Travaux mathématiques

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Special issue

dedicated to scientific contributions of the
Centre for Quantum Geometry of Moduli Spaces
QGM, Aarhus University, Denmark

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Editor

Martin Schlichenmaier

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UNIVERSITY OF LUXEMBOURG
Mathematics Research Unit, FSTC
Maison du Nombre
6, avenue de la Fonte
L-4364 Esch-sur-Alzette
Grand-Duchy of Luxembourg

Phone: (+352) 466 644 6236
Fax: (+352) 466 644 6237
Email: martin.schlichenmaier@uni.lu

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Address:
University of Luxembourg
Mathematics Research Unit, FSTC
Maison du Nombre
6, avenue de la Fonte
L-4364 Esch-sur-Alzette
Grand-Duchy of Luxembourg
Email: martin.schlichenmaier@uni.lu

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Quantum Geometry of Moduli Spaces

with applications to

TQFT and RNA folding

A volume dedicated to
Centre for Quantum Geometry of Moduli Spaces
QGM
at Aarhus University, Denmark

Edited by

Martin Schlichenmaier

Foreword

This volume of the *Travaux Mathématiques* is dedicated to the Centre for Quantum Geometry of Moduli Spaces (QGM) at Aarhus University, Denmark.

QGM was established in 2009 as a Center of Excellence funded by the Danish National Research Foundation. The research objective is to address fundamental mathematical problems at the interface between geometry and theoretical physics, in particular to further our understanding of quantum geometry of moduli spaces. One of its goals is to advance our knowledge about special kinds of quantum field theories such as Topological Quantum Field Theories and topological twisted theories. Furthermore, the mission of the centre is to explore applications of these theories and techniques in the understanding of macromolecular biology.

Based at Aarhus University and directed by Professor Jørgen Ellegaard Andersen, QGM hosts a strong team of high profile, internationally acclaimed researchers. With the continuous generation of ground-breaking results, the Centre together with its international collaborators are recognised throughout the mathematics community worldwide as one of the leading research institutions within its field of research.

The biannual *International School and Conference on Geometry and Quantization* (GEOQUANT) is now a well-established event in the scientific field. The main theme of the GEOQUANT conferences has strong ties to the research focus of QGM. This is further underlined by the fact that the next GEOQUANT conference will be hosted by QGM in August 2017. *Travaux Mathématiques* has several times before published volumes dedicated to the GEOQUANT conferences and intend to continue this tradition in connection with the upcoming GEOQUANT conference at QGM.

At the previous GEOQUANT school and conference at ICMAT in Madrid, Spain, 2015, the editor of this volume took, on the above background, the initiative to solicit articles for this volume of *Travaux Mathématiques* dedicated to the work of centre director Jørgen Ellegaard Andersen and other researchers at QGM. The intention is to give a snapshot of the very interesting spread of pure mathematics together with its application, which are being conducted currently by researchers at QGM.

Martin Schlichenmaier

Collection of Summaries

Jørgen Ellegaard Andersen and Niccolo Skovgård Poulsen
Coordinates for the Universal Moduli Space of Holomorphic Vector Bundles *(pp. 9–39)*

In this paper we provide two ways of constructing complex coordinates on the moduli space of pairs of a Riemann surface and a stable holomorphic vector bundle centered around any such pair. We compute the transformation between the coordinates to second order at the center of the coordinates. We conclude that they agree to second order, but not to third order at the center.

Jørgen Ellegaard Andersen and Jens-Jakob Kratmann Nissen
Asymptotic aspects of the Teichmüller TQFT *(pp. 41–95)*

We calculate the knot invariant coming from the Teichmüller TQFT introduced by Andersen and Kashaev a couple of years ago. Specifically we calculate the knot invariant for the complement of the knot 6_1 both in Andersen and Kashaev's original version of Teichmüller TQFT but also in their new formulation of the Teichmüller TQFT for the one-vertex H-triangulation of $(S^3, 6_1)$. We show that the two formulations give equivalent answers. Furthermore we apply a formal stationary phase analysis and arrive at the Andersen-Kashaev volume conjecture.

Furthermore we calculate the first examples of knot complements in the new formulation showing that the new formulation is equivalent to the original one in all the special cases considered.

Finally, we provide an explicit isomorphism between the Teichmüller TQFT representation of the mapping class group of a once punctured torus and a representation of this mapping class group on the space of Schwartz class functions on the real line.

Jørgen Ellegaard Andersen and Simone Marzioni
Level N Teichmüller TQFT and Complex Chern–Simons Theory *(pp. 97–146)*

In this manuscript we review the construction of the Teichmüller TQFT due to Andersen and Kashaev. We further upgrading it to a theory dependent on an extra odd integer N using results developed by Andersen and Kashaev in their work on complex quantum Chern-Simons theory. We also describe how this theory is related with quantum Chern–Simons Theory at level N with gauge group $\mathrm{PSL}(2, \mathbb{C})$.

Jørgen Ellegaard Andersen and William Elbæk Petersen
Construction of Modular Functors from Modular Categories *(pp. 147–211)*

In this paper we follow the constructions of Turaev's book, "Quantum invariants of knots and 3-manifolds" closely, but with small modifications, to construct a modular functor, in the sense of Kevin Walker, from any modular tensor category. We further show that this modular functor has duality and if the modular tensor category is unitary, then the resulting modular functor is also unitary. We further introduce the notion of a fundamental symplectic character for a modular tensor category. In the cases where such a character exists we show that compatibilities between the structures in a modular functor can be made strict in a certain sense. Finally we establish that the modular tensor categories which arise from quantum groups of simple Lie algebras all have natural fundamental symplectic characters.

Jørgen Ellegaard Andersen, Hiroyuki Fuji, Robert C. Penner, and Christian M. Reidys
The boundary length and point spectrum enumeration of partial chord diagrams using cut and join recursion *(pp. 213–232)*

We introduce the boundary length and point spectrum, as a joint generalization of the boundary length spectrum and boundary point spectrum introduced by Alexeev, Andersen, Penner and Zograf. We establish by cut-and-join methods that the number of partial chord diagrams filtered by the boundary length and point spectrum satisfies a recursion relation, which combined with an initial condition determines these numbers uniquely. This recursion relation is equivalent to a second order, non-linear, algebraic partial differential equation for the generating function of the numbers of partial chord diagrams filtered by the boundary length and point spectrum.

Jørgen Ellegaard Andersen, Hiroyuki Fuji, Masahide Manabe, Robert C. Penner, and Piotr Sułkowski
Partial Chord Diagrams and Matrix Models *(pp. 233–283)*

In this article, the enumeration of partial chord diagrams is discussed via matrix model techniques. In addition to the basic data such as the number of backbones and chords, we also consider the Euler characteristic, the backbone spectrum, the boundary point spectrum, and the boundary length spectrum. Furthermore, we consider the boundary length and point spectrum that unifies the last two types of spectra. We introduce matrix models that encode generating functions of partial chord diagrams filtered by each of these spectra. Using these matrix models, we derive partial differential equations – obtained independently

by cut-and-join arguments in an earlier work – for the corresponding generating functions.

Jørgen Ellegaard Andersen, Hiroyuki Fuji, Masahide Manabe, Robert C. Penner, and Piotr Sułkowski
Enumeration of Chord Diagrams via Topological Recursion and Quantum Curve Techniques *(pp. 285–323)*

In this paper we consider the enumeration of orientable and non-orientable chord diagrams. We show that this enumeration is encoded in appropriate expectation values of the β-deformed Gaussian and RNA matrix models. We evaluate these expectation values by means of the β-deformed topological recursion, and – independently – using properties of quantum curves. We show that both these methods provide efficient and systematic algorithms for counting of chord diagrams with a given genus, number of backbones and number of chords.

Travaux mathématiques, Volume 25 (2017), 9–39, © by the authors

Coordinates for the Universal Moduli Space of Holomorphic Vector Bundles

by Jørgen Ellegaard Andersen and Niccolo Skovgård Poulsen[1]

Abstract

In this paper we provide two ways of constructing complex coordinates on the moduli space of pairs of a Riemann surface and a stable holomorphic vector bundle centred around any such pair. We compute the transformation between the coordinates to second order at the center of the coordinates. We conclude that they agree to second order, but not to third order at the center.

1 Introduction

Fix $g, n > 1$ to be integers and let $d \in \{0, \dots n-1\}$. Let Σ be a closed oriented surface of genus g. Consider the universal moduli space $\mathcal{M}$ consisting of equivalence classes of pairs $(\phi : \Sigma \to X, E)$ where X is a Riemann Surface of genus g, $\phi : \Sigma \to X$ is a diffeomorphism and E is a semi-stable bundle over X of rank n and degree d. Let $\mathcal{M}^s$ be the open dense subset of $\mathcal{M}$ consisting of equivalence classes of such pairs $(\phi : \Sigma \to X, E)$ with E stable. The main objective of this paper is to provide coordinates in a neighbourhood of the equivalence class of any pair $(\phi : \Sigma \to X, E)$ in $\mathcal{M}^s$. There is an obvious forgetful map

$$\pi_{\mathcal{T}} : \mathcal{M} \to \mathcal{T}$$

where $\mathcal{T}$ is the Teichmüller space of Σ, whose fiber over $[\phi : \Sigma \to X] \in \mathcal{T}$ is the moduli space of semi-stable bundles for that Riemann surface structure on Σ. Let $\pi^s_{\mathcal{T}} : \mathcal{M}^s \to \mathcal{T}$ denote the restriction of $\pi_{\mathcal{T}}$ to $\mathcal{M}^s$, and we denote a point $[\phi : \Sigma \to X]$ in $\mathcal{T}$ by σ.

We recall that locally around any $\sigma \in \mathcal{T}$ there are the Bers coordinates [AhB]. Further, for any point $[E]$ in some fiber $(\pi^s_{\mathcal{T}})^{-1}(\sigma)$ we have the Zograf and Takhtadzhyan coordinates near $[E]$ along that fiber of $\pi_{\mathcal{T}}$ [TZ1].

In order to describe our coordinates on $\mathcal{M}^s$ we recall the Narasimhan-Seshadri theorem. Let $\tilde{\pi}_1(\Sigma)$ be the universal central $\mathbb{Z}/n\mathbb{Z}$ extension of $\pi_1(\Sigma)$ and let M

[1]Work supported in part by the center of excellence grant “Center for Quantum Geometry of Moduli Spaces” from the Danish National Research Foundation (DNRF95).

be the moduli space of representations of $\tilde{\pi}_1(\Sigma)$ to $\mathbf{U}(n)$ such that the central generator goes to $e^{2\pi i d/n}\mathrm{Id}$. Let M' be the subset of M consisting of equivalence classes of irreducible representations. The Narasimhan-Seshadri theorem gives us a diffeomorphism

$$\Psi : \mathcal{T} \times M' \to \mathcal{M}^s$$

which we use to induce a complex structure on $\mathcal{T} \times M'$ such that Ψ is complex analytic. We will now represent a point in $\mathcal{T}$ by a representation

$$\rho_0 : \tilde{\pi}_1(\Sigma) \to \mathbf{PSL}(2)$$

and denote the corresponding point in Teichmüller space by X_{ρ_0}. Here ρ_0 is really a representation of $\pi_1(\Sigma)$ pulled back to $\tilde{\pi}_1(\Sigma)$. A point in M' will be represented by a representation

$$\rho_E : \tilde{\pi}_1(\Sigma) \to \mathbf{U}(n)$$

which corresponds to the stable holomorphic bundle E on X_{ρ_0}.

We build complex analytic coordinates around any such $(\rho_0, \rho_E) \in \mathcal{T} \times M'$ by providing a complex analytic isomorphism from a small neighbourhood around 0 in the vector space $H^{0,1}(X_{\rho_0}, TX_{\rho_0}) \oplus H^{0,1}(X_{\rho_0}, \mathrm{End}E)$ to a small open subset containing (ρ_0, ρ_E) in $\mathcal{T} \times M'$.

The coordinates are given by constructing a certain family

$$\Phi^{\mu\oplus\nu} : \mathbf{H} \times \mathbf{GL}(n, \mathbb{C}) \to \mathbf{H} \times \mathbf{GL}(n, \mathbb{C}) \tag{1.1}$$

of bundle maps of the trivial $\mathbf{GL}(n, \mathbb{C})$-principal bundles over $\mathbf{H}$ indexed by pairs of sufficiently small elements

$$\mu \oplus \nu \in H^{0,1}(X_{\rho_0}, TX_{\rho_0}) \oplus H^{0,1}(X_{\rho_0}, \mathrm{End}E).$$

These bundle maps will uniquely determine representations $(\rho^\mu, \rho_E^{\mu\oplus\nu}) \in \mathcal{T} \times M'$ such that

$$\rho^\mu(\gamma) \times \rho_E^{\mu\oplus\nu}(\gamma) = \Phi^{\mu\oplus\nu} \circ (\rho_0(\gamma) \times \rho_E(\gamma)) \circ (\Phi^{\mu\oplus\nu})^{-1} \tag{1.2}$$

for all $\gamma \in \tilde{\pi}_1(X)$ by the following theorem. Pick a base point $z_0 \in \mathbf{H}$ and let $p_{\mathbf{GL}(n,\mathbb{C})}$ be the projection onto $\mathbf{GL}(n, \mathbb{C})$ of the trivial bundle $\mathbf{H} \times \mathbf{GL}(n, \mathbb{C})$.

Theorem 1.1. *For all sufficiently small* $\mu\oplus\nu \in H^{0,1}(X_{\rho_0}, TX_{\rho_0}) \oplus H^{0,1}(X_{\rho_0}, \mathrm{End}E)$ *there exist a unique bundle map* $\Phi^{\mu\oplus\nu}$ *such that*

1. $\Phi^{\mu\oplus\nu}$ *solves*

$$\bar{\partial}_{\mathbf{H}} \Phi^{\mu\oplus\nu} = \partial \Phi^{\mu\oplus\nu}(\mu \oplus \nu), \tag{1.3}$$

where ν *is considered a left-invariant vector field on* $\mathbf{GL}(n, \mathbb{C})$ *at each point in* $\mathbf{H}$.

2. *The base map extends to the boundary of* $\mathbf{H}$ *and fixes* $0, 1$ *and* ∞.
3. *The pair of representations* $(\rho^{\mu}, \rho_E^{\mu\oplus\nu})$ *defined by equation (1.2) represents a point in* $\mathcal{T} \times M'$.
4. $p_{\mathbf{GL}(n,\mathbb{C})}(\Phi^{\mu\oplus\nu}(z_0, e))$ *has determinant* 1 *and is positive definite.*

From this theorem we easily derive our main theorem of this paper.

Theorem 1.2. *Mapping all sufficiently small pairs*

$$\mu \oplus \nu \in H^{0,1}(X_\rho, TX_\rho) \oplus H^{0,1}(X_\rho, EndE)$$

to

$$(\rho^{\mu}, \rho_E^{\mu\oplus\nu}) \in \mathcal{T} \times M'$$

provides local analytic coordinates centered at $(\rho_0, \rho_E) \in \mathcal{T} \times M'$.

Our second coordinate construction provides fibered coordinates, which along $\mathcal{T}$ uses Bers' coordinates, [AhB], and which uses Zograf and Takhtadzhyan's coordinates [TZ1] along the fibers. We refer to section 4 for the precise description of these fibered coordinates.

Finally, we compare the two sets of coordinates by computing the infinitesimal transformation of the coordinates up to second order at the center of both coordinates.

Theorem 1.3. *The fibered coordinates and the universal coordinates agree to second order, but not the third order at the center of the coordinates.*

We refer to Theorem 5.5, for the details of how the two set of coordinates differ at third-order.

With these new coordinates we get a new tool to analyse the metric and the curvature of the moduli space. Here we have taken the first step in understanding the curvature by calculating the second variation of the metric in local coordinates, at the center point. We intend to return to the full calculation of the curvature in these coordinates in a future publication.

Remark 1.4. If we perform our construction using elements of $\mathcal{H}^{0,1}(X, (\mathrm{End}_0 E))$ where $(\mathrm{End}_0 E)$ is the subspace of traceless endomorphisms, we get coordinates on the universal $\mathbf{SU}(n)$ moduli space in a completely similar way.

The coordinate construction presented in this paper has been used and modified by the authors of this paper in [AP1], so as to give Kähler coordinates on the universal moduli space. In these modified coordinates, it was possible to verify an explicit expression for the Ricci potential for universal moduli space. Further in the paper [AP2], the result of this paper and that of [AP1] was used to compute explicitly the curvature of the Hitchin connection. This is of current great interest, since the Hitchin connection is know to be equivalent to the TUY connection [TUY] by the work of Laszlo [L] and by the work of Andersen and Ueno [AU1, AU2, AU3, AU4] the modular functor which underlies the conformal field theories studied by Tsuchiya, Ueno and Yamada [TUY] is equivalent to the Witten-Reshetikhin-Turaev TQFT [T, RT1, RT2]. This connection has been exploited significantly by Andersen and collaborators in a series of papers to obtain deep results about the WRT-TQFT [A1, A2, AGr1, AH, AMU, A3, A4, A5, AGa1, AB1, A6, AGL, AHi, A7].

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We would like to thank Peter Zograf for many interesting discussions.

2 The Complex Structure on $\mathcal{M}^s$ from a Differential Geometric Perspective

Recall that we endow the space $\mathcal{T} \times M$ with the structure of a complex manifold by using the Narasimhan-Seshadri theorem to provide us with the diffeomorphism

$$\Psi : \mathcal{T} \times M' \to \mathcal{M}^s$$

and then declaring it to be complex analytic. There is the following alternative construction of this complex manifold structure.

Recall the general setting of [AGL] in the context of geometric quantization and the Hitchin connection, namely $\tilde{\mathcal{T}}$ is a general complex manifold and $(\tilde{M}, \omega)$ is a general symplectic manifold. In that paper a construction of a complex structure on $\tilde{\mathcal{T}} \times \tilde{M}$ is provided via the following proposition. But first we need the following definition.

Definition 2.1. A family of Kähler structures on $(\tilde{M}, \omega)$ parametrized by $\tilde{\mathcal{T}}$ is called holomorphic if it satisfies:

$$V'[J] = V[J]' \quad V''[J] = V[J]''$$

for all vector fields V on $\tilde{\mathcal{T}}$. Here the single prime on V denotes projection on the $(1,0)$-part and the double prime on V denotes projection on the $(0,1)$-part of the vector field V. Further $V[J] \in T_\sigma \otimes (\bar{T}_\sigma)^* \oplus \bar{T}_\sigma \otimes T_\sigma^*$ and we let $V[J]'$ denote the projection on the first, and $V[J]''$ the projection on the second factor.

Proposition 2.2 ([AGL, Proposition 6.2])**.** *The family J_σ of Kähler structures on $\tilde{M}$ is holomorphic, if and only if the almost complex structure J given by*

$$J(V \oplus X) = IV \oplus J_\sigma X, \quad \forall (\sigma, [\rho_E]) \in \tilde{\mathcal{T}} \times \tilde{M}, \forall (V, X) \in T_{\sigma, [\rho_E]}(\tilde{\mathcal{T}} \times \tilde{M}),$$

is integrable.

The family of complex structures on M' considered in [H] see also [AGL], [A6] and [AGa1], given by the Hodge star, $-\star_\sigma$, $\sigma \in \mathcal{T}$, fulfills the requirements of the proposition with respect to the Atiyah-Bott symplectic form ω on M'. We will denote the complex structure which $\mathcal{T} \times M'$ has by J.

Proposition 2.3. *We have that the map*

$$\Psi : (\mathcal{T} \times M', J) \to \mathcal{M}^s$$

is complex analytic, e.g. J is in fact the complex analytic structure this space gets from the Narasimhan-Seshadri diffeomorphism Ψ.

Proof. In order to understand the complex structure of $\mathcal{T} \times M'$ from the algebraic geometric perspective we want to construct holomorphic horizontal sections of $\mathcal{T} \times M' \to \mathcal{T}$. We will use the universal property of the space of holomorphic bundles to show that the sections $\mathcal{T} \to \mathcal{T} \times \{\rho_E\} \subset \mathcal{T} \times M'$ are holomorphic for all $[\rho_E]$ in M'.

Our first objective is to construct a holomorphic family of vector bundles over Teichmüller space, where each bundle corresponds to the same unitary representation of $\tilde{\pi}_1(\Sigma)$. We start from the universal curve $\mathcal{T} \times \Sigma$ and its universal cover $\mathcal{T} \times \tilde{\Sigma}$. Both of these spaces are complex analytic, and we get the universal curve $\mathcal{T} \times \Sigma$ as the quotient of $\mathcal{T} \times \tilde{\Sigma}$ by the holomorphic $\pi_1(\Sigma)$ action.

This allows us to construct the vector bundles over $\mathcal{T}$ as the sheaf theoretic quotient of

$$\mathcal{T} \times \tilde{\Sigma} \times \mathbb{C}^n$$

by the $\tilde{\pi}_1(\Sigma)$ action, given by the $\pi_1(\Sigma)$-action on $\mathcal{T} \times \tilde{\Sigma}$, and the unitary action on $\mathbb{C}^n$ given by our fixed representation $\rho_E : \tilde{\pi}_1(\Sigma) \to \mathbf{U}(n)$ (see [MS] for details on this construction, where we simply just compose representations with the natural quotient map from $\pi_1(\Sigma - \{p\})$ to $\tilde{\pi}_1(\Sigma)$ to match up the setting of this paper to a special case of the setting in [MS]). The action is of course holomorphic, and so the quotient (fiberwise invariant sections over $\mathcal{T}$) is a family of Riemann surfaces with a holomorphic vector bundle over it of rank n and degree d. The universal property of $\mathcal{M}^s$ implies that this family therefore induced a holomorphic section

$$\iota_{\rho_E} : \mathcal{T} \to \mathcal{M}^s$$

by the universality of the moduli space $\mathcal{M}^s$. This shows that the horizontal sections are holomorphic submanifolds, and so the tangent space must split at every point as $I \oplus J_\sigma$. Here J_σ must be $-\star_\sigma$ since it comes from the structure of the fibers. $\square$

The conclusion is, that the algebraic complex structure on the moduli space of pairs of a Riemann surface and a holomorphic vector bundle over it and the complex structure from [AGL] on $\mathcal{T} \times M'$ are the same.

3 Coordinates for the Universal Moduli Space of Holomorphic Vector Bundles

In this section we prove Theorem 1.1.

We will need the composition of the map $\Phi^{\mu\oplus\nu}$ with the projection on each of the two factors, which we denote as follows:

$$\Phi_1^{\mu\oplus\nu} : \mathbf{H} \times \mathbf{GL}(n,\mathbb{C}) \to \mathbf{H},$$
$$\Phi_2^{\mu\oplus\nu} : \mathbf{H} \times \mathbf{GL}(n,\mathbb{C}) \to \mathbf{GL}(n,\mathbb{C}).$$

In fact $\Phi_1^{\mu\oplus\nu}$ is the projection onto $\mathbf{H}$ followed by the induced map on the base by (3.3) below.

The equation (1.3) is equivalent to the following two equations on $\Phi_i^{\mu\oplus\nu}$:

$$\bar\partial_{\mathbf{H}}\Phi_1^{\mu\oplus\nu}(z,g) = \mu\partial_{\mathbf{H}}\Phi_1^{\mu\oplus\nu}(z,g) \tag{3.1}$$
$$\bar\partial_{\mathbf{H}}\Phi_2^{\mu\oplus\nu}(z,g) = \mu\partial_{\mathbf{H}}\Phi_2^{\mu\oplus\nu}(z,g) + \partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_2^{\mu\oplus\nu}(z,g)\nu. \tag{3.2}$$

since $\partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_1^{\mu\oplus\nu}(z,g) = 0$. With this simplification the first equation is exactly Bers's equation for

$$\Phi_1^{\mu}(z) = \Phi_1^{\mu\oplus\nu}(z,g), \tag{3.3}$$

and so we can solve it using the techniques in [AhB], and we obtain a Riemann surface X_{ρ_μ} corresponding to a representation ρ_μ.

The second equation (3.2) we solve in two steps. First, we identify ν with an endomorphism valued 1-form using the standard identification of left invariant vector fields and elements of the Lie algebra. To solve the equation we consider the anti-holomorphic solution of the equation

$$\bar\partial_{\mathbf{H}}\Phi_-^{\nu}(z,e) = \partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_-^{\nu}(z,g)(\nu)|_{g=e} = \Phi_-^{\nu}(z,e)\cdot\nu$$

and extend it equivariantly to the rest of $\mathbf{H}\times\mathbf{GL}(n,\mathbb{C})$. We observe that $\partial_{\mathbf{H}}\Phi_-^{\nu} = 0$ since it is anti-holomorphic. And so it follows, by adding 0 to the defining equation of Φ_-^{ν} that:

$$\bar\partial_{\mathbf{H}}\Phi_-^{\nu}(z,g) = \partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_-^{\nu}(z,g)(\nu) + \mu\partial_{\mathbf{H}}\Phi_-^{\nu}(z,g).$$

The vector bundle on X_{ρ_μ} corresponding to the representation

$$\chi^{\nu}(\gamma) = \Phi_-^{\nu}(\rho_0(\gamma)z,e)\rho_E^{0\oplus 0}(\gamma)(\Phi_-^{\nu}(z,e))^{-1}$$

is stable, if $\mu\oplus\nu$ is small enough. This means, we can find a holomorphic gauge transformation on the universal cover of X_{ρ_μ}, $\Phi_+^{\mu\oplus\nu} : \mathbf{H} \to \mathbf{GL}(n,\mathbb{C})$, such that

$$\rho_E^{\mu\oplus\nu}(\gamma) = \Phi_+^{\mu\oplus\nu}(\rho_\mu(\gamma)z)\chi^{\mu\oplus\nu}(\gamma)(\Phi_+^{\mu\oplus\nu}(z))^{-1} \tag{3.4}$$

is an admissible $\mathbf{U}(n)$-representation and independent of z by the Narasimhan-Seshadri theorem [NSNS]. Now we use the basemap to define $\tilde{\Phi}_+^{\mu\oplus\nu} = \Phi_+^{\mu\oplus\nu} \circ \Phi_1^\mu$.

The following computation shows that the map $\tilde{\Phi}_+^{\mu\oplus\nu}$ is in the kernel of $\bar{\partial}_{\mathbf{H}} - \mu\partial_{\mathbf{H}}$:

$$
\begin{aligned}
(\bar{\partial}_{\mathbf{H}} - \mu\partial_{\mathbf{H}})\tilde{\Phi}_+^{\mu\oplus\nu} = {} & (\bar{\partial}_{\mathbf{H}}\Phi_+^{\mu\oplus\nu}) \circ \Phi_1^\mu \bar{\partial}_{\mathbf{H}}\bar{\Phi}_1^\mu + (\partial_{\mathbf{H}}\Phi_+^{\mu\oplus\nu}) \circ \Phi_1^\mu \bar{\partial}_{\mathbf{H}}\Phi_1^\mu \\
& - \mu(\bar{\partial}_{\mathbf{H}}\Phi_+^{\mu\oplus\nu}) \circ \Phi_1^\mu \partial_{\mathbf{H}}\bar{\Phi}_1^\mu - \mu(\partial_{\mathbf{H}}\Phi_+^{\mu\oplus\nu}) \circ \Phi_1^\mu \partial_{\mathbf{H}}\Phi_1^\mu
\end{aligned}
$$

We then use the differential equation $\bar{\partial}\Phi_1^\mu = \mu\partial\Phi_1^\mu$ and that $\bar{\partial}_{\mathbf{H}}\Phi_+^{\mu\oplus\nu} = 0$ to get that

$$
(\bar{\partial}_{\mathbf{H}} - \mu\partial_{\mathbf{H}})\tilde{\Phi}_+^{\mu\oplus\nu} = \partial_{\mathbf{H}}\Phi_+^{\mu\oplus\nu} \circ \Phi_1^\mu \mu\partial_{\mathbf{H}}\Phi_1^\mu - \mu\partial_{\mathbf{H}}\Phi_+^{\mu\oplus\nu} \circ \Phi_1^\mu \partial_{\mathbf{H}}\Phi_1^\mu = 0.
$$

Define $\Phi_2^{\mu\oplus\nu}(z,g) = \tilde{\Phi}_+^{\mu\oplus\nu}(z,g)\Phi_-^\nu(z,g)$. We see that $\Phi_2^{\mu\oplus\nu}$ fulfills equation (3.2) by the following calculation

$$
\begin{aligned}
\bar{\partial}_{\mathbf{H}}\Phi_2^{\mu\oplus\nu} &= (\bar{\partial}_{\mathbf{H}}\tilde{\Phi}_+^{\mu\oplus\nu})(\Phi_-^\nu) + (\tilde{\Phi}_+^{\mu\oplus\nu})(\bar{\partial}_{\mathbf{H}}\Phi_-^\nu) \\
&= (\bar{\partial}_{\mathbf{H}}\tilde{\Phi}_+^{\mu\oplus\nu})(\Phi_-^\nu) + (\tilde{\Phi}_+^{\mu\oplus\nu})(\partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_-^\nu \nu)
\end{aligned}
$$

since $\tilde{\Phi}_+^{\mu\oplus\nu} \in \ker(\bar{\partial}_{\mathbf{H}} - \mu\partial_{\mathbf{H}})$ we get that

$$
\bar{\partial}_{\mathbf{H}}\Phi_2^{\mu\oplus\nu} = (\mu\partial_{\mathbf{H}}\tilde{\Phi}_+^{\mu\oplus\nu})(\Phi_-^\nu) + (\tilde{\Phi}_+^{\mu\oplus\nu})(\partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_-^\nu \nu).
$$

To finish the calculation we use that Φ_+ and Φ_1 are independent of the $\mathbf{GL}(n,\mathbb{C})$ factor, and therefore so is $\tilde{\Phi}_+^{\mu\oplus\nu}$. Also $\Phi_-^{\mu\oplus\nu}$ is antiholomorphic so we have that

$$
\begin{aligned}
\bar{\partial}_{\mathbf{H}}\Phi_2^{\mu\oplus\nu} &= \mu\partial_{\mathbf{H}}(\tilde{\Phi}_+^{\mu\oplus\nu}\Phi_-^\nu) + \partial_{\mathbf{GL}(n,\mathbb{C})}(\tilde{\Phi}_+^{\mu\oplus\nu}\Phi_-^\nu)\nu \\
&= \mu\partial_{\mathbf{H}}\Phi_2^{\mu\oplus\nu} + (\partial_{\mathbf{GL}(n,\mathbb{C})}\Phi_2^{\mu\oplus\nu})\nu
\end{aligned}
$$

To show that we still get an admissible representation, we use that (3.4) is independent of which z we choose. This lets us conclude that

$$
\begin{aligned}
\rho_E^{\mu\oplus\nu}(\gamma) = {} & \Phi_+^{\mu\oplus\nu}(\rho_\mu(\gamma)\Phi_1^\mu(z))\chi^{\mu\oplus\nu}(\gamma)(\Phi_+^{\mu\oplus\nu}(\Phi_1^\mu(z)))^{-1} \\
= {} & \Phi_+^{\mu\oplus\nu}(\Phi_1^\mu(\rho_0(\gamma)(\Phi_1^\mu)^{-1}(\Phi_1^\mu(z))))\chi^{\mu\oplus\nu}(\gamma)(\Phi_+^{\mu\oplus\nu}(\Phi_1^\mu(z)))^{-1} \\
= {} & \tilde{\Phi}_+^{\mu\oplus\nu}(\rho_0(\gamma)z)\chi^{\mu\oplus\nu}(\gamma)(\tilde{\Phi}_+^{\mu\oplus\nu}(z))^{-1},
\end{aligned}
$$

and so

$$
\rho_E^{\mu\oplus\nu}(\gamma) = \Phi_2^{\mu\oplus\nu}(\rho_0(\gamma)z, g)\rho_E^{0\oplus 0}(\gamma)(\Phi_2^{\mu\oplus\nu}(z,g))^{-1}
$$

is an admissible $\mathbf{U}(n)$-representation. Finally, the requirement that $\Phi_2^{\mu\oplus\nu}(z_0, e)$ is a positive definite matrix of determinant 1 fixes all remaining indeterminacy as in [TZ1].

3.1 The Tangent Map from Kodaira-Spencer Theory

We will now analyse the tangential map of our coordinates. The only problematic part is what happens in the tangent directions parallel to the fibers. We can calculate the Kodaira-Spencer map of the family of representations $\rho_E^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}, t \in \mathbb{C}$. However, to ease the computation we first prove the following lemma.

Lemma 3.1. *We let X_{ρ_0} be a Riemann surface and ρ_0 the corresponding representation of $\pi_1(X_{\rho_0})$. For a family of representations of $R_t : \tilde{\pi}_1(X_{\rho_0}) \to \mathbf{U}(n)$, where*

$$R_t(\gamma) = \Upsilon(t, \rho_0(\gamma)z)\rho_E(\gamma)\Upsilon(t, z)^{-1}$$

with both ρ_0 and ρ_E independent of t and Υ any smooth map

$$\Upsilon : \mathbb{C} \times \mathbf{H} \to \mathbf{GL}(n, \mathbb{C}),$$

we have that the Kodaira-Spencer class's harmonic representative of the family R_t at $t = 0$ is:

$$P^{0,1}_{\rho_0,E}\left(Ad\Upsilon(0,z)\left(\frac{d}{dt}\bigg|_{t=0}\Upsilon(t,z)^{-1}\bar{\partial}_{\mathbf{H}}\Upsilon(t,z)\right)\right) \in H^{0,1}(X_0, EndE_{R_0}).$$

Here $P^{0,1}_{\rho_0,E}$ denotes the projection on the harmonic forms on X_{ρ_0} with values in $\mathrm{End}E_{R_0}$.

Proof. To compute the Kodaira-Spencer map we first consider $\frac{d}{dt}\big|_{t=0} R_t$ and note, this is an element of $H^1(X, \mathrm{End}(E))$. However, this cohomology group is isomorphic to $H^{0,1}(X, \mathrm{End}E)$. The isomorphism is constructed by finding a Čech chain with values in the sheaf $\Omega^1(\mathrm{End}E)$, say φ_i, such that

$$\delta^*(\phi)_{ij} = \phi_i - \phi_j = \frac{d}{dt}\bigg|_{t=0} R_t(\gamma_{ij}),$$

for open sets $U_i \cap U_j \neq \emptyset$ which are related by the transformation $\gamma_{ij} \in \tilde{\pi}_1(\Sigma)$ on the universal cover. Once ϕ_i has been found, $P^{0,1}_{\rho_0,E}(\bar{\partial}_{\mathbf{H}}\phi_i)$ will give a harmonic representative of the Kodaira-Spencer class.

