau(1)<\dots<\tau(i),\tau(i+1)<\dots<\tau(i+j),\tau(i+j+1)<\dots<\tau(i+j+k)$. The symbol $(\xi\leftrightarrow\psi,k\leftrightarrow l)$ denotes the sum just above it, switching ξ with ψ and k with l. The symbol $\circlearrowleft$ denotes cyclic permutations in ξ,ψ,ϕ.

Proof of Proposition 4.6. Given Remark 4.7, we just have to address the statement about Maurer-Cartan elements, which holds by [8, Theorem 4.14]. □

The $L_\infty[1]$-algebra structure is compatible with the $\Gamma(\wedge K^*)$-module structure, as described for instance in [19, Theorem 2.4 and 2.5]. We spell this out:

Lemma 4.8. *With respect to the (left) $\Gamma(\wedge K^*)$-module structure on $\Gamma(\wedge K^*\otimes G)[1]$, the multibrackets of the $L_\infty[1]$-algebra structure described in Proposition 4.6 have the following properties:*

$$l_1(\alpha\cdot\xi) = (d\alpha)|_{\wedge K}\cdot\xi + (-1)^a\alpha\cdot l_1(\xi)$$
$$l_2(\xi,\alpha\cdot\psi) = (-1)^k(\iota_\xi d\alpha)|_{\wedge K}\cdot\psi + (-1)^{ka}\alpha\cdot l_2(\xi,\psi)$$
$$l_3(\xi,\psi,\alpha\cdot\phi) = (-1)^{k+l+1}\iota_\xi\iota_\psi d\alpha\cdot\phi + (-1)^{(k+l+1)a}\alpha\cdot l_3(\xi,\psi,\phi).$$

for all $\alpha\in\Gamma(\wedge^a K^*)$ and $\xi\in\Gamma(\wedge^k K^*\otimes G)[1],\psi\in\Gamma(\wedge^l K^*\otimes G)[1],\phi\in\Gamma(\wedge K^*\otimes G)[1]$.

Proof. This follows from a computation using the characterization of the multibrackets in terms of the graded Lie algebra of vector fields on $T[1]M$, as described in Remark 4.7. Alternatively, one can derive it from the explicit formulas for l_1, l_2 and l_3 given above. □

Remark 4.9 (The involutive case)**.** When G is involutive, the trinary bracket l_3 vanishes and, after a degree shift, we obtain a dg Lie algebra. In that case, locally we can always choose "enough" sections $X\in\Gamma(K)$ whose flow preserves G (i.e. $[X,\Gamma(G)]\subset\Gamma(G)$), and if X_i are such sections we have

$$l_2(\xi,\psi)(X_1,\dots,X_{k+l}) = (-1)^{k(l+1)}\sum_{\tau\in S_{k,l}}(-1)^\tau\mathrm{pr}_G\Big[\xi(X_{\tau(l+1)},\dots,X_{\tau(l+k)})\,,\,\psi(X_{\tau(1)},\dots,X_{\tau(l)})\Big]$$

for any $\xi\in\Gamma(\wedge^k K^*\otimes G)[1]$ and $\psi\in\Gamma(\wedge^l K^*\otimes G)[1]$.

In the involutive case it is easy to check directly the following claim made in Proposition 4.6: the graph of $\phi \in \Gamma(K^* \otimes G)[1]$ is involutive iff ϕ satisfies the Maurer-Cartan equation of $(\Gamma(\wedge K^* \otimes G)[1], l_1, -l_2, l_3)$, i.e. iff

$$l_1(\phi) - \frac{1}{2} l_2(\phi, \phi) = 0.$$

Indeed, given X_1, X_2 as above, writing out $[X_1 + \phi(X_1), X_2 + \phi(X_2)]$ we obtain $[X_1, X_2]$ plus 3 terms lying in $\Gamma(G)$. Hence graph(ϕ) is involutive iff

$$0 = -\phi([X_1, X_2]) + [X_1, \phi(X_2)] - [X_2, \phi(X_1)] + [\phi(X_1), \phi(X_2)].$$

The first 3 terms combine to $(l_1(\phi))(X_1, X_2)$, while the last term is $-\frac{1}{2} l_2(\phi, \phi)(X_1, X_2)$.

4.3. **A strict morphism of $L_\infty[1]$-algebras.** As earlier, let η be a pre-symplectic form on M, and choose a complement G to $K = \ker(\eta)$. Let $Z \in \Gamma(\wedge^2 G)$ be the bivector field corresponding to the restriction of η to G, so $Z^\sharp := -(\eta^\sharp|_G)^{-1} \colon G^* \to G$. Recall from Corollary 3.22 that $F^2(\Omega(M)) = \Omega_{\mathrm{hor}}(M) \cdot \Omega_{\mathrm{hor}}(M)$ gives an $L_\infty[1]$-ideal of $\Omega_{\mathrm{hor}}(M)[2]$. This suggests the following result, where we use the notation $\Omega(K, G) := \Gamma(\wedge K^* \otimes G)$ and similarly for $\Omega(K, G^*)$:

Theorem 4.10. *The composition*

$$q[2] : \Omega_{\mathrm{hor}}(M)[2] \to {\Omega_{\mathrm{hor}}(M)[2]}\big/{F^2(\Omega(M))[2]} \cong \Omega(K, G^*)[1] \xrightarrow{Z^\sharp[1]} \Omega(K, G)[1]$$

is a strict morphism of $L_\infty[1]$-algebras, where the domain is the Koszul $L_\infty[1]$-algebra with multibrackets $\lambda_1, \lambda_2, \lambda_3$, see Theorem 3.17, and the target $\Omega(K, G)[1]$ is endowed with the multibrackets $l_1, -l_2, l_3$.

