Uncertainty Quantification - Sensitivity Analysis / Biomechanics

Paul Hauseux, Jack S. Hale and Stéphane P.A. Bordas

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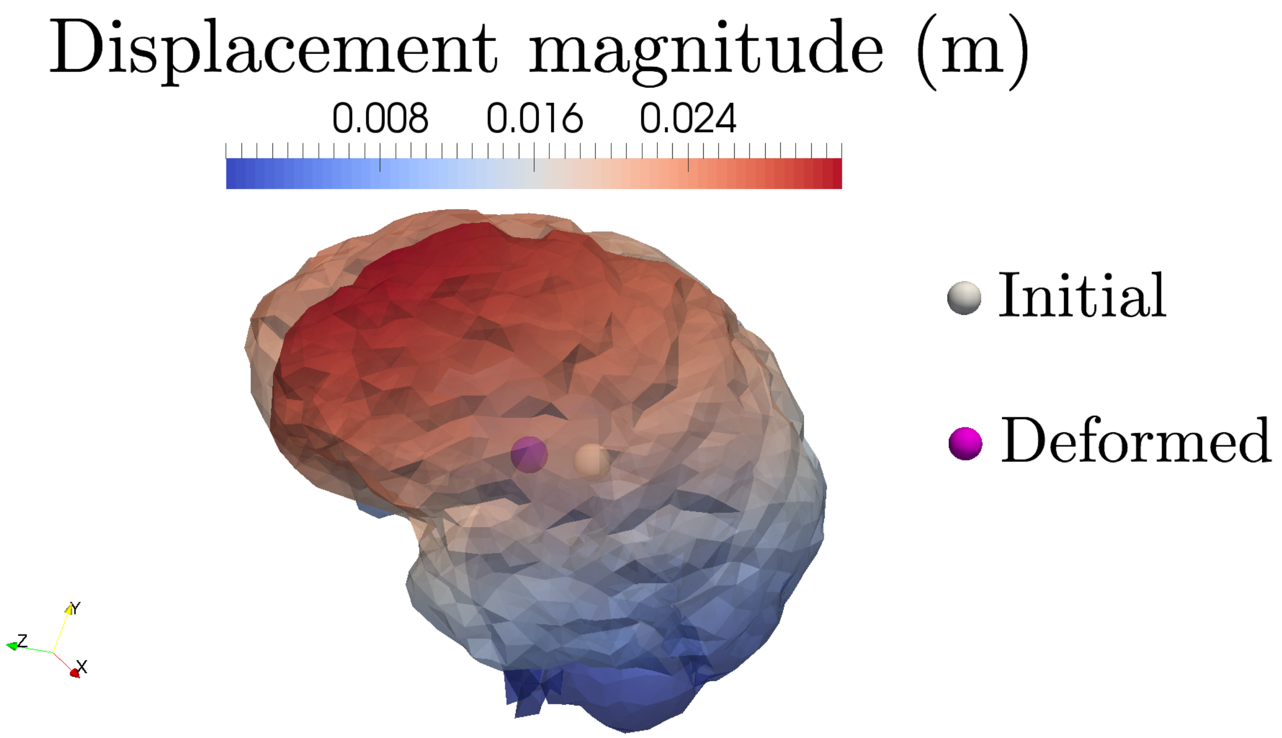
Context: Soft-tissue biomechanics simulations with uncertainty

- Uncertainty in parameters (material properties, loading, geometry, etc.) in biomechanics problems can influence the outcome of simulation results.



- **Objective: propagate and visualise this uncertainty** with *non* or *partially-intrusive* methods.

General framework

- Stochastic non-linear system: $F(\boldsymbol{u}, \boldsymbol{\omega}) = \mathbf{0}$
- Probability space: $(\Omega, \mathcal{F}, P)$
- Random parameters: $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_M)$

- Objective: provide statistical data for the solution of the problem.
- Integration (to determine the expected value of a quantity of interest):

$$E[\Psi(u(\omega))] = \int_\Omega \Psi(u(\omega)) dP(\omega)$$

Direct integration

Monte-Carlo method [Caflisch 1998]:

$$E[\Psi(u(\omega))] = \int_\Omega \Psi(u(\omega))dP(\omega) \simeq \sum_{z=1}^{Z} p_z \Psi(u(\omega_z))$$

Algorithm:

while $z < Z$:

- choose randomly ω_z.
- evaluate $\Psi(u(\omega_z))$.
- add the contribution to the sum.