We can now calculate that

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} R_t(\gamma_{ij}) &= \frac{d}{dt}\bigg|_{t=0} \Upsilon(t, \rho_0(\gamma)z)\rho_E(\gamma)\Upsilon(t, z)^{-1} \\
&= \frac{d}{dt}\bigg|_{t=0} \Upsilon(t, \rho_0(\gamma)z)\rho_E(\gamma)\Upsilon(0, z)^{-1} \\
&\quad + \Upsilon(0, \rho_0(\gamma)z)\rho_E(\gamma)\, \frac{d}{dt}\bigg|_{t=0} \Upsilon(t, z)^{-1} \\
&= \frac{d}{dt}\bigg|_{t=0} (\Upsilon(t, \rho_{\mathbf{H}}(\gamma_{ij})z)\Upsilon(0, \rho_0(\gamma_{ij})z)^{-1})R_0(\gamma_{ij})
\end{aligned}$$

$$\begin{aligned}
&- R_0(\gamma_{ij}) \left.\frac{d}{dt}\right|_{t=0} (\Upsilon(t,z)\Upsilon(0,z)^{-1})\\
&= R_0(\gamma_{ij})\delta(\left.\frac{d}{dt}\right|_{t=0} (\Upsilon(t,z)\Upsilon(0,z)^{-1}))_{ij}
\end{aligned}$$

The Kodaira-Spencer class is then:

$$\bar{\partial}_{\mathbf{H}} \left.\frac{d}{dt}\right|_{t=0} (\Upsilon(t,z)\Upsilon(0,z)^{-1}) = \mathrm{Ad}\Upsilon(0,z)\left(\left.\frac{d}{dt}\right|_{t=0} \Upsilon(t,z)^{-1}\bar{\partial}_{\mathbf{H}}\Upsilon(t,z)\right).$$

We compose with the harmonic projection to get the harmonic representative. □

We have the following proposition.

Proposition 3.2. *The Kodaira-Spencer map of* $\rho_E^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}, t \in \mathbb{C}$ *at* $\mu\oplus\nu \in H^{0,1}(X,TX)\oplus H^{0,1}(X,\mathrm{End}E)$,

$$KS_{\mu\oplus\nu}: H^{0,1}(X,TX)\oplus H^{0,1}(X,\mathrm{End}E) \to H^{0,1}(X_{\rho_\mu},TX_{\rho_\mu})\oplus H^{0,1}(X_{\rho_\mu},\mathrm{End}E_{\rho_E^{\mu\oplus\nu}})$$

is given by

$$KS_{\mu\oplus\nu}(\tilde{\mu}\oplus\tilde{\nu}) = P_\mu\tilde{\mu}^\mu \oplus P^{0,1}_{\mu\oplus\nu}\left((\Phi_1^\mu)_*^{-1}\left(Ad\Phi_2^{\mu\oplus\nu}\left(\tilde{\mu}(\Phi_2^{\mu\oplus\nu})^{-1}\partial\Phi_2^{\mu\oplus\nu}+\tilde{\nu}\right)\right)\right).$$

Here $\tilde{\mu}^\mu = (\frac{\tilde{\mu}}{1-|\mu|^2}\frac{\partial\Phi_1^\mu}{\overline{\partial\Phi_1^\mu}})\circ(\Phi_1^\mu)^{-1}$ *and* P_μ *and* $P^{0,1}_{\mu\oplus\nu}$ *are the* L^2*-projections on the harmonic forms* $H^{0,1}(X_{\rho_\mu},TX_{\rho_\mu})$ *respectively* $H^{0,1}(X_{\rho_\mu},\mathrm{End}E_{\rho_E^{\mu\oplus\nu}})$.

Proof. By using that the defining equation (1.2) for $\rho_E^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}$ is independent of z, we get that

$$\begin{aligned}
\rho_E^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}} &= \Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}(\rho_0(\gamma)z),e)\rho_E^{0\oplus0}(\gamma)\Phi_2^{\mu\oplus\nu}(z,e)^{-1}\\
&= \Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}((\Phi_1^\mu)^{-1}(\rho_\mu(\gamma)z),e)\rho_E^{0\oplus0}(\gamma)\\
&\quad\cdot\Phi_2^{\mu\oplus\nu}((\Phi_1^\mu)^{-1}(z),e)^{-1}.
\end{aligned}$$

And so to find the Kodaira-Spencer class, by Lemma 3.1 we only need to calculate:

$$\begin{aligned}
&\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\circ(\Phi_1^\mu)^{-1}\frac{d}{dt}|_{t=0}(\Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}\circ(\Phi_1^\mu)^{-1})^{-1}\bar{\partial}(\Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}}\circ(\Phi_1^\mu)^{-1})\\
&=\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\circ(\Phi_1^\mu)^{-1}\\
&\quad\cdot\frac{d}{dt}|_{t=0}((t\tilde{\mu}(\Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}})^{-1}\partial\Phi_2^{\mu\oplus\nu+t\tilde{\mu}\oplus\tilde{\nu}})\circ\Phi_1^\mu)^{-1}\bar{\partial}(\bar{\Phi}_1^\mu)^{-1}\\
&\quad+\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\circ(\Phi_1^\mu)^{-1}\frac{d}{dt}|_{t=0}(\nu+t\tilde{\nu})\circ(\Phi_1^\mu)^{-1}\bar{\partial}(\bar{\Phi}_1^\mu)^{-1}\\
&=(\Phi_1^\mu)_*^{-1}\left(\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\left(\tilde{\mu}(\Phi_2^{\mu\oplus\nu})^{-1}\partial\Phi_2^{\mu\oplus\nu}+\tilde{\nu}\right)\right).
\end{aligned}\tag{3.5}$$

Now to get the Kodaira-Spencer map we project on the harmonic $(0,1)$-forms and remark that in the Teichmüller directions we can apply the usual arguments from the classical case of Bers's coordinates. □

We see the map is injective and complex linear in both $\tilde{\mu}$ and $\tilde{\nu}$. Since we know $\mathcal{T} \times M'$ is a manifold the Implicit Function Theorem now implies that the coordinates we constructed are in fact holomorphic coordinates in a small neighbourhood. This completes the proof of Theorem 1.2.

4 The Fibered Coordinates

In this section we will fuse Zograf and Takhtadzhyan's coordinates with Bers's coordinates in a kind of fibered manner in order also to produce coordinates on $\mathcal{T} \times M'$, which are complex analytic with respect to J.

Since we trough any stable bundle have a copy of $\mathcal{T}$ embedded as a complex submanifold, we can construct fibered coordinates, once we identify the tangent spaces in the fiber direction locally along these copies of $\mathcal{T}$. We identify them by the maps

$$H^{0,1}(X_{\rho_0}, \mathrm{End} E_{\rho_E^0}) \ni \nu \to \nu^\mu = P_\mu^{0,1}((\Phi_1^\mu)_*^{-1}(\nu)) \in H^{0,1}(X_{\rho_\mu}, \mathrm{End} E_{\rho_E^{0\mu}}).$$

This identification gives us coordinates taking (μ, ν) to

$$(\rho_\mu, \rho_E^{\nu^\mu}) = \left(\Phi_1^\mu \circ \rho_0(\gamma) \circ (\Phi_1^\mu)^{-1}, f^{\nu^\mu}(\rho_\mu(\gamma)z)\rho_E^{0^0}(\gamma)(f^{\nu^\mu}(z))^{-1}\right).$$

These are complex coordinates, since ν^μ are local holomorphic sections of the tangent bundle.

Before we calculate the Kodaira-Spencer maps for these coordinate curves, we will need to understand the derivatives of $(\Phi_1^\mu)^{-1}$.

Lemma 4.1. *We have the following two identities for* $(\Phi_1^\mu)^{-1} : \mathbf{H} \to \mathbf{H}$

$$\bar{\partial}(\Phi_1^\mu)^{-1} = -\mu \circ (\Phi_1^\mu)^{-1} \bar{\partial}\overline{(\Phi_1^\mu)^{-1}}. \tag{4.1}$$

$$\bar{\partial}\overline{(\Phi_1^\mu)^{-1}} = \left(\frac{1}{1-|\mu|^2} \frac{1}{\overline{\partial \Phi_1^\mu}}\right) \circ (\Phi_1^\mu)^{-1}. \tag{4.2}$$

Proof. We consider the identity $\Phi_1^\mu \circ (\Phi_1^\mu)^{-1}(z) = z$. And we use the differential equation for Φ_1^μ which is

$$\bar{\partial}\Phi_1^\mu = \mu \partial \Phi_1^\mu$$

to calculate:

$$\begin{aligned} 0 = \bar{\partial}(\Phi_1^\mu \circ (\Phi_1^\mu)^{-1}) &= (\bar{\partial}\Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \bar{\partial}\overline{(\Phi_1^\mu)^{-1}} + (\partial \Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \bar{\partial}(\Phi_1^\mu)^{-1} \\ &= (\mu \partial \Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \bar{\partial}\overline{(\Phi_1^\mu)^{-1}} + (\partial \Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \bar{\partial}(\Phi_1^\mu)^{-1}. \end{aligned}$$

Now $\partial \Phi_1^\mu \neq 0$ for μ small, since Φ_1 is a continuous perturbation of the identity map $\mathrm{Id} : \mathbf{H} \to \mathbf{H}$. We then calculate that

$$-\mu \circ (\Phi_1^\mu)^{-1} \bar{\partial}\overline{(\Phi_1^\mu)^{-1}} = \bar{\partial}(\Phi_1^\mu)^{-1},$$

which is (4.1). We can use (4.1) to describe $\bar{\partial}\overline{(\Phi_1^\mu)^{-1}}$ by differentiating $\Phi_1^\mu \circ (\Phi_1^\mu)^{-1}(z) = z$:

$$\begin{aligned} 1 &= \partial(\Phi_1^\mu \circ (\Phi_1^\mu)^{-1}) = (\bar{\partial}\Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \partial\overline{(\Phi_1^\mu)^{-1}} + (\partial\Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \partial(\Phi_1^\mu)^{-1} \\ &= -\mu(\partial\Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \bar{\mu}\bar{\partial}\overline{(\Phi_1^\mu)^{-1}} + (\partial\Phi_1^\mu) \circ (\Phi_1^\mu)^{-1} \partial(\Phi_1^\mu)^{-1} \\ &= ((1-|\mu|^2)(\partial\Phi_1^\mu)) \circ (\Phi_1^\mu)^{-1} \partial(\Phi_1^\mu))^{-1}, \end{aligned}$$

and so conjugating and isolating $\bar{\partial}\overline{(\Phi_1^\mu)^{-1}}$ we find:

$$\bar{\partial}(\overline{\Phi_1^\mu})^{-1} = \left(\frac{1}{1-|\mu|^2} \frac{1}{\overline{\partial\Phi_1^\mu}} \right) \circ (\Phi_1^\mu)^{-1}$$

which proves (4.2). □

Let κ_μ be an (n, m)-tensor with values in the holomorphic bundle $E_{\rho_E^{0\mu}}$ on the Riemann surface X_{ρ_μ} i.e.

$$\kappa_\mu \in C^\infty(X_{\rho_\mu}, T^{-n}X_{\rho_\mu} \otimes \overline{T}^{-m} X_{\rho_\mu} \otimes \mathrm{End} E_{\rho_E^{0\mu}}).$$

Then we have that $(\Phi_1^\mu)_*(\kappa_\mu) = (\kappa_\mu \circ \Phi_1^\mu)(\partial\Phi_1^\mu)^n(\overline{\partial\Phi_1^\mu})^m$ and so

$$(\Phi_1^\mu)_*^{-1}(\kappa_0) = (\kappa_0 \circ (\Phi_1^\mu)^{-1})(\partial\Phi_1^\mu)^{-n}(\overline{\partial\Phi_1^\mu})^{-m}.$$

We have the families of unbounded operators

$$\begin{aligned} \bar{\partial}_{\mu, E_{\rho_E^{0\mu}}} &: L^2(X_{\rho_\mu}, \mathrm{End} E_{\rho_E^{0\mu}}) \to L^2(X_{\rho_\mu}, T^{0,1} \otimes \mathrm{End} E_{\rho_E^{0\mu}}), \\ \bar{\partial}^*_{\mu, E_{\rho_E^{0\mu}}} &: L^2(X_{\rho_\mu}, T^{0,1} \otimes \mathrm{End} E_{\rho_E^{0\mu}}) \to L^2(X_{\rho_\mu}, \mathrm{End} E_{\rho_E^{0\mu}}) \\ \Delta_{\mu, E_{\rho_E^{0\mu}}} &= \bar{\partial}^*\bar{\partial} : L^2(X_{\rho_\mu}, \mathrm{End} E_{\rho_E^{0\mu}}) \to L^2(X_{\rho_\mu}, \mathrm{End} E_{\rho_E^{0\mu}}), \end{aligned}$$

and the finite range operator

$$P^{0,1}_{\mu, E_{\rho_E^{0\mu}}} : L^2(X_{\rho_\mu}, T^{0,1} \otimes \mathrm{End} E_{\rho_E^{0\mu}}) \to L^2(X_{\rho_\mu}, T^{0,1} \otimes \mathrm{End} E_{\rho_E^{0\mu}})$$

given by

$$P^{0,1}_{\mu, E_{\rho_E^{0\mu}}} = I - \bar{\partial}_{\mu, E_{\rho_E^{0\mu}}} \Delta^{-1}_{0,\mu, E_{\rho_E^{0\mu}}} \bar{\partial}^*_{\mu, E_{\rho_E^{0\mu}}},$$

where $\Delta_{0,\mu,E_{\rho_E^{0\mu}}}$ is the restriction of $\Delta_{\mu,E_{\rho_E^{0\mu}}}$ to the orthogonal complement of the subspace consisting of constant functions tensor the identity, and $P^{0,1}$ is the projection on the harmonic $(0, 1)$-forms. We will also need the following results of Takhtajan and Zograf.

Lemma 4.2 ([TZ3]). *We have the following variational formulae for the derivative at X_{ρ_0}*

$$\frac{d}{dt}|_{t=0}(\Phi_1^{t\tilde{\mu}})_*\bar{\partial}_{t\tilde{\mu},E_{\rho_E^{0\mu}}}(\Phi_1^{t\tilde{\mu}})_*^{-1} = -\tilde{\mu}\partial_{0,E} \qquad \frac{d}{d\bar{t}}|_{t=0}(\Phi_1^{t\tilde{\mu}})_*\bar{\partial}_{t\tilde{\mu},E_{\rho_E^{0\mu}}}(\Phi_1^{t\tilde{\mu}})_*^{-1} = 0$$
$$\frac{d}{dt}|_{t=0}(\Phi_1^{t\tilde{\mu}})_*\bar{\partial}^*_{t\tilde{\mu},E_{\rho_E^{0\mu}}}(\Phi_1^{t\tilde{\mu}})_*^{-1} = 0 \qquad \frac{d}{dt}|_{t=0}(\Phi_1^{t\tilde{\mu}})_*\bar{\partial}^*_{t\tilde{\mu},E_{\rho_E^{0\mu}}}(\Phi_1^{t\tilde{\mu}})_*^{-1} = -\partial^*_{0,E}\overline{\tilde{\mu}}.$$

We further have at $(X_{\rho_\mu}, E_{\rho_E^{0\mu}})$, that

$$\begin{aligned}
&\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*P^{0,1}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}\\
&\qquad = (\Phi_1^{\mu})_*P^{0,1}_{\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}\\
&\qquad\quad (\Phi_1^{\mu})_*\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}\bar{\partial}^*_{\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}\\
&\qquad + (\Phi_1^{\mu})_*\bar{\partial}_{\mu,E_{\rho_E^{0\mu}}}\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}^*_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}\\
&\qquad\quad \frac{d}{dt}(\Phi_1^{\mu})_*P^{0,1}_{\mu,E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu})_*^{-1}.
\end{aligned}$$

Proof. The first identities are proven in [TZ3, Equation (2.6)] (without the End$E$ factor, which makes no difference), the last statement is seen straightforwardly as follows

$$\begin{aligned}
\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*P^{0,1}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1} &= \frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}\\
&(\Phi_1^{\mu+t\tilde{\mu}})_*\Delta^{-1}_{0,\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}^*_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}.
\end{aligned}$$

We can then use the following identities

$$\begin{aligned}
&\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\Delta^{-1}_{0,\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1} = -(\Phi_1^{\mu})_*\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}\\
&\qquad\qquad \frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\Delta_{0,\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\Phi_1^{\mu})_*\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1},\\
&\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\Delta_{0,\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}\\
&\qquad\qquad\qquad = \frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\Phi_1^{\mu})_*\bar{\partial}^*_{\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}\\
&\qquad\qquad\qquad + \frac{d}{dt}|_{t=0}(\Phi_1^{\mu})_*\bar{\partial}_{\mu,E_{\rho_E^{0\mu}}}(\Phi_1^{\mu})_*^{-1}(\Phi_1^{\mu+t\tilde{\mu}})_*\bar{\partial}^*_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}.
\end{aligned}$$

Now, putting this together and using that $P^{0,1}_{\mu,E_{\rho_E^{0\mu}}} = I - \bar{\partial}_{\mu,E_{\rho_E^{0\mu}}}\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}\bar{\partial}^*_{\mu,E_{\rho_E^{0\mu}}}$ we have the last identity. □

Proposition 4.3. *The Kodaira-Spencer map of the curve* $\rho_E^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}$ *at* $t=0$ *is*

$$\begin{aligned} KS_{\nu^\mu}(\tilde{\mu}\oplus\tilde{\nu}) =& P_\mu \tilde{\mu}^\mu \oplus P^{0,1}_{\nu^\mu}\Big(Ad(f^{\nu^\mu})((f^{\nu^\mu})^{-1}\cdot(\partial f^{\nu^\mu})\tilde{\mu}^\mu + \tilde{\nu}^\mu) \\ &+ Adf^{\nu^\mu}\ (\Phi_1^\mu)_*^{-1}\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_* P^{0,1}_{\mu+t\tilde{\mu}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\nu))(1-|\mu\circ(\Phi_1^\mu)^{-1}|^2)\Big)\Big) \end{aligned}$$

with $\tilde{\mu}^\mu = (\frac{\tilde{\mu}}{1-|\mu|^2}\frac{\partial\Phi_1^\mu}{\partial\Phi_1^\mu})\circ(\Phi_1^\mu)^{-1}$ *and* P_μ *and* $P^{0,1}_{\nu^\mu}$ *the* L^2*-projections on the harmonic forms* $\mathcal{H}^{0,1}(X_{\rho_\mu}, TX_{\rho_\mu})$ *respectively* $\mathcal{H}^{0,1}(X_{\rho_\mu}, \mathrm{End}E_{\rho_E^{\nu^\mu}})$

Proof. First, we observe that the Teichmüller direction is unchanged from the classical case. Now we want to use Lemma 3.1, and so using that $\rho_E^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}$ is independent of z we find that

$$\begin{aligned} \rho_E^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}(\gamma) &= f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}(\rho_{\mu+t\tilde{\mu}}(\gamma)z)\rho_E(\gamma)(f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}(z))^{-1} \\ &= f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}((\Phi_1^{\mu+t\tilde{\mu}}((\Phi_1^\mu)^{-1}(\rho_\mu(\gamma)z)))\rho_E(\gamma) \\ &\qquad (f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}((\Phi_1^{\mu+t\tilde{\mu}}((\Phi_1^\mu)^{-1}(z)))^{-1}. \end{aligned}$$

Next we have to calculate

$$\begin{aligned} &\mathrm{Ad}(f^{\nu^\mu})\frac{d}{dt}|_{t=0}\Big(\Big((f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}\circ\Phi_1^{\mu+t\tilde{\mu}})\circ(\Phi_1^\mu)^{-1}\Big)\,\bar{\partial}\Big((f^{\nu^{\mu+t\tilde{\mu}}})^{-1}\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}\Big)\Big) \\ &\quad= \mathrm{Ad}(f^{\nu^\mu})\frac{d}{dt}|_{t=0}\Big(\Big((\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}})\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}\Big)\,\overline{(\partial(\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}))}\Big) \\ &\qquad + \mathrm{Ad}(f^{\nu^\mu})\frac{d}{dt}|_{t=0}\Big(\Big(f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}}\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}\Big)^{-1} \\ &\qquad\quad \cdot(\partial f^{\nu^\mu})\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}(\overline{\partial}(\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}))\Big). \end{aligned}$$

For the first term we find that

$$\begin{aligned} &\mathrm{Ad}(f^{\nu^\mu})\frac{d}{dt}|_{t=0}\Big(\Big((\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}})\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}\Big)\,\overline{(\partial(\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1}))}\Big) = \\ &\quad= \frac{d}{dt}|_{t=0}(\mathrm{Ad}(f^{\nu^\mu})((\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}})\circ\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1} \\ &\qquad \cdot\overline{\Big((\partial\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1})\partial(\Phi_1^\mu)^{-1}+(\bar{\partial}\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1})\partial(\bar{\Phi}_1^\mu)^{-1}\Big)}. \end{aligned}$$

We can now rewrite the last factor using (4.1) and (4.2) and their conjugates to get that

$$\begin{aligned} (\partial\Phi_1^{\mu+t\tilde{\mu}}&\circ(\Phi_1^\mu)^{-1})\partial(\Phi_1^\mu)^{-1}+(\bar{\partial}\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1})\partial(\bar{\Phi}_1^\mu)^{-1} \\ &= (\partial\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1})\partial(\Phi_1^\mu)^{-1} \\ &\quad + (((\mu+t\tilde{\mu})\partial\Phi_1^{\mu+t\tilde{\mu}})\circ(\Phi_1^\mu)^{-1})(-\bar{\mu}\circ(\Phi_1^\mu)^{-1}\partial(\Phi_1^\mu)^{-1}). \\ &= (\partial\Phi_1^{\mu+t\tilde{\mu}}\circ(\Phi_1^\mu)^{-1})\partial(\Phi_1^\mu)^{-1}(1-(((\mu+t\tilde{\mu})\bar{\mu})\circ(\Phi_1^\mu)^{-1})). \end{aligned}$$

Using that $(\Phi_1^\mu)_*(\nu) = \nu \circ \Phi_1^\mu \overline{\partial \Phi_1^\mu}$ in the first term we find that

$$\begin{aligned}
&\mathrm{Ad}(f^{\nu^\mu}) \frac{d}{dt}|_{t=0} \left(\left((\nu + t\tilde{\nu})^{\mu+t\tilde{\mu}}) \circ \Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^\mu)^{-1} \right) (\overline{\partial(\Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^\mu)^{-1})}) \right) \\
&= \mathrm{Ad}(f^{\nu^\mu}) \frac{d}{dt}|_{t=0} \left((\Phi_1^\mu)_*^{-1} \left((\Phi_1^{\mu+t\tilde{\mu}})_* \left((\nu + t\tilde{\nu})^{\mu+t\tilde{\mu}} \right) (1 - |\mu|^2 - \bar{t}\mu\bar{\tilde{\mu}}) \right) \right) \\
&= \mathrm{Ad}(f^{\nu^\mu}) (\Phi_1^\mu)_*^{-1} \left((1 - |\mu|^2) \frac{d}{dt}|_{t=0} (\Phi_1^{\mu+t\tilde{\mu}})_* \left(P^{0,1}_{\mu+t\tilde{\mu}, E_{\rho_E^{0\mu+t\tilde{\mu}}}} (\Phi_1^{\mu+t\tilde{\mu}})_*^{-1} (\nu + t\tilde{\nu}) \right) \right) \\
&= \mathrm{Ad}(f^{\nu^\mu}) P^{0,1}_{\mu, E_{\rho_E^{0\mu}}} \left((\Phi_1^\mu)_*^{-1}(\tilde{\nu}) + \tilde{\mu}^\mu \partial_{\mu, E_{\rho_E^{0\mu}}} \Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}} \bar{\partial}^*_{\mu, E_{\rho_E^{0\mu}}} (\Phi_1^\mu)_*^{-1}(\nu) \right) (\Phi_1^\mu)_*^{-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left((1 - |\mu|^2) \right).
\end{aligned}$$

Here we have used the result from Lemma 4.2 to calculate the derivative of the projection.

For the second term we rewrite

$$\begin{aligned}
&\bar{\partial}(\Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^\mu)^{-1}) \\
&\quad = (\bar{\partial}\Phi_1^{\mu+t\tilde{\mu}}) \circ (\Phi_1^\mu)^{-1} \bar{\partial}\overline{(\Phi_1^\mu)^{-1}} + (\partial\Phi_1^{\mu+t\tilde{\mu}}) \circ (\Phi_1^\mu)^{-1} \bar{\partial}(\Phi_1^\mu)^{-1} \\
&\quad = ((\mu + t\tilde{\mu})\partial\Phi_1^{\mu+t\tilde{\mu}}) \circ (\Phi_1^\mu)^{-1} \bar{\partial}\overline{(\Phi_1^\mu)^{-1}} + (\partial\Phi_1^{\mu+t\tilde{\mu}}) \circ (\Phi_1^\mu)^{-1} \bar{\partial}(\Phi_1^\mu)^{-1}
\end{aligned} \tag{4.3}$$

using (4.2) and (4.1) in (4.3) and find that

$$\bar{\partial}(\Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^\mu)^{-1}) = \left(\frac{t\tilde{\mu}}{1 - |\mu|^2} \frac{\partial \Phi_1^{\mu+t\tilde{\mu}}}{\overline{\partial \Phi_1^\mu}} \right) \circ (\Phi_1^\mu)^{-1},$$

which implies that

$$\begin{aligned}
\mathrm{Ad}(f^{\nu^\mu}) \frac{d}{dt}|_{t=0} \Big(&\left(f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}} \circ \Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^\mu)^{-1} \right)^{-1} \\
&\cdot (\partial f^{\nu^\mu}) \circ \Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^\mu)^{-1} (\bar{\partial}(\Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^\mu)^{-1})) \Big) \\
&= \mathrm{Ad}(f^{\nu^\mu})((f^{\nu^\mu})^{-1} \cdot (\partial f^{\nu^\mu})(\tilde{\mu}^\mu),
\end{aligned}$$

where $\tilde{\mu}^\mu = \left(\frac{\tilde{\mu}}{1-|\mu|^2} \frac{\partial\Phi_1^\mu}{\overline{\partial\Phi_1^\mu}} \right) \circ (\Phi_1^\mu)^{-1}$. And so we have that

$$\begin{aligned}
&\mathrm{Ad}(f^{\nu^\mu}) \frac{d}{dt}|_{t=0} \left(\left((f^{(\nu+t\tilde{\nu})^{\mu+t\tilde{\mu}}} \circ \Phi_1^{\mu+t\tilde{\mu}}) \circ (\Phi_1^\mu)^{-1} \right) \bar{\partial} \left((f^{\nu^{\mu+t\tilde{\mu}}})^{-1} \circ \Phi_1^{\mu+t\tilde{\mu}} \circ (\Phi_1^\mu)^{-1} \right) \right) \\
&= \mathrm{Ad}(f^{\nu^\mu}) \big((\Phi_1^\mu)_*^{-1} \frac{d}{dt}|_{t=0} (\Phi_1^{\mu+t\tilde{\mu}})_* P^{0,1}_{\mu+t\tilde{\mu}, E_{\rho_E^{0\mu+t\tilde{\mu}}}} (\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\nu)))) \\
&\quad (1 - |\mu|^2) \circ (\Phi_1^\mu)^{-1}) \big) + \mathrm{Ad}(f^{\nu^\mu})\tilde{\nu}^\mu + \mathrm{Ad}(f^{\nu^\mu})((f^{\nu^\mu})^{-1} \cdot (\partial f^{\nu^\mu})(\tilde{\mu}^\mu)).
\end{aligned} \tag{4.4}$$

We have thus shown that composing with the projection gives us the harmonic representative. □

4.1 Comparison of the Two Tangent Maps and a proof of the first part of Theorem 1.3

We compare

$$P^{0,1}_{\mu\oplus\nu,}\left((\Phi_1^\mu)_*^{-1}\left(\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\left(\tilde{\mu}(\Phi_2^{\mu\oplus\nu})^{-1}\partial\Phi_2^{\mu\oplus\nu}+\tilde{\nu}\right)\right)\right)$$

and

$$P^{0,1}_{\nu^\mu}\left(\mathrm{Ad}(f^{\nu^\mu})((f^{\nu^\mu})^{-1}\cdot(\partial f^{\nu^\mu})\tilde{\mu}^\mu+\tilde{\nu}^\mu)\right.$$
$$\left.+\,\mathrm{Ad}f^{\nu^\mu}\,(\Phi_1^\mu)_*^{-1}\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*P^{0,1}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\nu))(1-|\mu\circ(\Phi_1^\mu)^{-1}|^2)\right).$$

First we observe that

$$\mathrm{Ad}f^{\nu^\mu}(\Phi_1^\mu)_*^{-1}\frac{d}{dt}|_{t=0}(\Phi_1^{\mu+t\tilde{\mu}})_*P^{0,1}_{\mu+t\tilde{\mu},E_{\rho_E^{0\mu+t\tilde{\mu}}}}(\Phi_1^{\mu+t\tilde{\mu}})_*^{-1}(\nu))(1-|\mu\circ(\Phi_1^\mu)^{-1}|^2)$$

vanishes to first order in ν and μ at the center, since we either differentiate with respect to μ and set $\nu=0$ or we differentiate with respect to ν and then we find, when we evaluate at $\mu=0$, that $\bar{\partial}^*_{0,E}\nu=0$, from the expression in Lemma 4.2.

Next we compare

$$(\Phi_1^\mu)_*^{-1}\left(\mathrm{Ad}\Phi_2^{\mu\oplus\nu}\left(\tilde{\mu}(\Phi_2^{\mu\oplus\nu})^{-1}\partial\Phi_2^{\mu\oplus\nu}\right)\right)$$

with

$$\mathrm{Ad}(f^{\nu^\mu})((f^{\nu^\mu})^{-1}\cdot(\partial f^{\nu^\mu})\tilde{\mu}^\mu).$$

We observe, that since $\partial I=0$ both $(\Phi_2^{\mu\oplus\nu})^{-1}\partial\Phi_2^{\mu\oplus\nu}$ and $(f^{\nu^\mu})^{-1}\cdot(\partial f^{\nu^\mu})$ vanish unless we differentiate it with respect to the moduli space direction or the Teichmüller direction. If we differentiate with respect to μ we get $\frac{\partial}{\partial\epsilon}\nu^{\epsilon\mu}$, but at $\nu=0$ this is 0. This means we can compare the two after evaluating $\mu=0$, and then we have $f^{\nu^0}=\Phi^{0\oplus\nu}$, and so they agree to first order.

The last terms to consider are $(\Phi_1^\mu)_*^{-1}\left(\mathrm{Ad}\Phi_2^{\mu\oplus\nu}(\tilde{\nu})\right)$ and $\mathrm{Ad}(f^{\nu^\mu})\tilde{\nu}^\mu$. Now, if we put $\mu=0$ the terms agree. If we differentiate with respect to μ, we can put $\nu=0$ first. We are differentiating a term of the form $\bar{\partial}_{\mu,E_{\rho_E^{0\mu}}}\Delta^{-1}_{0,\mu,E_{\rho_E^{0\mu}}}\bar{\partial}^*_{\mu,E_{\rho_E^{0\mu}}}(\Phi_1^\mu)_*^{-1}\tilde{\nu}$ with respect to μ. The result is an exact term which is killed by the harmonic projection $P^{0,1}$, plus a term containing $\bar{\partial}^*_{0,E}\nu=0$. This proves the first part of Theorem 1.3. The second part will be proved in the following section.

5 Variation of the Metric

In order to prove that our new coordinates are not the same as the fibered coordinates discussed above, we shall consider the variation of the metric in both set of coordinates and use the resulting formulae to demonstrate that they are not identical to third order.