Remark 4.11. i) For every element of $\Omega_{\mathrm{hor}}(M) = \Gamma(\wedge^\bullet K^* \otimes \wedge^{\geq 1} G^*)$, the first map in the above composition simply selects the component in $\Omega(K, G^*) = \Gamma(\wedge^\bullet K^* \otimes G^*)$. Hence q maps a Maurer-Cartan element $(\mu, \sigma) \in \Gamma(K^* \otimes G^*) \oplus \Gamma(\wedge^2 G^*)$ to the Maurer-Cartan element $Z^\sharp \mu^\sharp \in \Gamma(\mathrm{Hom}(K, G))$, where $\mu^\sharp \colon K \to G^*, X \mapsto \iota_X \mu$. Notice that, via Theorem 3.19 and Proposition 4.6, this is consistent with the fact that the kernel of (μ, σ) is the graph of $Z^\sharp \mu^\sharp$ (see Theorem 2.6) and with the well-known fact that the kernels of pre-symplectic forms are involutive.

ii) A consequence of Theorem 4.10 is that the map

$$Z^\sharp[2] : {\Omega_{\mathrm{hor}}(M)[2]}\big/{F^2\Omega(M)[2]} \cong \Omega(K, G^*)[1] \longrightarrow \Omega(K, G)[1]$$

is a strict isomorphism of $L_\infty[1]$-algebras. Here ${\Omega_{\mathrm{hor}}(M)[2]}\big/{F^2(\Omega(M))[2]}$ is endowed with the $L_\infty[1]$-algebra structure inherited from $\Omega_{\mathrm{hor}}(M)[2]$.

We need some preparation before giving the proof of Theorem 4.10. Notice that for $\alpha \in \Omega(K)$ and $\xi \in \Gamma(G^*)$, the element $\alpha \cdot \xi \in \Gamma(\wedge K^* \otimes G^*)$ is mapped to $\alpha \cdot Z^\sharp \xi \in \Omega(K, G)$ under q. In the next three statements, we write $\sharp$ instead of $Z^\sharp$, and suppress the degree shifts [2] for the sake of readability.

Proposition 4.12. *For all $\xi_1, \xi_2 \in \Gamma(G^*)$ and $\alpha_1, \alpha_2 \in \Gamma(\wedge K^*)$ we have*

$$q\left(\lambda_2(\alpha_1 \cdot \xi_1, \alpha_2 \cdot \xi_2)\right) = -l_2(\alpha_1 \cdot \sharp\xi_1, \alpha_2 \cdot \sharp\xi_2).$$

Proof. Using the biderivation property of the Koszul bracket $[\cdot, \cdot]_Z$ (see Remark 3.2) and Lemma 4.8 for l_2, we obtain

$$q([\alpha_1\xi_1, \alpha_2\xi_2]_Z) = \alpha_1[\xi_1, \alpha_2]_Z|_{\wedge K} \cdot \sharp\xi_2 - (-1)^{|\alpha_1||\alpha_2|}\alpha_2[\xi_2, \alpha_1]_Z|_{\wedge K} \cdot \sharp\xi_1 + \alpha_1\alpha_2 \cdot \sharp[\xi_1, \xi_2]_Z$$

$$(-1)^{|\alpha_1|} l_2(\alpha_1\sharp\xi_1, \alpha_2\sharp\xi_2) = \alpha_1(\iota_{\sharp\xi_1} d\alpha_2)|_{\wedge K} \cdot \sharp\xi_2 - (-1)^{|\alpha_1||\alpha_2|}\alpha_2(\iota_{\sharp\xi_2} d\alpha_1)|_{\wedge K} \cdot \sharp\xi_1 + \alpha_1\alpha_2 \cdot l_2(\sharp\xi_1, \sharp\xi_2).$$

Since $\lambda_2(\alpha_1 \cdot \xi_1, \alpha_2 \cdot \xi_2) = -(-1)^{|\alpha_1|}[\alpha_1\xi_1, \alpha_2\xi_2]_Z$, we have to prove that the two expressions above coincide. The right-most terms coincide by Lemma 1.13, hence it suffices to prove that $[\xi_1, \alpha_2]_Z = \iota_{\sharp\xi_1} d\alpha_2$. This identity is proven by induction over the degree of α_2, using that the case in which α_2 is a 1-form holds by Equation (4), page 4. □

Now we consider the trinary brackets.

Lemma 4.13. *For all $\xi_1, \xi_2 \in \Gamma(G^*)$ and $\alpha \in \Gamma(\wedge K^*)$ we have $\lambda_3(\xi_1, \xi_2, \alpha) = \iota_{\sharp\xi_2}\iota_{\sharp\xi_1} d\alpha$.*

Proof. By the definition in Proposition 3.5 we have $\lambda_3(\xi_1, \xi_2, \alpha) = \frac{1}{2}(\xi_1^\sharp \wedge \xi_2^\sharp \wedge \alpha^\sharp)[Z, Z]$. Now

$$\iota_{\sharp\xi_2}\iota_{\sharp\xi_1} d\alpha = -\iota_{[\sharp\xi_1, \sharp\xi_2]}\alpha = \frac{1}{2}\iota_{(\iota_{\xi_2}\iota_{\xi_1}[Z,Z])}\alpha,$$

using some Cartan calculus and the fact that $\sharp\xi_i \in \Gamma(G)$ in the first equality and Equation (5), page 6 in the second. Using the fact that $[Z, Z]$ is a section of $\wedge^2 G \otimes K$ (see Section 1.2), this concludes the proof. □

