Generalization of Czogała-Drewniak Theorem for n-ary semigroups

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Abstract We investigate n-ary semigroups as a natural generalization of binary semigroups. We refer it as a pair (X, F_n), where X is a set and an n-associative function $F_n : X^n \to X$ is defined on X. We show that if F_n is idempotent, n-associative function which is monotone in each of its variables, defined on an interval $I \subset \mathbb{R}$ and has a neutral element, then F_n is combination of the minimum and maximum operation. Moreover we can characterize the n-ary semigroups (I, F_n) where F_n has the previous properties.

1 Introduction

A function $F_n : X^n \to X$ is called *n-associative* if for every $x_1, \dots, x_{2n-1} \in X$ and for every $1 \le i \le n-1$ we have

$$F_n(F_n(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}) = F_n(x_1, \dots, x_i, F_n(x_{i+1}, \dots, x_{i+n}), x_{i+n+1}, \dots, x_{2n-1}).$$

Throughout this paper we assume that the underlying sets X are partially ordered sets (poset). However, some of the results only work for totally ordered sets. In our main results we investigate n-ary semigroups on arbitrary nonempty subintervals of the real numbers.

A set X endowed with an n-associative function $F_n : X^n \to X$ is called an *n-ary semigroup* and is denoted by (X, F_n). We say that (X, F_n) is a *totally (par-*

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tially) order based n-ary semigroup for emphasizing that X is totally (partially) ordered. Clearly, we obtain a generalisation of associative functions, which are the 2-associative functions using our terminology. The main purpose of this paper is to describe a class of n-ary semigroups. An n-ary semigroup is called *idempotent* if $F_n(a,\dots,a)=a$ for all $a\in X$. On a partially ordered set X we can define monotonicity of a function F_n. An n-associative function is called *monotone in the i'th variable* if for every $a_1,\dots,a_{i-1},a_{i+1},\dots,a_n$ the 1-variable functions $f_i(x):=F_n(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_n)$ are all order-preserving or all are order-reversing. An n-associative function is called *monotone* if it is monotone in each of its variables. Further we say that an n-associative function has *neutral element* denoted by $e\in X$ if for every $x\in X$ and $1\le i\le n$ we have $F(e,\dots,e,x,e,\dots,e)=x$, where x is substituted into the i'th coordinate.
Finally, we say that an n-ary semigroup (X,F_n) is *conservative* (or it is said to be *quasitrivial*) if for every $x_1,\dots,x_n\in X$ we have $F_n(x_1,\dots,x_n)\in\{x_1,\dots,x_n\}$. Such an n-variable function F_n is called a *choice function*. One might also say that F_n preserves all subsets of X. Ackerman [1] investigated conservative semigroups and also gave a characterization of them.

If we take $n=2$ we get the binary version of the definitions introduced above. The pair (X,F_2) is called a *semigroup*, where X is a set and the binary function $F_2:X^2\to X$ is (2-)associative.

2 Preliminary results

In this section we collect the previously known results that we use in order to prove our main results.

2.1 Binary case

Let $I\subset\mathbb{R}$ be a not necessarily bounded, nonempty interval and $\bar{I}$ be the closure of I. We also use the standard terminology of the extended reals $\overline{\mathbb{R}}=\mathbb{R}\cup\{\pm\infty\}$. Let $g:\bar{I}\to\bar{I}$ be a decreasing function. For every $x\in I$ let $g(x-0)$ and $g(x+0)$ denote the limit of g at x from the left and from the right, respectively. On the boundary we take the one sided limit of g. We denote by Γ_g the *completed graph* of g, which is a subset of $\bar{I}^2$ obtained by extending the graph of the function g in the following way. If $x\in I$ is a discontinuity point of g, then we add a vertical line segment between the points $(x,g(x-0))$ and $(x,g(x+0))$ to extend the graph of g. Formally,