Convergence

- Converge «in law»: 1% for 10000 realisations, slow but independent of the dimension !

$$||\mathbb{E}^{\mathrm{MC}}\left[\psi(\omega)\right] - \mathbb{E}\left[\psi(\omega)\right]||_{L^2(\Omega_p)} \sim \mathrm{N}(0,1)\sqrt{\frac{\mathbb{V}[\psi(\omega)]}{Z}}$$

- Necessity to improve the convergence.

Work done:

- Low discrepancy sequences (Sobol, Hamilton, …): quasi MCM [Caflisch 1998].
- Multi Level Monte-Carlo techniques [Giles 2015, Matthies 2008].
- MC methods by using sensitivity information (SD-MC) [Cao et. al 2004, Liu et al. 2013].

MC methods by using sensitivity information

Estimator [Cao et. al 2004, Liu et al. 2013]:

$$\mathbb{E}_1^{\mathrm{SD-MC}}[\psi(\omega)] := \frac{1}{Z}\sum_{z=1}^{Z}[\psi(\omega_z) - D[\psi(\bar{\omega})](\omega_z - \bar{\omega})]$$

This variance reduction method increases the accuracy of sampling methods. Here we only consider the case of the first-order sensitivity derivative enhanced Monte-Carlo method. By using sensitivity information computational workload can be reduced by one order of magnitude over commonly used schemes.

Main difficulty:

$$D[\psi(\bar{\omega})] \quad \textbf{??}$$

Numerical implementation

Implementation (DOLFIN/FEniCS) [Logg et al. 2012], advantages:

- UFL (Unified Form Language).
- Most existing FEM codes are not able to compute the tangent linear model and the sensitivity derivatives. However, it is possible with DOLFIN for a wide range of models with very little effort [Alnæs 2012, Farrell et al. 2013].
- Complex models with only few lines of Python code.

Parallel computing:

- Ipyparallel and mpi4py software tools to massively parallelise individual forward model runs across a cluster and to reduce the workload.

Python package for uncertainty quantification:

- Chaospy [Feinberg and Langtangen 2015] to provide different stochastic objects.

DOLFIN/FEniCS implementation: an example

- **Forward problem,** generalized Burgers equation with stochastic viscosity:

$$F(\nu, u; \tilde{u}) := \int_{\Omega_s} \nu \nabla u \cdot \nabla \tilde{u} - \frac{1}{2} \nabla u^2 \cdot \tilde{u} + \frac{1}{2} \nabla u \cdot \tilde{u} \, dx = 0 \quad \forall \tilde{u} \in H_0^1(\Omega_s)$$

```
nu_var = variable(Constant(omega))
F = nu_var*u_.dx(0)*u_t.dx(0)*dx + 0.5*u_.dx(0)*u_t*dx\
      - 0.5*(u_**2).dx(0)*u_t*dx
```

- The standard Newton method:

$$J(\nu, u^k; \delta u; \tilde{u}) = -F(\nu, u^k; \tilde{u}) \quad \forall \tilde{u} \in H_0^1(\Omega_s)$$

$$u^{k+1} = u^k + \delta u$$

```
J = derivative(F, u_, u)
solve(F == 0, u_, bcs, J=J)
```

DOLFIN/FEniCs implementation: an example

- **The tangent linear system:**

$$\underbrace{\frac{\partial F(\boldsymbol{u},\boldsymbol{\omega})}{\partial \boldsymbol{u}}}_{U\times U}\underbrace{\frac{d\boldsymbol{u}}{d\boldsymbol{\omega}}}_{U\times M}=-\underbrace{\frac{\partial F(\boldsymbol{u},\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}}_{U\times M}$$

U: size of the deterministic problem
M: number of random parameters

```
Fu = derivative(F, u, du)
Fd = - diff(F, omega)
dudomega = Function(V)
A, b = assemble_system(Fu, Fd, bcs=bcs)
solve(A, dudomega.vector(), b, “lu")
```

linear system to solve to evaluate du/dm !