Proposition 4.14. *For all $\xi_1, \xi_2, \xi_3 \in \Gamma(G^*)$ and $\alpha_1, \alpha_2, \alpha_3 \in \Gamma(\wedge K^*)$ we have*

$$q\left(\lambda_3(\alpha_1 \cdot \xi_1, \alpha_2 \cdot \xi_2, \alpha_3 \cdot \xi_3)\right) = l_3(\alpha_1\sharp\xi_1, \alpha_2\sharp\xi_2, \alpha_3\sharp\xi_3). \tag{23}$$

Proof. Using repeatedly the derivation property of λ_3 (Remark 3.4), the fact that $[Z, Z] \in \Gamma(\wedge^2 G \otimes K)$ (see Section 1.2), and the fact that λ_3 is graded symmetric, we obtain

$$\begin{aligned} \lambda_3(\alpha_1 \cdot \xi_1, \alpha_2 \cdot \xi_2, \alpha_3 \cdot \xi_3) \;=\; & (-1)^{|\alpha_1|}\alpha_1\alpha_2\lambda_3(\xi_1, \xi_2, \alpha_3)\xi_3 \\ & +(-1)^{|\alpha_2||\alpha_3|+|\alpha_1|+|\alpha_2|+|\alpha_3|}\alpha_1\alpha_3\lambda_3(\xi_3, \xi_1, \alpha_2)\xi_2 \\ & +(-1)^{|\alpha_1||\alpha_2|+|\alpha_1||\alpha_3|+|\alpha_3|}\alpha_2\alpha_3\lambda_3(\xi_2, \xi_3, \alpha_1)\xi_1. \end{aligned} \tag{24}$$

Applying repeatedly Lemma 4.8 for l_3, using the fact that l_3 applied to any three elements of $\Gamma(G)$ vanishes (by its very definition), and using three times Lemma 4.13, we see that $l_3(\alpha_1\sharp\xi_1, \alpha_2\sharp\xi_2, \alpha_3\sharp\xi_3)$ equals the image under q of the right-hand side of Equation (24). □

Proof of Theorem 4.10. Clearly the kernel of q is $F^2(\Omega(M))$, and each λ_i maps to zero in the quotient $\Omega_{\mathrm{hor}}(M)/F^2(\Omega(M))$ whenever one of the entries lies in $F^2\Omega(M)$, by Lemma 3.20. Hence, to check that the map q is a strict $L_\infty[1]$-algebra morphism, it is sufficient to restrict ourselves to elements of $\Gamma(\wedge K^* \otimes G^*)$. The restriction of q there

- intertwines λ_1 and l_1 by Remark 4.4 since the identification $\nu_K \cong G$ (given by the projection along K) identifies l_1 and the Bott connection,
- intertwines λ_2 and $-l_2$ by Proposition 4.12,
- intertwines λ_3 and l_3 by Proposition 4.14.

□

Example 4.15 (The four-dimensional torus)**.** As in Example3.28, we consider $M = (S^1)^{\times 4}$, with pre-symplectic form $\eta := d\theta_3 \wedge d\theta_4$. Its kernel K is spanned by $\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}$, and as (involutive) complementary subbundle G we take the span of $\frac{\partial}{\partial\theta_3}, \frac{\partial}{\partial\theta_4}$. With these choices, we have $\pi := \frac{\partial}{\partial\theta_3} \wedge \frac{\partial}{\partial\theta_4}$.

Let β be a horizontal 2-form. We use the abbreviations

$$\mathsf{MC}_\beta := \lambda_1(\beta) + \frac{1}{2}\lambda_2(\beta, \beta) = (d\beta + \frac{1}{2}[\beta, \beta]_\pi)[2]$$

and $\mathsf{MC}_{q(\beta)} := l_1(\beta) - \frac{1}{2}l_2(q(\beta), q(\beta))$. The identity $q(\mathsf{MC}_\beta) = \mathsf{MC}_{q(\beta)}$ holds since q is a strict morphism, and by Proposition 4.6, this expression vanishes iff graph$(q(\beta))$ – which coincides with $\ker(\exp_\eta(\beta))$ by Theorem 2.6 – is involutive. On the other hand, $q(\mathsf{MC}_\beta)$ vanishes iff

$(\mathsf{MC}_\beta)_{123} = 0 = (\mathsf{MC}_\beta)_{124}$, where we use the same notation for the components of MC_β as in Example 3.28. This is true since q has kernel $\Gamma(\wedge^\bullet K^* \otimes \wedge^{>1} G^*)$ and q restricted to $\Gamma(\wedge^\bullet K^* \otimes G^*)$ is an isomorphism. Altogether we see that the following is equivalent:

i) the kernel of $\exp_\eta(\beta)$ is involutive
ii) $(\mathsf{MC}_\beta)_{123} = 0 = (\mathsf{MC}_\beta)_{124}$.

We now reproduce this equivalence by a direct computation. In coordinates, a horizontal 2-form is an expression $\beta = \sum_{i<j} f_{ij} d\theta_i d\theta_j$ with $f_{12} = 0$. The kernel of $\exp_\eta(\beta)$ is spanned by the two column vectors appearing in (20). Using the computations done in Example 3.28, one sees that their Lie bracket is

$$-(\mathsf{MC}_\beta)_{124} \frac{\partial}{\partial\theta_3} + (\mathsf{MC}_\beta)_{123} \frac{\partial}{\partial\theta_4},$$

hence the kernel of $\exp_\eta(\beta)$ is involutive iff this Lie bracket vanishes.