5.1 Variation in the Universal Coordinates

In this section will calculate the second variation of the metric using the coordinates from Theorem 1.1. In the next section we will do the same for the fibered coordinates, and use this to show that the two sets of coordinates differ at third order. So first we consider the function $(\bar{\Phi}_2^{\epsilon(\mu\oplus\nu)})^T\Phi_2^{\epsilon(\mu\oplus\nu)}$. This transforms as a function on X with values in End$E$, our reference point. Now, to further understand this function, we look at $\frac{d}{d\epsilon}|_{\epsilon=0}(\bar{\Phi}_2^{\epsilon(\mu\oplus\nu)})^T\Phi_2^{\epsilon(\mu\oplus\nu)}$. Then we find that

$$\begin{aligned}\Delta\frac{d}{d\epsilon}|_{\epsilon=0}(\bar{\Phi}_2^{\epsilon(\mu\oplus\nu)})^T\Phi_2^{\epsilon(\mu\oplus\nu)} = \Delta(&\frac{d}{d\epsilon}|_{\epsilon=0}(\bar{\Phi}_+^{\epsilon(\mu\oplus\nu)})^T + \frac{d}{d\epsilon}|_{\epsilon=0}\Phi_-^{\epsilon(\mu\oplus\nu)} \\ &+ \frac{d}{d\epsilon}|_{\epsilon=0}(\bar{\Phi}_-^{\epsilon(\mu\oplus\nu)})^T + \frac{d}{d\epsilon}|_{\epsilon=0}\Phi_+^{\epsilon(\mu\oplus\nu)}),\end{aligned}$$

and since $\Phi_-^{\epsilon(\mu\oplus\nu)}$ is antiholomorphic we get that

$$\Delta\frac{d}{d\epsilon}|_{\epsilon=0}(\bar{\Phi}_2^{\epsilon(\mu\oplus\nu)})^T\Phi_2^{\epsilon(\mu\oplus\nu)} = \frac{d}{d\epsilon}|_{\epsilon=0}\Delta(\bar{\Phi}_+^{\epsilon(\mu\oplus\nu)})^T + \frac{d}{d\epsilon}|_{\epsilon=0}\Delta\Phi_+^{\epsilon(\mu\oplus\nu)}).$$

We now use that $(\bar{\partial} - \epsilon\mu\partial)\Phi_+^{\epsilon(\mu\oplus\nu)} = 0$ and $\Delta = y^{-2}\partial\bar{\partial}$ to see that

$$\Delta\frac{d}{d\epsilon}|_{\epsilon=0}(\bar{\Phi}_2^{\epsilon(\mu\oplus\nu)})^T\Phi_2^{\epsilon(\mu\oplus\nu)} = y^{-2}\bar{\mu}\bar{\partial}\bar{\partial}(\bar{\Phi}_+^{0(\mu\oplus\nu)})^T + y^{-2}\mu\partial\partial\Phi_+^{0(\mu\oplus\nu)}) = 0,$$

since $\Phi_+^0 = I$, and so the derivative is 0. This allows us to conclude, that $\frac{d}{d\epsilon}|_{\epsilon=0}(\bar{\Phi}_2^{\epsilon(\mu\oplus\nu)})^T\Phi_2^{\epsilon(\mu\oplus\nu)}$ is a constant multiple of the identity element in End$E$, and because of the determinant criteria in Theorem 1.1 we have

$$\begin{aligned}0 &= \frac{d}{d\epsilon}|_{\epsilon=0}(\det(\bar{\Phi}_2^{\epsilon(\mu\oplus\nu)})^T\Phi_2^{\epsilon(\mu\oplus\nu)}) \\ &= \mathrm{tr}|_{\epsilon=0}((\bar{\Phi}_2^0)^T\Phi_2^0)^{-1}\frac{d}{d\epsilon}|_{\epsilon=0}(\bar{\Phi}_2^{\epsilon(\mu\oplus\nu)})^T\Phi_2^{\epsilon(\mu\oplus\nu)} = \mathrm{tr}\frac{d}{d\epsilon}|_{\epsilon=0}(\bar{\Phi}_2^{\epsilon(\mu\oplus\nu)})^T\Phi_2^{\epsilon(\mu\oplus\nu)},\end{aligned}$$

and so $\frac{d}{d\epsilon}|_{\epsilon=0}(\bar{\Phi}_2^{\epsilon(\mu\oplus\nu)})^T\Phi_2^{\epsilon(\mu\oplus\nu)} = 0$. We see that this immediately implies that $\frac{d}{d\epsilon}|_{\epsilon=0}\partial\Phi_+^{\epsilon(\mu\oplus\nu)} = -\overline{\frac{d}{d\bar{\epsilon}}|_{\epsilon=0}\bar{\partial}\Phi_-^{\epsilon(\mu\oplus\nu)}}^T = 0$. We can study

$$\frac{d}{d\bar{\epsilon}}|_{\epsilon=0}(\bar{\Phi}_2^{\epsilon(\mu\oplus\nu)})^T\Phi_2^{\epsilon(\mu\oplus\nu)}$$

similarly and conclude that

$$\frac{d}{d\bar{\epsilon}}|_{\epsilon=0}\partial\Phi_+^{\epsilon(\mu\oplus\nu)} = -\overline{\frac{d}{d\epsilon}|_{\epsilon=0}\bar{\partial}\Phi_-^{\epsilon(\mu\oplus\nu)}}^T = -\bar{\nu}^T.$$

Now, we want to understand the variation of the $\bar{\partial}_{\mu,E_{\rho(\mu\oplus\nu)}}$-operator on functions and $\bar{\partial}^*_{\mu,E_{\rho(\mu\oplus\nu)}}$-operator on $(0,1)$-forms, since they play a central role in understanding the tangent spaces over the universal moduli space. We work on the universal cover and pull back our family of differential operators from the universal cover of $(X_{\rho_\mu}, E_{\rho(\mu\oplus\nu)})$ to that of (X_{ρ_0}, E), in terms of representations. Then $\bar\partial_{\mu, E_{\rho(\mu\oplus\nu)}}$ is just represented by $\bar\partial$ on $\mathbf{H}$

$$\begin{aligned}
&\frac{d}{d\epsilon}|_{\epsilon=0}\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu}(\Phi_1^{\epsilon\mu})_*\bar\partial(\Phi_1^{\epsilon\mu})_*^{-1}(\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu})^{-1}\\
&\qquad= \frac{d}{d\epsilon}|_{\epsilon=0}\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu}\frac{1}{1-|\epsilon\mu|^2}(\bar\partial-\mu\partial)(\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu})^{-1}\\
&\qquad= \frac{d}{d\epsilon}|_{\epsilon=0}\frac{1}{1-|\epsilon\mu|^2}\big(\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu}(\mathrm{ad}(\bar\partial-\epsilon\mu\partial)\Phi_2^{\epsilon\mu\oplus\nu})(\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu})^{-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + (\bar\partial-\epsilon\mu\partial)\big)\\
&\qquad= \frac{d}{d\epsilon}|_{\epsilon=0}\frac{1}{1-|\epsilon\mu|^2}(\epsilon\mathrm{ad}\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu}\nu+(\bar\partial-\epsilon\mu\partial))\\
&\qquad= \mathrm{ad}\nu-\mu\partial.
\end{aligned}\tag{5.1}$$

Likewise we find that the variation of $\bar\partial^* = -\rho^{-1}\partial$, where also the first derivative of density ρ is zero at the center point of our coordinates([W]). We begin by observing that on $(0,1)$-forms we have that

$$\begin{aligned}
(\Phi_1^{\epsilon\mu})_*\partial(\Phi_1^{\epsilon\mu})_*^{-1}\alpha &= (\Phi_1^{\epsilon\mu})_*\partial(\alpha\circ(\Phi_1^{\epsilon\mu})^{-1}\frac{1}{\overline{\bar\partial\Phi_1^{\epsilon\mu}\circ(\Phi_1^{\epsilon\mu})^{-1}}})\\
&= (\Phi_1^{\epsilon\mu})_*((\partial\alpha)\circ(\Phi_1^{\epsilon\mu})^{-1}\frac{\partial(\Phi_1^{\epsilon\mu})^{-1}}{\overline{\bar\partial\Phi_1^{\epsilon\mu}\circ(\Phi_1^{\epsilon\mu})^{-1}}}\\
&+ (\bar\partial\alpha)\circ(\Phi_1^{\epsilon\mu})^{-1}\frac{\partial\overline{(\Phi_1^{\epsilon\mu})^{-1}}}{\overline{\bar\partial\Phi_1^{\epsilon\mu}\circ(\Phi_1^{\epsilon\mu})^{-1}}}\\
&- \alpha\circ(\Phi_1^{\epsilon\mu})^{-1}\frac{\overline{(\partial(\bar\partial\Phi_1^{\epsilon\mu}))\circ(\Phi_1^{\epsilon\mu})^{-1}(\bar\partial\bar\Phi_1^{\epsilon\mu})^{-1}}}{(\overline{\bar\partial\Phi_1^{\epsilon\mu}\circ(\Phi_1^{\epsilon\mu})^{-1}})^2})\\
&- \alpha\circ(\Phi_1^{\epsilon\mu})^{-1}\frac{\overline{(\partial\partial\Phi_1^{\epsilon\mu})\circ(\Phi_1^{\epsilon\mu})^{-1}(\bar\partial\Phi_1^{\epsilon\mu})^{-1}}}{(\overline{\bar\partial\Phi_1^{\epsilon\mu}\circ(\Phi_1^{\epsilon\mu})^{-1}})^2})\\
&= \frac{1}{1-|\epsilon\mu|^2}(\partial-\bar\epsilon\bar\mu\bar\partial-\bar\epsilon(\bar\partial\bar\mu)) = \frac{1}{1-|\epsilon\mu|^2}(\partial-\overline{\epsilon\partial\mu})).
\end{aligned}$$

And so we find that

$$\begin{aligned}
&\frac{d}{d\bar\epsilon}|_{\epsilon=0}\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu}(\Phi_1^{\epsilon\mu})_*\bar\partial^*(\Phi_1^{\epsilon\mu})_*^{-1}(\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu})^{-1}\\
&\qquad= \frac{d}{d\bar\epsilon}|_{\epsilon=0}\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu}\frac{-\rho^{-1}}{1-|\epsilon\mu|^2}(\partial-\bar\epsilon\bar\partial\bar\mu)(\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu})^{-1}\\
&\qquad= \frac{d}{d\bar\epsilon}|_{\epsilon=0}\frac{-\rho^{-1}}{1-|\epsilon\mu|^2}\big(\mathrm{Ad}\Phi_2^{\epsilon\mu\oplus\nu}(\mathrm{Ad}(\partial-\bar\epsilon\bar\partial\bar\mu)\Phi_2^{\epsilon\mu\oplus\nu})^{-1}\\
&\qquad\qquad\qquad\qquad + (\bar\partial-\bar\epsilon\bar\partial\bar\mu)\big)
\end{aligned}$$

$$= \frac{d}{d\bar{\epsilon}}|_{\epsilon=0}\frac{-\rho^{-1}}{1-|\epsilon\mu|^2}(\mathrm{ad}(\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\epsilon(\mu\oplus\nu)}) - \bar{\epsilon}\bar{\mu}\rho^{-1}\mathrm{ad}\bar{\partial}\Phi_2^{\epsilon(\mu\oplus\nu)}$$
$$+ (\partial - \bar{\epsilon}\bar{\partial}\bar{\mu}))$$
$$= -*\mathrm{ad}\nu * + \bar{\mu}\bar{\partial}\rho^{-1}, \tag{5.2}$$

where the equality follows from the equation $\partial\mu = 2y^{-1}\mu$, and $\rho^{-1} = y^2$.

This is the first step in understanding the metric on the universal moduli space of pairs of a Riemann surface and a holomorphic bundle over it, given at a point (X, E) by identifying the tangent space with $\mathcal{H}^{0,1}(X, TX) \oplus \mathcal{H}^{0,1}(X, \mathrm{End}E)$. Two elements $\mu_1 \oplus \nu_1$ and $\mu_2 \oplus \nu_2$ can be paired as follows

$$g(\mu_1 \oplus \nu_1, \mu_2 \oplus \nu_2) = \int_\Sigma (\rho_X \mu_1 \bar{\mu}_2 + i\mathrm{tr}\nu_1 \wedge \star\bar{\nu}_2^T),$$

where ρ_X is the density of the hyperbolic metric corresponding to the complex structure on X. Since the term $\int_\Sigma \rho_X \mu_1 \bar{\mu}_2$, is independent of the bundle, nothing has changed compared to the situation on Teichmüller space. Let us examine the term $\int_\Sigma \mathrm{tr}\nu_1 \wedge \bar{(} - \star)\nu_2^T$. Since we are evaluation the metric on tangent vectors, $-\star$ will act by $-i$ and so we replace it in the following to avoid confusion.

In coordinates around (X, E) we have, using Proposition 3.2, that the metric is given by

$$g^{VB}_{\epsilon(\mu\oplus\nu)}(\mu_1 \oplus \nu_1, \mu_2 \oplus \nu_2)$$
$$= -i\int_\Sigma \mathrm{tr}P^{0,1}_{\epsilon(\mu\oplus\nu)}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_1$$
$$\wedge \overline{P^{0,1}_{\epsilon(\mu\oplus\nu)}((\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_2}^T$$
$$- i\int_\Sigma \mathrm{tr}P^{0,1}_{\epsilon(\mu\oplus\nu)}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})\mu_1(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\epsilon(\mu\oplus\nu)}$$
$$\wedge \overline{P^{0,1}_{\epsilon(\mu\oplus\nu)}((\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_2}^T$$
$$- i\int_\Sigma \mathrm{tr}P^{0,1}_{\epsilon(\mu\oplus\nu)}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_1$$
$$\wedge \overline{P^{0,1}_{\epsilon(\mu\oplus\nu)}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}(\mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)})\mu_2(\Phi_2^{\epsilon(\mu\oplus\nu)})\partial\Phi_2^{\epsilon(\mu\oplus\nu)}}^T$$
$$- i\int_\Sigma \mathrm{tr}P^{0,1}_{\epsilon(\mu\oplus\nu)}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})\mu_1(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\epsilon(\mu\oplus\nu)}$$
$$\wedge \overline{P^{0,1}_{\epsilon(\mu\oplus\nu)}((\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)})\mu_2(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\epsilon(\mu\oplus\nu)}}^T. \tag{5.3}$$

Now we can use that $P^{0,1}_{\epsilon(\mu\oplus\nu)}$ is self-adjoint with respect to the metric to rewrite the terms as follows

$$\int_\Sigma \mathrm{tr}P^{0,1}_{\epsilon(\mu\oplus\nu)}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_1 \wedge \overline{P^{0,1}_{\epsilon(\mu\oplus\nu)}((\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_2}^T \tag{5.4}$$

$$= \int_\Sigma \mathrm{trAd}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}}^T \Phi_2^{\epsilon(\mu\oplus\nu)})(1-|\epsilon\mu|^2)\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}$$
$$(\Phi_1^{\epsilon(\mu\oplus\nu)})_* P^{0,1}_{\epsilon(\mu\oplus\nu)}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_1 \wedge \overline{\nu_2}^T$$

Since $(\Phi_1^{\epsilon\mu})_*(d\bar z \wedge dz) = (|\partial\Phi_1^{\epsilon\mu}|^2 - |\bar\partial\Phi_1^{\epsilon\mu}|^2)d\bar z \wedge dz$. Also recall that

$$(\Phi_1^{\epsilon\mu})_*^{-1}\nu = \left(\frac{\nu}{\bar\partial\bar\Phi_1^{\epsilon\mu}}\right) \circ (\Phi_1^{\epsilon\mu})^{-1}$$

and $(\Phi_1^{\epsilon\mu})_* P^{0,1}_{\epsilon(\mu\oplus\nu)} h = (\bar\partial\bar\Phi_1^{\epsilon\mu})(P^{0,1}_{\epsilon(\mu\oplus\nu)} h) \circ \Phi_1^{\epsilon\mu}$.

From this it follows that

Lemma 5.1. *In the coordinates around (X, E) given by Theorem 1.1 we have that*

$$\frac{d}{d\epsilon}|_{\epsilon=0} g^{VB}_{\epsilon(\mu\oplus\nu)}(\mu_1 \oplus \nu_1, \mu_2 \oplus \nu_2) = i\int_X tr((\bar\mu_2\nu_1) \wedge \nu)$$
$$\frac{d}{d\bar\epsilon}|_{\epsilon=0} g^{VB}_{\epsilon(\mu\oplus\nu)}(\mu_1 \oplus \nu_1, \mu_2 \oplus \nu_2) = i\int_X tr((\mu_1\bar\nu^T) \wedge \bar\nu_2^T).$$

Proof. We calculate each term gathering the terms like (5.4). We have already seen that $\frac{d}{d\epsilon}|_{\epsilon=0}\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}}^T \Phi_2^{\epsilon(\mu\oplus\nu)} = 0$, and so these terms don't contribute. Now we consider the operators, where we have left out subscripts from the calculation as it should be clear where they live. The derivative of the projection is a sum of terms starting with an operator ending with $\bar\partial^*$ and ones which starts with $\bar\partial$ as is seen from the following calculation

$$\frac{d}{d\epsilon}|_{\epsilon=0}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_* P^{0,1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})$$
$$= \frac{d}{d\epsilon}|_{\epsilon=0}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*(-\bar\partial\Delta_0^{-1}\bar\partial^*)(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})$$
$$= \frac{d}{d\epsilon}|_{\epsilon=0}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*(-\bar\partial)(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})$$
$$\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*(\Delta_0^{-1})(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})$$
$$\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*(\bar\partial^*)(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})$$
$$= \frac{d}{d\epsilon}|_{\epsilon=0}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*(-\bar\partial)(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})\Delta_0^{-1}\bar\partial^*$$
$$+ \bar\partial\frac{d}{d\epsilon}|_{\epsilon=0}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*(\Delta_0^{-1})(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})\bar\partial^*$$
$$+ \bar\partial\Delta_0^{-1}\frac{d}{d\epsilon}|_{\epsilon=0}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*(\bar\partial^*)(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)}).$$

The first two terms are orthogonal to $\nu \in \mathcal{H}^{0,1}(X, \mathrm{End}E)$, and the second one applied to a harmonic from is 0. This means the contribution form the first term in (5.3) is 0.

Now all the remaining terms contain a $\partial\Phi_2^{\epsilon(\mu\oplus\nu)}$ which is 0 at $\epsilon = 0$. Hence the only contributions to the derivative arise when we derive these, and then we have that $\frac{d}{d\bar\epsilon}\partial\Phi_2^{\epsilon(\mu\oplus\nu)} = -\bar\nu^T$ and $\frac{d}{d\epsilon}\partial\Phi_2^{\epsilon(\mu\oplus\nu)} = 0$. Inserting this and setting $\epsilon = 0$ we find the formulas in the lemma. □

We proceed to calculate the second order derivatives of the metric. To do so, we need to calculate $\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}})^T\Phi_2^{\epsilon(\mu\oplus\nu)}$, $\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\epsilon(\mu\oplus\nu)}$ and the contribution from $\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}P^{0,1}\mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)}$. For the last term, we only need it when applied to a harmonic form and also it should not be orthogonal to a harmonic form.

We now calculate the three terms. For the first term, we begin by applying the Laplace operator on $\mathbf{H}$ to the expression.

$$\begin{aligned}
&\Delta\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}})^T\Phi_2^{\epsilon(\mu\oplus\nu)}\\
&\quad= y^2\bar\partial\partial\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}})^T\Phi_2^{\epsilon(\mu\oplus\nu)}\\
&\quad= \frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}y^2\bar\partial\partial((\overline{\Phi_+^{\epsilon(\mu\oplus\nu)}\Phi_-^{\epsilon\mu}})^T\Phi_+^{\epsilon(\mu\oplus\nu)}\Phi_-^{\epsilon\mu})\\
&\quad= \frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}y^2\big(\overline{\bar\partial\Phi_-^{\epsilon\mu}}^T\overline{\partial\Phi_+^{\epsilon(\mu\oplus\nu)}}^T\Phi_+^{\epsilon(\mu\oplus\nu)}\Phi_-^{\epsilon\mu} + \overline{\bar\partial\Phi_-^{\epsilon\mu}}^T\overline{\Phi_+^{\epsilon(\mu\oplus\nu)}}^T\bar\partial\Phi_+^{\epsilon(\mu\oplus\nu)}\Phi_-^{\epsilon\mu}\\
&\qquad\quad+ \overline{\bar\partial\Phi_-^{\epsilon\mu}}^T\overline{\Phi_+^{\epsilon(\mu\oplus\nu)}}^T\Phi_+^{\epsilon(\mu\oplus\nu)}\bar\partial\Phi_-^{\epsilon\mu} + \overline{\Phi_-^{\epsilon\mu}}^T\overline{\partial\Phi_+^{\epsilon(\mu\oplus\nu)}}^T\partial\Phi_+^{\epsilon(\mu\oplus\nu)}\Phi_-^{\epsilon\mu}\\
&\qquad\quad+ \overline{\Phi_-^{\epsilon\mu}}^T\overline{\Phi_+^{\epsilon(\mu\oplus\nu)}}^T\partial\Phi_+^{\epsilon(\mu\oplus\nu)}\bar\partial\Phi_-^{\epsilon\mu} + \overline{\Phi_-^{\epsilon\mu}}^T\overline{\Phi_+^{\epsilon(\mu\oplus\nu)}}^T\bar\partial\partial\Phi_+^{\epsilon(\mu\oplus\nu)}\Phi_-^{\epsilon\mu}\\
&\qquad\quad+ \overline{\Phi_-^{\epsilon\mu}}^T\overline{\bar\partial\partial\Phi_+^{\epsilon(\mu\oplus\nu)}}^T\Phi_+^{\epsilon(\mu\oplus\nu)}\Phi_-^{\epsilon\mu} + \overline{\Phi_-^{\epsilon\mu}}^T\overline{\bar\partial\Phi_+^{\epsilon(\mu\oplus\nu)}}^T\Phi_+^{\epsilon(\mu\oplus\nu)}\bar\partial\Phi_-^{\epsilon\mu}\big).
\end{aligned}$$

For all the terms where two different factors are differentiated we are only able to match the ϵ-derivatives in one way that is nonzero. We also have that $\bar\partial\partial\Phi_+^{\epsilon(\mu\oplus\nu)} = \partial\epsilon\mu\partial\Phi_+^{\epsilon(\mu\oplus\nu)}$, and so we need to derive it with respect to ϵ and $\bar\epsilon$ to get a nonzero contribution. For the same reason $\bar\partial\Phi_+^{\epsilon(\mu\oplus\nu)}$ needs to be differentiated twice to be nonzero. Since $\bar\partial\Phi_+^{\epsilon(\mu\oplus\nu)}$ is always paired with another term, we need to differentiate these terms and hence they will not contribute, thus we get that

$$\begin{aligned}
\Delta_0\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}})^T\Phi_2^{\epsilon(\mu\oplus\nu)} &= y^2(\overline{(\nu_2)}^T\overline{(-\bar\nu_1^T)}^T + 0 + \overline{(\nu_2)}^T\nu_1 + \overline{(-\bar\nu_1)^T}^T\\
&\qquad\cdot(-\bar\nu_2^T) + (-\bar\nu_2)^T\nu_1 - \mu_1\bar\nu_2^T - \bar\mu_2\nu_1 + 0)\\
&= y^2([\nu_1,\bar\nu_2^T] - \partial\mu_1\bar\nu_2^T - \partial\bar\mu_2\nu_1).
\end{aligned}$$

We conclude that

$$\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}})^T\Phi_2^{\epsilon(\mu\oplus\nu)} = \Delta_0^{-1}((-\star)\mathrm{ad}\nu_2\star\nu_1 - \star\partial\mu_1\bar\nu_2^T - \star\bar\partial\bar\mu_2\nu_1) + cI$$

for some constant c. Since the kernel of Δ_0 is the constant multiples of I. In what remains this term will not contribute, as we will be looking at
$\mathrm{ad}(\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}})^T\Phi_2^{\epsilon(\mu\oplus\nu)})$.

Next we calculate the second term $\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\epsilon(\mu\oplus\nu)}$. The calculation follows directly from the previous computation.

$$
\begin{aligned}
\bar\partial\Delta_0^{-1}((-\star)\mathrm{ad}\nu_2\star\nu_1 - \star\mu_1\bar\nu_2^T - \star\bar\mu_2\nu_1) &= \bar\partial\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}})^T\Phi_2^{\epsilon(\mu\oplus\nu)}\\
&= \frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}})^T\bar\partial\Phi_2^{\epsilon(\mu\oplus\nu)} + \frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{\partial\Phi_2^{\epsilon(\mu\oplus\nu)}})^T\Phi_2^{\epsilon(\mu\oplus\nu)}\\
&= \frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}})^T(\Phi_2^{\epsilon(\mu\oplus\nu)}\epsilon\nu + \epsilon\mu\partial\Phi_2^{\epsilon(\mu\oplus\nu)})\\
&\quad + \frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\epsilon(\mu\oplus\nu)}})^T\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}}^T\Phi_2^{\epsilon(\mu\oplus\nu)}\\
&= -\mu_1\bar\nu_2^T + \frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}(\overline{(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}\partial\Phi_2^{\epsilon(\mu\oplus\nu)}})^T,
\end{aligned}
$$

since $\frac{d}{d\epsilon}|_{\epsilon=0}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}})^T\Phi_2^{\epsilon(\mu\oplus\nu)} = 0$.

Finally we need to calculate the third term

$$
\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}|_{\epsilon=0}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon\mu})_*P^{0,1}(\Phi_1^{\epsilon\mu})_*^{-1}\mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)},
$$

but only where both $\bar\partial$ and $\bar\partial^*$ in $\bar\partial\Delta_0^{-1}\bar\partial^*$ has been differentiated. This is simplified by the fact that $\bar\partial$ only depending on ϵ and not $\bar\epsilon$ (see (5.1)). Using this and (5.2) we have that

$$
\begin{aligned}
&\frac{d}{d\epsilon_1}|_{\epsilon=0}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon\mu})_*\bar\partial(\Phi_1^{\epsilon\mu})_*^{-1}\mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)}\Delta_0^{-1}\\
&\qquad\qquad \frac{d}{d\bar\epsilon_2}|_{\epsilon=0}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon\mu})_*\bar\partial^*(\Phi_1^{\epsilon\mu})_*^{-1}\mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)}\\
&\qquad\qquad = (-\mu_1\partial + \mathrm{ad}\nu_1)\Delta_0^{-1}(\partial^*\bar\mu_2 - \star\mathrm{ad}\nu_2\star).
\end{aligned}
$$

Now we are ready to prove that

Theorem 5.2. *Consider the second variation of the metric in the coordinates on the universal moduli space of pairs of a Riemann surface and a holomorphic bundle on it. Then we have this second variation at the center is*

$$
\begin{aligned}
&\frac{d^2}{d\epsilon_1 d\bar\epsilon_2}\bigg|_{\epsilon=0} g_{\epsilon(\mu\oplus\nu)}\mu_3\oplus\nu_3, \mu_4\oplus\nu_4) =\\
&\qquad - i\int_\Sigma tr((-\mu_1\partial + ad\nu_1)\Delta_0^{-1}(\partial^*\bar\mu_2 - \star ad\nu_2\star)\nu_3\wedge\bar\nu_4^T
\end{aligned}
$$

$$
\begin{aligned}
&- i\int_\Sigma tr(ad\Delta_0^{-1}((-\star)ad\nu_2\star\nu_1 - \star(\partial\mu_1\bar{\nu}_2^T) - \star(\bar{\partial}\bar{\mu}_2\nu_1))\nu_3\wedge\bar{\nu}_4^T)\\
&+ i\int_\Sigma \mu_1\bar{\mu}_2 tr\nu_3\wedge\bar{\nu}_4^T\\
&- i\int_\Sigma tr(ad\nu_1+\mu_1\partial)\Delta_o^{-1}\bar{\partial}^*\mu_3\bar{\nu}_2^T\wedge\bar{\nu}_4^T\\
&- i\int_\Sigma tr\mu_3(\partial\Delta_0^{-1}(\star[\star v_1\nu_2] - \star(\partial\mu_1\bar{\nu}_2^T) - \star(\bar{\partial}\bar{\mu}_2\nu_1))\wedge\bar{\nu}_4^T\\
&- i\int_\Sigma tr\bar{\mu}_2\mu_3\nu_1\wedge\overline{\nu_4}^T\\
&- i\int_\Sigma tr\bar{\partial}\Delta_o^{-1}(-\star\ ad\nu_2\star+\partial^*\mu_2)\nu_3\wedge\bar{\mu}_4\nu_1\\
&- i\int_\Sigma tr\nu_3\wedge\overline{\mu_4(\partial\Delta_0^{-1}(\star[\star v_2\nu_1] - \star(\partial\mu_2\bar{\nu}_1^T) - \star(\bar{\partial}\bar{\mu}_1\nu_2))}^T\\
&- i\int_\Sigma tr\nu_3\wedge\overline{\bar{\mu}_1\mu_4\nu_2}^T - i\int_\Sigma tr\mu_3\nu_1\wedge\overline{\mu_4\nu_2}^T.
\end{aligned}
$$

Proof. Since we already have computed all the ingredients, we gather the results here.

$$
\begin{aligned}
-i\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2} vert_{\epsilon=0}&\int_\Sigma \mathrm{tr}P^{0,1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_3\\
&\quad\wedge\overline{P^{0,1}((\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_4}^T\\
&= -i\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}\int_\Sigma \mathrm{tr}\mathrm{Ad}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}}^T\Phi_2^{\epsilon(\mu\oplus\nu)})(1-|\epsilon\mu|^2))\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}\\
&\quad(\Phi_1^{\epsilon(\mu\oplus\nu)})_* P^{0,1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_3\wedge\overline{\nu_4}^T\\
&= -i\int_\Sigma \mathrm{tr}\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}\mathrm{Ad}(\overline{\Phi_2^{\epsilon(\mu\oplus\nu)}}^T\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_3\wedge\overline{\nu_4}^T\\
&- i\int_\Sigma \mathrm{tr}\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*\\
&\quad P^{0,1}(\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1}\mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_3\wedge\overline{\nu_4}^T\\
&- i\int_\Sigma \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(1-|\epsilon\mu|^2)\mathrm{tr}\nu_3\wedge\bar{\nu}_4^T\\
&= -i\int_\Sigma \mathrm{tr}(\mathrm{ad}(\Delta_0^{-1}((-\star)\mathrm{ad}\nu_2\star\nu_1 - \star(\partial\mu_1\bar{\nu}_2^T) - \star(\bar{\partial}\bar{\mu}_2\nu_1)))\nu_3\wedge\bar{\nu}_4^T)\\
&- i\int_\Sigma \mathrm{tr}((-\mu_1\partial+\mathrm{ad}\nu_1)\Delta_0^{-1}(\partial^*\bar{\mu}_2-\star\mathrm{ad}\nu_2\star)\nu_3\wedge\bar{\nu}_4^T\\
&+ i\int_\Sigma \mu_1\bar{\mu}_2\mathrm{tr}\nu_3\wedge\bar{\nu}_4^T.
\end{aligned}
$$

Now for the second term we have that

$$
\begin{aligned}
\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0} - i \int_\Sigma & \mathrm{tr} P^{0,1} (\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1} \mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)}) \mu_3 (\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1} \partial \Phi_2^{\epsilon(\mu\oplus\nu)} \\
& \qquad \wedge \overline{P^{0,1}((\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1} \mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)})\nu_4}^T \\
&= -i \int_\Sigma \mathrm{tr} P^{0,1} \mu_3 \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0} (\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1} \partial \Phi_2^{\epsilon(\mu\oplus\nu)} \wedge \overline{\nu_4}^T \\
&\quad - i \int_\Sigma \mathrm{tr} \frac{d}{d\epsilon_1}|_{\epsilon=0} \mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1} (\Phi_1^{\epsilon(\mu\oplus\nu)})_* P^{0,1} (\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1} \\
&\qquad \mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)}) \mu_3 \frac{d}{d\bar{\epsilon}_2}|_{\epsilon=0} (\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1} \partial \Phi_2^{\epsilon(\mu\oplus\nu)} \wedge \overline{\nu_4}^T \\
&= -i \int_\Sigma \mathrm{tr} P^{0,1} \mu_3 (\partial \Delta_0^{-1} ((-\star)\mathrm{ad}\nu_2 \star \nu_1 - \star(\bar\partial \bar\mu_2 \nu_1) - \star(\partial \mu_1 \bar\nu_2^T)) \\
&\qquad + \bar\mu_2 \nu_1) \wedge \overline{\nu_4}^T \\
&\quad - i \int_\Sigma \mathrm{tr}(-\mu_1 \partial + \mathrm{ad}\nu_1) \Delta_0^{-1} \bar\partial^* \mu_3 \bar\nu_2^T \wedge \overline{\nu_4}^T.
\end{aligned}
$$

And similarly

$$
\begin{aligned}
-i \int_\Sigma & \mathrm{tr} P^{0,1} (\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1} \mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)}) \nu_3 \\
& \qquad \wedge \overline{P^{0,1} (\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1} (\mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)}) \mu_4 (\Phi_2^{\epsilon(\mu\oplus\nu)}) \partial \Phi_2^{\epsilon(\mu\oplus\nu)}}^T \\
&= -i \int_\Sigma \mathrm{tr} \nu_3 \wedge \overline{P^{0,1} \mu_4 (\partial \Delta_0^{-1} ((-\star)\mathrm{ad}\nu_1 \star \nu_2 - \star(\bar\partial \bar\mu_1 \nu_2) - \star(\partial \mu_2 \bar\nu_1^T)) + \bar\mu_1 \nu_2)}^T \\
&\quad - i \int_\Sigma \mathrm{tr} \bar\partial \Delta_0^{-1} (-\partial^* \mu_2 - \star \mathrm{ad}\nu_2 \star) \nu_3 \wedge \overline{\mu_4} \nu_1.
\end{aligned}
$$

Finally there is not much choice in how to differentiate the following term

$$
\begin{aligned}
-i \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0} \int_\Sigma & \mathrm{tr} P^{0,1} (\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1} \mathrm{Ad}(\Phi_2^{\epsilon(\mu\oplus\nu)}) \mu_3 (\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1} \partial \Phi_2^{\epsilon(\mu\oplus\nu)} \\
& \qquad \wedge \overline{P^{0,1} ((\Phi_1^{\epsilon(\mu\oplus\nu)})_*^{-1} \mathrm{Ad}\Phi_2^{\epsilon(\mu\oplus\nu)}) \mu_4 (\Phi_2^{\epsilon(\mu\oplus\nu)})^{-1} \partial \Phi_2^{\epsilon(\mu\oplus\nu)}}^T \\
&= -i \int_\Sigma \mathrm{tr} P^{0,1} (\mu_3 \frac{d}{d\bar{\epsilon}_2}|_{\epsilon=0} \partial \Phi_2^{\epsilon(\mu\oplus\nu)} \wedge \overline{P^{0,1} \mu_4 \frac{d}{d\bar{\epsilon}_1}|_{\epsilon=0} \partial \Phi_2^{\epsilon(\mu\oplus\nu)}}^T \\
&= -i \int_\Sigma \mathrm{tr} \mu_3 \bar\nu_2^T \wedge \bar\mu_4 \nu_1.
\end{aligned}
$$

Collect all these results and we have the conclusion. □

5.2 The Variation of the Metric in Fibered Coordinates

Now for the fibered coordinates we can do the same computations. From the calculation of the Kodaira-Spencer map (Proposition 4.3) we know, that the metric in the moduli space of bundles direction is given by

$$
\begin{aligned}
&g_{VB}^{\epsilon\nu^{\epsilon\mu}}(\mu_1 \oplus \nu_1, \mu_2 \oplus \nu_2) = -i \int_\Sigma P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad} f^{\epsilon\nu^{\epsilon\mu}} \nu_1^{\epsilon\mu} \wedge \overline{P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad} f^{\epsilon\nu^{\epsilon\mu}} \nu_2^{\epsilon\mu}}^T \\
&- i \int_\Sigma P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad} f^{\epsilon\nu^{\epsilon\mu}} \nu_1^{\epsilon\mu} \wedge \overline{P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})(\mu_2^{\epsilon\mu}(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}})}^T \\
&- i \int_\Sigma P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad} f^{\epsilon\nu^{\epsilon\mu}} \nu_1^{\epsilon\mu} \\
&\wedge \overline{P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})((\Phi_1^{\epsilon\mu})_*^{-1}(1-|\epsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\mu_2})P^{0,1}(\Phi_1^{\epsilon\mu+t\mu_2})^{-1}\nu)}^T \\
&- i \int_\Sigma P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})(\mu_1^{\epsilon\mu}(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}}) \wedge \overline{P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad} f^{\epsilon\nu^{\epsilon\mu}} \nu_2^{\epsilon\mu}}^T \\
&- i \int_\Sigma P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})(\mu_1^{\epsilon\mu}(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}}) \wedge \overline{P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})(\mu_2^{\epsilon\mu}(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}})}^T \\
&- i \int_\Sigma P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})(\mu_1^{\epsilon\mu}(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}}) \\
&\wedge \overline{P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})((\Phi_1^{\epsilon\mu})_*^{-1}(1-|\epsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\mu_2})P^{0,1}(\Phi_1^{\epsilon\mu+t\mu_2})^{-1}\nu)}^T \\
&- i \int_\Sigma P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})((\Phi_1^{\epsilon\mu})_*^{-1}(1-|\epsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\mu_1})P^{0,1}(\Phi_1^{\epsilon\mu+t\mu_1})^{-1}\nu) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \wedge \overline{P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad} f^{\epsilon\nu^{\epsilon\mu}} \nu_2^{\epsilon\mu}}^T \\
&- i \int_\Sigma P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})((\Phi_1^{\epsilon\mu})_*^{-1}(1-|\epsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\mu_1})P^{0,1}(\Phi_1^{\epsilon\mu+t\mu_1})^{-1}\nu) \\
&\qquad\qquad\qquad \wedge \overline{P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})(\mu_2^{\epsilon\mu}(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}})}^T \\
&- i \int_\Sigma P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})((\Phi_1^{\epsilon\mu})_*^{-1}(1-|\epsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\mu_1})P^{0,1}(\Phi_1^{\epsilon\mu+t\mu_1})^{-1}\nu) \\
&\wedge \overline{P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})((\Phi_1^{\epsilon\mu})_*^{-1}(1-|\epsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\mu_2})P^{0,1}(\Phi_1^{\epsilon\mu+t\mu_2})^{-1}\nu)}^T \ .
\end{aligned}
$$

While these nine terms look intimidating, we can discard three of the terms, because $P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})((\Phi_1^{\epsilon\mu})_*^{-1}(1-|\epsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\mu_2})P^{0,1}(\Phi_1^{\epsilon\mu+t\mu_2})^{-1}\nu)$ vanishes to second-order and $P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})(\mu_2^{\epsilon\mu}(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}})$ vanishes to first-order, so terms containing both kind of factors or only the first kind of factors will vanish to higher order, than we are interested in. Now the first variation will be the same as in Section 5, but to calculate it we will have to work with slightly different expressions.