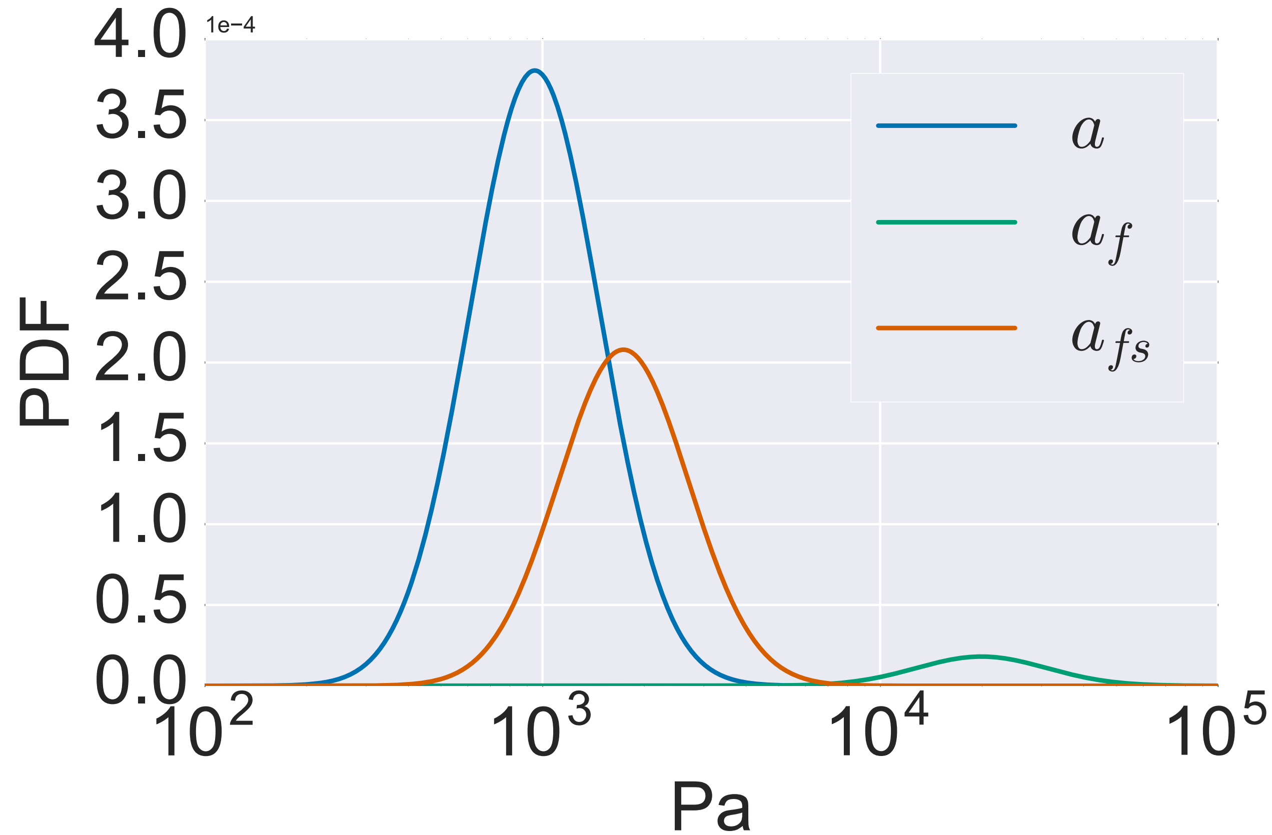
The complete implementation is only around 130 lines and the Docker image with the full software environment is included in: https://dx.doi.org/10.6084/m9.figshare.3561306 [Hauseux, P. and Hale, J.S. and Bordas, S. 2016]

Stochastic FE analysis of brain deformation

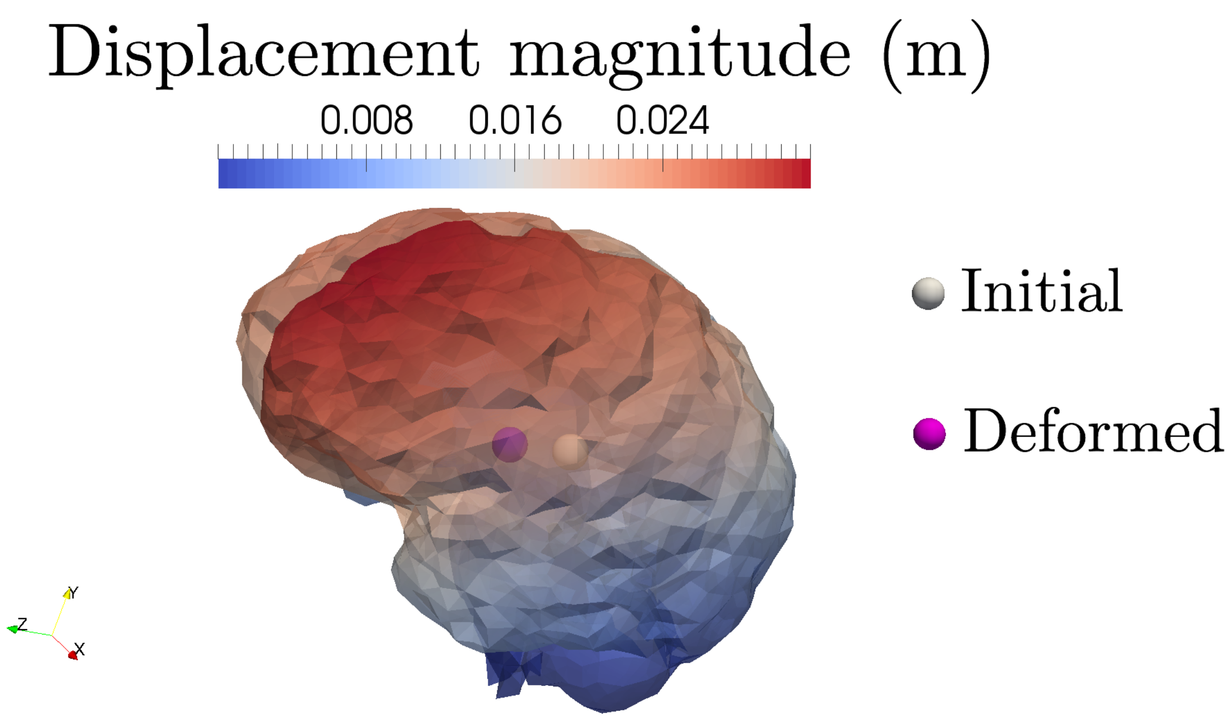
- Different hyper-elastic models implemented (Mooney-Rivlin, Neo-Hookean, Holzapfel and Ogden [Holzapfel and Ogden 2009]).
- Random variables/fields to model parameters [Adler 2007].
- Strain energy function for the Holzapfel and Ogden model:

$$\mathcal{W}_{iso} = \frac{a}{2b}\exp\left[b(\mathrm{I}_1 - 3)\right] + \sum_{i=f,s}\frac{a_i}{2b_i}\exp\left[b_i(\mathrm{I}_{4i}-1)^2\right] + \frac{a_{fs}}{2b_{fs}}\left(\exp\left[b_{fs}\mathrm{I}_{8fs}^2\right] - 1\right)$$

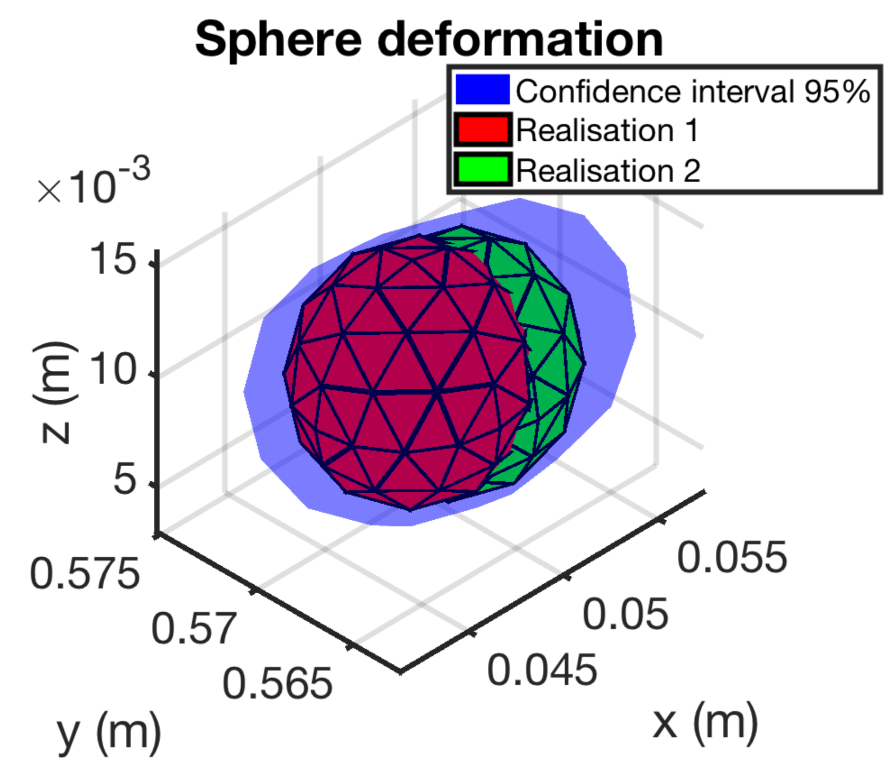
- for example 3RV:



Stochastic FE analysis of brain deformation
Numerical results (8 RV, Holzapfel model)



Brain deformation with random parameters
1 MC realisation.



Confidence interval 95%
MC simulations.