5. Obstructedness of deformations

We display examples showing that both the deformations of pre-symplectic forms and of foliations are formally obstructed. A deformation problem governed by an $L_\infty[1]$-algebra $(W, \{\lambda_k\}_{k\geq 1})$ is called *formally obstructed*, if there is a class in the zero-th cohomology $H^0(W)$ of the cochain complex (W, λ_1), such that one (or equivalently, any) representative $w \in W_0$ can not be extended to a formal curve of Maurer-Cartan elements. Obstructedness can be detected with the help of the Kuranishi map

$$\mathrm{Kr}\colon H^0(W) \to H^1(W), \quad [w] \mapsto [\lambda_2(w,w)], \tag{25}$$

for it is well-known that if the Kuranishi map is not identically zero, then the deformation problem is formally (and hence also smoothly) obstructed, see for example [13, Theorem 11.4].

5.1. **Obstructedness of pre-symplectic deformations.** We consider again Example 3.28, namely $M = (S^1)^{\times 4}$ with pre-symplectic form $\eta := d\theta_3 \wedge d\theta_4$ (so the kernel K is spanned by $\frac{\partial}{\partial\theta_1}$, $\frac{\partial}{\partial\theta_2}$), and choosing as complementary subbundle G the span of $\frac{\partial}{\partial\theta_3}, \frac{\partial}{\partial\theta_4}$ (so the Poisson bivector field corresponding to η reads $\pi := \frac{\partial}{\partial\theta_3} \wedge \frac{\partial}{\partial\theta_4}$).

In Section 3.3 we saw that the Koszul $L_\infty[1]$-algebra $(\Omega_{\mathrm{hor}}(M)[2], \lambda_1, \lambda_2, \lambda_3)$ governs the deformations of η as a pre-symplectic form. Recall that the multibrackets λ_i were defined in Proposition 3.5, with λ_1 being the de Rham differential.

The Kuranishi map $\mathrm{Kr}\colon H^0 \to H^1$ maps the class of any closed element $B \in \Omega^2_{\mathrm{hor}}(M)$ to the class of $[B,B]_\pi$. Now we choose

$$B := f(\theta_3) d\theta_1 \wedge d\theta_3 + g(\theta_4) d\theta_2 \wedge d\theta_4$$

for f, g smooth functions on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Clearly B is closed. In Example 3.28 we computed that

$$[B,B]_\pi = 2\big(f\partial_4 g d\theta_1 \wedge d\theta_2 \wedge d\theta_4 + g\partial_3 f d\theta_1 \wedge d\theta_2 \wedge d\theta_3\big),$$

where ∂_i denotes the partial derivative w.r.t. θ_i. While this 3-form is exact, we now show that it does not admit a primitive in $\Omega^2_{\mathrm{hor}}(M)$. This will show that the class it represents in the cohomology of $\Omega_{\mathrm{hor}}(M)$ is non-zero, i.e. that $\mathrm{Kr}([B]) \neq 0$.

For any $\alpha \in \Omega^2_{\mathrm{hor}}(M)$, we compute the integral of $d\alpha$ over $C_{[a,b]} := S^1 \times S^1 \times [a,b] \times \{0\}$ for all values of $a, b \in S^1$, where $[a,b]$ denotes the positively oriented arc joining these two points: $\int_{C_{[a,b]}} d\alpha = \int_{\partial C_{[a,b]}} \alpha = 0$, using Stokes' theorem and the fact that α has no $d\theta_1 \wedge d\theta_2$ component. On the other hand,

$$\int_{C_{[a,b]}} [B,B]_\pi = 2(2\pi)^2 g(0)[f(b) - f(a)],$$

which is non-vanishing for instance for $f = g = \cos$. Hence $[B, B]_\pi$ can not be equal to $d\alpha$ for any $\alpha \in \Omega^2_{\text{hor}}(M)$.

5.2. **Obstructedness of deformations of foliations.** As above take the manifold $M = (S^1)^{\times 4}$, take K the involutive distribution spanned by $\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}$, and choose as complementary subbundle G the span of $\frac{\partial}{\partial\theta_3}, \frac{\partial}{\partial\theta_4}$. As seen in Section 4.2, the deformations of the involutive distribution K (i.e. of the underlying foliation) are goverened by the $L_\infty[1]$-algebra $(\Gamma(\wedge K^* \otimes G)[1], l_1, -l_2, l_3)$.

The Kuranishi map $\text{Kr}\colon H^0 \to H^1$ maps the class of any l_1-closed $\Phi \in \Gamma(K^* \otimes G)$ to the class of $-l_2(\Phi, \Phi)$. We now take

$$\Phi = d\theta_1 \otimes f(\theta_3)\frac{\partial}{\partial\theta_4} - d\theta_2 \otimes g(\theta_4)\frac{\partial}{\partial\theta_3},$$

which is l_1-closed by eq. (26) below. We compute $l_2(\Phi, \Phi) \in \Gamma(\wedge^2 K^* \otimes G)$ by evaluating it on the frame $\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}$ of K and obtain

$$l_2(\Phi, \Phi) = (d\theta_1 \wedge d\theta_2) \otimes 2\big(f\partial_4 g\frac{\partial}{\partial\theta_3} - g\partial_3 f\frac{\partial}{\partial\theta_4}\big).$$