First we consider

$$
\frac{d}{d\epsilon}|_{\epsilon=0}((\overline{f^{\epsilon\nu^{\epsilon}\mu}})^T f^{\epsilon\nu^{\epsilon}\mu}) \circ \Phi_1^{\epsilon\mu} = 0
$$

and

$$\frac{d}{d\bar{\epsilon}}|_{\epsilon=0}((\overline{f^{\epsilon\nu^{\epsilon}\mu}})^T f^{\epsilon\nu^{\epsilon}\mu}) \circ \Phi_1^{\epsilon\mu} = 0.$$

Both of these follow from the computations in [TZ1], where it was shown that $\frac{d}{d\bar{\epsilon}}|_{\epsilon=0}((\overline{f^{\epsilon\nu}})^T f^{\epsilon\nu}) = 0$. Now composing with $\nu \to \nu^{\epsilon\mu}$ won't change it, and if we differentiate $\Phi_1^{\epsilon\mu}$ then we can set $\epsilon = 0$ in the rest of the terms and calculate $\frac{d}{d\epsilon} I \circ \Phi_1^{\epsilon\mu} = 0$. Now for a projection, the first derivative will either have harmonic forms in it's kernel or the image is in the orthogonal complement, hence the only contributions are from the terms

$$\int_\Sigma P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad} f^{\epsilon\nu^{\epsilon\mu}} \nu_1^{\epsilon\mu} \overline{P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})(\mu_2^{\epsilon\mu}(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}})}^T$$

and

$$\int_\Sigma P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})(\mu_1^{\epsilon\mu}(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}}) \overline{P^{0,1}_{\epsilon\nu^{\epsilon\mu}} \mathrm{Ad} f^{\epsilon\nu^{\epsilon\mu}} \nu_2^{\epsilon\mu}}^T.$$

And so we have, completely analogues to the previous section the following lemma.

Lemma 5.3. *In the fibered coordinates around (X, E) we have that:*

$$\frac{d}{d\epsilon}|_{\epsilon=0} g^{VB}_{\epsilon\nu^{\epsilon\mu}}(\mu_1 \oplus \nu_1, \mu_2 \oplus \nu_2) = i \int_X \bar{\mu}_2 tr(\nu_1 \nu)$$
$$\frac{d}{d\bar{\epsilon}}|_{\epsilon=0} g^{VB}_{\epsilon\nu^{\epsilon\mu}}(\mu_1 \oplus \nu_1, \mu_2 \oplus \nu_2) = i \int_X \mu_1 tr(\bar{\nu}^T \bar{\nu}_2^T)$$

Now for the second variation of the metric we need to calculate the two terms

$$\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}((\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}}) \circ \Phi_1^{\epsilon\mu}$$

and

$$\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}((f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}}) \circ \Phi_1^{\epsilon\mu}.$$

We calculate these the same way as we did with the previous set of coordinates.

$$\begin{aligned}\Delta_0 \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}((\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}}) \circ \Phi_1^{\epsilon\mu} &= y^2 \partial\bar{\partial} \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}((\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}}) \circ \Phi_1^{\epsilon\mu} \\ &= \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0} y^2 \partial\bar{\partial}((\overline{f_-^{\epsilon\nu^{\epsilon\mu}}})^T (\overline{f_+^{\epsilon\nu^{\epsilon\mu}}})^T f_+^{\epsilon\nu^{\epsilon\mu}} f_-^{\epsilon\nu^{\epsilon\mu}}) \circ \Phi_1^{\epsilon\mu}.\end{aligned}$$

Now we use

$$\begin{aligned}\bar{\partial}\partial(h \circ \Phi_1^{\epsilon\mu}) = &\partial\Phi_1^{\epsilon\mu}\partial\bar{\Phi}_1^{\epsilon\mu}(\partial\partial h) \circ \Phi_1^{\epsilon\mu} + \bar{\partial}\bar{\Phi}_1^{\epsilon\mu}\partial\Phi_1^{\epsilon\mu}(\partial\bar{\partial} h) \circ \Phi_1^{\epsilon\mu} \\ &+ \bar{\partial}\Phi_1^{\epsilon\mu}\partial\bar{\Phi}_1^{\epsilon\mu}(\bar{\partial}\partial h) \circ \Phi_1^{\epsilon\mu} + \partial\bar{\Phi}_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}(\bar{\partial}\bar{\partial} h) \circ \Phi_1^{\epsilon\mu}\end{aligned}$$

$$
\begin{aligned}
&= (\partial\Phi_1^{\epsilon\mu}\epsilon\mu\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}(\partial\partial h)\circ\Phi_1^{\epsilon\mu} + \bar{\partial}\bar{\Phi}_1^{\epsilon\mu}\partial\Phi_1^{\epsilon\mu}(\partial\bar{\partial}h)\circ\Phi_1^{\epsilon\mu}\\
&\quad + |\epsilon\mu|^2\partial\Phi_1^{\epsilon\mu}\partial\bar{\Phi}_1^{\epsilon\mu}(\bar{\partial}\partial h)\circ\Phi_1^{\epsilon\mu} + \overline{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}(\bar{\partial}\bar{\partial}h)\circ\Phi_1^{\epsilon\mu},
\end{aligned}
$$

for $h = (\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}})$, and since we know that $\partial f^{\epsilon\nu^{\epsilon\mu}}$ and $\bar{\partial} f^{\epsilon\nu^{\epsilon\mu}}$ vanish to first-order in ϵ, we only have the surviving terms

$$
\Delta_0\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}((\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}})\circ\Phi_1^{\epsilon\mu} = y^2(-\partial\mu_1\bar{\nu}_2^T - \bar{\partial}\bar{\mu}_2\nu_1 + [\nu_1,\bar{\nu}_2^T]),
$$

which is exactly like in the previous case. We proceed on to calculate $\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}((f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}})\circ\Phi_1^{\epsilon\mu}$, and so we study

$$
\begin{aligned}
\bar{\partial}\Delta_0^{-1}y^2(-\mu_1\partial\bar{\nu}_2^T - \bar{\mu}_2\bar{\partial}\nu_1 + [\nu_1,\bar{\nu}_2^T]) &= \bar{\partial}\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}((\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}})\circ\Phi_1^{\epsilon\mu}\\
&= \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(\bar{\partial}((\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}}))\circ\Phi_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}\\
&+ \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(\partial((\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}}))\circ\Phi_1^{\epsilon\mu}\bar{\partial}\Phi_1^{\epsilon\mu}\\
&= \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(((\overline{\partial f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}}))\circ\Phi_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}\\
&+ \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(((\overline{\bar{\partial} f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}}))\circ\Phi_1^{\epsilon\mu}\bar{\partial}\Phi_1^{\epsilon\mu}\\
&+ \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(((\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T \bar{\partial} f^{\epsilon\nu^{\epsilon\mu}}))\circ\Phi_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}\\
&+ \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(((\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T \partial f^{\epsilon\nu^{\epsilon\mu}}))\circ\Phi_1^{\epsilon\mu}\bar{\partial}\Phi_1^{\epsilon\mu}.
\end{aligned}
$$

Here the second and fourth term cancel, as is seen by using that two of the factors vanish to first-order in ϵ, which then give $\mu_1\bar{\nu}_2^T$ and $-\mu_1\bar{\nu}_2^T$ respectively.

$$
\begin{aligned}
&\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(((\overline{\partial f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}}))\circ\Phi_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}\\
&\qquad + \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(((\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T \bar{\partial} f^{\epsilon\nu^{\epsilon\mu}}))\circ\Phi_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}\\
&\quad = \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(((\overline{(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}}}^T \overline{f^{\epsilon\nu^{\epsilon\mu}}}^T f^{\epsilon\nu^{\epsilon\mu}}))\circ\Phi_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}\\
&\qquad + \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(((\overline{f^{\epsilon\nu^{\epsilon\mu}}})^T f^{\epsilon\nu^{\epsilon\mu}}\epsilon\nu^{\epsilon\mu}))\circ\Phi_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}\\
&\quad = \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(((\overline{(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}}}^T)\circ\Phi_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}\\
&\qquad + \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}(\epsilon\nu^{\epsilon\mu}))\circ\Phi_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu}
\end{aligned}
$$

$$= \frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}((\overline{(f^{\epsilon\nu^{\epsilon\mu}})^{-1}\partial f^{\epsilon\nu^{\epsilon\mu}}}^T) \circ \Phi_1^{\epsilon\mu}\bar{\partial}\bar{\Phi}_1^{\epsilon\mu} + \bar{\partial}\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1.$$

Now this is different from the previous coordinates. We need however to consider two things more. The first is the second variation of the harmonic projection. Since the only relevant part is the contribution where both $\bar{\partial}$ and $\bar{\partial}^*$ have been differentiated in $\bar{\partial}\Delta_0^{-1}\bar{\partial}^*$ and the coordinates agree to second-order nothing will have changed and we have it gives the following term

$$(-\mu_1\partial + \mathrm{ad}\nu_1)\Delta_0^{-1}(\partial^*\bar{\mu}_2 - \star\mathrm{ad}\nu_2\star).$$

The final term to consider is the new term in the formula for the metric, which is

$$\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}P^{0,1}_{\epsilon\nu^{\epsilon\mu}}\mathrm{Ad}(f^{\epsilon\nu^{\epsilon\mu}})((\Phi_1^{\epsilon\mu})_*^{-1}(1-|\epsilon\mu|^2)\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\tilde{\mu}})P^{0,1}(\Phi_1^{\epsilon\mu+t\tilde{\mu}})^{-1}\epsilon\nu),$$

as $\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\tilde{\mu}})P^{0,1}(\Phi_1^{\epsilon\mu+t\tilde{\mu}})^{-1}\epsilon\nu)$ vanishes to second-order this has to be differentiated twice

$$\begin{aligned}
&\frac{d}{d\bar{\epsilon}_2}|_{\epsilon=0}\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\tilde{\mu}})P^{0,1}(\Phi_1^{\epsilon\mu+t\tilde{\mu}})^{-1}\nu_1)\\
&\qquad = -(\frac{d}{d\bar{\epsilon}_2}|_{\epsilon=0}\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\tilde{\mu}})\bar{\partial}(\Phi_1^{\epsilon\mu+t\tilde{\mu}})^{-1}\\
&\qquad\qquad\qquad \Delta_0^{-1}\frac{d}{d\bar{\epsilon}_2}|_{\epsilon=0}\Phi_1^{\epsilon\mu+t\tilde{\mu}})\bar{\partial}^*(\Phi_1^{\epsilon\mu+t\tilde{\mu}})^{-1}\nu_1)\\
&\qquad - \bar{\partial}\frac{d}{d\bar{\epsilon}_2}|_{\epsilon=0}\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\tilde{\mu}})\Delta_0^{-1}\bar{\partial}^*(\Phi_1^{\epsilon\mu+t\tilde{\mu}})^{-1})\nu_1\\
&\qquad = -\tilde{\mu}\partial\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1 - \bar{\partial}\frac{d}{d\bar{\epsilon}_2}|_{\epsilon=0}\frac{d}{dt}|_{t=0}(\Phi_1^{\epsilon\mu+t\tilde{\mu}})\Delta_0^{-1}\bar{\partial}^*(\Phi_1^{\epsilon\mu+t\tilde{\mu}})^{-1}\nu_1.
\end{aligned}$$

We are now ready to gather all the contributions in the following theorem.

Theorem 5.4. *We have the following for the second variation of the metric at* (X, E) *in the fibered coordinates:*

$$\begin{aligned}
&\frac{d^2}{d\epsilon_1 d\bar{\epsilon}_2}|_{\epsilon=0}g^{VB}_{\epsilon\nu^{\epsilon\mu}}(\mu_3\oplus\nu_3, \mu_4\oplus\nu_4) =\\
&\qquad \int_\Sigma tr((-\mu_1\partial + ad\nu_1)\Delta_0^{-1}((-\star)ad\nu_2\star + \partial^*\bar{\mu}_2)\nu_3)\wedge\bar{\nu}_4^T\\
&\qquad - i\int_\Sigma tr(ad\Delta_0^{-1}((-\star)ad\nu_2\star\nu_1 - \star(\partial\mu_1\bar{\nu}_2^T) - \star(\bar{\partial}\bar{\mu}_2\nu_1))\nu_3\wedge\bar{\nu}_4^T)\\
&\qquad - i\int_\Sigma tr(ad\nu_1 - \mu_1\partial)\Delta_o^{-1}\bar{\partial}^*\mu_3\bar{\nu}_2^T\wedge\bar{\nu}_4^T\\
&\qquad + i\int_\Sigma \mu_1\bar{\mu}_2 tr\nu_3\wedge\bar{\nu}_4^T
\end{aligned}$$

$$
\begin{aligned}
&- i \int_\Sigma tr\mu_3(\partial\Delta_0^{-1}(\star[\star\nu_1\nu_2] - \star(\partial\mu_1\bar{\nu}_2^T) - \star(\bar{\partial}\bar{\mu}_2\nu_1)) \wedge \bar{\nu}_4^T \\
&+ i \int_\Sigma tr\mu_3\partial\Delta_0^{-1}\bar{\partial}^*\mu_1\bar{\nu}_2^T \wedge \overline{\nu_4}^T \\
&- i \int_\Sigma tr\bar{\partial}\Delta_o^{-1}(-\star\, ad\nu_2 \star + \partial^*\bar{\mu}_2)\nu_3 \wedge (-\bar{\mu}_4\nu_1) \\
&- i \int_\Sigma tr\nu_3 \wedge \overline{\mu_4(\partial\Delta_0^{-1}(\star[\star\nu_2\nu_1] - \star(\partial\mu_2\bar{\nu}_1^T) - \star(\bar{\partial}\bar{\mu}_1\nu_2))}^T \\
&+ i \int_\Sigma tr\nu_3 \wedge \overline{\mu_4\partial\Delta_0^{-1}\bar{\partial}^*\mu_2\bar{\nu}_1^T}^T - i \int_\Sigma tr\mu_3\nu_1 \wedge \overline{\mu_4\nu_2}^T \\
&+ i \int_\Sigma \mu_3\partial\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1 \wedge \bar{\nu}_4^T + i \int_\Sigma \nu_3 \wedge \overline{\mu_4\partial\Delta_0^{-1}\partial^*\bar{\mu}_1\nu_2}^T
\end{aligned}
$$

Comparing this to the previous coordinates (Theorem 5.2) we see that there are four terms here which we didn't have before and two we no longer have. The new terms are

$$
\begin{gathered}
i \int_\Sigma \mu_3\partial\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1 \wedge \bar{\nu}_4^T, \\
i \int_\Sigma \nu_3 \wedge \overline{\mu_4\partial\Delta_0^{-1}\partial^*\bar{\mu}_1\nu_2}^T, \\
i \int_\Sigma \mathrm{tr}\mu_3\partial\Delta_0^{-1}\bar{\partial}^*\mu_1\bar{\nu}_2^T \wedge \overline{\nu_4}^T
\end{gathered}
$$

and

$$
i \int_\Sigma \mathrm{tr}\nu_3 \wedge \overline{\mu_4\partial\Delta_0^{-1}\bar{\partial}^*\mu_2\bar{\nu}_1^T}^T,
$$

While the ones we no longer have are

$$
-i \int_\Sigma \mathrm{tr}\nu_3 \wedge \overline{\mu_1\bar{\mu}_4\nu_2}^T
$$

and

$$
-i \int_\Sigma \mathrm{tr}\bar{\mu}_2\mu_3\nu_1 \wedge \overline{\nu_4}^T.
$$

Now we have that for $\nu_1 = \nu_4$ and $\mu_2 = \mu_3$ and the rest 0 the difference between the two expressions are

$$
\begin{aligned}
- i \int_\Sigma \mu_3\partial\Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1 \wedge \bar{\nu}_4^T - i \int_\Sigma \mathrm{tr}\bar{\mu}_2\mu_3\nu_1 \wedge \overline{\nu_4}^T& \\
= -i \int_\Sigma \Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1 \wedge \overline{\partial^*\bar{\mu}_2\nu_1}^T \rho - i \int_\Sigma \mathrm{tr}\bar{\mu}_2\mu_2\nu_1 \wedge \overline{\nu_1}^T&
\end{aligned}
$$

Since Δ is a positive operator we have that $-i \int_\Sigma \Delta_0^{-1}\partial^*\bar{\mu}_2\nu_1 \wedge \overline{\partial^*\bar{\mu}_2\nu_1}^T \rho \geq 0$ and for obvious reasons $-i \int_\Sigma \mathrm{tr}\bar{\mu}_2\mu_2\nu_1 \wedge \overline{\nu_1}^T > 0$, if $\nu_1 \neq 0$ and $\mu_2 \neq 0$.

Theorem 5.5. *The coordinates of 1.1 and the Fibered coordinates agree to second order, but differ at third order in the derivatives at the center point.*

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Jørgen Ellegaard Andersen and Niccolo Skovgård Poulsen
Center for Quantum Geometry of Moduli Spaces
Department of Mathematics
University of Aarhus
DK-8000, Denmark

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Asymptotic aspects of the Teichmüller TQFT

by Jørgen Ellegaard Andersen and Jens-Jakob Kratmann Nissen[1]

Abstract

We calculate the knot invariant coming from the Teichmüller TQFT introduced by Andersen and Kashaev a couple of years ago. Specifically we calculate the knot invariant for the complement of the knot 6_1 both in Andersen and Kashaev's original version of Teichmüller TQFT but also in their new formulation of the Teichmüller TQFT for the one-vertex H-triangulation of $(S^3, 6_1)$. We show that the two formulations give equivalent answers. Furthermore we apply a formal stationary phase analysis and arrive at the Andersen-Kashaev volume conjecture.

Furthermore we calculate the first examples of knot complements in the new formulation showing that the new formulation is equivalent to the original one in all the special cases considered.

Finally, we provide an explicit isomorphism between the Teichmüller TQFT representation of the mapping class group of a once punctured torus and a representation of this mapping class group on the space of Schwartz class functions on the real line.

1 Introduction

Since discovered and axiomatised by Atiyah [At], Segal [S] and Witten [W], Topological Quantum Field Theories (TQFT's) have been studied extensively. The first constructions of such theories in dimension $2+1$ was given by Reshetikhin and Turaev [T, RT1, RT2] who obtained TQFT's through surgery and the combinatorial framework of Kirby calculus, and by Turaev and Viro [TV] using the framework of triangulations and Pachner moves. In both constructions the central algebraic ingredients comes from the category of finite dimensional representation of the quantum group $U_q(\mathfrak{sl}(2,\mathbb{C}))$ at roots of unity. Subsequently Blanchet, Habegger, Masbaum and Vogel gave a pure topological construction using Skein theory [BHMV1, BHMV2]. Recently it has been established by the first author and Ueno that this TQFT is equivalent to the one coming from conformal field theory [AU1, AU2, AU3, AU4] and further by the work of Laszlo [L] in the higher genus

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case with no marked point and the first author and Egsgaard [AE] in genus zero with marked points (for certain labels), that these TQFT's can be studied from the point of view of geometric quantization of the compact moduli space of flat SU(2) connections. The first author has extensively studied the asymptotics of this TQFT using this quantization of moduli spaces approach to this theory [A1, A2, AGr1, AH, AMU, A3, A4, A5, AGa1, AB1, A6, AGL, AHi, A7, AHJMMc].

A new line of development was initiated by Kashaev in [K1] where a state sum invariant of links in 3-manifolds was defined by using the combinatorics of charged triangulations. Here the charges are algebraic versions of dihedral angles of ideal hyperbolic tetrahedra in finite cyclic groups. The approach was subsequently developed further by Baseilhac, Benedetti and by Geer, Kashaev and Turaev [BB, GKT].

New challenges appear when one tries to construct combinatorial versions of Chern–Simons theory with non-compact gauge group such as $\mathrm{PSL}(2,\mathbb{R})$, which is the isometry group of 2-dimensional hyperbolic space. When one considers the corresponding classical moduli space of flat $\mathrm{PSL}(2,\mathbb{R})$-connections on a two dimensional surface, a connected component is identified with Teichmüller space, hence this Chern–Simons theory deserves the name Teichmüller TQFT.

Quantum Teichmüller theory corresponds to a specific classes of unitary mapping class representations on infinite dimensional Hilbert spaces [K2, FC]. Based on quantum Teichmüller theory several formal state-integral partition functions have been studied by Hikami, Dimofte, Gukov, Lenells, Zagier, Dijkgraaf, Fuji, Manabe [H1, H2, DGLZ, DFM] with the view to approach the Teichmüller TQFT. The question about convergence of the studied integrals however remained open until a mathematical rigorous version of Teichmüller TQFT was suggested by the first author and Kashaev in [AK1]. See also [AK1a, AK1b]. The convergence property of the Teichmüller TQFT is a property of the underlying combinatorial setting. An extra structure on the triangulations called a shape structure is imposed where each tetrahedron carries dihedral angles of an ideal hyperbolic tetrahedron. The dihedral angles provide absolute convergence and moreover they implement the complete symmetry with respect to change of edge orientation. The positivity condition of dihedral angles seems to impose restrictions on the construction of topologically invariant partition functions. In [KLV] Kashaev, Luo and Vartanov suggests a TQFT of Turaev–Viro type based on the combinatorics of shaped triangulations. As the absolute convergence of the partition function in this model is also based on positivity of dihedral angles, it is similar to the Teichmüller TQFT. A consequence is that as in the case of the Teichmüller TQFT the $2-3$ Pachner move is not immediately always applicable. However, in this model no other topological restrictions are needed. A new formulation of the Teichmüller TQFT was suggested in [AK2]. In the new formulation of Teichmüller TQFT both the $2-3$ and $3-2$ Pachner moves are applicable and as in the case of the TQFT of Turaev–Viro type [KLV] no other topological restrictions are needed.

Recently in [AK3] the first author of this paper and Kashaev have constructed quantum Chern–Simons theory for $\mathrm{PSL}(2,\mathbb{C})$ for all non-negative integer levels k and furthermore understood how it relates to geometric quantization of $\mathrm{PSL}(2,\mathbb{C})$-moduli spaces. They have proposed a general scheme which just requires a Pontryagin self-dual locally compact group, which is expected to lead to the construction of the $\mathrm{SL}(n,\mathbb{C})$ quantum Chern-Simons theory for all non-negative integer levels k. From the geometric quantization of moduli spaces viewpoint, the corresponding representations of the mapping class groups have been constructed in [AG]. This work is closely related to the work of Dimofte [Di] on the physics side. The Teichmüller TQFT is the complex quantum Chern–Simons TQFT for $\mathrm{PSL}(2,\mathbb{C})$ at level $k = 1$. See also [AM] in this volume.

Outline

We will review the construction of the charged tetrahedral operators which originates from Kashaev's quantization of Teichmüller space. The main ingredients in this theory are Penner's cell decomposition of decorated Teichmüller space and the associated Ptolemy groupoid [P] and Faddeev's quantum dilogarithm [F] which allows us to change polarization on Teichmüller space.

We recall how the partition function from the Teichmüller TQFT is defined using tetrahedral operators in both the original version [AK1] and in the new formulation [AK2].

We will then prove the equivalence of the two versions by direct calculations in several cases and elaborate on the Andersen-Kashaev *volume conjecture* arising in [AK1, Conj. 1.].

Following this, we will investigate the Teichmüller TQFT representation of the genus one, one parked point, mapping class group and prove that it is equivalent to an action of this same mapping class group acting on the space of Schwartz class functions on the real line.

Acknowledgements

We would like to thank Rinat Kashaev for many interesting discussions.

2 Teichmüller Space

As mentioned in the introduction quantum Chern–Simons theory with non-compact gauge group is of great interest. The gauge group in Teichmüller theory is $\mathrm{PSL}(2,\mathbb{R})$ which is the isometry group of 2 dimensional hyperbolic space.

Let M be a 3-manifold. Recall that the classical phase space of Chern–Simons theory with gauge group $\mathrm{PSL}(2,\mathbb{R})$ is given by the moduli space of flat connections

$$\mathcal{M} = \mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}).$$

It is natural to start from $M = \Sigma \times \mathbb{R}$, where $\pi_1(M) = \pi_1(\Sigma)$, so we can talk about the moduli space of flat connections on the surface

$$\mathcal{M}_\Sigma = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}).$$

We can write the moduli space as the disjoint union of connected components

$$\mathcal{M}_\Sigma = \bigsqcup_{-\chi(\Sigma)\le k\le \chi(\Sigma)} (\mathcal{M}_\Sigma)_k.$$

Teichmüller space is given by the connected component with the maximal index

$$\mathcal{T}_\Sigma = (\mathcal{M}_\Sigma)_{-\chi(\Sigma)}.$$

Let $\Sigma = \Sigma_{g,s}$ be a surface of finite type, i.e. Σ is an oriented genus g surface with s boundary components or punctures. Then, topologically, Teichmüller space is an open ball of dimension $6g - 6 + 2s$, i.e.

$$\mathcal{T}_\Sigma \cong \mathbb{R}^{6g-6+2s}.$$

Recall that $\mathcal{T}_\Sigma$ is a symplectic space, where the symplectic structure is given by the Weil–Petersson symplectic form.

2.1 Penner coordinate system on $\tilde{\mathcal{T}}_\Sigma$

Let Σ be an oriented genus g surface with $s > 0$ punctures and Euler characteristic $2 - 2g - s < 0$. We denote the set of punctures

$$V := \{P_1, \ldots, P_s\}.$$

Definition 2.1. A homotopy class of a path running between P_i and P_j is called an ideal arc. A set of ideal arcs obtained by taking a family X of disjointly embedded ideal arcs in Σ running between punctures and subject to the condition that each component of $\Sigma\backslash X$ is a triangle is called an ideal triangulation. Let Δ_Σ denote the set of all ideal triangulations.

Now take an ideal triangulation $\tau \in \Delta_\Sigma$ and calculate all λ-lengths with respect to a fixed configuration of horocycles. Let $E(\tau)$ denote the set of edges in τ. We impose the equivalence relation

$$\lambda \sim \lambda' : E(\tau) \to \mathbb{R}_{>0},$$

if there exists $f : V \to \mathbb{R}_{>0}$ such that

$$\lambda'(e) = f(v_1)f(v_2)\lambda(e) \quad e \in E(\tau),$$

where v_1 and v_2 are endpoints of e. Counting the number of edges and vertices in an ideal triangulation establishes the following

$$\mathbb{R}^{E(\tau)}_{>0}/\mathbb{R}^V_{>0} \cong \mathbb{R}^{6g-6+2s}.$$

The λ-lengths parametrizes the decorated Teichmüller space $\tilde{\mathcal{T}}_\Sigma$, which is a principal $\mathbb{R}^s_{>0}$ foliated fibration $\phi : \tilde{\mathcal{T}}_\Sigma \to \mathcal{T}_\Sigma$, where the fiber over a point of $\mathcal{T}_\Sigma$ is the space of all horocycles about the punctures of Σ.

Theorem 2.2 (Penner)**.** *(a) As a topological space the decorated Teichmüller space is homeomorphic to the set of positive numbers on edges given by λ-lengths*

$$\tilde{\mathcal{T}}_\Sigma \cong \mathbb{R}^{E(\tau)}_{>0}.$$

(b) Using the map ϕ which forgets the horocycles we can pull back the Weil–Petersson symplectic form to the decorated Teichmüller space. The pull-back satisfies the formula

$$\phi^*\omega_{WP} = \sum_{\substack{a \,\triangle\, b \\ c}} \frac{da \wedge db}{ab} + \frac{db \wedge dc}{bc} + \frac{dc \wedge da}{ca}.$$

(c) The mapping class group is contained in the groupoid generated by Ptolemy transformations. Suppose $a, b, c, d, e \in \tau \in \Delta_\Sigma$ are such that $\{a, b, e\}$ and $\{c, d, e\}$ bound distinct triangles. The operation that changes the ideal triangulation τ into τ^e, which consists of the ideal arcs of τ except e, which is replaced by e' such that triangles $\{a, b, e\}$ and $\{c, d, e\}$ are replaced by $\{b, c, e'\}$ and $\{a, d, e'\}$, is called an elementary move (see Figure 1). The six λ-lengths are related by one single equation

$$ee' = ac + bd. \tag{2.1}$$

Due to positivity this is a global coordinate change between parametrizations associated to two ideal triangulations. Two ideal triangulations are related through a sequence of flips. Composing the relations on Ptolemy transformations one obtains the relations between two coordinate systems.

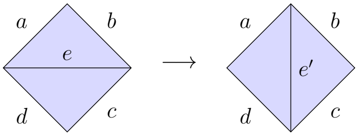


Figure 1: Elementary move

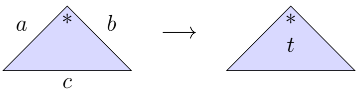


Figure 2: Ratio coordinates

2.2 Ratio Coordinates

Definition 2.3. An ideal triangulation with a choice of distinguished corner for each triangle is called a *decorated ideal triangulation* (d.i.t).

For an ideal triangle with sides having λ-lengths a, b, c we assign ratio coordinates according to Figure 2, where $t = \left(\frac{a}{c}, \frac{b}{c}\right) = (t_1, t_2)$. The pull back of the Weil–Petersson symplectic 2-form is then written in the very simple way

$$\phi^*\omega_{WP} = \sum_{\triangle t} \frac{dt_1 \wedge dt_2}{t_1 t_2} =: \sum_{\triangle t} \omega_t,$$

where the sum is over all triangles.

The d.i.t. τ_t obtained from τ by a change of distinguished corner of triangle t as indicated in Figure 3 is said to be obtained from τ by the *elementary change of decoration* in triangle t. The d.i.t. τ^e obtained from the d.i.t. τ by the elementary move along the i.a. e, where distinguished corners are as indicated in Figure 4, is said to be obtained from τ by the *decorated elementary move* along the i.a. e.

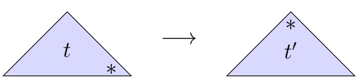


Figure 3: Elementary change of decoration.

It is easily seen that the coordinates u, v are related to the coordinates x, y. The relation is given by the two functions in x and y.

$$u = x \cdot y = (x_1 y_1, x_1 y_2 + x_2),$$

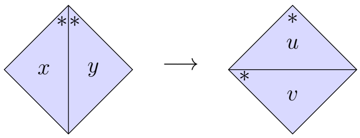


Figure 4: Decorated elementary move

$$v = x * y = \left(\frac{x_2 y_1}{x_1 y_2 + x_2}, \frac{y_2}{x_1 y_2 + x_2} \right).$$

We now observe that

$$\omega_x + \omega_y = \omega_u + \omega_v,$$

so that the change of coordinate with respect to this transformation

$$T : (x, y) \mapsto (u, v)$$

is a symplectomorphism of $\mathbb{R}^4_{>0}$.

3 Tetrahedral operator from quantum Teichmüller theory

We recall the main algebraic ingredients of quantum Teichmüller theory, following the approach of [K2, K3, K4]. Consider the canonical quantization of $T^*\mathbb{R}^n$ with the standard symplectic structure in the position representation. The Hilbert space we get is $L^2(\mathbb{R}^n)$. Position coordinates q_i and momentum coordinates p_i on $T^*\mathbb{R}^n$ upon quantization becomes selfadjoint unbounded operators $\mathbf{q}_i$ and $\mathbf{p}_i$ acting on $L^2(\mathbb{R}^n)$ via the formulae

$$\mathbf{q}_j(f)(t) = t_j f(t), \quad \mathbf{p}_j(f)(t) = \frac{1}{2\pi i} \frac{\partial}{\partial t_j} f(t), \quad \forall t \in \mathbb{R}^n,$$

satisfying the Heisenberg commutation relations

$$[\mathbf{q}_j, \mathbf{q}_k] = [\mathbf{p}_j, \mathbf{p}_k] = 0, \quad [\mathbf{p}_j, \mathbf{q}_k] = \frac{1}{2\pi i} \delta_{j,k}. \tag{3.1}$$

By the spectral theorem, one defines the operators

$$\mathbf{u}_i = e^{2\pi \mathrm{b} \mathbf{q}_i}, \quad \mathbf{v}_i = e^{2\pi \mathrm{b} \mathbf{p}_i}.$$

The commutation relations for $\mathbf{u}_i$ and $\mathbf{v}_j$ takes the form

$$[\mathbf{u}_j, \mathbf{u}_k] = [\mathbf{v}_j, \mathbf{v}_k] = 0, \quad \mathbf{u}_j \mathbf{v}_k = e^{2\pi \mathrm{b}^2 \delta_{j,k}} \mathbf{v}_k \mathbf{u}_j.$$

Consider the operations for $\mathbf{w}_j = (\mathbf{u}_j, \mathbf{v}_j), j \in \{1, 2\}$,

$$\mathbf{w}_1 \cdot \mathbf{w}_2 := (\mathbf{u}_1\mathbf{u}_2, \mathbf{u}_1\mathbf{v}_2 + \mathbf{v}_1), \tag{3.2}$$

$$\mathbf{w}_1 * \mathbf{w}_2 := (\mathbf{v}_1\mathbf{u}_2(\mathbf{u}_1\mathbf{v}_2 + \mathbf{v}_1)^{-1}, \mathbf{v}_2(\mathbf{u}_1\mathbf{v}_2 + \mathbf{v}_1)^{-1}). \tag{3.3}$$

Proposition 3.1 (Kashaev)**.** *Let $\psi(z)$ be some solution to the functional equation*

$$\psi\left(z + \frac{i\,\mathrm{b}}{2}\right) = \psi\left(z - \frac{i\,\mathrm{b}}{2}\right)(1 + e^{2\pi\,\mathrm{b}\,z}), \quad z \in \mathbb{C}. \tag{3.4}$$

Then, the operator

$$\mathbf{T} = \mathbf{T}_{12} := e^{2\pi i \mathbf{p}_1 \mathbf{q}_2} \psi\left(\mathbf{q}_1 - \mathbf{q}_2 + \mathbf{p}_2\right), \tag{3.5}$$

defines a continuous linear map from $\mathcal{S}(\mathbb{R}^4)$ to $\mathcal{S}(\mathbb{R}^4)$, which satisfies the equations

$$\mathbf{w}_1 \cdot \mathbf{w}_2 \mathbf{T} = \mathbf{T}\mathbf{w}_1, \quad \mathbf{w}_1 * \mathbf{w}_2 \mathbf{T} = \mathbf{T}\mathbf{w}_2. \tag{3.6}$$

For a proof of this proposition see [AK1] and Appendix B. One particular solution of (3.4) is given by Faddeev's quantum dilogarithm [F]

$$\psi(z) = \frac{1}{\Phi_{\mathrm{b}}(z)}. \tag{3.7}$$

The most important property of the operator (3.5) is the pentagon identity in $L^2(\mathbb{R}^3)$

$$\mathbf{T}_{12}\mathbf{T}_{13}\mathbf{T}_{23} = \mathbf{T}_{23}\mathbf{T}_{12}, \tag{3.8}$$

which follows from the five term identity (A.17) satisfied by Faddeev's quantum dilogarithm. The indices in (3.8) has the standard meaning. For example $\mathbf{T}_{13}$ is obtained from $\mathbf{T}_{12}$ by replacing $\mathbf{q}_2$ and $\mathbf{p}_2$ with $\mathbf{q}_3$ and $\mathbf{p}_3$ respectively and so forth.