Numerical results: convergence

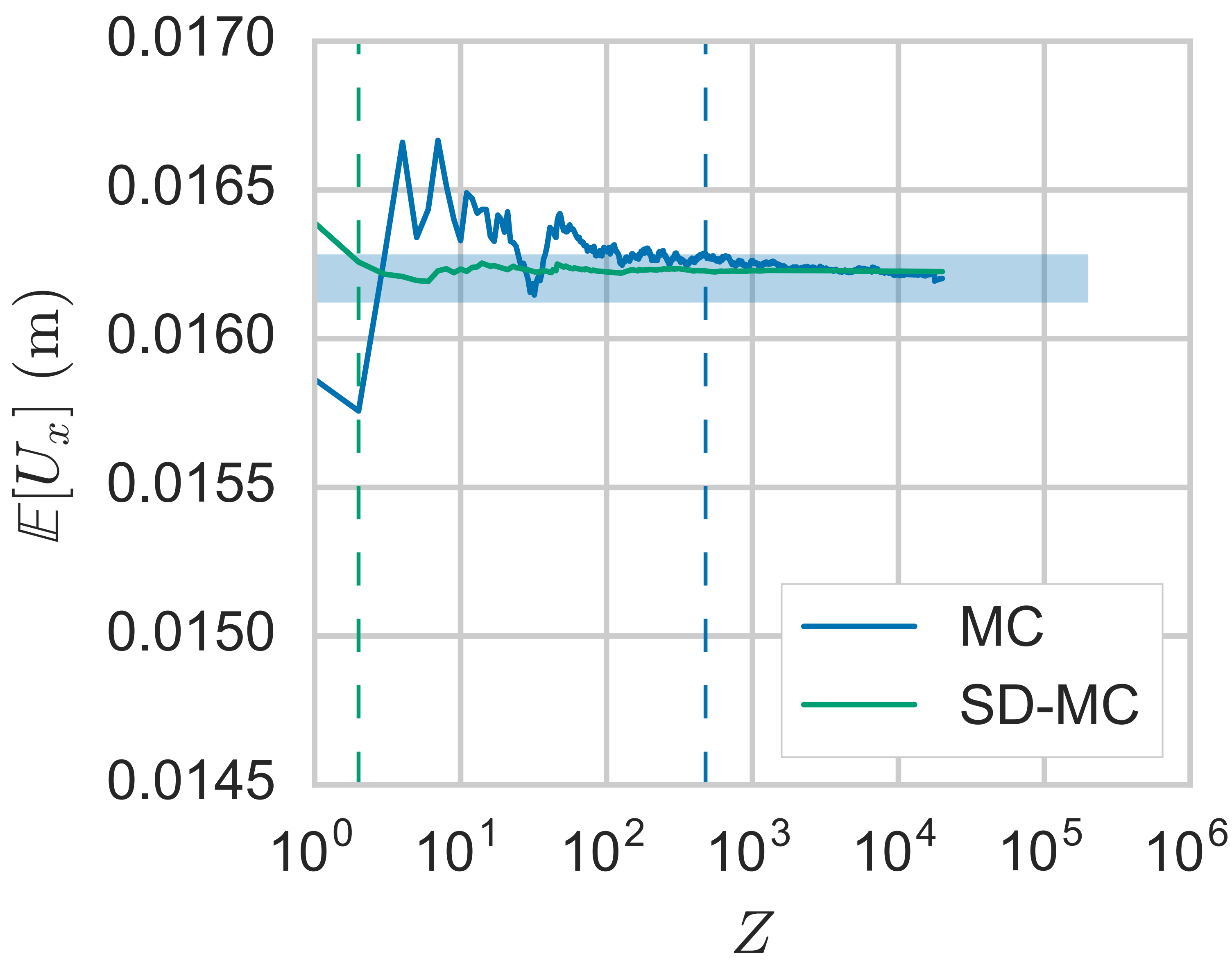