On the other hand, any $\xi \in \Gamma(K^* \otimes G)$ can be written as $\sum_{i=1}^2 \sum_{j=3}^4 d\theta_i \otimes h_{ij}\frac{\partial}{\partial\theta_j}$ for functions h_{ij} on $M = (S^1)^{\times 4}$, and one computes

$$l_1(\xi) = (d\theta_1 \wedge d\theta_2) \otimes \big[(\partial_1 h_{23} - \partial_2 h_{13})\frac{\partial}{\partial\theta_3} + (\partial_1 h_{24} - \partial_2 h_{14})\frac{\partial}{\partial\theta_4}\big]. \tag{26}$$

Notice that for every $a, c \in S^1$, the pullback of $l_1(\xi)$ to $N_{a,c} := S^1 \times S^1 \times \{a\} \times \{c\}$ is exact[5], hence $\int_{N_{a,c}} l_1(\xi) = 0$ for all a, c. But the pullback of $l_2(\Phi, \Phi)$ is a constant 2-form, since f and g do not depend on θ_1, θ_2, hence $\int_{N_{a,c}} l_2(\Phi, \Phi)$ is a (vector-valued) constant. This constant is non-zero for instance for $f = g = \cos$ and for $a, c \notin \frac{\pi}{2}\mathbb{Z}$. It follows that $l_2(\Phi, \Phi)$ is not equal to $l_1(\xi)$ for any $\xi \in \Gamma(K^* \otimes G)$. In other words, it follows that $\text{Kr}([\Phi]) \neq 0$.

Remark 5.1. Φ is the image of B under the strict $L_\infty[1]$-morphism q of Theorem 4.10. Hence the induced map in cohomology maps $\text{Kr}([B])$ to $\text{Kr}([\Phi])$. This and the fact that $\text{Kr}([\Phi]) \neq 0$ implies that the result we obtained in Subsection 5.1, namely that $\text{Kr}([B]) \neq 0$.

Appendix A. Cartan calculus of multivector fields

We recall the Cartan calculus on manifolds. Let M be a manifold and Y a multivector field on M of degree k. Associated with Y, we have the following operators on $\Omega(M)$:

- *Contraction*: $\iota_Y : \Omega^\bullet(M) \to \Omega^{\bullet - k}(M)$, which for $Y = f$ a function is ordinary multiplication, for a vector field X is defined by

  $$(\iota_X\omega)(X_1, \cdots, X_{r-1}) := \omega(X, X_1, \cdots, X_{r-1}),$$

  and then extended to all multivector fields by the rule

  $$\iota_{X\wedge\tilde{X}} = \iota_X \circ \iota_{\tilde{X}}.$$

- *Lie derivative*: $\mathcal{L}_Y : \Omega^\bullet(M) \to \Omega^{\bullet-k+1}(M)$, which is defined as the graded commutator $[\iota_Y, d] = \iota_Y \circ d - (-1)^{|Y|} d \circ \iota_Y$, with d the de Rham differential.

These operations obey the following commutator rules:

(1) $[\mathcal{L}_Y, d] = 0$ and $[\iota_Y, \iota_{\tilde{Y}}] = 0$,

[5] A primitive is $(h_{13}d\theta_1 + h_{23}d\theta_2) \otimes \frac{\partial}{\partial\theta_3} + (h_{14}d\theta_1 + h_{24}d\theta_2) \otimes \frac{\partial}{\partial\theta_4}$.

(2) $[\mathcal{L}_Y, \iota_{\tilde{Y}}] = \iota_{[Y,\tilde{Y}]}$, for $[\cdot,\cdot]$ the Schouten-Nijenhuis bracket of multivector fields, and
(3) $[\mathcal{L}_Y, \mathcal{L}_{\tilde{Y}}] = \mathcal{L}_{[Y,\tilde{Y}]}$.

Appendix B. Reminder on L_∞-algebras, higher derived brackets, and Koszul brackets

B.1. **L_∞- and $L_\infty[1]$-algebras.** We briefly review the basic background about L_∞- and $L_\infty[1]$-algebras. Let V be a graded vector space.