3.1 Oriented triangulated pseudo 3-manifolds

Consider the disjoint union of finitely many copies of the standard 3-simplices in $\mathbb{R}^3$, each having totally ordered vertices. The order of the vertices induces an orientation on edges. Identify some codimension-1 faces of this union in pairs by vertex order preserving and orientation reversing affine homeomorphisms called *gluing homeomorphisms*. The quotient space X is a specific CW-complex with oriented edges which will be called an oriented *triangulated pseudo 3-manifold.* For $i \in \{0, 1, 2, 3\}$, we denote by $\Delta_i(X)$ the set of i-dimensional cells in X. For any $i > j$, we denote

$$\Delta_i^j(X) = \{(a, b) \mid a \in \Delta_i(X),\ b \in \Delta_j(a)\}$$

with natural projection maps

$$\phi_{i,j} : \Delta_i^j(X) \to \Delta_i(X), \quad \phi^{i,j} : \Delta_i^j(X) \to \Delta_j(X).$$

We also have canonical boundary maps

$$\partial_i : \Delta_j(X) \to \Delta_{j-1}(X), \quad 0 \le i \le j,$$

which in the case of a j-dimensional simplex $S = [v_0, v_1, \dots, v_j]$ with ordered vertices $v_0, v_1, \dots, v_j$ in $\mathbb{R}^3$ takes the form

$$\partial_i S = [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_j], \quad i \in \{0, \dots, j\}\,.$$

3.2 Shaped 3-manifolds

Let X be an oriented triangulated pseudo 3-manifold.

Definition 3.2. A *shape structure* on X is an assignment to each edge of each tetrahedron of X a positive number called the dihedral angle,

$$\alpha_X : \Delta_3^1(X) \to \mathbb{R}_+$$

so that the sum of the three angles at the edges from each vertex of each tetrahedron is π. An oriented triangulated pseudo 3-manifold with a shape structure will be called a shaped pseudo 3-manifold.

It is straightforward to see that the dihedral angles at opposite edges are equal.

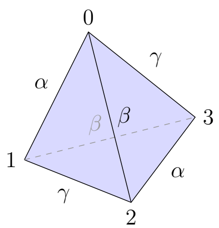


Figure 5: Labeling of edges by dihedral angles.

Definition 3.3. To each shape structure on X, we associate a *Weight function*

$$\omega_X : \Delta_1(X) \to \mathbb{R}_+,$$

which to each edge of X associates the total sum of dihedral angles around it

$$\omega_X(e) = \sum_{(T,e)\in\Delta_3^1(X)} \alpha_X(T,e).$$

An edge of a shaped pseudo 3-manifold X will be called *balanced* if it is internal and $\omega_X(e) = 2\pi$. We call a shaped pseudo 3-manifold *fully balanced* if all edges of X are balanced.

3.3 Shape gauge transformation

In the space of shape structures on a pseudo 3-manifold there is a gauge group action. The gauge group is generated by the total dihedral angles around internal edges acting through the Neumann–Zagier Poisson bracket. See [AK1] for further details.

3.4 Geometric interpretation of the five term identity

For an operator $\mathbf{T}$ we denote the integral kernel of the operator as $\langle x_0, x_2 \mid \mathbf{T} \mid x_1, x_3\rangle$. Then the pentagon identity can be written in the following way

$$\langle x, y, z \mid \mathbf{T}_{12}\mathbf{T}_{13}\mathbf{T}_{23} \mid u, v, w\rangle = \langle x, y, z \mid \mathbf{T}_{23}\mathbf{T}_{12} \mid u, v, w\rangle \tag{3.9}$$

Decomposition of unity gives for the left hand side of (3.9)

$$\langle x, y, z \mid \mathbf{T}_{12}\mathbf{T}_{13}\mathbf{T}_{23} \mid u, v, w\rangle = \int \langle x, y \mid \mathbf{T} \mid \alpha_1, \alpha_2\rangle\langle \alpha_1, z \mid \mathbf{T} \mid u, \beta_3\rangle \langle \alpha_2, \beta_3 \mid \mathbf{T} \mid v, w\rangle d\alpha_1 d\alpha_2 d\beta_3.$$

Decomposing of unity for the right hand side gives

$$\langle x, y, z \mid \mathbf{T}_{23}\mathbf{T}_{12} \mid u, v, w\rangle = \int \langle y, z \mid \mathbf{T} \mid \gamma_2, w\rangle\langle x, \gamma_2 \mid \mathbf{T} \mid u, v\rangle d\gamma_2.$$

To make the correspondence between the pentagon identity and the 3-2 Pachner move precise, we label each vertex of a tetrahedron T with a number $i \in \{0, 1, 2, 3\}$. The numbers on vertices induce an orientation on edges, i.e. we put arrows on the edges pointing in the direction from the smaller to the bigger label on vertices. The number at a vertex corresponds to the number of incoming edges, see Figure 6.

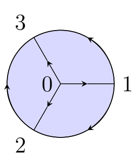


Figure 6: Interpretation of a positively oriented tetrahedron.

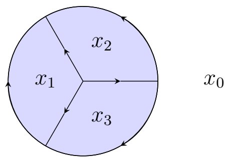


Figure 7: A tetrahedron in state x.

3.5 States

A state of a tetrahedron T with totally ordered vertices $\{0, 1, 2, 3\}$ is a map

$$x : \Delta_2(X) \to \mathbb{R}.$$

A tetrahedron in state x is illustrated in Figure 7, where $x_i := x(\partial_i T)$.

We identify a tetrahedron T in state x as in Figure 7 with the integral kernel $\langle x_0, x_2 \mid \mathbf{T} \mid x_1, x_3 \rangle$. This gives a geometric interpretation of the pentagon identity (3.9) as the 2-3 Pachner move as illustrated in Figure 8 and Figure 9. The

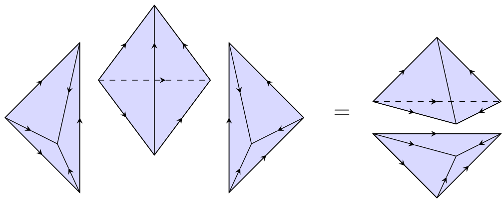


Figure 8: Decomposition of tetrahedra in the 2-3 Pachner move.

integrations corresponds to gluing of faces as illustrated in Figure 9.

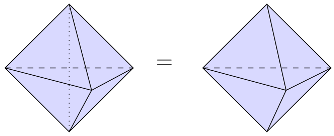


Figure 9: 3-2 Pachner move.

3.6 Integral kernel

Let us calculate the integral kernel of the operator $\mathbf{T}$

$$\langle x_0, x_2 \mid \mathbf{T} \mid x_1, x_3\rangle \equiv \mathbf{T}f(x,y) = \int \langle x, y \mid \mathbf{T} \mid u, v\rangle f(u,v)dudv,$$

where $\mathbf{T}$ is the operator given by (3.5).

$$\begin{aligned}
\langle x_0, x_2 \mid \mathbf{T} \mid x_1, x_3\rangle &= \langle x_0, x_2 \mid e^{2\pi i \mathbf{p}_1\mathbf{q}_2}\psi\left(\mathbf{q}_1 - \mathbf{q}_2 + \mathbf{p}_2\right) \mid x_1, x_3\rangle \\
&= e^{x_2\frac{\partial}{\partial x_0}}\langle x_0, x_2 \mid \psi\left(\mathbf{q}_1 - \mathbf{q}_2 + \mathbf{p}_2\right) \mid x_1, x_3\rangle \\
&= \langle x_0 + x_2, x_2 \mid \psi\left(\mathbf{q}_1 - \mathbf{q}_2 + \mathbf{p}_2\right) \mid x_1, x_3\rangle \\
&= \int \langle x_0 + x_2, x_2 \mid e^{2\pi i(\mathbf{q}_1 - \mathbf{q}_2 + \mathbf{p}_2)y} \mid x_1, x_3\rangle \tilde{\psi}(y)\, dy \\
&= \int e^{2\pi i y x_1}\delta(x_1 - x_0 - x_2)\langle x_2 \mid e^{2\pi i(\mathbf{p}_2 - \mathbf{q}_2)y} \mid x_3\rangle \tilde{\psi}(y)\, dy \\
&= \int e^{2\pi i x_1 y}\delta(x_1 - x_0 - x_2)\tilde{\psi}(y)e^{-2\pi i x_3 y}\langle x_2 + y \mid x_3\rangle\, dy \\
&= \int e^{2\pi i x_1 y}\delta(x_1 - x_0 - x_2)\tilde{\psi}(y)e^{-2\pi i x_3 y}\delta(x_2 + y - x_3)e^{\pi i y^2}\, dy \\
&= e^{2\pi i x_1(x_3 - x_2)}\delta(x_1 - x_0 - x_2)\tilde{\psi}(x_3 - x_2)e^{-2\pi i x_3(x_3 - x_2) + \pi i(x_3 - x_2)^2} \\
&= \delta(x_1 - x_0 - x_2)\tilde{\psi}'(x_3 - x_2)e^{2\pi i x_0(x_3 - x_2)},
\end{aligned}$$

where

$$\tilde{\psi}'(x) := \tilde{\psi}(x)e^{-\pi i x^2}, \quad \text{and} \quad \tilde{\psi}(x) := \int_{\mathbb{R}} \psi(y)e^{2\pi i x y}\, dy.$$

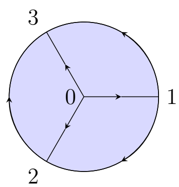
3.7 Positively and negatively oriented tetrahedra

In an oriented triangulated 3-manifold there are two possibilities for the orientation of tetrahedra. The orientation follows from Figure 10. To a negatively oriented tetrahedron the integral kernel associated to it in the geometric interpretation is the complex conjugate of that of a positively oriented tetrahedron.

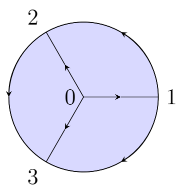
3.8 Charged Tetrahedral Operators and Pentagon Identity

To ensure that the Fourier integral is absolutely convergent charges on the operator $\mathbf{T}$ are introduced. For any positive real a and c such that $b := \frac{1}{2} - a - c$ is also positive, define the charged $\mathbf{T}$-operators

$$\mathbf{T}(a,c) := e^{-\pi i c_{\mathrm{b}}^2(4(a-c)+1)/6}e^{4\pi i c_{\mathrm{b}}(c\mathbf{q}_2 - a\mathbf{q}_1)}\mathbf{T}e^{-4\pi i c_{\mathrm{b}}(a\mathbf{p}_2 + c\mathbf{q}_2)} \tag{3.10}$$



Positively oriented tetrahedron.



Negatively oriented tetrahedron.

Figure 10: Orientations on tetrahedra.

and

$$\bar{\mathbf{T}}(a,c) := e^{\pi i c_{\mathrm{b}}^2(4(a-c)+1)/6} e^{-4\pi i c_{\mathrm{b}}(a\mathbf{p}_2+c\mathbf{q}_2)} \bar{\mathbf{T}} e^{4\pi i c_{\mathrm{b}}(c\mathbf{q}_2-a\mathbf{q}_1)}$$

where $\bar{\mathbf{T}} := \mathbf{T}^{-1}$ and $c_{\mathrm{b}} := \frac{i}{2}(\mathrm{b}+\mathrm{b}^{-1})$. We have the equality

$$\mathbf{T}(a,c) = e^{2\pi i \mathbf{p}_1\mathbf{q}_2}\psi_{a,c}(\mathbf{q}_1 - \mathbf{q}_2 + \mathbf{p}_2)$$

where

$$\psi_{a,c}(x) := \psi(x - 2c_{\mathrm{b}}(a+c)) e^{-4\pi i c_{\mathrm{b}} a(x - c_{\mathrm{b}}(a+c))} e^{-\pi i c_{\mathrm{b}}^2(4(a-c)+1)/6}.$$

The Fourier transformation formula for Faddeev's quantum dilogarithm (A.19) leads to the identities

$$\begin{aligned}
\tilde{\psi}'_{a,c}(x) &= e^{-\frac{\pi i}{12}}\psi_{c,b}(x),\\
\overline{\psi_{a,c}(x)} &= e^{-\frac{\pi i}{6}} e^{\pi i x^2}\psi_{c,a}(-x) = e^{-\frac{\pi i}{12}}\tilde{\psi}_{b,c}(-x),\\
\overline{\tilde{\psi}'_{a,c}(x)} &= e^{\frac{\pi i}{12}}\overline{\psi_{c,b}(x)} = e^{-\frac{\pi i}{12}} e^{\pi i x^2}\psi_{b,c}(-x).
\end{aligned}$$

From the formulas above we obtain that

$$\langle x_0, x_2 \mid \mathbf{T}(a,c) \mid x_1, x_3\rangle = \delta(x_1 - x_0 - x_2)\tilde{\psi}'_{a,c}(x_3 - x_2) e^{2\pi i x_0(x_3 - x_2)}, \tag{3.11}$$

$$\langle x, y \mid \bar{\mathbf{T}}(a,c) \mid u, v\rangle = \overline{\langle u, v \mid \mathbf{T}(a,c) \mid x, y\rangle}. \tag{3.12}$$

Proposition 3.4 (Andersen–Kashaev [AK1])**.** *The charged pentagon identity is satisfied*

$$\mathbf{T}_{12}(a_4,c_4)\mathbf{T}_{13}(a_2,c_2)\mathbf{T}_{23}(a_0,c_0) = e^{\pi i c_{\mathrm{b}}^2 P_e/3}\mathbf{T}_{23}(a_1,c_1)\mathbf{T}_{12}(a_3,c_3), \tag{3.13}$$

where

$$P_e = 2(c_0 + a_2 + c_4) - \frac{1}{2}$$

and $a_0, a_1, a_2, a_3, a_4, c_0, c_1, c_2, c_3, c_4 \in \mathbb{R}$ *are such that*

$$a_1 = a_0 + a_2,\ a_3 = a_2 + a_4,\ c_1 = c_0 + a_4,\ c_3 = a_0 + c_4,\ c_2 = c_1 + c_3.$$

3.9 Partition function

For a tetrahedron $T = [v_0, v_1, v_2, v_3]$ with ordered vertices v_0, v_1, v_2, v_3, we define its sign

$$\operatorname{sign}(T) = \operatorname{sign}(\det(v_1 - v_0, v_2 - v_0, v_3 - v_0)).$$

For (T, α, x) an oriented tetrahedron with shape structure α in state x, define the partition function taking values in the space of tempered distributions by the formula

$$Z_\hbar(T, \alpha, x) := \begin{cases} \langle x_0, x_2 \mid \mathbf{T}(a, c) \mid x_1, x_3 \rangle, & \text{if } \operatorname{sign}(T) = 1 \\ \langle x_1, x_3 \mid \bar{\mathbf{T}}(a, c) \mid x_0, x_2 \rangle, & \text{if } \operatorname{sign}(T) = -1. \end{cases} \tag{3.14}$$

where

$$x_i := x(\partial_i T)$$

and

$$a = \frac{1}{2\pi} \alpha_T(T, e_{01}), \quad c = \frac{1}{2\pi} \alpha_T(T, e_{03}).$$

For a closed oriented triangulated pseudo 3-manifold X with shape structure α, we associate the partition function

$$Z_\hbar(X, \alpha) := \int_{x \in \mathbb{R}^{\Delta_2(X)}} \prod_{T \in \Delta_3(X)} Z_\hbar(T, \alpha, x)\, dx. \tag{3.15}$$

Theorem 3.5 (Andersen–Kashaev [AK1]). *If $H_2(X \backslash \Delta_0(X), \mathbb{Z}) = 0$, then the quantity $|Z_\hbar(X, \alpha)|$ is well defined in the sense that the integral is absolutely convergent, and*

1. *it depends on only the gauge reduced class of α;*
2. *it is invariant under $2-3$ Pachner moves.*

The definition of the partition function (3.15) can be extended to manifolds having boundary eventually giving rise to a TQFT, see [AK1].

3.10 Invariants of knots in 3-manifolds

By considering one-vertex ideal triangulations of complements of hyperbolic knots in compact oriented closed 3-manifolds, we obtain knot invariants.

Another possibility is to consider a one-vertex Hamiltonian triangulation (H-triangulation) of pairs (a closed 3-manifold M, a knot K in M), i.e., a one-vertex triangulation of M, where the knot is represented by one edge, with degenerate shape structures, where the weight on the knot approaches zero and where simultaneously the weights on all other edges approach the balanced value 2π. This limit by itself is divergent as a simple pole (after analytic continuation to complex angles) in the weight of the knot, but the residue at this pole is a knot invariant which is a direct analogue of Kashaev's invariants [K1], which were at the origin of the hyperbolic volume conjecture.

In [AK1] the first author and Rinat Kashaev have set forth the following conjecture:

Conjecture 3.6 (Andersen and Kashaev [AK1])**.** *Let M be a closed oriented 3-manifold. For any hyperbolic knot $K \subset M$, there exists a smooth function $J_{M,K}(\hbar, x)$ on $\mathbb{R}_{>0} \times \mathbb{R}$ which has the following properties.*

(1) For any fully balanced shaped ideal triangulation X of the complement of K in M, there exists a gauge invariant real linear combination of dihedral angles λ, a (gauge non-invariant) real quadratic polynomial of dihedral angles ϕ such that

$$Z_\hbar(X) = e^{i\frac{\phi}{\hbar}} \int_{\mathbb{R}} J_{M,K}(\hbar, x) e^{-\frac{x\lambda}{\sqrt{\hbar}}} dx.$$

(2) For any one vertex shaped H-triangulation Y of the pair (M, K) there exists a real quadratic polynomial of dihedral angles ϕ such that

$$\lim_{\omega_Y \to \tau} \Phi_{\mathrm{b}}\left(\frac{\pi - \omega_Y(K)}{2\pi i\sqrt{\hbar}}\right) Z_\hbar(Y) = e^{i\frac{\phi}{\hbar} - i\pi/12} J_{M,K}(\hbar, 0),$$

where $\tau : \Delta_1(Y) \to \mathbb{R}$ takes the value 0 on the knot K and the value 2π on all other edges.

(3) The hyperbolic volume of the complement of K in M is recovered as the limit

$$\lim_{\hbar \to 0} 2\pi\hbar \log |J_{M,K}(\hbar, 0)| = -\operatorname{vol}(M \backslash K).$$

Theorem 3.7. *Conjecture 3.6 is true for the pair $(S^3, 6_1)$ with*

$$J_{S^3,6_1}(\hbar, x) = \chi_{6_1}(x). \tag{3.16}$$

The function $\chi_{6_1}(x)$ is defined to be:

$$\chi_{6_1}(x) = \int_{\mathbb{R}^2} \frac{e^{2\pi i(x^2 + \frac{1}{2}y^2 + 2xy)e^{4\pi i c_{\mathrm{b}} z}}}{\Phi_{\mathrm{b}}(x+y)\Phi_{\mathrm{b}}(x+z+c_{\mathrm{b}})\Phi_{\mathrm{b}}(y)\Phi_{\mathrm{b}}(z-x-y)} dydz.$$

See [N] for a calculation of the invariant of an ideal triangulation of the complement of the knot 6_1.

4 New formulation

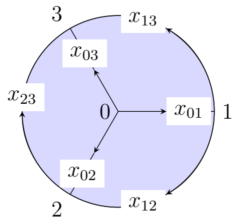
In this section we recall the new formulation of the Teichmüller TQFT introduced by Andersen and Kashaev in [AK2].

4.1 States and Boltzmann weights

Let $T \subset \mathbb{R}^3$ be a tetrahedron with shape structure α_T and vertex ordering mapping

$$v : \{0, 1, 2, 3\} \to \Delta_0(T).$$

A *state* of a tetrahedron T is a map $x \colon \Delta_1(T) \to \mathbb{R}$. Pictorially, a positive tetrahedron T in state x looks as follows



More generally, a state of a triangulated pseudo 3-manifold X is a map

$$y : \Delta_1(X) \to \mathbb{R}.$$

For any state y define the *Boltzmann weight*

$$B(T, x) = g_{\alpha_1,\alpha_3}(y_{02} + y_{13} - y_{03} - y_{12}, y_{02} + y_{13} - y_{01} - y_{23})$$

if T is positive and complex conjugate otherwise. Here $y_{ij} \equiv y(v_i v_j), \alpha_i \equiv \alpha_T(v_0 v_i)/2\pi$,

$$g_{a,c}(s,t) = \sum_{m\in\mathbb{Z}} \tilde{\psi}'_{a,c}(s+m) e^{\pi i t(s+2m)}. \tag{4.1}$$

Theorem 4.1 (Andersen–Kashaev [AK1]). *Let X be a levelled shaped triangulated oriented pseudo 3-manifold. Then, the quantity*

$$Z_\hbar^{new}(X) := e^{\pi i l_X/4\hbar} \int_{[0,1]^{\Delta_1(X)}} \left(\prod_{T\in\Delta_3(X)} B\left(T, y\big|_{\Delta_1(T)}\right)\right) dx \tag{4.2}$$

admits an analytic continuation to a meromorphic function of the complex shapes, which is invariant under all shaped $2-3$ and $3-2$ Pachner moves (along balanced edges).

Conjecture 4.2. *The proposed model in Theorem 4.1 is equivalent to the Teichmüller TQFT from [AK1].*

Theorem 4.3. *The new formulation of the Teichmüller TQFT is equivalent to the original formulation for the pairs $(S^3, 3_1)$, $(S^3, 4_1)$, $(S^3, 5_2)$ and $(S^3, 6_1)$.*

5 Calculations of specific knot complements

In the following calculations we encode an oriented triangulated pseudo 3-manifold X into a diagram where a tetrahedron T is represented by an element

(5.1) $T \quad = $ 

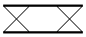
where the vertical segments, ordered from left to right, correspond to the faces $\partial_0 T$,$\partial_1 T$, $\partial_2 T$,$\partial_3 T$ respectively. When we glue tetrahedron along faces, we illustrate this by joining the corresponding vertical segments.

We will further use the notation

$$\nu_{a,c} := e^{-\pi i c_{\mathrm{b}^2}(4(a-c)+1)/6}.$$

5.1 The complement of the figure-8-knot

Let X be the following oriented triangulated pseudo 3-manifold,

(5.2) 

which represented the usual diagram for the complement of the figure eight knot. Choosing an orientation, the diagram consists of one positive tetrahedron T_+ and one negative T_-. $\partial X = \emptyset$ and combinatorially we have $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{e_0, e_1\}$. The gluing of the tetrahedra is vertex order preserving which means that edges are glued together in the following manner.

$$e_0 = x^+_{01} = x^+_{03} = x^+_{23} = x^-_{02} = x^-_{12} = x^-_{13} =: x,$$
$$e_1 = x^+_{02} = x^+_{13} = x^+_{12} = x^-_{01} = x^-_{03} = x^-_{23} =: y.$$

That this diagram represents the complement of the figure eight know means that the topological space $X\backslash\{*\}$ is homeomorphic to the complement of the figure-eight knot. The set $\Delta^1_3(X)$ consists of the elements $(T_\pm, e_{j,k})$ for $0 \leq j < k \leq 3$. We fix a shape structure

$$\alpha_X : \Delta^1_3(X) \to \mathbb{R}_+$$

by the formulae

$$\alpha_X(T_\pm, e_{0,1}) = 2\pi a_\pm, \quad \alpha_X(T_\pm, e_{0,2}) = 2\pi b_\pm, \quad \alpha_X(T_\pm, e_{0,3}) = 2\pi c_\pm,$$

where $a_\pm + b_\pm + c_\pm = \frac{1}{2}$. This result in the following weight functions

$$\omega_X(e_0) = 2a_+ + c_+ + 2b_- + c_-, \quad \omega_X(e_1) = 2b_+ + c_+ + 2a_- + c_-.$$

In the completely balanced case these equations correspond to

$$a_+ - b_+ = a_- - b_-.$$

The Boltzmann weights are given by the functions

$$B\left(T_+, x_{|_{\Delta_1(T_+)}}\right) = g_{a_+,c_+}(y-x, 2(y-x)),$$
$$B\left(T_-, x_{|_{\Delta_1(T_-)}}\right) = \overline{g_{a_-,c_-}(x-y, 2(x-y))}.$$

We calculate the partition function for the Teichmüller TQFT using the new formulation

$$\begin{aligned}
Z_\hbar^{\mathrm{new}}(X) &= \int_{[0,1]^2} \sum_{m,n\in\mathbb{Z}} \tilde{\psi}'_{a_+,c_+}(y-x+m)\overline{\tilde{\psi}'_{a_-,c_-}(x-y+n)}e^{4\pi i(y-x)(m+n)}\, dxdy \\
&= \int_{[0,1]} \sum_{m,n\in\mathbb{Z}} \tilde{\psi}'_{a_+,c_+}(y+m)\overline{\tilde{\psi}'_{a_-,c_-}(-y+n)}e^{4\pi iy(m+n)}dy \\
&= \sum_{m,n\in\mathbb{Z}} \int_{[m,m+1]} \tilde{\psi}'_{a_+,c_+}(y)\overline{\tilde{\psi}'_{a_-,c_-}(-y+m+n)}e^{4\pi i(y-m)(m+n)}dy \\
&= \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \tilde{\psi}'_{a_+,c_+}(y)\overline{\tilde{\psi}'_{a_-,c_-}(-y+p)}e^{4\pi iyp}dy \\
&= e^{-\frac{\pi i}{6}} \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \psi_{c_+,b_+}(y)\psi_{b_-,c_-}(y-p)e^{\pi i(y-p)^2}e^{4\pi iyp}dy \\
&= e^{-\frac{\pi i}{6}} \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \psi(y-2c_{\mathrm{b}}(c_+ + b_+))\psi(y-p-2c_{\mathrm{b}}(b_- + c_-))e^{\pi iy^2}e^{\pi ip^2}e^{2\pi iyp} \\
&\quad \times e^{-4\pi ic_{\mathrm{b}}c_+(y-c_{\mathrm{b}}(c_+ + b_+))}e^{-4\pi ic_{\mathrm{b}}b_-(y-p-c_{\mathrm{b}}(b_- + c_-))} \\
&\quad \times e^{-\pi i(4(c_+ - b_+)+1)/6}e^{-\pi i(4(b_- - c_-)+1)/6}dy.
\end{aligned}$$

We set $Y = y - 2c_{\mathrm{b}}(c_+ + b_+)$. Assuming that we are in the completely balanced case we have that

$$-b_- - c_- + c_+ + b_+ = -b_+ + b_-.$$

Furthermore we have $y^2 = Y^2 + 4c_{\mathrm{b}}^2(c_+ + b_+)^2 + 4c_{\mathrm{b}}Y(c_+ + b_+)$. Implementing this we get the following expression

$$\begin{aligned}
Z_\hbar^{\mathrm{new}}(X) &= \nu_{c_+,b_+}\nu_{b_-,c_-}e^{-\frac{\pi i}{6}} \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \psi(Y-p-2c_{\mathrm{b}}(b_+ - b_-))\psi(Y) \\
&\quad \times e^{\pi i(Y^2+4c_{\mathrm{b}}^2(c_+ + b_+)^2+4c_{\mathrm{b}}Y(c_+ + b_+))}e^{\pi ip^2} \\
&\quad \times e^{2\pi i(Y+2c_{\mathrm{b}}(c_+ + b_+))p} \\
&\quad \times e^{-4\pi ic_{\mathrm{b}}c_+(Y+c_{\mathrm{b}}(c_+ + b_x))}e^{-4\pi ic_{\mathrm{b}}b_-(Y-p-c_{\mathrm{b}}(b_- + c_-)+2c_{\mathrm{b}}(c_+ + b_+))}dY \\
&= \nu_{c_+,b_+}\nu_{b_-,c_-}e^{-\frac{\pi i}{6}} \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \frac{1}{\Phi_{\mathrm{b}}(Y-p-2c_{\mathrm{b}}(b_+ - b_-))}\frac{1}{\Phi_{\mathrm{b}}(Y)} \\
&\quad \times e^{\pi iY^2}e^{\pi ip^2}e^{-4\pi ic_{\mathrm{b}}Y(-c_+ - b_+ + c_+ + b_-)}e^{2\pi iYp}
\end{aligned}$$

$$\times\ e^{-4\pi i c_{\mathrm{b}} p(-(c_+ + b_+) - b_-)}$$
$$\times\ e^{4\pi i c_{\mathrm{b}}^2((c_+ + b_+)^2 - c_+(c_+ + b_+) - b_-(b_- + c_- - 2(c_+ + b_+))} dY.$$

Now set

$$u = 2c_{\mathrm{b}}(b_+ - b_-) \tag{5.3}$$

and

$$v = 2b_- + c_- = 2b_+ + c_+, \tag{5.4}$$

and use the formula

$$\Phi_{\mathrm{b}}(z)\Phi_{\mathrm{b}}(-z) = \zeta_{inv}^{-1} e^{\pi i z^2},$$

together with the calculation

$$b_- + b_+ + c_+ = b_- + b_+ - 2b_+ + 2b_- + c_- = -(b_+ - b_-) + (2b_- + c_-).$$

to get that

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) = {} & \nu_{c_+,b_+}\nu_{b_-,c_-}\zeta_{inv} e^{-\frac{\pi i}{6}} \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \frac{\Phi_{\mathrm{b}}(p+u-Y)}{\Phi_{\mathrm{b}}(Y)} e^{-\pi i (Y^2+u^2+p^2-2Yu-2Yp+2up)} \\ & \times\ e^{\pi i Y^2} e^{\pi i p^2} e^{2\pi i Y u} e^{2\pi i Y p} dY \\ & \times\ e^{-2\pi i p u} e^{2\pi i p v} \\ & \times\ e^{4\pi i c_{\mathrm{b}}^2((c_+ + b_+)^2 - c_+(c_+ + b_+) - b_-(b_- + c_- - 2(c_+ + b_+))}. \end{aligned}$$

Using the balance condition and formulas (5.3) and (5.4) we get the equality

$$\begin{aligned} & -4\pi i c_{\mathrm{b}}^2\{(c_+ + b_+)^2 + b_-(-b_- - c_- + 2c_+ + 2b_+) + c_+(c_+ + b_+)\} = \\ & -4\pi i c_{\mathrm{b}}^2\{(-(b_+ - b_1)^2) - c_+ b_+ + c_+ b_- + b_- c_+ - b_- c_-\} = \\ & -2\pi i c_{\mathrm{b}}\{-(c_+ + 2b_-)u\} + \pi i u^2 = \\ & -2\pi i c_{\mathrm{b}}\{-(2b_- + c_-)u + 2(b_+ - b_-))u\} + \pi i u^2 = \pi i (uv - u^2). \end{aligned}$$

We get the following expression for the partition function

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) = {} & \nu_{c_+,b_+}\nu_{b_-,c_-}\zeta_{inv} e^{-\frac{\pi i}{6}} \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \frac{\Phi_{\mathrm{b}}(p+u-Y)}{\Phi_{\mathrm{b}}(Y)} e^{-\pi i u^2} \\ & \times\ e^{4\pi i Y u} e^{4\pi i Y p} e^{-4\pi i p u} e^{2\pi i p v} e^{\pi i (uv-u^2)} dY \\ = {} & \nu_{c_+,b_+}\nu_{b_-,c_-}\zeta_{inv} e^{-\frac{\pi i}{6}} \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \frac{\Phi_{\mathrm{b}}(p+u-Y)}{\Phi_{\mathrm{b}}(Y)} \\ & \times\ e^{4\pi i Y u} e^{4\pi i Y p} e^{-4\pi i p u} e^{2\pi i p v} e^{\pi i u v} e^{-2\pi i u^2} dY \end{aligned}$$

$$= \nu_{c_+,b_+}\nu_{b_-,c_-}\zeta_{inv}e^{-\frac{\pi i}{6}}\sum_{p\in\mathbb{Z}}\int_{\mathbb{R}}\frac{\Phi_{\mathrm{b}}(p+u-Y)}{\Phi_{\mathrm{b}}(Y)}e^{2\pi i(u+p)(2Y-u-p)}dY e^{\pi i v(2p+u)}.$$

Using the Weil-Gel'fand-Zak transform we see that the partition function has the form

$$Z_\hbar^{\mathrm{new}}(X) = \nu_{c_+,b_+}\nu_{b_-,c_-}\zeta_{inv}e^{-\frac{\pi i}{6}}W(\chi_{4_1})(u,v).$$

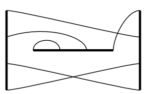
Where the function $\chi_{4_1}(x) = \int_{\mathbb{R}-i0}\frac{\Phi_{\mathrm{b}}(x-y)}{\Phi_{\mathrm{b}}(y)}e^{2\pi i x(2y-x)}dy$. The function $\chi_{4_1}(x)$ is exactly the function $J_{S^3,4_1}(\hbar,x)$ from [AK1, Thm. 5]. It should be noted that this result is connected to Hikami's invariant. Andersen and Kashaev observes in [AK1] that the expression

$$\frac{1}{2\pi\,\mathrm{b}}\chi_{4_1}\left(-\frac{u}{\pi\,\mathrm{b}},\frac{1}{2}\right),$$

where $\chi_{4_1}(x,\lambda) = \chi_{4_1}(x)e^{4\pi i c_{\mathrm{b}}\lambda}$ is equal to the formal derived expression in [H2].