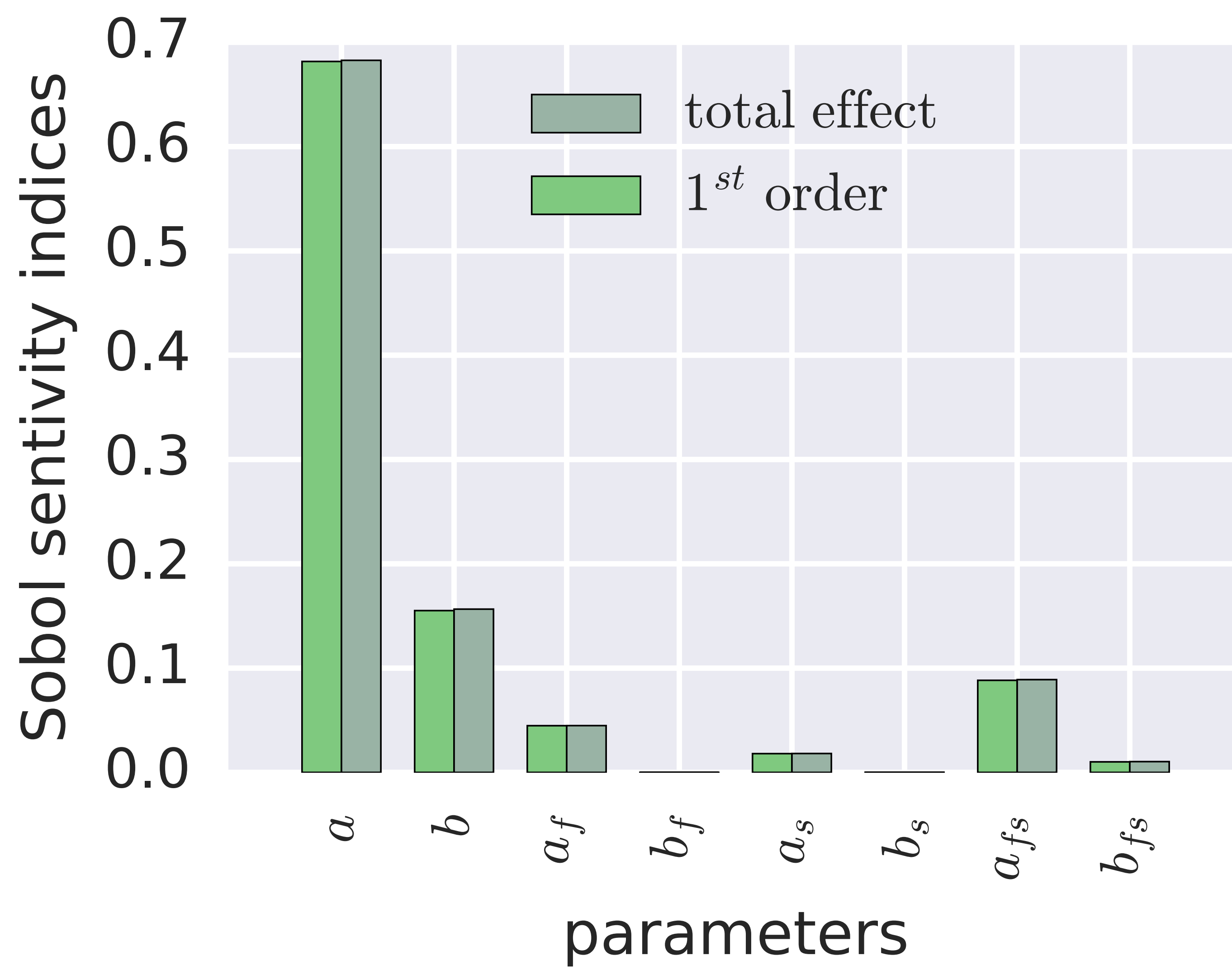