- For every $r \in \mathbb{Z}$, we have the degree shift endofunctor $[r]$, which maps a graded vector space V to $V[r]$, whose component $(V[r])^i$ in degree $i \in \mathbb{Z}$ is V^{i+r}.
- We denote by S_n the symmetric group on n letters. Given an integer $n \geq 1$ and an ordered partition $i_1 + \cdots + i_k = n$, we denote by $S_{i_1,\dots,i_k} \subset S_n$ the set of $(i_1,\dots,i_k)$-*unshuffles*, i.e., permutations $\sigma \in S_n$ such that $\sigma(i) < \sigma(i+1)$ for $i \neq i_1, i_1+i_2, \dots, i_1+\cdots+i_{k-1}$.
- $\overline{T}(V) = \bigoplus_{n\geq 1} T^n(V)$ is the n-fold tensor product of V with itself. The symmetric group S_n acts on $T^n(V)$ by $\sigma(x_1\otimes\cdots\otimes x_n) = \varepsilon(\sigma)x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}$, where $\varepsilon(\sigma) = \varepsilon(\sigma; x_1,\dots,x_n)$ is the usual *Koszul sign*. We denote the space of coinvariants by $\bigodot^n V$, and by $x_1\odot\cdots\odot x_n$ the image of $x_1\otimes\cdots\otimes x_n$ under the natural projection $T^n(V) \to \bigodot^n V$.
- The reduced symmetric coalgebra over V is the space $\bigodot V = \bigoplus_{n\geq 1}\bigodot^n V$, with the unshuffle coproduct
$$\overline{\Delta}(x_1\odot\cdots\odot x_n) = \sum_{i=1}^{n-1}\sum_{\sigma\in S_{i,n-i}} \varepsilon(\sigma)(x_{\sigma(1)}\odot\cdots\odot x_{\sigma(i)})\otimes(x_{\sigma(i+1)}\odot\cdots\odot x_{\sigma(n)}).$$
This is the cofree, coassociative, cocommutative and locally conilpotent graded coalgebra over V.
- Let (C,Δ) be a graded coalgebra. A map $Q : (C,\Delta)\to(C,\Delta)$ of degree 1 is a codifferential if $Q\circ Q = 0$ and $\Delta\circ Q = (Q\otimes \mathrm{id} + \mathrm{id}\otimes Q)\circ\Delta$ hold true.
- An $L_\infty[1]$-algebra structure on V is a codifferential Q of the graded coalgebra $(\bigodot V, \overline{\Delta})$.
- A morphism of $L_\infty[1]$-algebras from $L_\infty[1]$-algebra V to $L_\infty[1]$-algebra W is a morphism of the corresponding dg coalgebras $F : (\bigodot V, \overline{\Delta}, Q_V) \to (\bigodot W, \overline{\Delta}, Q_W)$.
- An $L_\infty[1]$-algebra structure Q on V is determined by its Taylor coefficients $(Q_n)_{n\geq 1}$, which are the maps given by
$$\bigodot^n V \longrightarrow \bigodot V \xrightarrow{Q} (\bigodot V)[1] \xrightarrow{p} V[1].$$
Moreover, a morphism F of $L_\infty[1]$-algebras from V to W is determined by its Taylor coefficients $F_n : \bigodot^n V \to W$, which are defined in the same manner as the Taylor coefficients of an $L_\infty[1]$-algebra structure.
- A morphism of $L_\infty[1]$-algebras is called an isomorphism if the corresponding morphism of dg coalgebras is invertible. It is called strict if all its structure maps except for the first one vanish.
- A graded subspace W of an $L_\infty[1]$-algebra V is an $L_\infty[1]$-subalgebra if the corresponding structure maps Q_n map $\odot^n W$ to W. Similarly, W is an $L_\infty[1]$-ideal if Q_n maps $W\odot\odot^{n-1}V$ to W.
- The category of dg Lie algebras embeds into the category of $L_\infty[1]$-algebra via
$$(L, d, [\cdot,\cdot]) \mapsto (\bigodot(L[1]), Q),$$
where Q is the coderivation whose non-trivial Taylor coefficients are $Q_1(a[1]) = -(da)[1]$ and $Q_2(a[1]\otimes b[1]) = (-1)^{|a|}([a,b])[1]$.

- Finally, we define the structure of an L_∞-algebra on V to be an $L_\infty[1]$-algebra structure on $V[1]$.

The main example of an L_∞-algebra (respectively $L_\infty[1]$-algebra) in this paper is the *Koszul L_∞-algebra ($L_\infty[1]$-algebra)* associated to a pre-symplectic manifold, cf. Section 3.3.

B.2. **Higher derived brackets.** The formalism of higher derived brackets from [20, 21] is a mechanism to construct L_∞-algebras from certain input data. The input data are a graded Lie algebra $(V, [\cdot,\cdot])$, together with

- a splitting as a graded vector space $V = \mathfrak{a} \oplus \mathfrak{h}$, where $\mathfrak{a}$ is an abelian subalgebra and $\mathfrak{h}$ is a Lie subalgebra,
- a Maurer-Cartan element $X \in V$, i.e. an element of degree 1 such that $[X, X] = 0$ holds.

There is a compatibility condition, which requires that X lies in $\mathfrak{h}$.

The higher derived brackets associated to these data are the maps defined by

$$Q_n : \odot^n \mathfrak{a} \to \mathfrak{a}[1], \quad a_1 \odot \cdots \odot a_n \mapsto \mathrm{pr}_{\mathfrak{a}}\Big([\cdots [X, a_1], \cdots, a_n]\Big),$$

where $\mathrm{pr}_{\mathfrak{a}}$ denotes the projection from V to $\mathfrak{a}$ along $\mathfrak{h}$.

The following result is proven in [20]:

Theorem B.1. *The maps $(Q_n)_{n\geq 1}$ equip $\mathfrak{a}$ with the structure of an $L_\infty[1]$-algebra.*

B.3. **BV_∞-algebras and Koszul brackets.** We collect some useful facts from the literature about commutative BV_∞-algebras and Koszul brackets. We follow mostly [9, 10] and the exposition in [1, Section 4.2.1]. We refer the reader for details and proofs to these sources.

First recall the notion of differential operators on a graded commutative algebra A: One defines recursively the set $\mathsf{DO}_k(A) \subset \mathrm{End}(A)$ of differential operators of order $\leq k$ on A by

$$\mathsf{DO}_{-1}(A) = \{0\}, \quad \text{and} \quad \mathsf{DO}_k(A) = \{f \in \mathrm{Hom}(A, A) \,|\, [f, \ell_a] \subset \mathsf{DO}_{k-1}(A) \,\forall a \in A\},$$

where ℓ_a denotes left multiplication, i.e. $\ell_a(b) := ab$.

