PROBABILITY SIGNATURES OF MULTISTATE SYSTEMS MADE UP OF TWO-STATE COMPONENTS

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The structure signature of a system made up of n components having continuous and i.i.d. lifetimes was defined in the eighties by Samaniego as the n-tuple whose k-th coordinate is the probability that the k-th component failure causes the system to fail. More recently, a bivariate version of this concept was considered as follows. The joint structure signature of a pair of systems built on a common set of components having continuous and i.i.d. lifetimes is a square matrix of order n whose (k, l)-entry is the probability that the k-th failure causes the first system to fail and the l-th failure causes the second system to fail. This concept was successfully used to derive a signature-based decomposition of the joint reliability of the two systems. In this talk we will show an explicit formula to compute the joint structure signature of two or more systems and extend this formula to the general non-i.i.d. case, assuming only that the distribution of the component lifetimes has no ties. Then we will discuss a condition on this distribution for the joint reliability of the systems to have a signature-based decomposition. Finally we will show how these results can be applied to the investigation of the reliability and signature of multistate systems made up of two-state components. This talk is based on the research paper [7].

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1. Introduction

Consider a system $\mathcal{S} = (C, \phi, F)$, where C is a set $[n] = \{1, \ldots, n\}$ of nonrepairable components, $\phi\colon \{0,1\}^n \to \{0,1\}$ is a structure function, and F is the joint c.d.f. of the component lifetimes $T_1, \ldots, T_n$, defined by

$$F(t_1, \ldots, t_n) \ = \ \Pr(T_1 \leqslant t_1, \ldots, T_n \leqslant t_n)\,, \qquad t_1, \ldots, t_n \geqslant 0.$$

We assume that the system $\mathcal{S}$ is *semicoherent*, which means that the function ϕ is nondecreasing in each variable and satisfies the conditions $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = 1$, where $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$.

Samaniego [11] defined the *signature* of any system $\mathcal{S}$ whose components have continuous and i.i.d. lifetimes as the n-tuple $\mathbf{s} = (s_1, \ldots, s_n)$ where

$$s_k \ = \ \Pr(T_{\mathcal{S}} = T_{k:n}), \qquad k = 1, \ldots, n,$$

Here $T_{\mathcal{S}}$ is the system lifetime and $T_{k:n}$ is the k-th smallest component lifetime, that is, the k-th order statistic of the component lifetimes.

It is well-known that the signature $\mathbf{s}$ is independent of the joint distribution F of the i.i.d component lifetimes. A combinatorial interpretation of this vector and a formula for its computation are given by Boland [1]. Here we thus see $\mathbf{s}$ as a combinatorial object associated with (C, ϕ) and we call it *structure signature*.

This original definition of signature can be extended to the general non-i.i.d. case, assuming only that the joint distribution function F has no ties, i.e., $\Pr(T_i = T_j) = 0$ for $i \neq j$ (we will always make this assumption). The *probability signature* of a system $\mathcal{S}$ is the n-tuple $\mathbf{p} = (p_1, \ldots, p_n)$ whose k-th coordinate is defined by

$$p_k \ = \ \Pr(T_{\mathcal{S}} = T_{k:n}), \qquad k = 1, \ldots, n.$$

Thus, for a given system (C, ϕ, F), we can consider the two objects $\mathbf{s}$ and $\mathbf{p}$. The first one is associated with (C, ϕ) and can be computed for instance via Boland's formula, or by its definition after replacement of F by the c.d.f. corresponding to any i.i.d. distribution of lifetimes.

The probability signature $\mathbf{p}$ depends on the joint c.d.f. F of the component lifetimes through the *relative quality function* $q\colon 2^{[n]} \to [0,1]$, which is defined as

$$q(A) \ = \ \Pr\Big(\max_{i \in C \smallsetminus A} T_i < \min_{j \in A} T_j\Big), \qquad A \subseteq C,$$

with the convention that $q(\varnothing) = q(C) = 1$. We have indeed (see [6])

$$p_k \ = \ \sum_{\substack{A \subseteq C \\ |A| = n-k+1}} q(A)\,\phi(A) - \sum_{\substack{A \subseteq C \\ |A| = n-k}} q(A)\,\phi(A), \qquad (1)$$

which generalizes Boland's formula under the sole assumption that the joint distribution function F has no ties.