$$\Gamma_g=\{(x,y)\in\bar{I}^2 : g(x+0)\le y\le g(x-0)\}.$$

On the infimum and the supremum of $\bar{I}$, the extended graph Γ_g defined with the sets

$$\{(\inf\overline{I},y)\in\overline{I}^2 : g(\inf\overline{I}+0)\le y\le \sup\overline{I}\},$$

$$\{(\sup\overline{I},y)\in\overline{I}^2 : \inf\overline{I}\le y\le g(\sup\overline{I}-0)\},$$

respectively. It is easy to show that Γ_g is a closed set. We call Γ_g *(id-)symmetric* if Γ_g is symmetric to the line $x=y$. These definitions was introduced in [10] and [11].

The following theorem gives a description of idempotent, monotone, (2-ary) semigroups with neutral elements. These semigroups were first investigated by Czogała and Drewniak [2], where the authors only dealt with closed, bounded subintervals of $\mathbb{R}$ but the statement holds for any nonempty interval as it was mentioned in [6]. On the other hand, instead of monotonicity it was assumed that the binary function is monotone increasing. However, Lemma 4 shows that monotonicity implies monotone increasingness in this case.

Theorem 1. *Let I be an arbitrary nonempty real interval. If a function $F_2 : I^2\to I$ is associative, idempotent, monotone and has a neutral element $e\in I$, then there exists a monotone decreasing function $g:\overline{I}\to\overline{I}$, with $g(e)=e$, such that for every $x,y\in I$*

$$F_2(x,y)=\begin{cases}\min(x,y), & \text{if } y<g(x)\\ \max(x,y), & \text{if } y>g(x)\\ \min(x,y) \text{ or } \max(x,y), & \text{if } y=g(x).\end{cases}$$

Now we present a full characterization of idempotent, monotone increasing, (2-ary) semigroups with neutral elements. First this was proved by Martin, Mayor and Torrens [10]. The statement of their theorem contained a small error in its condition, but essentially it was correct. In the original paper [10] the results worked on the closed unit interval $[0,1]$ and there was given the following condition for g, instead of the symmetry of Γ_g. The function $g:[0,1]\to[0,1]$ satisfies

$$\inf\{y : g(y)=g(x)\}\le (g\circ g)(x)\le \sup\{y : g(y)=g(x)\} \text{ for all } x\in[0,1]. \quad (1)$$

It was proved in [11] that Theorem 2 holds if F_2 is commutative and shown that condition (1) is not equivalent to the (id)-symmetry of Γ_g. Recently, Theorem 2 was reproved in an alternative way in [5] for any nonempty subinterval of $\mathbb{R}$.

From now on, we denote $(g\circ g)(x)$ by $g^2(x)$.

Theorem 2. *Let $I\subseteq\mathbb{R}$ be an arbitrary, nonempty interval. A function $F_2\colon I^2\to I$ is associative, idempotent, monotone and has a neutral element $e\in I$ if and only if there exists a decreasing function $g:\overline{I}\to\overline{I}$ with $g(e)=e$ ($e\in I$) such that the completed graph Γ_g is (id)-symmetric and for every $x,y\in I$*

$$F_2(x,y)=\begin{cases}\min(x,y), & \text{if } y<g(x) \text{ or } y=g(x) \text{ and } x<g^2(x)\\ \max(x,y), & \text{if } y>g(x) \text{ or } y=g(x) \text{ and } x>g^2(x)\\ \min(x,y) \text{ or } \max(x,y), & \text{if } y=g(x) \text{ and } x=g^2(x).\end{cases} \quad (2)$$

Moreover, $F_2(x,y) = F_2(y,x)$ except perhaps the set of points $(x,y) \in I^2$ satisfying $y = g(x)$ and $x = g^2(x) = g(y)$.