Following [10] we define

Definition B.2. *A* commutative BV_∞-algebra $(A, d = \Delta_0, \Delta_1, \dots)$ *of (odd) degree r is a commutative, unital dg algebra $(A, d, \cdot)$, equipped with a family of endomorphisms $(\Delta_i)_{i\geq 0}$ of degree $1 - i(r+1)$ such that:*

1. *for all $i \geq 1$, the endomorphism Δ_i is a differential operator of order $\leq i+1$, which annihilates the unit 1_A,*
2. *if we adjoin a central variable t of degree $r+1$, the operator*

$$\Delta := \Delta_0 + t\Delta_1 + t^2\Delta_2 + \cdots : A[[t]] \to A[[t]]$$

squares to zero.

We refer to Δ as the BV-operator.

It is well-known that to every commutative BV_∞-algebra of degree r, one can associate an $L_\infty[1]$-algebra structure on $A[r+1]$, see [20, 1]. The Taylor coefficients of this $L_\infty[1]$-algebra structure are given by the Koszul brackets associated to the operators Δ_i: One associates to the endomorphism Δ_i a sequence of operations $\mathcal{K}(\Delta_i)_n : \odot^n A \to A$ defined iteratively by $\mathcal{K}(\Delta_i)_1 = \Delta_i$ and

$$\begin{aligned}\mathcal{K}(\Delta_i)_n(a_1 \odot \cdots \odot a_n) \quad = \quad & +\mathcal{K}(\Delta_i)_{n-1}(a_1 \odot \cdots \odot a_{n-2} \odot a_{n-1}a_n) \\ & -\mathcal{K}(\Delta_i)_{n-1}(a_1 \odot \cdots \odot a_{n-1})a_n \\ & -(-1)^{|a_{n-1}||a_n|}\mathcal{K}(\Delta_i)_{n-1}(a_1 \odot \cdots \odot a_{n-1} \odot a_n)a_{n-1},\end{aligned}$$

The following result was noticed in [20], we follow the exposition from [1, Proposition 4.2.21]:

Proposition B.3. *Given a commutative BV_∞-algebra $(A, d, = \Delta_0, \Delta_1, \dots,)$ of degree r, the sequence of maps*

$$(\Delta_0, \mathcal{K}(\Delta_1)_2, \mathcal{K}(\Delta_2)_3, \dots)$$

equips $A[r+1]$ with the structure of an $L_\infty[1]$-algebra.

Appendix C. Reminder on Dirac geometry

C.1. **Dirac linear algebra.** Let V be a finite-dimensional, real vector space. We denote by $\mathbb{V}$ the direct sum $V \oplus V^*$ and by $\langle \cdot, \cdot \rangle$ the following non-degenerate pairing on $\mathbb{V}$:

$$\langle (v, \xi), (w, \chi) \rangle := \xi(w) + \chi(v).$$

Definition C.1. *A subspace $W \subset \mathbb{V}$ is called* isotropic *if for all $w, w' \in W$ we have $\langle w, w' \rangle = 0$. It is* Lagrangian *if it is isotropic and* $\dim(W) = \dim(V)$. *Two subspaces W and $\tilde{W} \subset \mathbb{V}$ are* transverse*, if $W \oplus \tilde{W} = \mathbb{V}$.*

Given an element $Z \in \wedge^2 V$, we can consider $\mathrm{graph}(Z) := \{(\iota_\xi Z, \xi) \mid \xi \in V^*\} \subset \mathbb{V}$, a Lagrangian subspace. We denote the linear map $\xi \mapsto \iota_\xi Z = Z(\xi, \cdot)$ from V^* to V by $Z^\sharp$, and similarly we define $\beta^\sharp : V \to V^*$ for $\beta \in \wedge^2 V^*$.

Remark C.2. Every $\beta \in \wedge^2 V^*$ defines an orthogonal transformation $\mathfrak{t}_\beta$ of $(\mathbb{V}, \langle \cdot, \cdot \rangle)$, by

$$(v, \xi) \mapsto (v, \xi + \beta^\sharp(v)).$$

Similarly, every $Z \in \wedge^2 V$ gives rise to an orthogonal transformation $\mathfrak{t}_Z$, which takes (v, ξ) to $(v + Z^\sharp(\xi), \xi)$. In particular, elements of $\wedge^2 V^*$ and $\wedge^2 V$ act on the set of Lagrangian subspaces of $\mathbb{V}$.

Remark C.3. Suppose L, R are transverse Lagrangian subspaces of $\mathbb{V}$. There is a canonical isomorphism

$$R \cong L^*, r \mapsto \langle r, \cdot \rangle|_L.$$

Since R is transverse to L, any subspace of $\mathbb{V}$ transverse to R is the graph of a linear map $L \to R$. Any *Lagrangian* subspace transverse to R is the graph of a linear map $L \to R$ such that, composing with the canonical isomorphism above, we obtain a *skew-symmetric* linear map $L \to L^*$ (i.e. the sharp map associated to an element of $\wedge^2 L^*$).

C.2. **Dirac geometry.** Let us briefly recall the basic constituencies of Dirac geometry. We denote by $\mathbb{T}M$ the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$. It comes equipped with a non-degenerate pairing

$$\langle (X, \alpha), (Y, \beta) \rangle = \alpha(Y) + \beta(X)$$

and the Dorfman bracket

$$[\![(X, \alpha), (Y, \beta)]\!] = ([X, Y], \mathcal{L}_X \beta - \iota_Y d\alpha).$$

Together with the projection to TM, this makes $\mathbb{T}M$ into an example of Courant algebroid.

