Generalization of Czogała-Drewniak Theorem for n-ary semigroups

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Notation: If $F : I^2 \to I$ is associative, then we also say that the pair (I, F) is a (2-ary) semigroup.

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3. F is continuous.

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Theorem (Czogała,Drewniak, 1984)

Let $I = [a, b]$ be a closed real interval. If a function $F : I^2 \to I$ is associative, idempotent, monotone which has a neutral element $e \in I$,

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$$F(x, y) = \begin{cases} \min(x, y), & \text{if } y < g(x) \\ \max(x, y), & \text{if } y > g(x) \\ \min(x, y) \text{ or } \max(x, y), & \text{if } y = g(x) \end{cases} \quad (1)$$

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Lemma

If F is associative, idempotent and monotone (in each variable) then it is monotone increasing (in each variable).

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This property was introduced by Bernard De Baets et al. They called a function *id-symmetric* if the 'extended' graph is symmetric w.r.t. the diagonal.

Characterization of associative, idempotent, monotone increasing functions with neutral element

Theorem (Martín-Mayor-Torrens,'03; Ruiz-Aguilera-Torrens-De Baets-Fodor, '10)

Let $I \subseteq \mathbb{R}$ be a closed interval. The function $F : I^2 \to I$ is associative, monotone increasing, idempotent and has a neutral element $e \in I$

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Moreover, in this case F must be commutative except perhaps on the set of points (x,y) such that $y = g(x)$ and $x = g(y)$.

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- F_n has *neutral element* e if for every $x \in X$ and $1 \leq i \leq n$ we have $F(e, \ldots, e, x, e, \ldots, e) = x$, where x is substituted into the i'th coordinate.

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An important construction: Let (X, F_2) be a binary semigroup and

$$F_n(x_1, \ldots, x_n) := \underbrace{F_2 \circ F_2 \circ \ldots \circ F_2}_{n-1}(x_1, \ldots, x_n) \\ = F_2(x_1, F_2(x_2, \ldots, F_2(x_{n-1}, x_n))).$$

Then F_n is n-associative.

Dudek-Mukhin's results

Theorem (Dudek-Mukhin, 2006)

If an n-associative F_n has a neutral element e, then F_n is derived from an associative function $F_2 : X^2 \to X$ where $F_2(a,b) = F_n(a,e,\ldots,e,b)$. (i.e: $F_n = \underbrace{F_2 \circ \cdots \circ F_2}_{n-1}$.)

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By the definition of F_2, the element e is also a neutral element of F_2.

Monotonicity

Let X be a partially ordered set. A function $F_n : X^n \to X$ is called *monotone in the i'th variable* if for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ the 1-variable functions $f_i(x) := F_n(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ are all order-preserving or all are order-reversing.

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Lemma

Let $(X, \leq)$ be a partially ordered set, (X, F_2) be a semigroup and F_n be derived from F_2. If F_2 has any of the following properties

1. *monotonicity,*
2. *idempotent,*
3. *has a neutral element,*

then F_n has also.

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By a previous lemma, if F_2 is monotone, idempotent, associative, then F_2 is monotone increasing in each variable. Easily, F_n is also monotone increasing in each variable.

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Example

Let $G_n(x_1, \ldots, x_n) = \sum_{i=1}^{n}(-1)^i x_i$.

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By a previous lemma, if F_2 is monotone, idempotent, associative, then F_2 is monotone increasing in each variable. Easily, F_n is also monotone increasing in each variable.

Example

Let $G_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i x_i$. Then G_n is n-associative and is not derived from a binary function.

Counter example on modular lattices

Example

Let $k \geq 3$ and $X = \{m, M\} \cup Z_{k-1}$, where Z_{k-1} is the cyclic group of order $k-1$.

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$$F_2(x,y) = \begin{cases} M, & \text{if } x = M \text{ or } y = M \\ m, & \text{if } x = m \text{ or } y = m \text{ and } x, y < M \\ xy, & \text{if } x, y \in Z_{k-1}. \end{cases}$$

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Then F_2 is associative and monotone increasing but non-idempotent. The identity element e of Z_{k-1} is the neutral element of (X, F_2). The function F_k is k-associative, monotone and e is the neutral element and idempotent.

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Then F_2 is associative and monotone increasing but non-idempotent. The identity element e of Z_{k-1} is the neutral element of (X, F_2). The function F_k is k-associative, monotone and e is the neutral element and idempotent.

The question is open for distributive lattices.

Generalization of Czogała-Drewniak theorem

We denote $\min(a_1, \ldots, a_n)$ and $\max(a_1, \ldots, a_n)$ by $\wedge_{i=1}^{n} a_i$ and $\vee_{i=1}^{n} a_i$, respectively.

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A function $F_2 : [0, 1]^2 \to [0, 1]$ is a *uninorm*, if it is associative, commutative, monotone and have a neutral element.

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A function $F_2 : [0,1]^2 \to [0,1]$ is a *uninorm*, if it is associative, commutative, monotone and have a neutral element. Now we introduce *n-ary uninorms*, which are n-associative, commutative, monotone functions with neutral elements.

Theorem

The function $U_n : [0,1]^n \to [0,1]$ *is an idempotent n-ary uninorm on* $[0,1]$ *with neutral element* $e \in [0,1]$ *if and only if there exists a decreasing id-symmetric function* $g : [0,1] \to [0,1]$ *with fixed point* e *such that*

$$U_n(a_1, \ldots, a_n) = \begin{cases} \wedge_{i=1}^{n} a_i, & \textit{if } \wedge_{i=1}^{n} a_i < g(\vee_{i=1}^{n} a_i) \\ & \textit{or } \vee_{i=1}^{n} a_i < g(\wedge_{i=1}^{n} a_i) \\ \vee_{i=1}^{n} a_i & \textit{if } \wedge_{i=1}^{n} a_i > g(\vee_{i=1}^{n} a_i) \\ & \textit{or } \vee_{i=1}^{n} a_i > g(\wedge_{i=1}^{n} a_i) \\ \wedge_{i=1}^{n} a_i \textit{ or } \vee_{i=1}^{n} a_i & \textit{if } g(\vee_{i=1}^{n} a_i) = \wedge_{i=1}^{n} a_i \\ & \textit{and } g(\wedge_{i=1}^{n} a_i) = \vee_{i=1}^{n} a_i \end{cases}$$

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Moreover, if $g(\vee_{i=1}^n a_i) = \wedge_{i=1}^n a_i$ *and* $g(\wedge_{i=1}^n a_i) = \vee_{i=1}^n a_i$*, then the value of* $U_n(a_1, \dots, a_n)$ *can be chosen to be* $\wedge_{i=1}^n a_i$ *or* $\vee_{i=1}^n a_i$ *arbitrarily and independently from other points.*

Further developments

A function $F_n : X^n \to X$ is called *conservative* if for every $x_1, \ldots, x_n \in X$

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Corollary

Let I be a closed interval. For $n \geq 2$ let $F_n : I^n \to I$ be n-associative, monotone increasing, idempotent n-ary semigroup and has a neutral element $e \in I$. Then F_n is conservative.

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The conservative, symmetric, monotone increasing n-ary semigroups have been analyzed in a recent paper with Jean-Luc Marichal and Jimmy Devillet.

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Thank you for your kind attention!