Recent results on conservative and symmetric n-ary semigroups

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n-ary semigroups

Definition

Let X be an arbitrary set. An operation $F\colon X^n \to X$ is said to be (n-)associative if

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\begin{aligned}
& F(x_1,\ldots,x_{i-1},F(x_i,\ldots,x_{i+n-1}),x_{i+n},\ldots,x_{2n-1}) \\
= \quad & F(x_1,\ldots,x_i,F(x_{i+1},\ldots,x_{i+n}),x_{i+n+1},\ldots,x_{2n-1})
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$$F(F(x,y),z) = F(x,F(y,z))$$

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For a nonempty set X and an associative function $F : X^n \to X$ the pair (X,F) is called n-ary semigroup.

Other important definitions

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Neutral element

Definition

Let $F: X^n \to X$ be an operation.

- *An element $e \in X$ is said to be a neutral element of F if*

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Example

$F(x_1, x_2, x_3) \equiv x_1 + x_2 + x_3 \pmod 2$ on $X = \mathbb{Z}_2$.

Connectivity and neutral element

Example (More generally)

$F(x_1, \ldots, x_n) \equiv x_1 + \cdots + x_n \pmod{(n-1)}$ on $X = \mathbb{Z}_{n-1}$ $(n \geq 3)$.

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Corollary

Any conservative operation $F\colon X^n \to X$ has at most one isolated point.

Proposition

Let $F\colon X^n \to X$ be a conservative operation and let $e \in X$. If $(n \cdot e)$ is isolated for F, then e is a neutral element.

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Counter example

Let $X = \{a, b, e\}$ and let $F: X^3 \to X$ be defined as

$$F(x, y, z) = \begin{cases} a, & \text{if there are more } a\text{'s than } b\text{'s among } x, y, z, \\ b, & \text{if there are more } b\text{'s than } a\text{'s among } x, y, z, \\ e, & \text{otherwise.} \end{cases}$$

The operation F is conservative and has e as the neutral element. However, we have $F(e, e, e) = F(a, b, e)$ and hence the point (e, e, e) is not isolated for F.

Why are neutral elements so important?

Definition

Let $F: X^n \to X$ and $H: X^2 \to X$ be associative operations. F is said to be derived from (or reducible to) H if $F(x_1, \ldots, x_n) = x_1 \circ \cdots \circ x_n$ for all $x_1, \ldots, x_n \in X$, where $x \circ y = H(x, y)$.

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Theorem (Dudek-Mukhin)

Let X be a nonempty set. A function $F : X^n \to X$ can be derived from an associative function $H : X^2 \to X$ if and only if F has a neutral element or there can be adjoin a neutral element to X for F.

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Corollary

If $F: X^n \to X$ is associative and has a neutral element $e \in X$, then F is derived from the associative operation $H: X^2 \to X$ defined by $H(x, y) = F(x, (n-2) \cdot e, y)$.

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In this presentation we extend the investigation:

Proposition

Let X be a chain and $F : X^n \to X$ be an associative, reflexive, nondecreasing function that has a neutral element. Then F is conservative.

Proposition (Martin-Mayor-Torrens, Couceiro-Devillet-Marichal)

Let X be a chain. If $G\colon X^2 \to X$ is conservative, symmetric, and nondecreasing, then it is associative.

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- (ii) $F((n-1)\cdot x, y) = F(x, (n-1)\cdot y)$ *for all* $x, y \in X$.
- (iii) *There exists a conservative and nondecreasing operation $G\colon X^2 \to X$ such that*

$$F(x_1, \ldots, x_n) = G(\textstyle\bigwedge_{i=1}^n x_i, \bigvee_{i=1}^n x_i), \qquad x_1, \ldots, x_n \in X. \qquad (1)$$

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Moreover, the operation G is unique, symmetric, and associative in assertion (iii).

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Back to the neutral element

Proposition

Let X be a chain and let $e \in X$. Assume that $F: X^n \to X$ is of the form (1)*, where $G: X^2 \to X$ is conservative, nondecreasing and symmetric. Then the following assertions are equivalent.*

(i) *e is a neutral element of F.*

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(iii) *The point (e, e) is isolated for G.*

(iv) *The point $(n \cdot e)$ is isolated for F.*

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Corollary

Let X and F as above. Then F has a neutral element iff there exists an isolated point for F.

The single-peaked ordering

Proposition (Ackerman)

Let X be a set and $H : X^2 \to X$ be an associative, conservative, symmetric function, then there exists a linear ordering $\leq$ on X such that F is the maximum operation on $(X, \leq)$.

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Corollary

An operation $F\colon X^n \to X$ is conservative, symmetric, associative, and derived from a conservative and associative operation $H\colon X^2 \to X$ iff there exists a linear ordering $\leq$ on X such that F is the maximum operation on $(X, \leq)$, i.e.,

$$F(x_1, \ldots, x_n) = x_1 \vee_{\leq} \cdots \vee_{\leq} x_n, \qquad x_1, \ldots, x_n \in X. \tag{2}$$

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Definition

In this case if $(X, \leq)$ is a chain, then we say that new ordering $\leq$ is *single-peaked* *w.r.t. $\leq$.*

Example

Consider the real operation $F\colon [0,1]^2 \to [0,1]$ defined as

$$F(x,y) = \begin{cases} x \vee y, & \text{if } x + y \geq 1, \\ x \wedge y, & \text{otherwise} . \end{cases} \quad (3)$$

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Denoting the single-peaked linear ordering on $[0,1]$ by $\preceq$, then

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So for every $x \in [0,1]$, there is no $y \in [0,1]$ such that $x \prec y \prec 1 - x$ or $1 - x \prec y \prec x$.

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Denoting the single-peaked linear ordering on $[0,1]$ by $\preceq$, then

$$x \preceq y \quad \Leftrightarrow \quad (y \leq x < 1 - y \ \text{ or } \ 1 - y \leq x \leq y), \qquad x, y \in [0,1].$$

So for every $x \in [0,1]$, there is no $y \in [0,1]$ such that $x \prec y \prec 1 - x$ or $1 - x \prec y \prec x$. From this observation one can show that the chain $([0,1], \preceq)$ cannot be embedded into the reals $(\mathbb{R}, \leq)$.

Thank you for your kind attention.

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