2.2 n-ary case

An important construction of n-ary semigroups is the following. Let (X,F_2) be a binary semigroup. Let $F_n := \underbrace{F_2 \circ F_2 \circ \ldots \circ F_2}_{n-1}$, where

$$\begin{aligned} F_n(x_1,\ldots,x_n) &= \underbrace{F_2 \circ F_2 \circ \ldots \circ F_2}_{n-1}(x_1,\ldots,x_n) \\ &= F_2(x_1, F_2(x_2,\ldots,F_2(x_{n-1},x_n))). \end{aligned}$$

The last equality is one of the possible evaluation of the composition. By associativity any evaluation gives the same value.

We obtain an n-associative function $F_n \colon X^n \to X$ and an n-ary semigroup (X,F_n). In this case we say that (X,F_n) *is derived* from the binary semigroup (X,F_2). Generally we simply say that F_n is derived from F_2.

Dudek and Mukhin [3] have found the exact condition when an n-ary semigroup (X,F_n) is derived from a binary one.

Proposition 1 ([3]). *If (X,F_n) is an n-ary semigroup with a neutral element e, then F_n can be derived from a binary semigroup denoted by F_2, where*

$$F_2(a,b) = F_n(a,e,\ldots,e,b). \tag{3}$$

As an application of Proposition 1 the authors of [3] obtained that X is an n-ary group which is derived from a group if and only if it contains a neutral element. An n-ary semigroup (X,F_n) is called an *n-ary group* if for $i \in \{1,\ldots,n\}$ and every $n-1$ elements $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n$ in X and every $a \in X$ there exists a unique $b \in X$ with $F_n(x_1,\ldots,x_{i-1},b,x_{i+1},\ldots,x_n) = a$. It is easy to see from the definition that ordinary groups are exactly the 2-ary groups. Clearly, a function F_n derived from a semigroup F_2 is n-associative but not every n-ary semigroup can be obtained in this way. We can easily construct n-ary groups which are not derived from binary groups if n is odd. Indeed, let $G_n(x_1,\ldots,x_n) = \sum_{i=1}^n (-1)^i x_i$. It is easy to verify that G_n is n-associative and we obtain an n-ary group. Moreover G_n is clearly monotone and there is no neutral element for G_n. For further examples see [9].

3 From n-ary to binary semigroups

The main purpose of this section is to derive properties from the n-ary semigroup to the corresponding binary semigroup and vice versa. The results of this section are also preparations for proving Theorem 3.

The following lemma is an easy consequence of the definitions.

Lemma 1. *Let $(X,\leq)$ be a partially ordered set, (X,F_2) be a semigroup and F_n be derived from F_2. If F_2 has any of the following properties*

1. *monotonicity,*
2. *idempotent,*
3. *has a neutral element,*

then the n-associative F_n also has.

From now on we focus on the possible reverse of the cases of Lemma 1.

First we investigate the *neutral element* property. By Proposition 1, if F_n has a neutral element, then F_n is derived from F_2 which is defined by equation (3).

Remark 1. By the definition (3) of F_2, if e is a neutral element of F_n, then e is also a neutral element of F_2. Indeed, $F_2(e,a) = F_n(e,\dots,e,a) = a = F_n(a,e,\dots,e) = F_2(a,e)$ for every $a \in X$.

For *monotonicity* the following statement have been proved for more general settings. On the other hand, by Remark 2, it turns out that this weaker condition implies that F_n is monotone in each of its variables.

Lemma 2. *Let $F_n : X^n \to X$ be an n-associative function on the partially ordered set X. Assume F_n is idempotent and monotone in the first and the last coordinates and derived from an associative function F_2. Then F_2 is monotone.*

Remark 2. As a consequence of Lemma 1 and Lemma 2 we have that if F_n is n-associative, idempotent and monotone in the first and the last variables on a poset X and derived from F_2, then F_n is monotone in each of its variables.

We can verify *idempotency* only for totally ordered sets. In Example 1 we show that this requirement is essential.