5.2 One vertex H-triangulation of the figure-8-knot

Let X be represented by the diagram

(5.5) 

where the figure-eight knot is represented by the edge of the central tetrahedron connecting the maximal and next to maximal vertices. Choosing an orientation, the diagram consists of two positive tetrahedra T_1, T_3 and one negative T_2. $\partial X = \emptyset$ and combinatorially we have $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{x, y, z, x'\}$. The gluing of the tetrahedra is vertex order preserving which means that edges are glued together in the following manner.

$$\begin{aligned} x &= x_{01}^1 = x_{03}^1 = x_{02}^2 = x_{02}^3 = x_{03}^3,\\ y &= x_{02}^1 = x_{12}^1 = x_{13}^1 = x_{01}^2 = x_{03}^2 = x_{23}^2 = x_{23}^2,\\ z &= x_{23}^1 = x_{12}^2 = x_{13}^2 = x_{12}^3 = x_{13}^3,\\ x' &= x_{01}^3. \end{aligned}$$

This results in the following equations for the dihedral angles when we balance all but one edge

$$b_1 + a_3 = b_2, \quad a_1 = a_2 + a_3.$$

In the limit where we let $a_3 \to 0$ we get the equations

$$b_1 = b_2, \quad a_1 = a_2.$$

The Boltzmann weights are given by the functions

$$\begin{aligned}
B\left(T_1, x_{|_{\Delta_1(T_1)}}\right) &= g_{a_1,c_1}(y-x, 2y-x-z),\\
B\left(T_2, x_{|_{\Delta_1(T_2)}}\right) &= \overline{g_{a_2,c_2}(x-y, x+z-2y)},\\
B\left(T_3, x_{|_{\Delta_1(T_3)}}\right) &= g_{a_3,c_3}(0, x+z-x'-y).
\end{aligned}$$

So we get that

$$\begin{aligned}
Z_\hbar^{\text{new}}(X) = \int_{[0,1]^4} \sum_{m,n,l\in\mathbb{Z}} & \tilde{\psi}'_{a_1,c_1}(y-x+m) e^{\pi i(2y-x-z)(y-x+2m)}\\
& \overline{\tilde{\psi}'_{a_2,c_2}(x-y+n)} e^{-\pi i(x+z-2y)(x-y+2n)}\\
& \tilde{\psi}'_{a_3,c_3}(l) e^{2\pi i(x+z-x'-y)l}\, dxdydzdx'.
\end{aligned}$$

Integration over x' removes one of the sums since $\int_0^1 e^{-2\pi i x' l} dx' = \delta(l)$. Hence

$$\begin{aligned}
Z_\hbar^{\text{new}}(X) &= \tilde{\psi}'_{a_3,c_3}(0) \int_{[0,1]^3} \sum_{m,n\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(y-x+m) e^{\pi i(2y-x-z)(y-x+2m)}\\
&\qquad \overline{\tilde{\psi}'_{a_2,c_2}(x-y+n)} e^{-\pi i(x+z-2y)(x-y+2n)}\, dxdydz\\
&= \tilde{\psi}'_{a_3,c_3}(0) \int_{[0,1]^3} \sum_{m,n\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(y-x+m) \overline{\tilde{\psi}'_{a_2,c_2}(x-y+n)}\\
&\qquad e^{2\pi i(2y-x)(m+n)} e^{-2\pi i z(m+n)}\, dxdydz.
\end{aligned}$$

Now integration over z gives $\int_0^1 e^{-2\pi i z(m+n)} dz = \delta(n+m)$. So the partition function takes the form

$$Z_\hbar^{\text{new}}(X) = \tilde{\psi}'_{a_3,c_3}(0) \int_{[0,1]^2} \sum_{m\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(y-x+m) \overline{\tilde{\psi}'_{a_2,c_2}(x-y-m)}\, dxdy,$$

We make the shift $y \mapsto y+x$ to get the expression

$$\begin{aligned}
Z_\hbar^{\text{new}}(X) &= \tilde{\psi}'_{a_3,c_3}(0) \int_{[0,1]^2} \sum_{m\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(y+m) \overline{\tilde{\psi}'_{a_2,c_2}(-y-m)}\, dxdy\\
&= \tilde{\psi}'_{a_3,c_3}(0) \int_{[0,1]} \sum_{m\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(y+m) \overline{\tilde{\psi}'_{a_2,c_2}(-y-m)}\, dy\\
&= \tilde{\psi}'_{a_3,c_3}(0) \int_{\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(y) \overline{\tilde{\psi}'_{a_2,c_2}(-y)}\, dy\\
&= e^{-\frac{\pi i}{6}} \tilde{\psi}'_{a_3,c_3}(0) \int_{\mathbb{Z}} \psi_{c_1,b_1}(y) \psi_{b_2,c_2}(y) e^{\pi i y^2}.
\end{aligned}$$

We set $Y = y - 2c_{\mathrm{b}}(c_1 + b_1) = y - c_{\mathrm{b}}(1 - 2a_1)$. Assuming that we are in the case where all but one edge is balanced we have $a_1 = a_2$

$$y^2 = Y^2 + c_{\mathrm{b}}^2(1 - 2a_1)^2 + 2c_{\mathrm{b}}Y(1 - 2a_1).$$

Implementing this we get the following expression

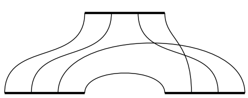
$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) = e^{-\frac{\pi i}{6}}\tilde{\psi}'_{a_3,c_3}(0) \int_{\mathbb{Z}} & \psi(Y)\psi(Y) e^{\pi i(Y^2 + c_{\mathrm{b}}^2(1-2a_1)^2 + 2c_{\mathrm{b}}Y(1-2a_1))} \\ & e^{-4\pi i c_{\mathrm{b}} c_1 (Y + c_{\mathrm{b}}(1/2 - a_1))} \nu_{c_1,b_1} \\ & e^{-4\pi i c_{\mathrm{b}} b_2 (Y + c_{\mathrm{b}}(1/2 - a_1))} \nu_{b_2,c_2} dy \\ = e^{-\frac{\pi i}{6}} \nu_{c_1,b_1}\nu_{b_2,c_2}\tilde{\psi}'_{a_3,c_3}(0) \int_{\mathbb{Z}-0i} & \frac{1}{\Phi(Y)^2} e^{\pi i Y^2}\, dy\, e^{\frac{i\phi}{\hbar}}. \end{aligned}$$

This result corresponds exactly to the partition function in the original formulation, see [AK1, Chap. 11]. I.e. in the limit where $a_3 \to 0$ we get the renormalised partition function

$$\tilde{Z}_\hbar^{\mathrm{new}}(X) := \lim_{a_3 \to 0} \Phi_{\mathrm{b}}(2c_{\mathrm{b}}a_3 - c_{\mathrm{b}}) Z_\hbar^{\mathrm{new}}(X) = \frac{e^{-\pi i/12}}{\nu(c_3)} \chi_{4_1}(0).$$

5.3 The complement of the knot 5_2

Let X be represented by the diagram

(5.6) 

Choosing an orientation the diagram consists of three positive tetrahedra. We denote T_1, T_2, T_3 the left, the right an top tetrahedra respectively. The combinatorial data in this case are $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{e_0, e_1, e_2\}$, $\Delta_2(X) = \{f_0, f_1, f_2, f_3, f_4, f_5\}$ and $\Delta_3(X) = \{T_1, T_2, T_3\}$.

The edges are glued in the following manner

$$\begin{aligned} e_0 &= x_{02}^1 = x_{12}^1 = x_{13}^2 = x_{23}^2 = x_{01}^3 = x_{23}^3 =: x, \\ e_1 &= x_{03}^1 = x_{23}^1 = x_{02}^2 = x_{03}^2 = x_{03}^3 = x_{13}^3 = x_{12}^3 =: y \\ e_2 &= x_{01}^1 = x_{13}^1 = x_{01}^2 = x_{12}^2 = x_{02}^3 =: z. \end{aligned}$$

We impose the condition that all edges are balanced which exactly corresponds to the two equations

$$2a_3 = a_1 + c_2, \quad b_3 = c_1 + b_2.$$

The Boltzmann weights are given by the equations

$$\begin{aligned}
B\left(T_1, x_{|\Delta_1(T_1)}\right) &= g_{a_1,c_1}(z-y, x-y),\\
B\left(T_2, x_{|\Delta_1(T_2)}\right) &= g_{a_2,c_2}(x-z, y-z),\\
B\left(T_2, x_{|\Delta_1(T_2)}\right) &= g_{a_3,c_3}(z-y, z+y-2x).
\end{aligned}$$

We calculate the following function

$$\begin{aligned}
Z_\hbar^{\text{new}}(X) &= \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} g_{a_1,c_1}(z-y, x-y) g_{a_2,c_2}(x-z, y-z)\\
&\qquad\qquad \times g_{a_3,c_3}(z-y, z+y-2x)\, dxdydz\\
&= \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(z-y+j) e^{\pi i(x-y)(z-y+2j)} \tilde{\psi}'_{a_2,c_2}(x-z+k)\\
&\quad e^{\pi i(y-z)(x-z+2k)} \times \tilde{\psi}'_{a_3,c_3}(z-y+l) e^{\pi i(z+y-2x)(z-y+2l)}\, dxdydz.
\end{aligned}$$

Shift $x \mapsto x+z$,

$$\begin{aligned}
Z_\hbar^{\text{new}}(X) = \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} & \tilde{\psi}'_{a_1,c_1}(z-y+j) e^{\pi i(x+z-y)(z-y+2j)} \tilde{\psi}'_{a_2,c_2}(x+k) e^{\pi i(y-z)(x+2k)}\\
&\times \tilde{\psi}'_{a_3,c_3}(z-y+l) e^{\pi i(y-2x-z)(z-y+2l)}\, dxdydz.
\end{aligned}$$

Shift $z \mapsto z+y$

$$\begin{aligned}
Z_\hbar^{\text{new}}(X) &= \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(z+j) e^{\pi i(x+z)(z+2j)} \tilde{\psi}'_{a_2,c_2}(x+k) e^{\pi i(-z)(x+2k)}\\
&\quad \times \tilde{\psi}'_{a_3,c_3}(z+l) e^{\pi i(-2x-z)(z+2l)}\, dxdydz\\
&= \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(z+j) \tilde{\psi}'_{a_2,c_2}(x+k) \tilde{\psi}'_{a_3,c_3}(z+l)\\
&\quad \times e^{\pi i(x+z)(z+2j)} e^{\pi i(-z)(x+2k)} e^{\pi i(-2x-z)(z+2l)}\, dxdydz\\
&= \int_{[0,1]^3} \sum_{j,k,l\in\mathbb{Z}} \tilde{\psi}'_{a_1,c_1}(z+j) \tilde{\psi}'_{a_2,c_2}(x+k) \tilde{\psi}'_{a_3,c_3}(z+l)\\
&\quad \times e^{2\pi i(x(j-2l-z)+z(j-k-l))}\, dxdydz.
\end{aligned}$$

Integration over y contributes nothing. We now shift $x \mapsto x-k$ and integrate over the interval $[-k, -k+1]$.

$$Z_\hbar^{\text{new}}(X) = \sum_{j,k,l\in\mathbb{Z}} \int_{[0,1]} \int_{[-k,-k+1]} \tilde{\psi}'_{a_1,c_1}(z+j) \tilde{\psi}'_{a_2,c_2}(x) \tilde{\psi}'_{a_3,c_3}(z+l)$$

$$\begin{aligned}
&\times e^{2\pi i((x-k)(j-2l-z)+z(j-k-l)}\ dxdz\\
=&\sum_{j,k,l\in\mathbb{Z}} e^{2\pi ik(2l-j)}\int_{[0,1]}\int_{[-k,-k+1]}\tilde{\psi}'_{a_1,c_1}(z+j)\tilde{\psi}'_{a_2,c_2}(x)\tilde{\psi}'_{a_3,c_3}(z+l)\\
&\times e^{2\pi i(x(j-2l-z)+z(j-l))}\ dxdz\\
=&\sum_{j,l\in\mathbb{Z}}\int_{[0,1]}\tilde{\psi}'_{a_1,c_1}(z+j)\tilde{\psi}'_{a_3,c_3}(z+l)e^{2\pi iz(j-l)}\int_{\mathbb{R}}\tilde{\psi}'_{a_2,c_2}(x)e^{-2\pi ix(z+2l-j)}dx\,dz\\
=&e^{-\frac{\pi i}{12}}\sum_{j,l\in\mathbb{Z}}\int_{[0,1]}\tilde{\psi}'_{a_1,c_1}(z+j)\tilde{\psi}'_{a_3,c_3}(z+l)e^{2\pi iz(j-l)}\int_{\mathbb{R}}\psi_{c_2,b_2}(x)e^{-2\pi ix(z+2l-j)}dx\,dz\\
=&e^{-\frac{\pi i}{4}}\sum_{j,l\in\mathbb{Z}}\int_{[0,1]}\psi_{c_1,b_1}(z+j)\psi_{c_3,b_3}(z+l)\tilde{\psi}_{c_2,b_2}(z+2l-j)e^{2\pi iz(j-l)}dz.
\end{aligned}$$

We set $m=j-l$.

$$\begin{aligned}
Z_\hbar^{\mathrm{new}}(X)=&e^{-\frac{\pi i}{4}}\sum_{l,m\in\mathbb{Z}}\int_{[0,1]}\psi_{c_1,b_1}(z+l+m)\psi_{c_3,b_3}(z+l)\tilde{\psi}_{c_2,b_2}(z+l-m)e^{2\pi izm}\ dz\\
=&e^{-\frac{\pi i}{4}}\sum_{l,m\in\mathbb{Z}}\int_{[l,l+1]}\psi_{c_1,b_1}(z+m)\psi_{c_3,b_3}(z)\tilde{\psi}_{c_2,b_2}(z-m)e^{2\pi izm}\ dz\\
=&e^{-\frac{\pi i}{3}}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\psi_{c_1,b_1}(z+m)\psi_{c_3,b_3}(z)\psi_{b_2,a_2}(z-m)e^{\pi i(z-m)^2}e^{2\pi izm}\ dz\\
=&e^{-\frac{\pi i}{3}}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\psi_{c_1,b_1}(z+m)\psi_{c_3,b_3}(z)\psi_{b_2,a_2}(z-m)e^{\pi i(z^2+m^2)}\ dz.
\end{aligned}$$

$$\begin{aligned}
Z_\hbar^{\mathrm{new}}(X)=e^{-\frac{\pi i}{3}}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}&\psi(z+m-c_{\mathrm{b}}(1-2a_1))e^{-4\pi ic_{\mathrm{b}}c_1\{(z+m)-c_{\mathrm{b}}(1/2-a_1)\}}\\
&e^{-\pi ic_{\mathrm{b}}^2(4(c_1-b_1)+1)/6}\\
&\psi(z-m-c_{\mathrm{b}}(1-2c_2))e^{-4\pi ic_{\mathrm{b}}c_1\{(z+m)-c_{\mathrm{b}}(1/2-c_2)\}}\\
&e^{-\pi ic_{\mathrm{b}}^2(4(c_1-b_1)+1)/6}\\
&\psi(z-c_{\mathrm{b}}(1-2a_3))e^{-4\pi ic_{\mathrm{b}}c_3\{(z+m)-c_{\mathrm{b}}(1/2-a_3)\}}\\
&e^{-\pi ic_{\mathrm{b}}^2(4(c_3-b_3)+1)/6}e^{\pi iz^2}e^{\pi ip^2}\ dz.
\end{aligned}$$

Set $w=z-c_{\mathrm{b}}(1-2a_3)$

$$\begin{aligned}
Z_\hbar^{\mathrm{new}}(X)=e^{-\frac{\pi i}{3}}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}-i0}&\psi(w+m+2c_{\mathrm{b}}(a_1-a_3))\psi(w-m+2c_{\mathrm{b}}(c_2-a_3))\psi(w)\\
&\times e^{\pi ip^2}e^{\pi iw^2}e^{4\pi ic_{\mathrm{b}}^2(1/2-a_3)^2}e^{4\pi ic_{\mathrm{b}}w(1/2-a_3)}\\
&e^{-4\pi ic_{\mathrm{b}}c_1\{w+p+c_{\mathrm{b}}(1-2a_3)-c_{\mathrm{b}}(1/2-a_1)\}}
\end{aligned}$$

$$\begin{aligned}
&e^{-4\pi i c_{\mathrm{b}} c_1\{w-p+c_{\mathrm{b}}(1-2a_3)-c_{\mathrm{b}}(1/2-c_2)\}}\\
&e^{-4\pi i c_{\mathrm{b}} c_1\{w+c_{\mathrm{b}}(1/2-a_3)\}}\nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}\ dw.
\end{aligned}$$

Simplify by setting $u = 2c_{\mathrm{b}}(a_1 - a_3)$. Using $c_1 + b_2 + c_3 + a_3 - 1/2 = 0$ we are left with

$$\begin{aligned}
Z_\hbar^{\mathrm{new}}(X) =&e^{-\frac{\pi i}{3}}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}-i0}\psi(w+m+u)\psi(w-m-u)\psi(w)\\
&e^{\pi i w^2}e^{\pi i m^2}e^{4\pi i c_{\mathrm{b}}(b_2-c_1)m}\\
&e^{-4\pi i c_{\mathrm{b}}^2\{-b_3^2-b_3c_3+c_1(b_3+c_3)+b_2(b_3+c_3)+(c_1-b_2)(a_1-a_3)\}}\ dw\\
&\nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}.
\end{aligned}$$

Let $v = 2c_{\mathrm{b}}(a_1 - c_1 + b_2 - a_3)$, then Note that

$$4\pi i c_{\mathrm{b}}(b_2-c_1)p = 4\pi i c_{\mathrm{b}}(a_1-c_1+b_2-a_3)p - 4\pi i c_{\mathrm{b}}(a_3-a_1) = 2\pi i(vp-up),$$

$$-b_3^2 - b_3c_3 + c_1(b_3+c_3) + b_2(b_3+c_3) = 0,$$

and

$$\begin{aligned}
-4\pi i c_{\mathrm{b}}^2((c_1-b_2)(a_1-a_3)) &= 4\pi i c_{\mathrm{b}}^2((a_1-c_1+b_2-a_3)(a_1-a_3)-(a_1-a_3)(a_1-a_3))\\
&= \pi i(vu-u^2).
\end{aligned}$$

$$\begin{aligned}
Z_\hbar^{\mathrm{new}}(X) =&e^{-\frac{\pi i}{3}}e^{\pi i u v}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}-i0}\frac{e^{\pi i w^2}e^{-\pi i m^2}e^{-\pi i u^2}}{\Phi_{\mathrm{b}}(w+m+u)\Phi_{\mathrm{b}}(w-m-u)\Phi_{\mathrm{b}}(w)}dw e^{2\pi i v m}\\
&\nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}\\
=&e^{-\frac{\pi i}{3}}e^{\pi i u v}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}-i0}\frac{e^{\pi i(w+(u+m))(w-(u+m))}}{\Phi_{\mathrm{b}}(w+m+u)\Phi_{\mathrm{b}}(w-m-u)\Phi_{\mathrm{b}}(w)}dw e^{2\pi i v m}\\
&\nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}\\
=&W\chi_{5_2}(u,v)\nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}.
\end{aligned}$$

Where $\chi_{5_2}(u)$ is given by the formula

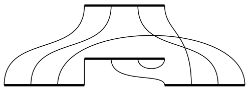
$$\chi_{5_2}(u) = e^{-\frac{\pi i}{3}}\int_{\mathbb{R}-i0}\frac{e^{\pi i(w-u)(w+u)}}{\Phi_{\mathrm{b}}(w+m+u)\Phi_{\mathrm{b}}(w-m-u)\Phi_{\mathrm{b}}(w)}\ dw.$$

Again the function χ_{5_2} is that of [AK1], which again is related to Hikami's invariant, in particular Hikami's formally derived expression in [H2, (4.10)] is equal to $e^{\pi i\frac{c_{\mathrm{b}}2}{3}}\frac{1}{2\pi\,\mathrm{b}}\chi_{5_2}(\frac{-u}{\pi\,\mathrm{b}},\frac{1}{2})$, where $\chi_{5_2} := \chi_{5_2}(x)e^{4\pi i c_{\mathrm{b}} x\lambda}$.

5.4 One vertex H-triangulation of $(S^3, 5_2)$

Let X be represented by the diagram

(5.7)



Choosing an orientation, the diagram consists of four positive tetrahedra T_0, T_1, T_2, T_3. $\partial X = \emptyset$ and combinatorially we have $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{x, y, z, w, x'\}$. The gluing of the tetrahedra is vertex order preserving which means that edges are glued together in the following manner.

$$\begin{aligned}
x &= x^0_{03} = x^0_{13} = x^1_{01} = x^3_{12} = x^3_{02}, \\
y &= x^1_{03} = x^1_{12} = x^1_{13} = x^2_{02} = x^2_{03} = x^3_{03} = x^3_{23}, \\
z &= x^0_{01} = x^1_{02} = x^2_{01} = x^2_{12} = x^3_{01} = x^3_{13} \\
v &= x^0_{02} = x^0_{12} = x^1_{23} = x^2_{13} = x^2_{23}, \\
x' &= x^0_{23}.
\end{aligned}$$

This results in the following equations for the dihedral angles, when we balance all edges but one edge.

$$a_3 = a_1 - a_0 = c_2, \quad a_0 + b_1 = b_2 + c_3, \quad a_1 + a_2 + b_3 = \frac{1}{2} + c_1.$$

The Boltzmann weights are given by the functions

$$\begin{aligned}
B\left(T_0, x_{|_{\Delta_1(T_0)}}\right) &= g_{a_0,c_0}(0, v + x - z - x'), \\
B\left(T_1, x_{|_{\Delta_1(T_1)}}\right) &= g_{a_1,c_1}(z - y, z + y - x - v), \\
B\left(T_2, x_{|_{\Delta_1(T_2)}}\right) &= g_{a_2,c_2}(v - z, y - z), \\
B\left(T_3, x_{|_{\Delta_1(T_3)}}\right) &= g_{a_3,c_3}(z - y, x - y).
\end{aligned}$$

The partition function is represented by the integral

$$\begin{aligned}
Z^{\mathrm{new}}_\hbar(X) = \int_{[0,1]^5} \sum_{m,n,k,p \in \mathbb{Z}} & \tilde{\psi}'_{a_0,c_0}(m) e^{\pi i (v+x-z-x')(2m)} \\
& \tilde{\psi}'_{a_1,c_1}(z - y + n) e^{\pi i (z+y-x-v)(z-y+2n)} \\
& \tilde{\psi}'_{a_2,c_2}(v - z + k) e^{\pi i (y-z)(v-z+2k)} \\
& \tilde{\psi}'_{a_3,c_3}(z - y + p) e^{\pi i (x-y)(z-y+2p)}\, dx' dx dy dz dv.
\end{aligned}$$

Integration over x' removes one of the sums since $\int_0^1 e^{-2\pi i x' m} dx' = \delta(m)$. Hence

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) = \tilde{\psi}'_{a_0,c_0}(0) \int_{[0,1]^4} \sum_{n,k,p\in\mathbb{Z}} & \tilde{\psi}'_{a_1,c_1}(z-y+n) e^{\pi i(z+y-x-v)(z-y+2n)} \\ & \tilde{\psi}'_{a_2,c_2}(v-z+k) e^{\pi i(y-z)(v-z+2k)} \\ & \tilde{\psi}'_{a_3,c_3}(z-y+p) e^{\pi i(x-y)(z-y+2p)} \, dx' dx dy dz dv. \end{aligned}$$

Now integration over x gives $\int_0^1 e^{-2\pi i x(n-p)} dx = \delta(n-p)$. Implementing this and shifting the variable $v \mapsto v+z$, the partition function takes the form

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) = \tilde{\psi}'_{a_0,c_0}(0) \int_{[0,1]^3} \sum_{n,k\in\mathbb{Z}} & \tilde{\psi}'_{a_1,c_1}(z-y+n) e^{\pi i(y-v)(z-y+2n)} \\ & \tilde{\psi}'_{a_2,c_2}(v+k) e^{\pi i(y-z)(v+2k)} \\ & \tilde{\psi}'_{a_3,c_3}(z-y+n) e^{-\pi i y(z-y+2n)} \, dy dz dv. \end{aligned}$$

We make the shift $z \mapsto z+y$ to get the expression

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) = \tilde{\psi}'_{a_0,c_0}(0) \int_{[0,1]^3} \sum_{n,k\in\mathbb{Z}} & \tilde{\psi}'_{a_1,c_1}(z+n) e^{\pi i(y-v)(z+2n)} \\ & \tilde{\psi}'_{a_2,c_2}(v+k) e^{-\pi i z(v+2k)} \\ & \tilde{\psi}'_{a_3,c_3}(z+n) e^{-\pi i y(z+2n)} \, dy dz dv, \end{aligned}$$

which is independent of y so we can remove the integration over this variable. We integrate over the variable v.

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \int_{[0,1]} \tilde{\psi}'_{a_2,c_2}(v+k) e^{-2\pi i v(z+n)} dv e^{-2\pi i z k} = & \sum_{k\in\mathbb{Z}} \int_k^{k+1} \tilde{\psi}'_{a_2,c_2}(v) e^{-2\pi i v(z+n)} dv \\ & e^{-2\pi i z k} e^{2\pi i k(z+n)} \\ = & e^{-\frac{\pi i}{12}} \int_{\mathbb{R}} \psi_{c_2,b_2}(v) e^{-2\pi i z(z+n)} dv \\ = & e^{-\frac{\pi i}{12}} \tilde{\psi}_{c_2,b_2}(z+n) \\ = & e^{-\frac{\pi i}{6}} e^{\pi i(z+n)^2} \psi_{b_2,a_2}(z+n). \end{aligned}$$

We therefore get the expression

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) &= e^{-\frac{\pi i}{3}} \tilde{\psi}'_{a_0,c_0}(0) \int_{[0,1]} \sum_{n\in\mathbb{Z}} \psi_{c_1,b_1}(z+n) \psi_{b_2,a_2}(z+n) \psi_{c_3,b_3}(z+n) e^{\pi i(z+n)^2} dz \\ &= e^{-\frac{\pi i}{3}} \tilde{\psi}'_{a_0,c_0}(0) \int_{\mathbb{R}} \psi_{c_1,b_1}(z) \psi_{b_2,a_2}(z) \psi_{c_3,b_3}(z) e^{\pi i(z)^2} \end{aligned}$$

We set $Z = z - 2c_{\mathrm{b}}(c_1 + b_1) = y - c_{\mathrm{b}}(1 - 2a_1)$. Assuming that we are in the case where all but the edge representing the knot is balanced, i.e. $a_0 \to 0$, we have $a_1 = c_2 = a_3$.

$$z^2 = Z^2 + c_{\mathrm{b}}^2(1-2a_1)^2 + 2c_{\mathrm{b}}Z(1-2a_1).$$

Implementing this we get the expression.

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) = e^{-\frac{\pi i}{3}}\tilde{\psi}'_{a_0,c_0}(0)\int_{\mathbb{R}} &\psi(Z)\psi(Z)\psi(Z)e^{\pi i(Z^2+c_{\mathrm{b}}^2(1-2a_1)^2+2c_{\mathrm{b}}Z(1-2a_1))} \\ &e^{-4\pi i c_{\mathrm{b}} c_1(Z+c_{\mathrm{b}}(1/2-a_1))}\nu_{c_1,b_1} \\ &e^{-4\pi i c_{\mathrm{b}} b_2(Z+c_{\mathrm{b}}(1/2-c_2))}\nu_{b_2,a_2} \\ &e^{-4\pi i c_{\mathrm{b}} c_2(Z+c_{\mathrm{b}}(1/2-a_3))}\nu_{c_3,b_{\mathbb{Z}}3}dz. \end{aligned}$$

$$\begin{aligned} Z_\hbar^{\mathrm{new}}(X) &= \nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}e^{-\frac{\pi i}{3}}e^{\frac{\phi i}{\hbar}}\tilde{\psi}'_{a_0,c_0}(0)\int_{\mathbb{R}}\psi(Z)^3 e^{\pi i Z^2}\,dz \\ &= \nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}e^{-\frac{\pi i}{3}}e^{\frac{\phi i}{\hbar}}\tilde{\psi}'_{a_0,c_0}(0)\int_{\mathbb{R}}\frac{e^{\pi i Z^2}}{\Phi_{\mathrm{b}}(Z)^3}\,dz \end{aligned}$$

Because the combination of dihedral angles in front of Z sums to 0.

$$-4\pi i c_{\mathrm{b}}Z(c_1 + b_2 + c_3 - \frac{1}{2} + a_1) = -4\pi i c_{\mathrm{b}}Z(a_1 + b_1 + c_1 - \frac{1}{2}) = 0$$

This corresponds to the partition function in the original formulation, see [AK1].

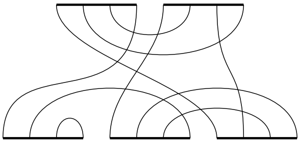
In this case the renormalised partition function takes the form

$$\tilde{Z}_\hbar^{\mathrm{new}}(X) = \lim_{a_0\to 0}\Phi_{\mathrm{b}}Z_\hbar^{\mathrm{new}}(X) = \frac{e^{i\pi/4}}{\nu_{c_0,0}}\chi_{5_2}(0).$$

5.5 One-vertex H-triangulation of $(S^3, 6_1)$

Let X be represented by the diagram

(5.8)



This one vertex H-triangulation of $(S^3, 6_1)$ consists of 5 tetrahedra T_1 and T_3 which are negatively oriented tetrahedra and T_2, T_4, T_5 which are positively oriented tetrahedra. We denoted the tetrahedra as follows. In the bottom we have T_1, T_2, T_3 from left to right and on top we have T_4, T_5 from right to left.

In the diagram the knot 6_1 is represented by the edge connecting the maximal and next to maximal vertex of T_1. We impose a shape structure on the triangulation and balance all but the one edge representing the knot. We get the following equations on the shape parameters

$$a_3 = a_1 + c_2, \quad a_3 + a_4 = a_1 + a_5, \quad a_1 + c_2 = c_4 + c_5,$$
$$\frac{1}{2} + b_3 + c_5 = a_2 + a_3 + a_4, \quad 1 = a_2 + c_3 + c_4 + a_5.$$

We calculate the partition function for the Teichmüller TQFT using the original formulation of the theory. In this formulation the states are assigned to each face of each tetrahedron according to the diagram (5.8).

$$Z_\hbar(X) = \int_{\mathbb{R}^{10}} \overline{\langle w, t \mid T_{a_1,c_1} \mid u, t\rangle} \, \langle z, q \mid T_{a_2,c_2} \mid v, u\rangle \, \overline{\langle x, q \mid T_{a_3,c_3} \mid r, v\rangle}$$
$$\langle s, y \mid T_{a_4,c_4} \mid r, z\rangle \, \langle w, x \mid T_{a_5,c_5} \mid y, s\rangle \, d\bar{x}$$

$$\begin{aligned} Z_\hbar(X) = \int_{\mathbb{R}^{10}} & \delta(w+t-u)\delta(z+q-v)\delta(x+q-r)\delta(s+y-r)\delta(w+x-y) \\ & \overline{\tilde{\psi}'_{a_1,c_1}(0)} e^{-2\pi i w(0)} \\ & \tilde{\psi}'_{a_2,c_2}(u-q) e^{2\pi i z(u-q)} \\ & \overline{\tilde{\psi}'_{a_3,c_3}(v-q)} e^{-2\pi i x(v-q)} \\ & \tilde{\psi}'_{a_4,c_4}(x-y) e^{2\pi i s(z-y)} \\ & \tilde{\psi}'_{a_5,c_5}(s-x) e^{2\pi i w(s-x)} \, dqdrdsdtdudvdxdwdzdy \end{aligned}$$

Integrating over five variables t, v, r, y, w yields the expression

$$\begin{aligned} Z_\hbar(X) = \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^5} & \tilde{\psi}'_{a_2,c_2}(u-q) e^{2\pi i z(u-q)} \\ & \times \overline{\tilde{\psi}'_{a_3,c_3}(z)} e^{-2\pi i x z} \\ & \times \tilde{\psi}'_{a_4,c_4}(z+s-x-q) e^{2\pi i s(z+s-x-q)} \\ & \times \tilde{\psi}'_{a_5,c_5}(s-x) e^{2\pi i (q-s)(s-x)} \, dqdsdudxdz. \end{aligned}$$

We integrate over the variable u using the Fourier transform.

$$e^{-\frac{\pi i}{12}} \int_{\mathbb{R}} \psi_{c_2,b_2}(u-q) e^{2\pi i z(u-q)} du = e^{-\frac{\pi i}{12}} \tilde{\psi}_{c_2,b_2}(-z) = e^{-\frac{\pi i}{6}} e^{\pi i z^2} \psi_{b_2,a_2}(-z).$$

Using formulas from Section 3.8 we can write

$$Z_\hbar(X) = e^{-\frac{3\pi i}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^4} \psi_{b_2,a_2}(-z) e^{\pi i z^2} \psi_{b_3,c_3}(-z) e^{\pi i z^2}$$

$$\tilde{\psi}'_{a_4,c_4}(z+s-x-q)\tilde{\psi}'_{a_5,c_5}(s-x) \\ e^{2\pi i(sz-xz-qx)}\,dqdsdxdz.$$

Integration over the variable q gives

$$\begin{aligned}\int_{\mathbb{R}} \tilde{\psi}'_{a_4,c_4}(z+s-x-q)e^{-2\pi iqx}dq &= e^{-\frac{\pi i}{12}}\tilde{\psi}_{c_4,b_4}(-x)e^{2\pi i(x^2-xz-sx)} \\ &= e^{-\frac{\pi i}{6}}\psi_{b_4,a_4}(-x)e^{2\pi i(\frac{3}{2}x^2-xz-sx)}\end{aligned}$$

$$Z_\hbar(X) = e^{-\frac{5\pi i}{12}}\overline{\tilde{\psi}'_{a_1,c_1}(0)}\int_{\mathbb{R}^2}\psi_{b_2,a_2}(-z)\psi_{b_3,c_3}(-z)\psi_{b_4,a_4}(-x)\tilde{\psi}'_{a_5,c_5}(s-x) \\ e^{2\pi i(sz-2xz+z^2+\frac{3}{2}x^2-sx)}\,dxdsdz$$

Integration over s now gives

$$\begin{aligned}e^{-\frac{\pi i}{12}}\int_{\mathbb{R}}\psi_{c_5,b_5}(s-x)e^{-2\pi is(x-z)}ds &= e^{-\frac{\pi i}{12}}\tilde{\psi}_{c_5,b_5}(x-z)e^{-2\pi i(x^2-xz)} \\ &= e^{-\frac{\pi i}{6}}\psi_{b_5,a_5}(x-z)e^{\pi i(x-z)^2}e^{-2\pi i(x^2-xz)}.\end{aligned}$$

So the partition function takes the form

$$Z_\hbar(X) = e^{-\frac{7\pi i}{12}}\overline{\tilde{\psi}'_{a_1,c_1}(0)}\int_{\mathbb{R}^2}\psi_{b_2,a_2}(-z)\psi_{b_3,c_3}(-z)\psi_{b_4,a_4}(-x)\psi_{b_5,a_5}(x-z) \\ e^{2\pi i(-2xz+\frac{3}{2}z^2+x^2)}\,dxdz.$$

which is equivalent to

$$Z_\hbar(X) = e^{-\frac{7\pi i}{12}}\overline{\tilde{\psi}'_{a_1,c_1}(0)}\int_{\mathbb{R}^2}\psi_{b_2,a_2}(z)\psi_{b_3,c_3}(z)\psi_{b_4,a_4}(-x)\psi_{b_5,a_5}(x+z) \\ e^{2\pi i(2xz+\frac{3}{2}z^2+x^2)}\,dxdz. \tag{5.9}$$

Set $\tilde{z} = z - c_{\mathrm{b}}(1-2c_2)$ and $-\tilde{x} = -x - c_{\mathrm{b}}(1-2c_4)$. Then

$$z - c_{\mathrm{b}}(1-2a_3) = \tilde{z} + c_{\mathrm{b}}(1-2c_2) - c_{\mathrm{b}}(1-2a_3) = \tilde{z},$$

because $a_3 \to c_2$ in the limit where $a_1 \to 0$. Furthermore we have

$$x + z - c_{\mathrm{b}}(1-2c_5) = \tilde{x} - c_{\mathrm{b}}(1-2c_4) + \tilde{z} + c_{\mathrm{b}}(1-2c_2) - c_{\mathrm{b}}(1-2c_5) = \tilde{x} + \tilde{z} - c_{\mathrm{b}}$$

because

$$c_4 + c_5 - c_2 \to 0$$

when $a_1 \to 0$. We can now write the partition function in the following way

$$
\begin{aligned}
Z_\hbar(X) = e^{-\frac{7\pi i}{12}} \int_{\mathbb{R}^2} & \psi(\tilde{z})\psi(\tilde{z})\psi(-\tilde{x})\psi(\tilde{x}+\tilde{z}-c_{\mathrm{b}})\overline{\tilde{\psi}'_{a_1,c_1}(0)} \\
& e^{-4\pi i(-\tilde{z}-c_{\mathrm{b}}(1-2c_2))(\tilde{x}-c_{\mathrm{b}}(1-2c_4))+3\pi i(-\tilde{z}-c_{\mathrm{b}}(1-2c_2))^2+2\pi i(\tilde{x}-c_{\mathrm{b}}(1-2c_4))^2} \\
& e^{-4\pi i c_{\mathrm{b}} b_2(\tilde{z}+c_{\mathrm{b}}(1/2-c_2))}\nu_{a_2,b_2} \\
& e^{-4\pi i c_{\mathrm{b}} b_3(\tilde{z}+c_{\mathrm{b}}(1-2c_2)-c_{\mathrm{b}}(1/2-a_3))}\nu_{b_3,c_3} \\
& e^{-4\pi i c_{\mathrm{b}} b_4(-\tilde{x}-c_{\mathrm{b}}(1/2-c_4))}\nu_{b_4,a_4} \\
& e^{-4\pi i c_{\mathrm{b}} b_5(\tilde{x}+\tilde{z}-c_{\mathrm{b}}(1-2c_4)+c_{\mathrm{b}}(1-2c_2)-c_{\mathrm{b}}(1/2-c_5))}\nu_{b_5,a_5} d\tilde{x}d\tilde{z}
\end{aligned}
$$

In front of $\tilde{z}$ in the exponent we have the factor

$$
\begin{aligned}
& 4\pi i c_{\mathrm{b}}(-1+2c_4+3/2-3c_2-b_2-b_3-b_5) \\
&= 4\pi i c_{\mathrm{b}}(1/2+2c_4-2c_2-b_2-a_3-b_3-b_5) \\
&= 4\pi i c_{\mathrm{b}}(1/2+2c_4-c_2-1/2+a_2-1/2+c_3-1/2+a_5+c_5) \\
&= 4\pi i c_{\mathrm{b}}(-1+1+c_4-c_2+c_5) = 0.
\end{aligned}
$$

In front of $\tilde{x}$ in the exponent we also have the factor 0 since

$$
\begin{aligned}
-b_5+b_4+1-2c_2-1+2c_4 &= -\frac{1}{2}+a_5+c_5+c_4+b_4+c_4-2c_2 \\
&= -\frac{1}{2}+a_5+\frac{1}{2}-a_4-c_2 \\
&= a_5-a_4-a_3 = 0.
\end{aligned}
$$

This gives us the partition function

$$
\begin{aligned}
Z_\hbar(X) = e^{i\frac{\phi}{\hbar}} e^{-\frac{7\pi i}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} & \psi(\tilde{z})\psi(\tilde{z})\psi(-\tilde{x})\psi(\tilde{x}+\tilde{z}-c_{\mathrm{b}}) \\
& e^{2\pi i(\frac{3}{2}\tilde{z}^2+\tilde{x}^2+2\tilde{x}\tilde{z})} d\tilde{x}d\tilde{z},
\end{aligned}
$$

where ϕ is a real quadratic polynomial of dihedral angles. Finally, we do the shift $\tilde{x} \mapsto \tilde{x}-\tilde{z}+c_{\mathrm{b}}$ and get the expression

$$
Z_\hbar(X) = \zeta_{inv}^2 e^{2\pi i c_{\mathrm{b}}^2} e^{i\frac{\phi}{\hbar}} e^{-\frac{7\pi i}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} \frac{\Phi_{\mathrm{b}}(\tilde{z})\Phi_{\mathrm{b}}(\tilde{x})}{\Phi_{\mathrm{b}}(-\tilde{z})\Phi_{\mathrm{b}}(\tilde{x}-\tilde{z}-c_{\mathrm{b}})} e^{\pi i\tilde{x}^2+4\pi c_{\mathrm{b}} x} d\tilde{x}d\tilde{z}.
$$

which exactly corresponds to the result for an H-triangulation of the 6_1 knot in [KLV].