Navarro et al. [9, 10] proposed to analyze the joint behavior of several systems $\mathcal{S}_1 = (C, \phi_1, F), \ldots, \mathcal{S}_m = (C, \phi_m, F)$ built on a common set of components. To simplify our presentation we will henceforth restrict ourselves to the case of two systems. In this case, under the assumption that the lifetimes are i.i.d. and continuous, Navarro et al. [10] defined the *joint structure signature* of the systems $\mathcal{S}_1$ and $\mathcal{S}_2$ as the square matrix $\mathbf{s}$ of order n whose (k, l)-entry is the probability

$$s_{k,l} = \Pr(T_{\mathcal{S}_1} = T_{k:n} \text{ and } T_{\mathcal{S}_2} = T_{l:n}), \qquad k, l = 1, \ldots, n.$$

This definition can also be extended directly to the general dependent setting, assuming only that the function F has no ties : the *joint probability signature* of two systems $\mathcal{S}_1 = (C, \phi_1, F)$ and $\mathcal{S}_2 = (C, \phi_2, F)$ is the square matrix $\mathbf{p}$ of order n whose (k, l)-entry is the probability

$$p_{k,l} = \Pr(T_{\mathcal{S}_1} = T_{k:n} \text{ and } T_{\mathcal{S}_2} = T_{l:n}), \qquad k, l = 1, \ldots, n.$$

Thus, with a given pair of systems we can associate the two matrices $\mathbf{s}$ and $\mathbf{p}$. Since $\mathbf{s}$ can be computed using the general formulas obtained for $\mathbf{p}$, simply by replacing F by the c.d.f corresponding to any i.i.d. distribution of lifetimes, we will concentrate on the results for $\mathbf{p}$.

We will provide a formula for the computation of the joint probability signature $\mathbf{p}$ by introducing a bivariate version of the relative quality function.

In the general non-i.i.d. setting, we will provide and discuss a necessary and sufficient condition on the function F for the joint reliability of two systems

$$\overline{F}_{\mathcal{S}_1, \mathcal{S}_2}(t_1, t_2) = \Pr(T_{\mathcal{S}_1} > t_1 \text{ and } T_{\mathcal{S}_2} > t_2), \qquad t_1, t_2 \geqslant 0,$$

to have a signature-based decomposition, thus extending the results of Navarro et al. [10] in that direction.

We will then apply these results to the investigation of the signature and reliability of multistate systems made up of two-state components in the general dependent setting. We will show how such a system can be decomposed in several two-state system built on the same set of components. This decomposition will provide a connection between the concept of signature that naturally emerges in the framework of such multistate systems and the concept of joint signature of several systems.

2. The joint probability signature

The results on the joint probability signature of two systems are more easily stated using the concept of *joint tail probability signature*. It is the square matrix $\overline{\mathbf{P}}$ of order $n+1$ whose (k,l)-entry is the probability

$$\overline{P}_{k,l} \;=\; \Pr(T_{\mathcal{S}_1} > T_{k:n} \text{ and } T_{\mathcal{S}_2} > T_{l:n}), \qquad k,l = 0,\ldots,n.$$

The conversion formulas between $\mathbf{p}$ and $\overline{\mathbf{P}}$ are given by the next result.

Proposition 1. *We have* $\overline{P}_{k,l} = \sum_{i=k+1}^{n}\sum_{j=l+1}^{n} p_{i,j}$ *, for* $k,l = 0,\ldots,n$*, and* $p_{k,l} = \overline{P}_{k-1,l-1} - \overline{P}_{k,l-1} - \overline{P}_{k-1,l} + \overline{P}_{k,l}$ *, for* $k,l = 1,\ldots,n$*.*

In order to compute the matrix $\overline{\mathbf{P}}$ from ϕ_1, ϕ_2, and F, we introduce the bivariate version of the relative quality function.