Definition C.4. *An* almost Dirac structure *on M is a Lagrangian, i.e. maximally isotropic, subbundle $L \subset (\mathbb{T}M, \langle \cdot, \cdot \rangle)$. An almost Dirac structure L is called integrable when its space of sections $\Gamma(L)$ is closed with respect to the Dorfman bracket $[\![\cdot, \cdot]\!]$. A* Dirac structure *is an almost Dirac structure which is integrable.*

Remark C.5. Let L, R be transverse Dirac structures on M. As seen above, almost Dirac structures transverse to R are in bijection with elements of $\Gamma(\wedge^2 L^*)$. We now recall a result of Liu-Weinstein-Xu [12] establishing when such an almost Dirac structure is integrable. Recall that every Dirac structure, with the restricted Dorfman bracket and anchor, is a Lie algebroid.

Since L is a Lie algebroid, it induces a differential d_L on $\Gamma(\wedge L^*)$. Further[6], since $L^* \cong R$ is a Lie algebroid, it induces a graded Lie bracket $[\cdot,\cdot]_{L^*}$ on $\Gamma(\wedge L^*)[1]$. Together with d_L and $[\cdot,\cdot]_{L^*}$, the graded vector space $\Gamma(\wedge L^*)[1]$ becomes a dg Lie algebra. The main result of [12] is: for all $\varepsilon \in \Gamma(\wedge^2 L^*)$, the graph $L_\varepsilon = \{v + \iota_v\varepsilon : v \in L\}$ is a Dirac structure iff ε satisfies the Maurer-Cartan equation, that is

$$d_L\varepsilon + \frac{1}{2}[\varepsilon,\varepsilon]_{L^*} = 0.$$

References

[1] R. Bandiera, *Higher Deligne groupoids, derived brackets and deformation problems in holomorphic Poisson geometry*, PhD-thesis at Università di Roma La Sapienza (2014).

[2] R. Bandiera and M. Manetti, *On coisotropic deformations of holomorphic submanifolds*, J. Math. Sci. Univ. Tokyo **22** (2015), no. 1, 1–37.

[3] V. Dotsenko, S. Shadrin, B. Vallette, *De Rham cohomology and homotopy Frobenius manifolds*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 3, 535–547.

[4] D. Fiorenza and M. Manetti, *Formality of Koszul brackets and deformations of holomorphic Poisson manifolds*, Homology Homotopy Appl. **14** (2012), no. 2, 63–75.

[5] Y. Frégier and M. Zambon, *Simultaneous deformations and Poisson geometry*, Compos. Math. **151** (2015), no. 9, 1763–1790.

[6] M. Gualtieri, M. Matviichuk and G. Scott, *Deformations of Dirac structures via L_∞ algebras*, preprint, `arXiv:1702.08837`.

[7] J. Huebschmann, *Higher homotopies and Maurer-Cartan algebras: quasi-Lie-Rinehart, Gerstenhaber, and Batalin-Vilkovisky algebras*, in: The Breadth of symplectic and Poisson geometry, in: Progr. Math., **232** (2005), pp. 237–302.

[8] X. Ji, *Simultaneous deformations of a Lie algebroid and its Lie subalgebroid*, J. Geom. Phys., Vol. **84** (2014), 8–29.

[9] J.-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, Astérisque, (Numero Hors Serie) (1985) 257–271.

[10] O. Kravchenko, *Deformations of Batalin-Vilkovisky algebras*, Banach Center Publications (2000), Vol. **51**, Issue 1, 131–139.

[11] H. V. Lê, Y.-G. Oh, A. G. Tortorella, and L. Vitagliano. *Deformations of Coisotropic Submanifolds in Abstract Jacobi Manifolds*, to appear in J. Symplectic Geom., available at `arXiv:1410.8446`.

[12] Z-J. Liu, A. Weinstein, P. Xu, *Manin triples for Lie bialgebroids*, J. Differential Geom. **45** (1997), no. 3, 547–574.

[13] Y.-G. Oh and J.-S. Park, *Deformations of coisotropic submanifolds and strong homotopy Lie algebroids*, Invent. Math. **161** (2005), no. 2, 287–360.

[14] D. Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, Ph.D. thesis, UC Berkeley (1999), available at `math.DG/9910078`.

[15] W.-D. Ruan, *Deformation of integral coisotropic submanifolds in symplectic manifolds*, J. Symplectic Geom. **3** (2005), no. 2, 161–169.

[16] F. Schätz and M. Zambon, *Deformations of coisotropic submanifolds for fibrewise entire Poisson structures*, Lett. Math. Phys. **103** (2013), no. 7, 777–791.

[17] F. Schätz and M. Zambon, *Equivalences of coisotropic submanifolds*, J. Symplectic Geom. **15** (2017), 107–149, available at `arXiv:1411.3227`.

[18] P. Šĕvera and A. Weinstein, *Poisson geometry with a 3-form background*, Noncommutative geometry and string theory (Yokohama, 2001). Progr. Theoret. Phys. Suppl. No. **144** (2001), 145–154.

[19] L. Vitagliano, *On the strong homotopy Lie-Rinehart algebra of a foliation*, Commun. Contemp. Math. **16** (2014), no. 6, 1450007, 49 pp.

[20] Th. Voronov, *Higher derived brackets and homotopy algebras*, J. Pure Appl. Algebra **202** (2005), no. 1-3, 133–153.

[21] Th. Voronov, *Higher derived brackets for arbitrary derivations*, Travaux mathématiques. Fasc. **XVI**, 163–186, Univ. Luxemb., Luxembourg, 2005.

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[6]The Lie algebroid structures on L and L^* are compatible in the sense that the pair (L, L^*) forms a Lie bialgebroid.

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