5.6 One vertex H-triangulation of $(S^3, 6_1)$ – New formulation

We here calculate the partition function for the H-triangulation of the knot 6_1 using the new formulation of the TQFT from quantum Teichmüller theory.

The gluing pattern of faces and edges in diagram (5.8) results in the following states

$$\begin{aligned}
x &:= x^1_{02} = x^1_{03} = x^2_{01} = x^2_{02} = x^3_{01},\\
y &:= x^2_{03} = x^2_{13} = x^3_{02} = x^3_{03} = x^3_{13} = x^4_{02} = x^4_{03} = x^5_{03},\\
z &:= x^2_{23} = x^3_{12} = x^4_{12} = x^5_{01},\\
v &:= x^1_{12} = x^1_{13} = x^3_{23} = x^4_{23} = x^5_{12} = x^5_{13},\\
w &:= x^1_{23} = x^2_{12} = x^4_{01} = x^4_{13} = x^5_{02} = x^5_{23},\\
x' &:= x^1_{01}.
\end{aligned}$$

The Boltzmann weights for the five tetrahedron are given by

$$\overline{g_{a_1,c_1}(0, x+v-x'-w)}, \quad g_{a_2,c_2}(x-w, y-z), \quad \overline{g_{a_3,c_3}(y-z, 2y-x-v)},$$
$$g_{a_4,c_4}(w-z, y-v), \quad g_{a_5,c_5}(w-y, v-z+p).$$

$$\begin{aligned}
Z^{\mathrm{New}}_\hbar(X) = \int_{[0,1]^6} \sum_{k,l,m,n,p\in\mathbb{Z}} & 3\overline{\tilde{\psi}'_{a_1,c_1}(k)}\\
& \tilde{\psi}'_{a_2,c_2}(x-w+l)e^{\pi i(y-z)(x-w+2l)}\\
& \overline{\tilde{\psi}'_{a_3,c_3}(y-z+m)}e^{-\pi i(2y-x-v)(y-z+2m)}\\
& \tilde{\psi}'_{a_4,c_4}(w-z+n)e^{\pi i(y-v)(w-z+2n)}\\
& \tilde{\psi}'_{a_5,c_5}(w-y+p)e^{\pi i(v-z)(w-y+p)}dxdydzdvdwdx'
\end{aligned}$$

Integration over x' gives $\delta(k)$, which removes one of the sums

$$\begin{aligned}
Z^{\mathrm{New}}_\hbar(X) = \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{[0,1]^5} \sum_{l,m,n,p\in\mathbb{Z}} & \tilde{\psi}'_{a_2,c_2}(x-w+l)e^{\pi i(y-z)(x-w+2l)}\\
& \overline{\tilde{\psi}'_{a_3,c_3}(y-z+m)}e^{-\pi i(2y-x-v)(y-z+2m)}\\
& \tilde{\psi}'_{a_4,c_4}(w-z+n)e^{\pi i(y-v)(w-z+2n)}\\
& \tilde{\psi}'_{a_5,c_5}(w-y+p)e^{\pi i(v-z)(w-y+p)}dxdydzdvdw.
\end{aligned}$$

We do a shift $x \to x+w$

$$\begin{aligned}
Z^{\mathrm{New}}_\hbar(X) = \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{[0,1]^5} \sum_{l,m,n,p\in\mathbb{Z}} & \tilde{\psi}'_{a_2,c_2}(x+l)e^{\pi i(y-z)(x+2l)}\\
& \overline{\tilde{\psi}'_{a_3,c_3}(y-z+m)}e^{-\pi i(2y-x-w-v)(y-z+2m)}\\
& \tilde{\psi}'_{a_4,c_4}(w-z+n)e^{\pi i(y-v)(w-z+2n)}\\
& \tilde{\psi}'_{a_5,c_5}(w-y+p)e^{\pi i(v-z)(w-y+p)}dxdydzdvdw.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}} e^{2\pi i l(y-z)} \int_{[0,1]} \tilde{\psi}'_{a_2,c_2}(x+l)e^{-2\pi i x(z-y-m)}dx\\
&= e^{-\frac{\pi i}{12}} \sum_{l\in\mathbb{Z}} \int_l^{l+1} \psi_{c_2,b_2}(x)e^{-2\pi i x(z-y-m)}dx\\
&= e^{-\frac{\pi i}{12}} \int_{\mathbb{R}} \psi_{c_2,b_2}(x)e^{-2\pi i x(z-y-m)}dx\\
&= e^{-\frac{\pi i}{12}} \tilde{\psi}_{c_2,b_2}(z-y-m) = e^{-\frac{\pi i}{6}} \psi_{b_2,a_2}(z-y-m)e^{\pi i(z-y-m)^2}.
\end{aligned}$$

$$\begin{aligned}
Z_\hbar^{\text{New}}(X) = e^{-\frac{\pi i5}{12}}\overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{[0,1]^3} \sum_{m,n,p\in\mathbb{Z}} &\psi_{b_2,a_2}(z-y-m)\psi_{b_3,c_3}(z-y-m)\\
&\psi_{c_4,b_4}(w-z+n)\psi_{c_5,b_5}(w-y+p)\\
&e^{2\pi i(z-y-m)^2}e^{-\pi i(2y-w-v)(y-z+2m)}\\
&e^{(y-v)(w-z+2n)}e^{\pi i(v-z)(w-y+2p)}\, dzdw.
\end{aligned}$$

Integration over v removes yet another sum. I.e. $n = p+m$. We shift $z \mapsto z+y$ and $w \mapsto w+y$ and see that the function is independent of y which yields the expression

$$\begin{aligned}
Z_\hbar^{\text{New}}(X) = e^{-\frac{\pi i5}{12}}\overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{[0,1]^3} \sum_{m,p\in\mathbb{Z}} &\psi_{b_2,a_2}(z-m)\psi_{b_3,c_3}(z-m)\\
&\psi_{c_4,b_4}(w-z+p+m)\psi_{c_5,b_5}(w+p)\\
&e^{2\pi i(z-m)^2}e^{\pi i w(-z+2m)}e^{-\pi i z(w+2p)}\, dzdw.
\end{aligned}$$

$$\begin{aligned}
Z_\hbar^{\text{New}}(X) = e^{-\frac{\pi i5}{12}}\overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} &\psi_{b_2,a_2}(z)\psi_{b_3,c_3}(z)\\
&\psi_{c_4,b_4}(w-z)\psi_{c_5,b_5}(w)\\
&e^{2\pi i z^2}e^{\pi i(w-m)(-z+m)}e^{-\pi i(z+m)(w+p)}\, dzdw.
\end{aligned}$$

Now let $f(z) := \psi_{b_2,a_2}(z)\psi_{b_3,c_3}(z)$. We then calculate

$$\begin{aligned}
Z_\hbar^{\text{New}}(X) &= e^{-\frac{\pi i5}{12}}\overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} f(z)\psi_{c_4,b_4}(w-z)\psi_{c_5,b_5}(w)e^{2\pi i z^2}e^{-2\pi i wz}\, dzdw\\
&= e^{-\frac{\pi i5}{12}}\overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} f(z)\psi_{b_3,c_3}(z)\psi_{c_4,b_4}(w)\psi_{c_5,b_5}(w+z)e^{-2\pi i wz}\, dzdw\\
&= e^{-\frac{\pi i5}{12}}\overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} f(z)\psi_{c_4,b_4}(w)\tilde{\psi}_{c_5,b_5}(x)e^{2\pi i((x-z)(w+z)+z^2)}\, dxdzdw
\end{aligned}$$

$$
\begin{aligned}
&= e^{-\frac{\pi i 5}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} f(z)\psi_{c_4,b_4}(w)\tilde{\psi}_{c_5,b_5}(x+z) e^{2\pi i (x(w+z)+z^2)}\, dxdzdw \\
&= e^{-\frac{\pi i 6}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} f(z)\psi_{c_4,b_4}(w)\psi_{b_5,a_5}(x+z) e^{2\pi i (wx+2xz+\frac{3}{2}z^2+\frac{x^2}{2})}\, dxdzdw \\
&= e^{-\frac{\pi i 6}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} f(z)\tilde{\psi}_{c_4,b_4}(-x)\psi_{b_5,a_5}(x+z) e^{2\pi i (2xz+\frac{3}{2}z^2+\frac{x^2}{2})}\, dzdw \\
&= e^{-\frac{\pi i 7}{12}} \overline{\tilde{\psi}'_{a_1,c_1}(0)} \int_{\mathbb{R}^2} f(z)\psi_{b_4,a_4}(-x)\psi_{b_5,a_5}(x+z) e^{2\pi i (2xz+\frac{3}{2}z^2+x^2)}\, dzdw
\end{aligned}
$$

This is the exact same expression as in (5.9) and the two formulations coincide.

5.7 Volume of $(S^3, 6_1)$

Theorem 5.1. *The hyperbolic volume of the complement of* 6_1 *in* S^3 *is recovered as the following limit*

$$
\lim_{\hbar\to 0} 2\pi\hbar \log |J_{s^3,6_1}(\hbar,0)| = -\mathrm{Vol}(S^3\backslash 6_1). \tag{5.10}
$$

Proof. We consider the expression

$$
J_{S^3,6_1}(\hbar,0) = \int_{\mathbb{R}^2} \frac{\Phi_{\mathrm{b}}(x)\Phi_{\mathrm{b}}(z)}{\Phi_{\mathrm{b}}(-x)\Phi_{\mathrm{b}}(z-x-c_{\mathrm{b}})} e^{\pi i z^2 - 4\pi i c_{\mathrm{b}} z} dxdz. \tag{5.11}
$$

Using the quasi-classical asymptotic behaviour of Faddeev's quantum dilogarithm shown in Corollary A.6 we can approximate in the following manner. For b close to zero the integral in (A.12) is approximated by the double contour integral

$$
\begin{aligned}
J_{S^3,6_1}(\hbar,0) &= \frac{1}{(2\pi\,\mathrm{b})^2} \int_{\mathbb{R}^2} \frac{\Phi_{\mathrm{b}}\left(\frac{x}{2\pi\,\mathrm{b}}\right)\Phi_{\mathrm{b}}\left(\frac{z}{2\pi\,\mathrm{b}}\right)}{\Phi_{\mathrm{b}}\left(\frac{-x}{2\pi\,\mathrm{b}}\right)\Phi_{\mathrm{b}}\left(\frac{z-x-c_{\mathrm{b}}}{2\pi\,\mathrm{b}}\right)} e^{-\frac{z^2}{4\pi i\,\mathrm{b}^2}+\frac{y}{\mathrm{b}^2}} dxdz \\
&\sim \frac{1}{(2\pi\,\mathrm{b})^2} \int_{\mathbb{R}^2} e^{\frac{1}{2\pi i\,\mathrm{b}^2}(2\,\mathrm{Li}_2(-e^x)+\mathrm{Li}_2(-e^z)-\mathrm{Li}_2(e^{z-x})-\frac{1}{2}z^2+2\pi i z+\frac{1}{2}x^2)} dxdz \\
&= \frac{1}{(2\pi\,\mathrm{b})^2} \int_{\mathbb{R}^2} e^{\frac{1}{2\pi i\,\mathrm{b}^2}V(x,z)} dxdz,
\end{aligned}
$$

where the potential V is given by

$$
V(x,z) = 2\,\mathrm{Li}_2(-e^x) + \mathrm{Li}_2(-e^z) - \mathrm{Li}_2(e^{z-x}) - \frac{1}{2}z^2 + 2\pi i z + \frac{1}{2}x^2. \tag{5.12}
$$

It is easily seen that we can treat b^2 as $\hbar$. Therefore, we look for stationary points of the potential V

$$
\frac{\partial V(x,z)}{\partial x} = -2\log(1+e^x) - \log(1-e^{z-x}) + x = \log \frac{e^x}{(1+e^x)^2(1-e^{z-x})}. \tag{5.13}
$$

$$\frac{\partial V(x,z)}{\partial z} = -\log(1+e^z) + \log(1-e^{z-x}) - z + 2\pi i = \log\frac{1-e^{z-x}}{(1+e^z)e^z}. \tag{5.14}$$

Stationary points are given by solutions to the equations

$$e^x = (1+e^x)^2(1-e^{z-x}), \tag{5.15}$$
$$(1+e^z)e^z = 1-e^{z-x}. \tag{5.16}$$

From (5.16) we see that

$$e^x = \frac{1}{e^{-z}-1-e^z}.$$

Inserting in (5.15) we get the equation

$$\frac{1}{e^{-z}-1-e^z} = \left(\frac{e^{-z}-e^z}{e^{-z}-1-e^z}\right)^2(1+e^z)e^z \iff 1-t-t^2 = (1-t^2)^2(1+t),$$

where we set $e^z = t$.

Numerical solutions for the last equation are given by

$$t_1 = -1,39923 - 0,32564i, \qquad t_3 = 0,899232 - 0,400532i,$$
$$t_2 = -1,39923 + 0,32564i, \qquad t_4 = 0,899232 + 0,400532i.$$

The maximal contribution to the integral comes from the point t_2. The saddle point method lets us obtain the following limit

$$\lim_{\hbar^2\to 0} 2\pi\hbar|J_{S^3,6_1}(\hbar,0)| = -3.1632...\cdot I = -\mathrm{Vol}(S^3\backslash 6_1)$$

□

6 The Teichmüller TQFT representation of the mapping class group $\Gamma_{1,1}$

We will here give a representation for the mapping class group of the once punctured torus by the use of the new formulation of the Teichmüller TQFT.

The framed mapping class group $\Gamma_{1,1}$ is generated by the standard elements S and T. See e.g. Section 6 in [AU3] for a description of these elements (they of course maps to the standard S and T matrix once mapped to the mapping class group of the torus). Hence we just need to understand how these to elements are represented by the Teichmüller TQFT. To this end, we build a cobordism $(M,\mathbb{T}^2,\mathbb{T}^{2\prime})$ from one triangulation of $\mathbb{T}^2$ to the image of this triangulation under the action of S and likewise for the action of T. We triangulate the torus $\mathbb{T}^2 = S^1\times S^1$ according to Figure 11. In this triangulation opposite arrows are identified

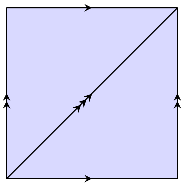


Figure 11: Triangulated torus.

and this gives us a triangulation with two triangles and three edges. We build the cobordism for the action of S according to Figure 12 and the cobordism for T according to Figure 13. We see that on each boundary component we have three edges. The cobordisms that we build are given shaped triangulations. We can choose the dihedral angles such that they are all positive. And we are able to compose these cobordisms.

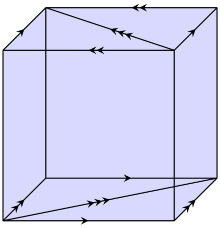


Figure 12: The cobordism for the operator S.

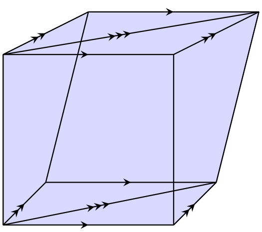


Figure 13: The cobordism for the operator T.

For each edge in these triangulations we assign a state variable. We abuse notation and label an edge and a state variable by the same letter. We assign a multiplier to each edge. As we will see below in Lemma 8.1 and Lemma 9.1 it turns out, that all internal edges have trivial multiplier. Further we emphasise that there is a direction on each of the two boundary tori where the multiplier is trivial.

The Teichmüller TQFT gives an operator between the vector spaces associated to each of the boundary components. We will see that we get representations

$$\rho : \Gamma_{1,1} \to \mathcal{B}(C^\infty(\mathbb{T}^3, \mathcal{L}')),$$

of the mapping class group $\Gamma_{1,1}$ into bounded operators on the smooth sections $C^\infty(\mathbb{T}^3, \mathcal{L}')$. However, we will show below that we actually get representations into $\mathcal{B}(\mathcal{S}(\mathbb{R}))$, bounded operators on the Schwartz space $\mathcal{S}(\mathbb{R})$.

Theorem 6.1. *The Teichmüller TQFT provides us with representations (dependent on h)*

$$\tilde{\rho} : \Gamma_{1,1} \to \mathcal{B}(\mathcal{S}(\mathbb{R}))$$

of the mapping class group $\Gamma_{1,1}$ into bounded operators on the Schwarz space $\mathcal{S}(\mathbb{R})$. In particular we get operators $\tilde{\rho}(S), \tilde{\rho}(T) : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ according to the diagram (6.1).

(6.1)
$$\begin{array}{ccc}
\mathcal{S}(\mathbb{R}) & \xrightarrow{\tilde{\rho}(S),\tilde{\rho}(T)} & \mathcal{S}(\mathbb{R}) \\
\Big\downarrow{\scriptstyle W} & & \Big\downarrow{\scriptstyle W} \\
C^\infty(\mathbb{T}^2, \mathcal{L}) & \longrightarrow & C^\infty(\mathbb{T}^2, \mathcal{L}) \\
\Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^*} \\
\pi^*\left(C^\infty(\mathbb{T}^2, \mathcal{L})\right) & \longrightarrow & \pi^*\left(C^\infty(\mathbb{T}^2, \mathcal{L})\right) \\
\cap & & \cap \\
C^\infty(\mathbb{T}^3, \mathcal{L}') & \xrightarrow[\rho(T)]{\rho(S)} & C^\infty(\mathbb{T}^3, \mathcal{L}')
\end{array}$$

where $\mathcal{L}' = \pi^*\mathcal{L}$.

Proof. We know that the Weil–Gel'fand–Zak transformation gives an isomorphism from the Schwarz space to smooth sections of the complex line bundle $\mathcal{L}$ over the 2-torus. If a section of $C^\infty(\mathbb{T}^2, \mathcal{L})$ is pulled back to $\pi^*\left(C^\infty(\mathbb{T}^2, \mathcal{L})\right)$ we show in Lemma 8.2 and Lemma 9.2 that the operators $\rho(S), \rho(T)$ acting on $C^\infty(\mathbb{T}^3, \mathcal{L}')$ take this pull back of a section in $\pi^*\left(C^\infty(\mathbb{T}^2, \mathcal{L})\right)$. In Lemma 8.1 and 9.1 we prove that the multipliers on internal edges are trivial. Further we show that the multipliers on the two boundary tori are trivial in the direction $(1, 1, 1)$. We can therefore integrate over the fibre in this direction. We then use the inverse WGZ transformation. In other words we have shown that the operators $\rho(S), \rho(T)$ induce operators $\tilde{\rho}(S), \tilde{\rho}(T) : \mathcal{S}(\mathbb{R}) \to S(\mathbb{R})$ given by

$$\tilde{\rho}(S) = W^{-1} \circ \int_{F_{z'}} \circ\, \rho(S) \circ \pi^* \circ W,$$

$$\tilde{\rho}(T) = W^{-1} \circ \int_{F_{z'}} \circ\, \rho(S) \circ \pi^* \circ W.$$

□

Remark 6.2. Above we obtained a representation for the mapping class group $\Gamma_{1,1}$. We do not in a similar manner get a representation for the mapping class group $\Gamma_{1,0}$. The reason is that not all edges in the cobordisms can be balanced without turning to negative angles.

7 Line bundles over the two boundary tori

Let us here describe how the line bundles we pull back looks like.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $\pi(x_1, x_2, x_3) = (ax_1 + bx_2 + cx_3, \alpha x_1 + \beta x_2 + \gamma x_3)$. Recall that we have the relation on multipliers

$$e_\lambda^{\pi^*}(x, y, z) = e_{\pi(\lambda)}(\pi(x, y, z)). \tag{7.1}$$

Note that the map π sends $\lambda_{x_1} = (1, 0, 0)$, $\lambda_{x_2} = (0, 1, 0)$, $\lambda_{x_3} = (0, 0, 1)$ to the following elements of $\mathbb{R}^2$

$$\pi(\lambda_{x_1}) = (a, \alpha), \quad \pi(\lambda_{x_2}) = (b, \beta), \quad \pi(\lambda_{x_3}) = (c, \gamma).$$

The equation (7.1) gives the following relations:

In the λ_{x_1}-direction

$$\begin{aligned} e^{2\pi i(x_3 - x_2)} =& e_{(1,0,0)(x_1,x_2,x_3)} = e_{(a,\alpha)}(ax_1 + bx_2 + cx_3, \alpha x_1 + \beta x_2 + \gamma x_3) \\ =& e_{(a,0)}(ax_1 + bx_2 + cx_3, \alpha x_1 + \beta x_2 + \gamma x_3) \\ & e_{(0,\alpha)}(ax_1 + bx_2 + cx_3, \alpha(x_1 + 1) + \beta x_2 + \gamma x_3) \\ =& e^{-\pi i a(\alpha x_1 + \beta x_2 + \gamma x_3)} e^{\pi i(ax_1 + bx_2 + cx_3)} \\ =& e^{\pi i((\alpha b - a\beta)x_2 + (\alpha c - a\gamma)x_3)}, \end{aligned}$$

In the λ_{x_2}-direction

$$e^{2\pi i(x_1 - x_3)} = e_{(0,1,0)(x_1,x_2,x_3)} = e^{\pi i((\beta a - \alpha b)x_1 + (\beta c - b\gamma)x_3)},$$

In the λ_{x_3}-direction

$$e^{2\pi i(x_2 - x_1)} = e_{(0,0,1)(x_1,x_2,x_3)} = e^{\pi i((\gamma a - \alpha c)x_1 + (\gamma b - c\beta)x_2)}.$$

In other words we only need to solve the three equations

$$\alpha b - a\beta = -2, \quad \alpha c - a\gamma = 2, \quad \beta c - b\gamma = -2. \tag{7.2}$$

One particular solution is $a = -2, b = 0, c = 2, \alpha = 0, \beta = -1, \gamma = 1$ which gives the map

$$\pi(x_1, x_2, x_3) = (-2x_1 + 2x_3, -x_2 + x_3).$$

8 The operator $\rho(S)$

The operator $\rho(S)$ can be viewed as the cobordism X_S which is triangulated into 6 tetrahedra $T_1, \dots, T_6$ where T_1, T_3, T_4, T_6 have positive orientation and the tetrahedra T_2, T_5 have negative orientation. See the gluing pattern in figure 14.

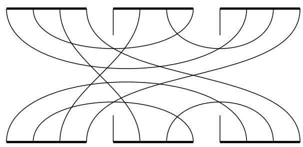


Figure 14: Gluing pattern for the operator X_S

In the triangulation we have ten edges $x_1, x_2 \dots, x_7, x'_1, x'_2, x'_3$. To each of the edges on the boundary we associate the a weight function:

$$\omega_{X_S}(x_1) = 2\pi(a_1 + a_5 + c_3), \quad \omega_{X_S}(x_2) = 2\pi(a_4 + c_5 + a_6), \quad \omega_{X_S}(x_3) = 2\pi(b_5 + b_6),$$
$$\omega_{X_S}(x'_1) = 2\pi(a_1 + c_2 + a_3), \quad \omega_{X_S}(x'_2) = 2\pi(a_2 + c_3 + a_4), \quad \omega_{X_S}(x'_3) = 2\pi(b_2 + b_3).$$

and to the edges x_4, x_5, x_6, x_7 we associate the weight functions:

$$\omega_{X_S}(x_4) = 2\pi(a_1 + c_2 + b_4 + c_5 + c_6), \quad \omega_{X_S}(x_5) = 2\pi(c_1 + b_3 + b_4 + a_5 + a_6),$$
$$\omega_{X_S}(x_6) = 2\pi(b_1 + a_2 + a_3 + c_4 + b_6), \quad \omega_{X_S}(x_7) = 2\pi(b_1 + c_2 + c_3 + c_4 + b_5).$$

8.1 Boltzmann weights

The Boltzmann weights assigned to the tetrahedra are

$$\begin{aligned}
B\left(T_1, x_{|\Delta_1(T_1)}\right) &= g_{a_1,c_1}(x_7 + x_6 - x_4 - x_5, x_7 + x_6 - x'_1 - x_1),\\
B\left(T_2, x_{|\Delta_1(T_2)}\right) &= \overline{g_{a_2,c_2}(x'_3 + x_4 - x'_1 - x_7, x'_3 + x_4 - x'_2 - x_6)},\\
B\left(T_3, x_{|\Delta_1(T_3)}\right) &= g_{a_3,c_3}(x'_3 + x_5 - x'_2 - x_7, x'_3 + x_5 - x'_1 - x_6)\\
B\left(T_4, x_{|\Delta_1(T_4)}\right) &= g_{a_4,c_4}(x_5 + x_4 - x_7 - x_6, x_5 + x_4 - x'_2 - x_2)\\
B\left(T_5, x_{|\Delta_1(T_5)}\right) &= \overline{g_{a_5,c_5}(x_7 + x_3 - x_4 - x_2, x_5 + x_4 - x_5 - x_1)}\\
B\left(T_6, x_{|\Delta_1(T_6)}\right) &= g_{a_6,c_6}(x_6 + x_3 - x_4 - x_1, x_6 + x_3 - x_5 - x_2).
\end{aligned}$$

Lemma 8.1. *The multipliers corresponding to the edges are calculated to be 1 for the internal edges x_4, x_5, x_6, x_7. And the multipliers for the remaining 6 edges are calculated to be*

$$e_{\lambda_{x_1}}(\boldsymbol{x}) = e^{2\pi i(x_3 - x_2)}, \quad e_{\lambda_{x_2}}(\boldsymbol{x}) = e^{2\pi i(x_1 - x_3)}, \quad e_{\lambda_{x_3}}(\boldsymbol{x}) = e^{2\pi i(x_2 - x_1)},$$

$$e_{\lambda_{x_1'}}(\boldsymbol{x}) = e^{2\pi i(x_2'-x_3')}, \quad e_{\lambda_{x_2'}}(\boldsymbol{x}) = e^{2\pi i(x_3'-x_1')}, \quad e_{\lambda_{x_3'}}(\boldsymbol{x}) = e^{2\pi i(x_1'-x_2')},$$

where $\boldsymbol{x}$ *denotes the tuple* $\boldsymbol{x} = (x_1, x_2, x_3, x_1', x_2', x_3')$.

Proof. Let us here just calculate the multiplier for the direction x_4. The rest follows by analogous calculations. The edge x_4 is an edge in the tetrahedra T_1, T_2, T_4, T_5, T_6 each contributing to the multiplier. The contribution from T_1 corresponds to the multiplier

$$\begin{aligned} e_{\lambda_{x_4}}(x_1, x_2, \ldots, x_7, x_1', x_2', x_3') &= e_{-(1,0)}(x_5 + x_4 - x_7 - x_6, x_5 + x_4 - x_2' - x_2) \\ &= e^{\pi i(x_7 + x_6 - x_1' - x_1)}. \end{aligned}$$

The contribution from T_2 is

$$\begin{aligned} e_{\lambda_{x_4}}(x_1, x_2, \ldots, x_7, x_1', x_2', x_3') &= \overline{e_{(1,1)}(x_3' + x_4 - x_1' - x_7, x_3' + x_4 - x_2' - x_6)} \\ &= -e^{-\pi i(x_2' + x_6 - x_1' - x_7)}. \end{aligned}$$

The contribution from T_4 is

$$\begin{aligned} e_{\lambda_{x_4}}(x_1, x_2, \ldots, x_7, x_1', x_2', x_3') &= e_{(1,1)}(x_5 + x_4 - x_7 - x_6, x_5 + x_4 - x_2' - x_2) \\ &= -e^{\pi i(x_2' + x_2 - x_6 - x_7)}. \end{aligned}$$

The contribution from T_5 is

$$e_{\lambda_{x_4}}(x_1, x_2, \ldots, x_7, x_1', x_2', x_3') = -e^{-\pi i(x_7 + x_3 - x_5 - x_1)}.$$

The contribution from T_6 is

$$e_{\lambda_{x_4}}(x_1, x_2, \ldots, x_7, x_1', x_2', x_3') = -e^{\pi i(x_6 + x_3 - x_5 - x_2)}.$$

Multiplying these contributions gives $e^0 = 1$. □

We remark that the multiplier on each boundary component in direction $(1, 1, 1)$ is trivial.

We are interested in how the operator $\rho(S)$ acts. We express the operator $\rho(S)$ in terms of the integral kernel K_S. The operator $\rho(S)$ acts on sections in the following manner

$$\rho(S)(s)(x_1', x_2', x_3') = \int_{[0,1]^3} K_S(x_1', x_2', x_3', x_1, x_2, x_3) s(x_1, x_2, x_3)\, dx_1 dx_2 dx_3. \tag{8.1}$$

We want to show that the operator S takes the pull back of a section to the pull back of a section. Using integration by parts it is enough to check that the sum of partial derivatives disappear.