Definition 1. *The* joint relative quality function *associated with the joint c.d.f.* F *is the symmetric function* $q\colon 2^{[n]}\times 2^{[n]} \to [0,1]$ *defined by*

$$q(A,B) \;=\; \Pr\Big(\max_{i\in C\smallsetminus A} T_i < \min_{j\in A} T_j \;\textit{ and }\; \max_{i\in C\smallsetminus B} T_i < \min_{j\in B} T_j\Big),$$

with the convention that $q(A,\varnothing) = q(A,C) = q(A)$ *for every* $A \subseteq C$ *and* $q(\varnothing,B) = q(C,B) = q(B)$ *for every* $B \subseteq C$*.*

In many situations, including the i.i.d. case, but also the exchangeable case, the joint relative quality function can be easily computed, and reduces to the function $q_0\colon 2^{[n]}\times 2^{[n]} \to [0,1]$ defined by

$$q_0(A,B) \;=\; \begin{cases} \frac{(n-|A|)!\,(|A|-|B|)!\,|B|!}{n!} & \text{if } B\subseteq A,\\ \frac{(n-|B|)!\,(|B|-|A|)!\,|A|!}{n!} & \text{if } A\subseteq B,\\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

We can now show how to compute the matrix $\overline{\mathbf{P}}$ from ϕ_1, ϕ_2, and q.

Theorem 1. *For every* $k,l \in \{0,\ldots,n\}$ *we have*

$$\overline{P}_{k,l} \;=\; \sum_{|A|=n-k}\;\sum_{|B|=n-l} q(A,B)\phi_1(A)\phi_2(B). \tag{3}$$

The same formula holds for the computation of the joint tail structure signature, just replacing q by q_0. It is also clear that Theorem 1 easily generalizes to the case of m systems.

3. Signature-based decomposition of the joint reliability

In [11], a signature-based decomposition of the reliability of a system (C,ϕ,F) was derived, assuming that the lifetimes have a continuous i.i.d. distribution. This decomposition has the form

$$\overline{F}_{\mathcal{S}}(t) \ = \ \sum_{k=1}^{n} s_k \, \overline{F}_{k:n}(t)\,, \tag{4}$$

where $\overline{F}_{\mathcal{S}}(t) = \Pr(T_{\mathcal{S}} > t)$ is the reliability of the system at time t and $\overline{F}_{k:n}(t)$ is the reliability at time t of the k-out-of n system built on the same components, i.e. $\overline{F}_{k:n}(t) = \Pr(T_{k:n} > t)$. Several conditions were analyzed in the literature for the existence of this decomposition. It was proved by Marichal et al. [8] that its existence at time t for all coherent or semicoherent systems is governed by the exchangeability of the state variables of the components at time t (the state variable of component $j \in C$ at time t is defined by $X_j(t) = \mathrm{Ind}(T_j > t)$. The state vector at time t is then $\mathbf{X}(t) = (X_1(t), \ldots, X_n(t))$).

We will now analyze the existence of an analogous decomposition for the case of two systems in the general dependent case, thus extending the results of [10]. To this aim we will consider the joint distribution of the state vectors $\mathbf{X}(t_1)$ and $\mathbf{X}(t_2)$ at times $t_1 \geqslant 0$ and $t_2 \geqslant 0$. To simplify the notation we regard these two vectors together as a single object, namely a $2 \times n$ random array $\binom{\mathbf{X}(t_1)}{\mathbf{X}(t_2)}$.

The existence of a signature-based decomposition of reliability at times t_1, t_2 is actually governed by the exchangeability of th columns of the array $\binom{\mathbf{X}(t_1)}{\mathbf{X}(t_2)}$. The columns of this array are exchangeable if and only if

$$\Pr\Big(\begin{pmatrix}\mathbf{X}(t_1)\\ \mathbf{X}(t_2)\end{pmatrix} = \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\Big) \ = \ \Pr\Big(\begin{pmatrix}\mathbf{X}(t_1)\\ \mathbf{X}(t_2)\end{pmatrix} = \begin{pmatrix}\sigma(\mathbf{x})\\ \sigma(\mathbf{y})\end{pmatrix}\Big) \tag{5}$$

for any $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$ and any permutation σ of $\{1, \ldots, n\}$.