Lemma 3. *Let $F_n : X^n \to X$ be an n-associative function on a **totally ordered** set. Assume F_n is idempotent and monotone in each variable and derived from an associative function F_2. Then F_2 is idempotent as well.*

Example 1. For $k \geq 3$ we construct a k-ary semigroup (X,F_k), which is derived from a non-idempotent semigroup (X,F_2), where F_2 is monotone in both of its variables and have a neutral element.

Let $X = \{m,M\} \cup Z_{k-1}$, where Z_{k-1} is the cyclic group of order $k-1$. We define a partial ordering on X in the following way. M and m are the largest and smallest elements of X, respectively. The elements of Z_{k-1} are mutually incomparable but they are all larger than m and smaller than M. The set X endowed with this partial ordering is a modular lattice. Further we build up an associative function F_2:

$$F_2(x,y) = \begin{cases} M, & \text{if } x = M \text{ or } y = M \\ m, & \text{if } x = m \text{ or } y = m \text{ and } x,y < M \\ xy, & \text{if } x,y \in Z_{k-1}. \end{cases}$$

It is easy to verify that F_2 is associative and monotone increasing in both of its variables. The identity element e of Z_{k-1} is the neutral element of (X,F_2). One can define F_{k-1} and F_k as before. By Lemma 1 the functions F_{k-1} and F_k are $(k-1)$- and k-associative functions, respectively. Both of them are monotone having neutral element. Finally, it is easy to check that F_{k-1} is not idempotent since $F_{k-1}(a,\dots,a) = e$ for every $a \in Z_{k-1}$ while $F_k(x,\dots,x) = x$ for every $x \in X$. Since F_{k-1} is non-idempontent, F_2 cannot be idempotent by Lemma 1 (ii).

We note that the cyclic group Z_{k-1} might be substituted by any nontrivial group whose exponent divides $k-1$.

Remark 3. We note that for distributive lattices the statement of Lemma 3 seems true, but a potential proof would be basically different from the one of Lemma 3.

The following easy lemma provides that monotonicity implies *monotone increasingness* for partially order based, idempotent semigroups.

Lemma 4. *Let (X,F_2) be a partially order based semigroup, where $F_2 : X^2 \to X$ is idempotent and monotone in each variable, then F_2 is monotone increasing in each variable.*

Remark 4. Now we obtain some examples showing that we cannot omit any of the conditions of Lemma 4.

1. Let $F_2(x,x) = x$ for $x \in \mathbb{R}$ and $F_2(x,y) = 0$ if $x,y \in \mathbb{R}, x \neq y$. Then F_2 is associative and idempotent, but not monotone in each variable.
2. Let $F_2(x,y) = 2x - y$ for $x,y \in \mathbb{R}$. Then F_2 is idempotent and monotone in each variable, but not associative and clearly not monotone increasing.
3. Let $F_2(x,y) = -x$, if $x,y > 0$, and $F_2(x,y) = 0$ otherwise. Then F_2 is associative, since $F_2(x,F_2(y,z)) = F_2(F_2(x,y),z)) = 0$ and F_2 is monotone decreasing in each variable but F_2 is not idempotent.

Corollary 1. *If (X,F_n) is a totally order based n-ary semigroup, where F_n is idempotent and monotone in the first and in the last variables and derived from F_2, then F_n is monotone* increasing *in* each *variable. Moreover, F_k is* monotone increasing *for every $k \geq 2$.*

Using the results of this section we get the following proposition.

Proposition 2. *Let (X,F_n) be a totally order based n-ary semigroup, which is monotone, idempotent and has a neutral element. Then F_n is derived from a binary semigroup (X,F_2), where F_2 is also monotone idempotent and it also has a neutral element. Moreover, F_n is monotone increasing in each variables.*

As a consequence of Proposition 2 we can prove the following.

Lemma 5. *Let (X,F_n) be a totally order based n-ary semigroup derived from (X,F_2), where F_2 is idempotent, associative, monotone increasing and have a neutral element on X. Then*

$$F_n(a,y_1,\ldots,y_{n-2},b) = F_2(a,b)$$
$$F_n(b,y_1,\ldots,y_{n-2},a) = F_2(a,b)$$

for every $a \le y_1,\ldots,y_{n-2} \le b$.