Fig. Center of the sphere: expected value of the displacement in the x direction as a function of Z.

Global sensitivity analysis

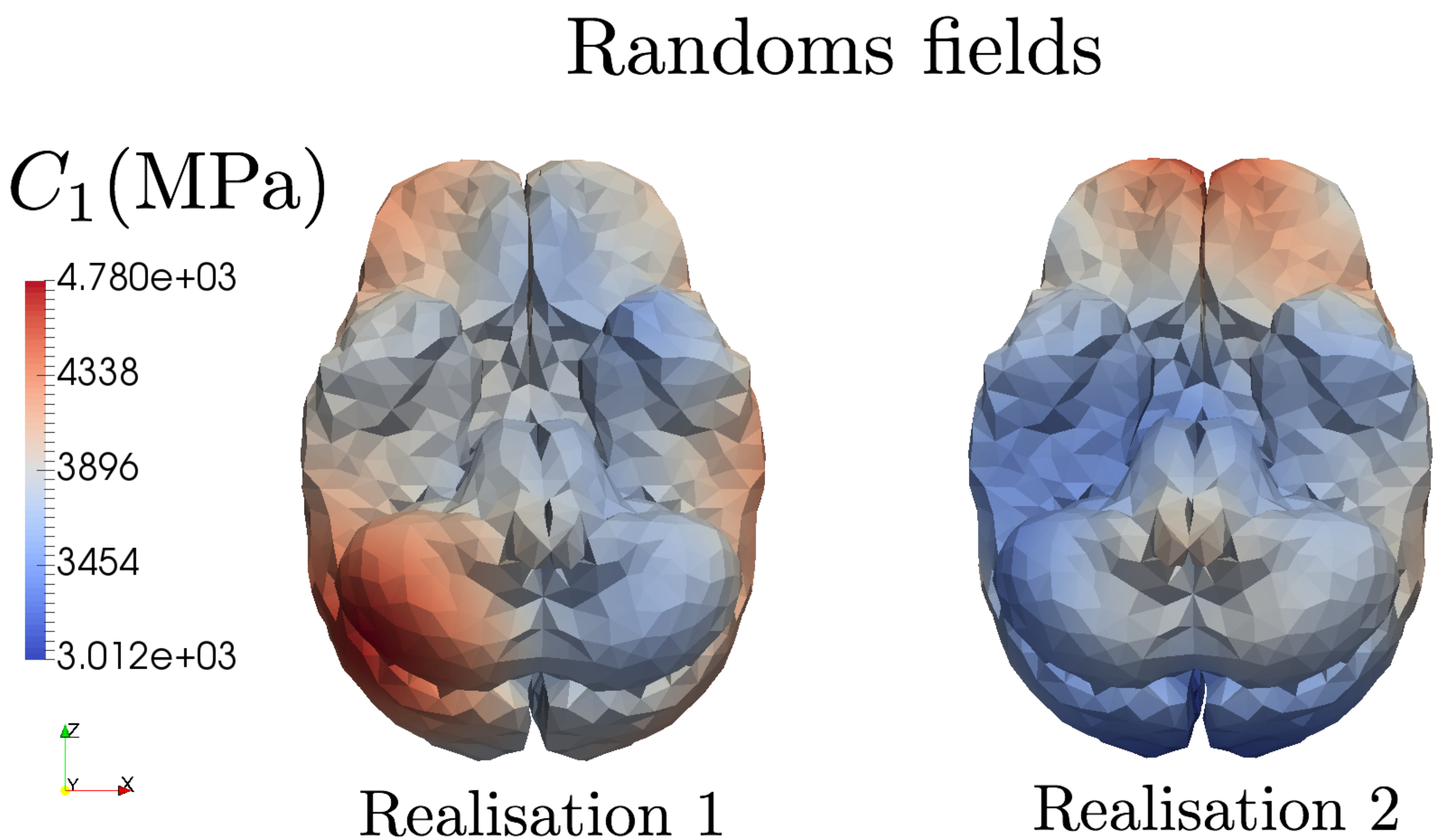
- Sobol sensitivity indices [Sobol 2015, Saltelli 2002]



Quantity of interest: displacement magnitude of the target.

Random Fields

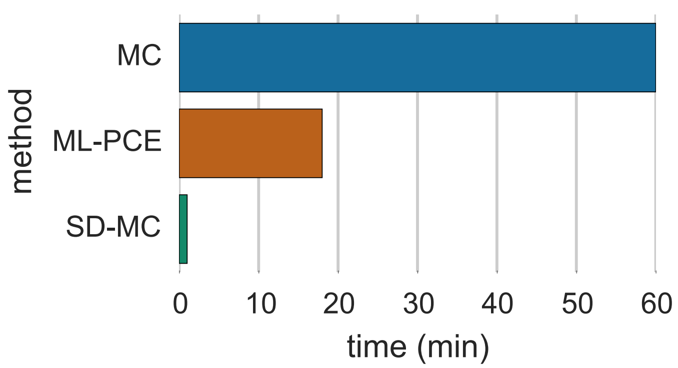
- Different methods: Karhunen–Loève expansion [Adler 2007], Fast Fourier transform [Nowak 2004].