We can now state the main result concerning the signature-based decomposition of the reliability of two systems. This decomposition is given in terms of reliabilities of pairs of k-out-of-n systems, namely

$$\overline{F}_{k:n,l:n}(t_1, t_2) \ = \ \Pr(T_{k:n} > t_1 \text{ and } T_{l:n} > t_2).$$

Proposition 2. *Let $t_1, t_2 \geqslant 0$ be fixed. If the joint c.d.f. F satisfies condition* (5) *for any nonzero $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$, then we have*

$$\overline{F}_{\mathcal{S}_1,\mathcal{S}_2}(t_1, t_2) \ = \ \sum_{k=1}^{n} \sum_{l=1}^{n} s_{k,l} \, \overline{F}_{k:n,l:n}(t_1, t_2) \tag{6}$$

for any semicoherent systems $\mathcal{S}_1$ and $\mathcal{S}_2$, where $s_{k,l}$ is the (k,l) entry of the joint structure signature. Conversely, if $n \geqslant 3$ and if (6) *holds for any coherent systems $\mathcal{S}_1$ and $\mathcal{S}_2$ (at times t_1, t_2), then the joint c.d.f. F satisfies condition* (5) *for any nonzero $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$.*

Equation (5) provides an intermediate condition between the exchangeability of the component lifetimes and the exchangeability of the component states.

4. Applications to multistate systems

If $m \geqslant 1$ is an integer, an $(m+1)$-state system is a triple $\mathcal{S} = (C, \phi, F)$, where C and F are as above and where $\phi\colon \{0,1\}^n \to \{0,\dots,m\}$ is the structure function that expresses the state of the system in terms of the states of its components.

The system is assumed to be semicoherent, i.e., ϕ is nondecreasing in each variable and satisfies the boundary conditions $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = m$. We assume again that the c.d.f. F has no ties. Using the component states as above, we can express the system state at time t by $X_{\mathcal{S}}(t) = \phi(\mathbf{X}(t))$.

Since the system has $m+1$ possible states, its "lifetime" can be described by m random variables that represent the times at which the state of the system strictly decreases. We introduce these random variables, denoted $T_{\mathcal{S}}^{\geqslant 1}, \dots, T_{\mathcal{S}}^{\geqslant m}$, by means of the conditions

$$T_{\mathcal{S}}^{\geqslant k} > t \quad \Leftrightarrow \quad \phi(\mathbf{X}(t)) \geqslant k\,, \qquad k = 1, \dots, m.$$

Thus defined, $T_{\mathcal{S}}^{\geqslant k}$ is the time at which the system ceases to be at a state $\geqslant k$ and deteriorates to a state $< k$. In this setting it is also useful and natural to introduce the reliability function (called reliability of the system at states $\geqslant k$)

$$\overline{F}_{\mathcal{S}}^{\geqslant k}(t) = \Pr(T_{\mathcal{S}}^{\geqslant k} > t), \qquad t \geqslant 0, \tag{7}$$

for $k = 1, \dots, m$. The (overall) reliability function is then

$$\overline{F}_{\mathcal{S}}(t_1, \dots, t_m) = \Pr(T_{\mathcal{S}}^{\geqslant 1} > t_1\,, \dots,\, T_{\mathcal{S}}^{\geqslant m} > t_m), \qquad t_1, \dots, t_m \geqslant 0. \tag{8}$$

For simplicity let us now consider the special case where $m = 2$. All the results can be adapted directly in the general case. The *probability signature* of a 3-state system $\mathcal{S} = (C, \phi, F)$ is the square matrix $\mathfrak{p}$ of order n whose (k,l)-entry is the probability

$$\mathfrak{p}_{k,l} = \Pr(T_{\mathcal{S}}^{\geqslant 1} = T_{k:n} \text{ and } T_{\mathcal{S}}^{\geqslant 2} = T_{l:n}), \qquad k, l = 1, \dots, n, \tag{9}$$

where $T_{1:n} \leqslant \cdots \leqslant T_{n:n}$ are the order statistics of the component lifetimes.

Also, the *tail probability signature* of a 3-state system $\mathcal{S} = (C, \phi, F)$ is the square matrix $\overline{\mathfrak{P}}$ of order $n+1$ whose (k,l)-entry is the probability

$$\overline{\mathfrak{P}}_{k,l} = \Pr(T_{\mathcal{S}}^{\geqslant 1} > T_{k:n} \text{ and } T_{\mathcal{S}}^{\geqslant 2} > T_{l:n}), \qquad k, l = 0, \dots, n. \tag{10}$$

Note that these concepts were already introduced in [4] and [3] as the "bivariate signature" and "bivariate tail signature", in the special case where the component lifetimes are i.i.d. and continuous (see also [5] for an earlier work).

The key observation to connect these concepts defined in the framework of multistate systems to the concept of joint probability signatures for several systems is that we can always uniquely decompose a multistate system as a sum of two-state systems. Actually, this idea was already suggested in another form by Block and Savits [2, Theorem 2.8].