4 Main results

If (X,F_n) is an n-ary semigroup having a neutral element e, then one can assign a semigroup by $F_2(a,b) = F_n(a,e,\ldots,e,b)$ for every $a,b \in X$ as it was defined in equation (3). This operation will be denoted by $\mathscr{F}$. One of our main theoretic result is the following:

Theorem 3. *For any totally ordered set X the operation $\mathscr{F}$ creates bijection between the set of idempotent, monotone, associative functions on X having neutral elements and the set of n-associative, idempotent, monotone functions on X having neutral elements.*

We get the following as an easy consequence of our investigation.

Theorem 4. *Let I be a nonempty interval. For $n \ge 2$ let $F_n : I^n \to I$ be n-associative, monotone increasing, idempotent n-ary semigroup and has a neutral element $e \in I$. Then F_n is conservative.*

Applying Theorem 2 and Theorem 3 we can obtain a practical method to calculate the value of $F_n(a_1,\ldots,a_n)$ for any $a_1,\ldots,a_n \in I$, where $I \subset \mathbb{R}$ is a nonempty interval.

For every decreasing function $g : \bar{I} \to \bar{I}$ a pair $(a,b) \in I^2$ is called *critical* if $g(a) = b$ and $g(b) = a$. By Theorem 2 and Lemma 4, for every idempotent, monotone semigroup (X,F_2) with neutral element there exists a unique decreasing function g satisfying (2). Theorem 2 shows also that F_2 commutes in every non-critical pair $(x,y) \in I^2$ (i.e. $F_2(x,y) = F_2(y,x)$). Since for a critical pair (a,b) the value of $F_2(a,b)$ and $F_2(b,a)$ can be independently chosen from g we have two cases. A pair (a,b) is called *extra-critical* if critical and $F_2(a,b) \ne F_2(b,a)$. We note that being critical or extra-critical are both symmetric relations.

Finally, in order to simplify notation and give a compact way to express a value of F_n we introduce the following. The set of entries $\{a_1,\ldots,a_n\}$ of F_n is denoted by A. The smallest and largest element of A is denoted by c and d, respectively. Further there exist $1 \le i \le j \le n$ such that $a_i,a_j \in \{c,d\}$ and $a_k \notin \{c,d\}$ for every $1 \le k < i$ and $j < k \le n$. We write $e_1 = a_i$ and $e_2 = a_j$.

Theorem 5. *Let $F_n : I^n \to I$ be an n-associative, idempotent function with neutral element. Assume that F_n is monotone in its first and last coordinates. If (c,d) is not an extra-critical pair, then $F_n(a_1,\dots,a_n) = F_2(c,d)$.*
If (c,d) is an extra-critical pair, then $F_n(a_1,\dots,a_n) = F_2(e_1,e_2)$.

Now we point out three important consequences of Theorem 5. First we generalise Czogala-Drewniak's theorem (Theorem 1) as follows.

Theorem 6. *Let I be an arbitrary nonempty real interval. If a function $F_n : I^n \to I$ is n-associative, idempotent, monotone and has a neutral element $e \in I$, then there exits a monotone decreasing function $g : \overline{I} \to \overline{I}$ with $g(e) = e$ ($e \in I$) such that Γ_g is symmetric and*

$$F_n(a_1,\dots,a_n) = \begin{cases} c, & \text{if } c < g(d) \\ d, & \text{if } c > g(d) \\ c \text{ or } d, & \text{if } c = g(d), \end{cases}$$

where c and d denote the minimum and the maximum of set $A = \{a_1,\dots,a_n\} \subset \mathbb{R}$, respectively.