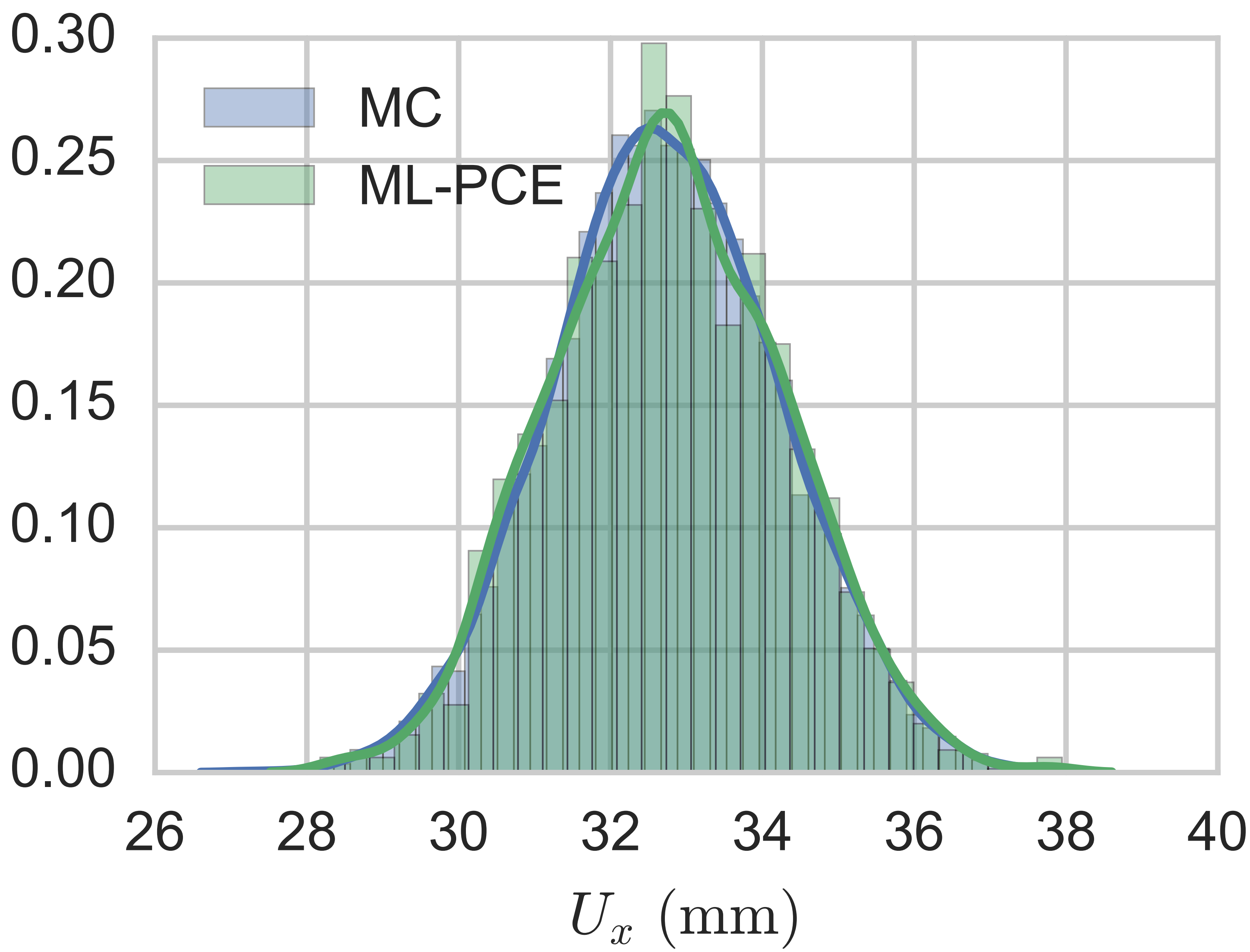


Two realisations of RF, with a log-normal distribution, for the parameter C_1 (in MPa).

Numerical results (Mooney-Rivlin solid)
ML Monte-Carlo technique: ML-PCE

- Monte Carlo method with use of Polynomial Chaos Expansion to improve the convergence [Matthies 2008, Hauseux 2016].





Histogram (MC and ML-PCE methods).

Future work for UQ

Stochastic modelling:

- Random fields generation with PDEs [Lindgren 2011]. Seeding white noise onto a mesh. Riesz representation theorem.

Multi Level Monte Carlo (MLMC) methods :

- By using Multi Level techniques [Giles 2015] the computational workload can be reduced by performing most simulations with low accuracy at a correspondingly low cost and few simulations at high accuracy and high cost.
- Combine MLMC with sensitivity derivatives (derives the discrete adjoint and tangent linear models).
- Implement various applications to illustrate the advantages of the method.
- Adjoint extension function space setting.

Malliavin calculus [Warren 2012].

Conclusion

Stochastic modelling:

- Random variables/fields to model parameters with a degree of uncertainty: application to brain deformation.

P*artially-intrusive* Monte-Carlo methods to propagate uncertainty:

- By using sensitivity information and multi-level methods with polynomial chaos expansion we demonstrate that computational workload can be reduced by one order of magnitude over commonly used schemes.

- Global and local sensitivity analysis.

Numerical implementation:

- Implementation: DOLFIN [Logg et al. 2012] and chaospy [Feinberg and Langtangen 2015].

- Non-linear hyper-elastic models (Mooney-Rivlin, Neo-Hookean, Holzapfel and Ogden [Holzapfel and Ogden 2009]).

- Ipyparallel and mpi4py to massively parallelise individual forward model runs accros a cluster.