Proposition 3. *Any semicoherent structure function* $\phi\colon\{0,1\}^n \to \{0,\dots,m\}$ *decomposes in a unique way as a sum*

$$\phi = \sum_{k=1}^{m} \phi_{\langle k\rangle}, \tag{11}$$

where $\phi_{\langle k\rangle}\colon\{0,1\}^n \to \{0,1\}$ $(k=1,\dots,m)$ *are semicoherent structure functions such that* $\phi_{\langle 1\rangle} \geqslant \cdots \geqslant \phi_{\langle m\rangle}$ *(i.e.,* $\phi_{\langle 1\rangle}(\mathbf{x}) \geqslant \cdots \geqslant \phi_{\langle m\rangle}(\mathbf{x})$ *for all* $\mathbf{x}\in\{0,1\}^n$*).*

This proposition naturally leads to the following definition and to the subsequent decomposition principle.

Definition 2. *Given a semicoherent* $(m+1)$*-state system* $\mathcal{S} = (C,\phi,F)$*, with Boolean decomposition* $\phi = \sum_k \phi_{\langle k\rangle}$*, we define the semicoherent systems* $\mathcal{S}_k = (C,\phi_{\langle k\rangle},F)$ *for* $k=1,\dots,m$*.*

Decomposition Principle. *Any semicoherent* $(m+1)$*-state system* $\mathcal{S}$ *made up of two-state components can be additively decomposed into* m *semicoherent two-state systems* $\mathcal{S}_1,\dots,\mathcal{S}_m$ *constructed on the same set of components, with the property that for any* $k\in\{1,\dots,m\}$ *the lifetime of* $\mathcal{S}_k$ *is the time at which* $\mathcal{S}$ *deteriorates from a state* $\geqslant k$ *to a state* $< k$*.*

Setting $m=2$ and considering the systems $\mathcal{S}_1$ and $\mathcal{S}_2$ we denote by $T_{\mathcal{S}_k}$ the random lifetime of $\mathcal{S}_k$. We can immediately derive the following important theorem that connects the different lifetimes introduced so far as well as the joint probability signatures of the systems $\mathcal{S}_1$ and $\mathcal{S}_2$ and the probability signatures of the system $\mathcal{S}$.

Theorem 2. *We have* $T_{\mathcal{S}}^{\geqslant k} = T_{S_k}$ *and* $\overline{F}_{\mathcal{S}}^{\geqslant k} = \overline{F}_{S_k}$ *for* $k=1,\dots,m$*. Moreover we have* $\mathfrak{p} = \mathbf{p}$*,* $\overline{\mathfrak{P}} = \overline{\mathbf{P}}$*, and* $\overline{F}_{\mathcal{S}} = \overline{F}_{\mathcal{S}_1,\mathcal{S}_2}$*.*

This theorem allows us to interpret the results concerning joint signatures of several systems in the context of signatures of multistate systems. For instance we obtain a formula for the probability signature.

Theorem 3. *For every* $k,l\in\{0,\dots,n\}$ *we have*

$$\overline{\mathfrak{P}}_{k,l} = \sum_{\substack{A\subseteq C\\|A|=n-k}} \sum_{\substack{B\subseteq C\\|B|=n-l}} q(A,B)\,\phi_{\langle 1\rangle}(A)\phi_{\langle 2\rangle}(B).$$

Concerning a possible signature-based decomposition, we derive the following proposition. This result generalizes to the non-i.i.d. setting a recent result obtained by Gertsbakh et al. [4, Theorem 1] and Da and Hu [3, Theorem 7.2.3].

Proposition 4. *If, for any $t_1, t_2 \geqslant 0$, the joint c.d.f. F satisfies condition* (5) *for any nonzero $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$ and any permutation σ on $[n]$, then we have*

$$\overline{F}_{\mathcal{S}}(t_1, t_2) = \sum_{k=1}^{n} \sum_{l=1}^{n} s_{k,l}\, \overline{F}_{k:n,l:n}(t_1, t_2), \tag{12}$$

where the coefficients $s_{k,l}$ correspond to the structure signature of the pair of systems $\mathcal{S}_1$ and $\mathcal{S}_2$.

Many other results concerning the signatures of multistate systems can be easily derived using the Theorem 3.

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