We note that a generalization of Theorem 2 is essentially stated in Theorem 5. In [11] the authors investigated idempotent uninorms, which are idempontent, associative, commutative, monotone functions with a neutral element and defined on $[0,1]$. We introduce *n-ary uninorms*, which are *n*-associative, commutative, monotone functions with neutral element. Here we show a generalization of [11, Theorem 3] for *n*-ary operations.

Theorem 7. *An n-ary operator U_n is an idempotent n- ary uninorm on $[0,1]$ with neutral element $e \in [0,1]$ if and only if there exists a decreasing function $g : [0,1] \to [0,1]$ with fixed point e and with symmetric graph Γ_g such that*

$$U_n(a_1,\dots,a_n) = \begin{cases} c & \text{if } c < g(d)) \text{ or } d < g(c) \\ d & \text{if } c > g(d) \text{ or } d > g(c) \\ c \text{ or } d & \text{if } c = g(d) \text{ and } d = g(c), \end{cases} \tag{4}$$

where c and d are as in Theorem 6. Moreover, if (c,d) is a critical pair ($c = g(d), d = g(c)$), then the value of $U_n(a_1,\dots,a_n)$ can be chosen to be c or d arbitrarily and independently from other critical pairs.

One may extend the concept of associativity for string functions ([4], [8]). Let us define

$$X^* = \bigcup_{n\in\mathbb{N}} X^n$$

to be the space of finite length words over the alphabet X. A multivariate function $F : X^* \to X$ is *associative* if it satisfies

$$F(\mathbf{x},\mathbf{x}') = F(F(\mathbf{x}),F(\mathbf{x}'))$$

for all $\mathbf{x}, \mathbf{x}' \in X^*$. It is easy to check that $F|_{X^n}$ is n-associative for every $n \in \mathbb{N}$. Idempotency, monotonicity and the neutral element properties of F can be defined as they hold for every $n \in \mathbb{N}$.

Theorem 8. *Let I be a nonempty real interval. Then $F : I^* \to I$ is associative, idempotent, monotone and has a neutral element if and only if there is a decreasing function $g : \bar{I} \to \bar{I}$ with symmetric completed graph Γ_g such that $F|_{X^2}$ satisfies* (2). *Furthermore F must be monotone increasing in each variable.*

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References

1. N. L. Ackerman, A characterization of quasitrivial n-semigroups, preprint.
2. E. Czogala, J. Drewniak, Associative monotonic operations in fuzzy set theory, *Fuzzy Sets and Systems* **12** (1984) 249-269.
3. W.A.Dudek, V.V. Mukhin, On n-ary semigroups with adjoint neutral element, *Quasigroups and Related Systems* **14** (2006) 163-168.
4. M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions.* Encyclopedia of Mathematics and its Applications, vol. 127. Cambridge University Press, Cambridge, 2009.
5. J. Devillet, G. Kiss, J.-L. Marichal, B. Teheux, On reflexive, monotone functions that are associative and bisymmetric, *in preparation*.
6. J. Fodor An extension of Fung-Fu's theorem, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **4**(3) (1996) 235-243.
7. G. Kiss, G. Somlai, Characterization of n-associative, monotone, idempotent functions on an interval which have neutral elements, *Semigroup forum*, in press, arXiv:1609.00279.
8. E. Lehtonen, J.-L. Marichal, B. Teheux, Associative string functions, *Asian-European Journal of Mathematics* **7(4)** (2014), 1450059.
9. J-L. Marichal, P. Mathonet, A description of n-ary semigroups polynomial-derived from integral domains, *Semigroup Forum* **83**(2) (2011) 241-249.
10. J. Martin, G. Mayor, J. Torrens, On locally internal monotonic operations, *Fuzzy Sets and Systems* **137** (2003) 27-42.
11. D. Ruiz-Aguilera, J. Torrens, B. De Baets, J. Fodor, Some Remarks on the Characterization of Idempotent Uninorms, Chapter of *Computational Intelligence for Knowledge-Based Systems Design* (2010) 425-434.