Lemma 8.2. *The sum of the partial derivatives of K_S disappears. I.e.*

$$\frac{\partial K_S}{\partial x'_1}+\frac{\partial K_S}{\partial x'_2}+\frac{\partial K_S}{\partial x'_3}+\frac{\partial K_S}{\partial x_1}+\frac{\partial K_S}{\partial x_2}+\frac{\partial K_S}{\partial x_3}=0.$$

Proof. Let

$$\begin{aligned}I_3^{n,m,k,j}(x_1,x_2,x_3,x'_2,x'_3):=\int_{[0,1]^2}&\tilde{\psi}'_{a_1,c_1}(x_7+k)\tilde{\psi}'_{a_4,c_4}(-x_7+n)\\&e^{2\pi i x_7(x_2-x_3+x_7+x_5+2n-m+k-j)}\\&e^{2\pi i(x'_3-x'_2-x_3+x_1+k-j)}dx_5dx_7\end{aligned}$$

The partial derivatives of I_3 with respect to x_1,x_2,x_3,x'_2,x'_3 are easily calculated to be

$$\begin{aligned}\frac{\partial}{\partial x_1}I_3^{n,m,k,j}(x_1,x_2,x_3,x'_2,x'_3)&=2\pi i x_5 I_3(x_1,x_2,x_3,x'_2,x'_3)=:I'_3(x_1,x_2,x_3,x'_2,x'_3),\\\frac{\partial}{\partial x_2}I_3^{n,m,k,j}(x_1,x_2,x_3,x'_2,x'_3)&=2\pi i x_7 I_3(x_1,x_2,x_3,x'_2,x'_3)=:I''_3(x_1,x_2,x_3,x'_2,x'_3),\\\frac{\partial}{\partial x_3}I_3^{n,m,k,j}(x_1,x_2,x_3,x'_2,x'_3)&=-I'_3(x_1,x_2,x_3,x'_2,x'_3)-I''_3(x_1,x_2,x_3,x'_2,x'_3),\\\frac{\partial}{\partial x'_2}I_3^{n,m,k,j}(x_1,x_2,x_3,x'_2,x'_3)&=-I'_3(x_1,x_2,x_3,x'_2,x'_3),\\\frac{\partial}{\partial x'_3}I_3^{n,m,k,j}(x_1,x_2,x_3,x'_2,x'_3)&=I'_3(x_1,x_2,x_3,x'_2,x'_3).\end{aligned}$$

The partial derivatives of I_2 with respect to the variables x_2,x_3,x'_1,x'_3 are

$$\begin{aligned}\frac{\partial}{\partial x_2}I_2^{k,l,n,p}(x_2,x_3,x'_1,x'_3)=&\frac{e^{2\pi i(x'_1-x'_3-x_3+x_2+2(k,l,n,p))}(x'_1-x'_3-x_3+x_2+2(k,l,n,p))}{(x'_1-x'_3-x_3+x_2+2(k,l,n,p))^2}\\&-\frac{(e^{2\pi i(x'_1-x'_3-x_3+x_2+2(k,l,n,p))}-1)}{2\pi i(x'_1-x'_3-x_3+x_2+2(k,l,n,p))^2}\\=&:I'_2(x_2,x_3,x'_1,x'_3),\\\frac{\partial}{\partial x_3}I_2^{k,l,n,p}(x_2,x_3,x'_1,x'_3)=&-I'_2(x_2,x_3,x'_1,x'_3),\\\frac{\partial}{\partial x'_1}I_2^{k,l,n,p}(x_2,x_3,x'_1,x'_3)=&I'_2(x_2,x_3,x'_1,x'_3),\\\frac{\partial}{\partial x'_3}I_2^{k,l,n,p}(x_2,x_3,x'_1,x'_3)=&-I'_2(x_2,x_3,x'_1,x'_3).\end{aligned}$$

The partial derivatives of I_1 with respect to the variables x_2,x_3,x'_1,x'_3 are

$$\frac{\partial}{\partial x_2}I_1^{l,p}(x_2,x_3,x'_1,x'_3)=-\frac{e^{2\pi i(x'_3-x'_1-x_2+x_3+2(m+q))}(x'_3-x'_1-x_2+x_3+2(m+q))}{(x'_3-x'_1-x_2+x_3+2(m+q))^2}$$

$$
\begin{aligned}
&+\frac{(e^{2\pi i(x_3'-x_1'-x_2+x_3+2(m+q))}-1)}{2\pi i(x_3'-x_1'-x_2+x_3+2(m+q))^2}\\
&=I_1'(x_2,x_3,x_1',x_3'),\\
\frac{\partial}{\partial x_3}I_1^{l,p}(x_2,x_3,x_1',x_3') &= -I_1'(x_2,x_3,x_1',x_3'),\\
\frac{\partial}{\partial x_1'}I_1^{l,p}(x_2,x_3,x_1',x_3') &= I_1'(x_2,x_3,x_1',x_3'),\\
\frac{\partial}{\partial x_3'}I_1^{l,p}(x_2,x_3,x_1',x_3') &= -I_1'(x_2,x_3,x_1',x_3').
\end{aligned}
$$

The rest of the terms in K_S all depends on pairs of the variables $x_1, x_2, x_3, x_1', x_2', x_3'$ with opposite sign, summing all contributions together therefore shows that the sum of the partial derivatives disappears. □

9 The operator $\rho(T)$

The operator $\rho(T)$ is the TQFT operator associated to the cobordism X_T which is triangulated into 6 tetrahedra $T_1, \ldots, T_6$ where T_1, T_4, T_5 have negative orientation and the tetrahedra T_2, T_3, T_6 have positive orientation. See Figure 15.

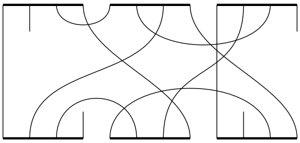


Figure 15: Gluing pattern for the operator X_T.

In the triangulation we have ten edges $x_1, x_2 \ldots, x_7, x_1', x_2', x_3'$. The weight functions corresponding to this triangulation for the edges $x_1, x_2, x_3, x_1', x_2', x_3'$ are

$$
\begin{aligned}
&\omega_{Y_T}(x_1) = 2\pi(c_3+a_6), && \omega_{Y_T}(x_2) = 2\pi(b_2+a_3+b_6), && \omega_{Y_T}(x_3) = 2\pi(b_3+b_5+c_6),\\
&\omega_{Y_T}(x_1') = 2\pi(a_1+c_4), && \omega_{Y_T}(x_2') = 2\pi(b_1+a_4+b_5), && \omega_{Y_T}(x_3') = 2\pi(c_1+b_2+b_4).
\end{aligned}
$$

and to the edges x_4, x_5, x_6, x_7 we associate the weight functions

$$
\begin{aligned}
&\omega_{Y_T}(x_4) = 2\pi(a_1+c_2+c_5+a_6), && \omega_{Y_T}(x_5) = 2\pi(b_1+a_2+b_3+b_4+a_5+b_6),\\
&\omega_{Y_T}(x_6) = 2\pi(c_1+a_2+a_3+a_4+a_5+c_6), && \omega_{Y_T}(x_7) = 2\pi(b_2+c_3+c_4+c_5).
\end{aligned}
$$

The Boltzmann weights assigned to the tetrahedra are

$$
B\left(T_1, x_{|_{\Delta_1(T_1)}}\right) = \overline{g_{a_1,c_1}(x_5+x_2'-x_3'-x_6, x_5+x_2'-x_1'-x_4)},
$$

$$
\begin{aligned}
B\left(T_2, x_{|\Delta_1(T_2)}\right) &= g_{a_2,c_2}(x_3' + x_2 - x_7 - x_4, x_3' + x_2 - x_5 - x_6),\\
B\left(T_3, x_{|\Delta_1(T_3)}\right) &= g_{a_3,c_3}(x_5 + x_3 - x_7 - x_1, x_5 + x_3 - x_6 - x_2)\\
B\left(T_4, x_{|\Delta_1(T_4)}\right) &= \overline{g_{a_4,c_4}(x_3' + x_5 - x_7 - x_1', x_3 + x_5 - x_2' - x_6)}\\
B\left(T_5, x_{|\Delta_1(T_5)}\right) &= \overline{g_{a_5,c_5}(x_2' + x_3 - x_7 - x_4, x_2' + x_3 - x_6 - x_5)}\\
B\left(T_6, x_{|\Delta_1(T_6)}\right) &= g_{a_6,c_6}(x_2 + x_5 - x_3 - x_6, x_2 + x_5 - x_4 - x_1).
\end{aligned}
$$

Lemma 9.1. *The multipliers corresponding to the edges are calculated to be 1 for the internal edges x_4, x_5, x_6, x_7. And the multipliers for the remaining 6 edges are calculated to be*

$$
\begin{aligned}
&e_{\lambda_{x_1}}(\boldsymbol{x}) = e^{2\pi i(x_3 - x_2)}, \quad e_{\lambda_{x_2}}(\boldsymbol{x}) = e^{2\pi i(x_1 - x_3)}, \quad e_{\lambda_{x_3}}(\boldsymbol{x}) = e^{2\pi i(x_2 - x_1)},\\
&e_{\lambda_{x_1'}}(\boldsymbol{x}) = e^{2\pi i(x_2' - x_3')}, \quad e_{\lambda_{x_2'}}(\boldsymbol{x}) = e^{2\pi i(x_3' - x_1')}, \quad e_{\lambda_{x_3'}}(\boldsymbol{x}) = e^{2\pi i(x_1' - x_2')},
\end{aligned}
$$

where $\boldsymbol{x}$ denotes the tuple $\boldsymbol{x} = (x_1, x_2, x_3, x_1', x_2', x_3')$.

Proof. The proof is straight forward verification. The computations are analogue to the calculations in the proof of Lemma 8.1. □

Again, in order to check that the operator $\rho(T)$ takes the pull back of a section to a pull back of a section we show the following Lemma.

Lemma 9.2. *The sum of the partial derivatives of K_T disappears. I.e.*

$$
\frac{\partial K_T}{\partial x_1'} + \frac{\partial K_T}{\partial x_2'} + \frac{\partial K_T}{\partial x_3'} + \frac{\partial K_T}{\partial x_1} + \frac{\partial K_T}{\partial x_2} + \frac{\partial K_T}{\partial x_3} = 0
$$

Proof. In each term of the expression for K_T there is an equal number of variables one half having positive coefficient and the other half having negative coefficient. Therefore the sum of the partial differentials must equal zero. □

Appendices

A Faddeev's quantum dilogarithm

The quantum dilogarithm function $\mathrm{Li}_2(x;q)$, studied by Faddeev and Kashaev [FK] and other authors, is the function of two variables defined by the series

$$
\mathrm{Li}_2(x;q) = \sum_{n=1}^{\infty} \frac{x^n}{n(1-q^n)}, \tag{A.1}
$$

where $x, q \in \mathbb{C}$, with $|x|, |q| < 1$. It is connected to the classical Euler dilogarithm Li_2 given by $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ in the sense that it is a q-deformation of the classical one in the following manner

$$\lim_{\epsilon \to 0} \left(\epsilon \, \mathrm{Li}_2(x, e^{-\epsilon})\right) = \mathrm{Li}_2(x), \quad |x| < 1. \tag{A.2}$$

Indeed using the expansion $\frac{1}{1-e^{-t}} = \frac{1}{t} + \frac{1}{2} + \frac{t}{12} - \frac{t^3}{720} + \dots$ we obtain a complete asymptotic expansion

$$\mathrm{Li}_2(x, e^{-\epsilon}) = \mathrm{Li}_2(x)\epsilon^{-1} + \frac{1}{2}\log\left(\frac{1}{1-x}\right) + \frac{x}{1-x}\frac{\epsilon}{12} - \frac{x+x^2}{(1-x)^3}\frac{\epsilon^3}{720} + \dots \tag{A.3}$$

as $\epsilon \to 0$ with fixed $x \in \mathbb{C}$, $|x| < 1$.

The second quantum dilogarithm $(x; q)_\infty$ defined for $|q| < 1$ and all $x \in \mathbb{C}$ is given as the function

$$(x; q)_\infty = \prod_{i=0}^{\infty} (1 - xq^i). \tag{A.4}$$

This second quantum dilogarithm is related to the first by the formula

$$(x; q)_\infty = \exp(-\mathrm{Li}_2(x; q)). \tag{A.5}$$

This is easily proven by a direct calculation

$$-\log{(x; q)_\infty} = \sum_{i=0}^{\infty} \log(1 - xq^i) = \sum_{i=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n} x^n q^{in} = \sum_{n=1}^{\infty} \frac{x^n}{n(1-q^n)} = \mathrm{Li}_2(x; q). \tag{A.6}$$

Proposition A.1. *The function $(x; q)_\infty$ and its reciprocal have the Taylor expansions*

$$(x; q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n}{(q)_n} q^{\frac{n(n-1)}{2}} x^n, \qquad \frac{1}{(x; q)_\infty} \sum_{n=0}^{\infty} \frac{1}{(q)_n} x^n, \tag{A.7}$$

around $x = 0$, where

$$(q)_n = \frac{(q; q)_\infty}{(q^{n+1}; q)_\infty} = (1-q)(1-q^2) \cdot (1-q^n).$$

The proofs of these formulas follows easily from the recursion formula $(x; q)_\infty = (1-x)(qx; x)_\infty$, which together with the initial value $(0; q)_\infty = 1$ determines the power series for $(x; q)_\infty$ uniquely.

Yet another famous result for the function $(x; q)_\infty$, which can be proven by use of the Taylor expansion and the identity $\sum_{m-n=k} \frac{q^{mn}}{(q)_m (q)_n} = \frac{1}{(q)_\infty}$, is the Jacobi triple product formula

$$(q;q)_\infty(x;q)_\infty(qx^{-1};q)_\infty = \sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{k(k-1)}{2}} x^k, \tag{A.8}$$

which relates the quantum dilogarithm function to the classical Jacobi theta-function.

The quantum dilogarithm functions introduced are related to yet another quantum dilogarithm function named after Faddeev.

Faddeev's quantum dilogarithm

Definition A.2. Faddeev's quantum dilogarithm function is the function in two complex arguments z and b defined by the formula

$$\Phi_{\mathrm{b}}(z) = \exp\left(\int_C \frac{e^{-2izw}dw}{4\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}\right), \tag{A.9}$$

where the contour C runs along the real axis, deviating into the upper half plane in the vicinity of the origin.

Proposition A.3. *Faddeev's quantum dilogarithm function $\Phi_{\mathrm{b}}(z)$ is related to the function $(x;q)_\infty$, where $|q|<1$, in the following sense. When $\Im(\mathrm{b}^2)>0$, the integral can be calculated explicitly*

$$\Phi_{\mathrm{b}}(z) = \frac{\left(e^{2\pi(z+c_{\mathrm{b}})\mathrm{b}};q^2\right)_\infty}{\left(e^{2\pi(z-c_{\mathrm{b}})\mathrm{b}};\tilde{q}^2\right)_\infty} \tag{A.10}$$

where $q\equiv e^{i\pi\mathrm{b}^2}$ and $\tilde{q}\equiv e^{-\pi i\mathrm{b}^{-2}}$.

Proof. We consider the integrand of the integral $I(z,\mathrm{b}) = \frac{1}{4}\int_C \frac{e^{-2izw}}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}dw$. The integrand has poles at $w=\pi in\,\mathrm{b}$ and $w=\pi in\,\mathrm{b}^{-1}$. The residue at c of a fraction i.e. $f(x)=\frac{g(x)}{h(x)}$ can be calculated as $\operatorname{Res} f(c) = \frac{g(c)}{h'(c)}$ when c is a simple pole. Therefore we get by the residue theorem

$$\begin{aligned}
I(z,\mathrm{b}) &= \frac{\pi i}{2}\sum_{n=1}^\infty \frac{e^{2\pi z\,\mathrm{b}\,n}}{\pi in\,\mathrm{b}(-1)^n\sinh(\pi in\,\mathrm{b}^2)} + \frac{e^{2\pi z\,\mathrm{b}^{-1}n}}{\pi in\,\mathrm{b}(-1)^n\sinh(\pi in\,\mathrm{b}^{-2})}\\
&= \sum_{n=1}^\infty \frac{e^{\pi in}e^{2\pi z\,\mathrm{b}\,n}}{n(e^{\pi in\,\mathrm{b}^2}-e^{-\pi in\,\mathrm{b}^2})} + \frac{e^{\pi in}e^{2\pi z\,\mathrm{b}^{-1}n}}{n(e^{\pi in\,\mathrm{b}^{-2}}-e^{-\pi in\,\mathrm{b}^{-2}})}\\
&= \sum_{n=1}^\infty -\frac{\left(e^{2\pi z\,\mathrm{b}+\pi i+\pi i\,\mathrm{b}^2}\right)^n}{n(1-e^{2\pi i\,\mathrm{b}^2 n})} + \frac{\left(e^{2\pi z\,\mathrm{b}^{-1}-\pi i-\pi i\,\mathrm{b}^{-2}}\right)^n}{n(1-e^{-2\pi i\,\mathrm{b}^{-2}n})}\\
&= \sum_{n=1}^\infty -\frac{e^{2\pi(z+c_{\mathrm{b}})\,\mathrm{b}\,n}}{n(1-e^{2\pi i\,\mathrm{b}^2 n})} + \frac{e^{2\pi(z-c_{\mathrm{b}})\,\mathrm{b}^{-1}n}}{n(1-e^{-2\pi i\,\mathrm{b}^{-2}n})}\\
&= \log\left(e^{2\pi(z+c_{\mathrm{b}})\,\mathrm{b}};q^2\right)_\infty - \log\left(e^{2\pi(z-c_{\mathrm{b}})\,\mathrm{b}};\tilde{q}^2\right)_\infty.
\end{aligned}$$

□

Functional equations

Proposition A.4. *Faddeev's quantum dilogarithm function satisfies the two functional equations*

$$\frac{1}{\Phi_{\mathrm{b}}(z+i\mathrm{b}/2)}=\frac{1}{\Phi_{\mathrm{b}}(z-i\mathrm{b}/2)}\left(1+e^{2\pi\mathrm{b}z}\right), \tag{A.11}$$

$$\Phi_{\mathrm{b}}(z)\Phi_{\mathrm{b}}(-z)=\zeta_{inv}^{-1}e^{i\pi z^2}, \tag{A.12}$$

where $\zeta_{inv}=e^{i\pi(1+2c_{\mathrm{b}}^2)/6}$.

Proof. Let us first prove (A.11). We have

$$\begin{aligned}\frac{\Phi_{\mathrm{b}}(z-i\mathrm{b}/2)}{\Phi_{\mathrm{b}}(z+i\mathrm{b}/2)}&=\exp\int_C\frac{e^{-2i(z-i\mathrm{b}/2)w}-e^{-2i(z+i\mathrm{b}/2)w}}{4\sinh(w\mathrm{b})\sinh(w/\mathrm{b})w}\,dw\\&=\exp\int_C\frac{e^{-2izw}\left(e^{-\mathrm{b}w}-e^{\mathrm{b}w}\right)}{4\sinh(w\mathrm{b})\sinh(w/\mathrm{b})w}\,dw\\&=\exp\left(-\frac{1}{2}\int_C\frac{e^{-2izw}}{\sinh(w/\mathrm{b})w}\right)\,dw.\end{aligned}$$

Let $a>0$. Let $\varepsilon=1$ if $\Im(-2iz)\geq 0$ and $\varepsilon=-1$ otherwise. Put $\delta_a^-=[-a,i\varepsilon a]$ and $\delta_a^-=[i\varepsilon a,a]$. The integrals $\int_{\delta_{a^\pm}}\frac{e^{-2izw}}{2\sinh(w/\mathrm{b})w}\,dw$ converge to zero as $a\to\infty$. Therefore

$$\int_C\frac{e^{-2izw}}{\sinh(w/\mathrm{b})w}\,dw=\epsilon 2\pi i\left(c_\epsilon+\sum_{n=1}^{\infty}\mathrm{Res}_{w=\epsilon i\pi\mathrm{b}n}\left\{\frac{e^{-2izw}}{\sinh(w/\mathrm{b})w}\right\}\right),$$

where $c_1=0$ and $c_{-1}=\mathrm{Res}_{w=0}\left\{\frac{e^{2izw}}{\sinh(w/\mathrm{b})w}\right\}=-2iz\mathrm{b}$. For $n\in\mathbb{Z}\setminus\{0\}$ we have

$$\mathrm{Res}_{w=\pi in\mathrm{b}\epsilon}\left\{\frac{e^{-2izw}}{\sinh(w/\mathrm{b})w}\right\}=\frac{(-1)^n e^{2z\pi\mathrm{b}\epsilon n}}{\pi in}$$

so

$$\int_C\frac{e^{-2izw}}{\sinh(w/\mathrm{b})w}\,dw=(\epsilon-1)2\pi z\mathrm{b}-2\log(1+e^{2z\pi\mathrm{b}\epsilon}),$$

giving the first result.

To prove equation (A.12) let us choose the path $C=(-\infty,-\epsilon]\cup\epsilon\exp([\pi i,0])\cup[\epsilon,\infty)$ and let $\epsilon\to 0$. The rest is just calculations

$$\log\Phi_{\mathrm{b}}(z)\Phi_{\mathrm{b}}(-z)=\frac{1}{2}\int_C\frac{\cos(2wz)}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}dw$$

Note that

$$\frac{1}{2}\int_{(-\infty,-\epsilon]}\frac{\cos(2wz)}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}dw = -\frac{1}{2}\int_{[\epsilon,\infty)}\frac{\cos(2wz)}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}dw.$$

i.e. it is enough to collect the half residue around $w = 0$ of the remaining integral

$$\begin{aligned}\frac{1}{2}\int_{\epsilon([\pi i,0])}\frac{\cos(2wz)}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}dw &= \frac{\pi i}{2}\,\mathrm{Res}_{w=0}\frac{\cos(2wz)}{\sinh(w\,\mathrm{b})\sinh(w/\,\mathrm{b})w}\\ &= \frac{\pi i}{2}\left(\frac{b^2+b^{-2}}{6}+2z^2\right)\\ &= e^{-\pi i(1+2c_\mathrm{b}^2)/6}e^{\pi i z^2}.\end{aligned}$$

□

Zeros and poles

The functional equation (A.11) shows that $\Phi_\mathrm{b}(z)$, which in its initial domain of definition has no zeroes and poles, extends (for fixed b with $\Im\,\mathrm{b}^2 > 0$) to a meromorphic function in the variable z to the entire complex plane with essential singularity at infinity and with characteristic properties:

$$(\Phi_\mathrm{b}(z))^{\pm 1} = 0 \iff z = \mp(c_\mathrm{b} + mi\,\mathrm{b} + ni\,\mathrm{b}). \tag{A.13}$$

The behaviour at infinity depends on the direction along which the limit is taken

$$\Phi_\mathrm{b}(z)\big|_{|z|\to\infty} \approx \begin{cases} 1 & |\arg z| > \frac{\pi}{2} + \arg\mathrm{b}, \\ \zeta_{inv}^{-1}e^{\pi i z^2} & |\arg z| < \frac{\pi}{2} - \arg\mathrm{b} \\ \frac{(\tilde{q}^2;\tilde{q}^2)_\infty}{\Theta(i\,\mathrm{b}^{-1}z;-\mathrm{b}^{-2})} & |\arg z - \frac{\pi}{2}| < \arg\mathrm{b} \\ \frac{\Theta(i\,\mathrm{b}\,z;\mathrm{b}^2)}{(q^2;q^2)_\infty} & |\arg z + \frac{\pi}{2}| < \arg\mathrm{b} \end{cases} \tag{A.14}$$

where

$$\Theta(z;\tau) \equiv \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2 + 2\pi i z n}, \quad \Im\,\tau > 0. \tag{A.15}$$

Unitarity

When b is real or on the unit circle

$$(1-|\mathrm{b}|)\Im\,\mathrm{b} = 0 \quad\Rightarrow\quad \overline{\Phi_\mathrm{b}(z)} = \frac{1}{\Phi_\mathrm{b}(\overline{z})}. \tag{A.16}$$

Quantum Pentagon Identity

In terms of specifically normalised selfadjoint Heisenberg momentum and position operators acting as unbounded operators on $L^2(\mathbb{R})$ by the formulae

$$\mathbf{q}f(x) = xf(x), \quad \mathbf{p}f(x) = \frac{1}{2\pi i}f(x),$$

the following pentagon identity for unitary operators is satisfied [FK]

$$\Phi_{\mathrm{b}}(\mathbf{p})\Phi_{\mathrm{b}}(\mathbf{q}) = \Phi_{\mathrm{b}}(\mathbf{q})\Phi_{\mathrm{b}}(\mathbf{p}+\mathbf{q})\Phi_{\mathrm{b}}(\mathbf{p}). \tag{A.17}$$

Fourier transformation formulae for Faddeev's quantum dilogarithm

The quantum pentagon identity (A.17) is equivalent to the integral identity

$$\int_{\mathbb{R}+i\varepsilon} \frac{\Phi_{\mathrm{b}}(x+u)}{\Phi_{\mathrm{b}}(x-c_{\mathrm{b}})} e^{-2\pi i w x}\, dx = \frac{\Phi_{\mathrm{b}}(u)\Phi_{\mathrm{b}}(c_{\mathrm{b}}-w)}{\Phi_{\mathrm{b}}(u-w)} e^{\frac{\pi i}{12}(1-4c_{\mathrm{b}}^2)}, \tag{A.18}$$

where $\Im\,\mathrm{b}^2 > 0$. From here we get the Fourier transformation formula for the quantum dilogarithm formally sending $u \to -\infty$ by the use of (A.10) and (A.16)

$$\int_{\mathbb{R}+i\varepsilon} \Phi_{\mathrm{b}}(x+c_{\mathrm{b}}) e^{2\pi i w x} = \frac{1}{\Phi_{\mathrm{b}}(-w-c_{\mathrm{b}})} e^{-\frac{\pi i}{12}(1-4c_{\mathrm{b}}^2)}. \tag{A.19}$$

Quasi-classical limit of Faddeev's quantum dilogarithm

Proposition A.5. *For fixed x and $\mathrm{b} \to 0$ we have the following asymptotic expansion*

$$\log \Phi_{\mathrm{b}}\left(\frac{x}{2\pi\,\mathrm{b}}\right) = \sum_{n=0}^{\infty} (2\pi i\,\mathrm{b})^{2n-1} \frac{B_{2n}(1/2)}{(2n)!} \frac{\partial^{2n}\,\mathrm{Li}_2(-e^x)}{\partial x^{2n}}, \tag{A.20}$$

where $B_{2n}(1/2)$ are the Bernoulli polynomials B_{2n} evaluated at $1/2$.

Proof. From (A.11) we have that

$$\log\left(\frac{\Phi_{\mathrm{b}}\left(\frac{x-i\pi\,\mathrm{b}^2}{2\pi\,\mathrm{b}}\right)}{\Phi_{\mathrm{b}}\left(\frac{x+i\pi\,\mathrm{b}^2}{2\pi\,\mathrm{b}}\right)}\right) = \log(1+e^x).$$

The left hand side yields

$$\log \Phi_{\mathrm{b}}\left(\frac{x-i\pi\,\mathrm{b}^2}{2\pi\,\mathrm{b}}\right) - \log \Phi_{\mathrm{b}}\left(\frac{x+i\pi\,\mathrm{b}^2}{2\pi\,\mathrm{b}}\right) = -2\sinh(i\pi b^2 \partial/\partial x) \log \Phi_{\mathrm{b}}\left(\frac{x}{2\pi\,\mathrm{b}}\right),$$

where we have used the fact that

$$f(x+y) = e^{y\frac{\partial}{\partial x}}(f)(x),$$

which is just the Taylor expansion of f around x. While the right hand side can be written in the following manner

$$\log(1+e^x) = \frac{\partial}{\partial x}\int_{-\infty}^{x} \log(1+e^z)\,dz = -\frac{\partial}{\partial x}\,\mathrm{Li}_2(-e^x).$$

Using the expansion

$$\frac{z}{\sinh(z)} = \sum_{n=0}^{\infty} B_{2n}(1/2)\frac{(2z)^{2n}}{(2n)!}$$

gives exactly (A.20). □

Corollary A.6. *For fixed x and $\mathrm{b}\to 0$ one has*

$$\Phi_{\mathrm{b}}\left(\frac{x}{2\pi\,\mathrm{b}}\right) = \exp\left(\frac{1}{2\pi i b^2}\,\mathrm{Li}_2(-e^x)\right)\left(1+O(b^2)\right). \tag{A.21}$$

B The Tetrahedral Operator

In order to prove Proposition 3.1 we make use of the following formulae

Lemma B.1. *Suppose x and y are operators in an algebra such that*

$$z = [x,y],\, [x,z] = 0.$$

Then

$$\begin{aligned} f(x)y &= yf(x) + zf'(x) \\ e^x f(x) &= f(y+z)e^x, \end{aligned}$$

for every power series such that $f(x), f'(x)$ and $f(y+z)$ can be defined in the same operator algebra.

Proof. Let $f(x) = \sum_{j=0}^{\infty} a_j x^j$. Then,

$$[f(x),y] = \sum_{j=0}^{\infty} a_j[x^j,y] = \sum_{j=0}^{\infty} a_j \sum_{k=0}^{j-1} x^k[x,y]x^{j-k-1} = \sum_{j=0}^{\infty} a_j j z x^{j-1} = zf'(x).$$

which shows the first equation. The second equation follows from this when we set $f(x) = e^x y^{l-1}$

$$e^x y^l = ye^x y^{l-1} = (y-z)e^x y^{l-l} = \cdots = (y+z)^l e^x,$$

and from here we get that

$$e^x f(y) = e^x \sum_{j=0}^{\infty} a_j y^j = \sum_{j=0}^{\infty} a_j (y+z)^j e^x = f(y+z)e^x.$$

□

Proof of Proposition 3.1. The equations in (3.6) follows from the system of equations

$$\begin{aligned}
&\mathbf{T}\mathbf{q}_1 = (\mathbf{q}_1 + \mathbf{q}_2)\mathbf{T}, \\
&\mathbf{T}(\mathbf{p}_1 + \mathbf{p}_2) = (\mathbf{p}_1 + \mathbf{q}_2)\mathbf{T}, \\
&\mathbf{T}(\mathbf{p}_1 + \mathbf{q}_2) = (\mathbf{p}_1 + \mathbf{q}_2)\mathbf{T}, \\
&\mathbf{T}e^{2\pi\,\mathrm{b}\,\mathbf{p}_1} = (e^{2\pi\,\mathrm{b}\,\mathbf{p}_1} + e^{2\pi\,\mathrm{b}(\mathbf{q}_1+\mathbf{p}_2)})\mathbf{T},
\end{aligned}$$

where $\mathbf{T} = e^{2\pi i \mathbf{p}_1\mathbf{q}_2}\psi(\mathbf{q}_1 - \mathbf{q}_2 + \mathbf{p}_2)$. We prove them one by one below using Lemma B.1.

$$\begin{aligned}
\mathbf{T}\mathbf{q}_1 &= e^{2\pi i \mathbf{p}_1\mathbf{q}_2}\psi(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2)\mathbf{q}_1 \\
&= e^{2\pi i \mathbf{p}_1\mathbf{q}_2}\mathbf{q}_1\psi(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2) \\
&= (\mathbf{q}_1 e^{2\pi i \mathbf{p}_1\mathbf{q}_2} + \mathbf{q}_2 e^{2\pi i \mathbf{p}_1\mathbf{q}_2})\psi(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2) \\
&= (\mathbf{q}_1 + \mathbf{q}_2)\mathbf{T}.
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}(\mathbf{p}_1 + \mathbf{p}_2) &= e^{2\pi i \mathbf{p}_1\mathbf{q}_2}\psi(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2)(\mathbf{p}_1 + \mathbf{p}_2) \\
&= e^{2\pi i \mathbf{p}_1+\mathbf{q}_2}(\mathbf{p}_1 + \mathbf{p}_2)\psi(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2) \\
&= \left\{\mathbf{p}_1 e^{2\pi i \mathbf{p}_1\mathbf{q}_2} + \mathbf{p}_2 e^{2\pi i \mathbf{p}_1\mathbf{q}_2} - \mathbf{p}_1 e^{2\pi i \mathbf{p}_1\mathbf{q}_2}\right\}\psi(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2) \\
&= \mathbf{p}_2\mathbf{T},
\end{aligned}$$

where the second equality is true since $[\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2, \mathbf{p}_1 + \mathbf{p}_2] = 0$.

$$\begin{aligned}
\mathbf{T}(\mathbf{p}_1 + \mathbf{q}_2) &= e^{2\pi i \mathbf{p}_1\mathbf{q}_2}\psi(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2)(\mathbf{p}_1 + \mathbf{q}_2) \\
&= e^{2\pi i \mathbf{p}_1\mathbf{q}_2}(\mathbf{p}_1 + \mathbf{q}_2)\psi(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2) \\
&= (\mathbf{p}_1 + \mathbf{q}_2)\mathbf{T},
\end{aligned}$$

where second equality is true since $[q_1 + p_2 - q_2, p_1 + q_2] = 0$.

$$\begin{aligned}
\mathbf{T}e^{2\pi\,\mathrm{b}\,\mathbf{p}_1} &= e^{2\pi i \mathbf{p}_1\mathbf{q}_2}\psi(\mathbf{q}_1 - \mathbf{q}_2 + \mathbf{p}_2)e^{2\pi\,\mathrm{b}\,\mathbf{p}_1} \\
&= \psi(\mathbf{q}_1 - \mathbf{p}_1 + \mathbf{p}_2)e^{2\pi i \mathbf{p}_1\mathbf{q}_2}e^{2\pi\,\mathrm{b}\,\mathbf{p}_1} \\
&= \psi(\mathbf{q}_1 - \mathbf{p}_1 + \mathbf{p}_2)e^{2\pi\,\mathrm{b}\,\mathbf{p}_1}e^{2\pi i \mathbf{p}_1\mathbf{q}_2} \\
&= e^{2\pi\,\mathrm{b}\,\mathbf{p}_1}\psi(\mathbf{q}_1 - \mathbf{p}_1 + \mathbf{p}_2 + i\,\mathrm{b})e^{2\pi i \mathbf{p}_1\mathbf{q}_2} \\
&= e^{2\pi\,\mathrm{b}\,\mathbf{p}_1}\left(1 + e^{2\pi\,\mathrm{b}(\mathbf{q}_1 - \mathbf{p}_1 + \mathbf{p}_2 + \frac{i\,\mathrm{b}}{2})}\right)\psi(\mathbf{q}_1 - \mathbf{p}_1 + \mathbf{p}_2)e^{2\pi i \mathbf{p}_1\mathbf{q}_2} \\
&= \left(e^{2\pi\,\mathrm{b}\,\mathbf{p}_1} + e^{2\pi i\,\mathrm{b}(\mathbf{q}_1+\mathbf{p}_2)}\right)\mathbf{T},
\end{aligned}$$

where in the last equality we use the *Baker–Campbell–Hausdorff formula.* □

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Jørgen Ellegaard Andersen and Jens-Jakob Kratmann Nissen
Center for Quantum Geometry of Moduli Spaces
Department of Mathematics
University of Aarhus
DK-8000, Denmark

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Level N Teichmüller TQFT and Complex Chern–Simons Theory

by Jørgen Ellegaard Andersen and Simone Marzioni[1]

Abstract

In this manuscript we review the construction of the Teichmüller TQFT due to Andersen and Kashaev. We further upgrading it to a theory dependent on an extra odd integer N using results developed by Andersen and Kashaev in their work on complex quantum Chern-Simons theory. We also describe how this theory is related with quantum Chern–Simons Theory at level N with gauge group $\mathrm{PSL}(2, \mathbb{C})$.

1 Introduction

In this paper we review Andersen and Kashaev's construction of the Teichmüller TQFT from [AK1] making it dependent on an extra odd integer N, called the level. The original work of [AK1] corresponds to the choice $N = 1$, and emerged as an extension to a 3–dimensional theory of the representations one obtains from Quantum Teichmüller Theory [K3]. In particular, it defines a class of quantum invariants for hyperbolic knots, dependent on a continuous parameter b. The level N Teichmüller TQFT is an analogously upgrade of representations in Quantum Teichmüller theory and it depends on a pair of parameters (b, N), one continuous and one discrete. Such quantum theory is related to the level N Chern–Simons theory with gauge group $\mathrm{PSL}(2, \mathbb{C})$ via the level N Weil-Gel'fand-Zak transform. Such a relation was proposed in [AK3], and here we show it in a more tight way for the four punctured sphere. One of the main ingredient in the construct of the Teichmüller TQFT is the *quantum dilogarithm*, that is a function $\mathrm{D}_\mathrm{b} \colon \mathbb{R} \times \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, satisfying some particular properties. Such functions were introduced in [AK3], which for $N = 1$ is Faddeev's original quantum dilogarithm. The theory we get has different and interesting unitarity behaviour depending on the pair of parameters (b, N): for level $N = 1$ the theory is unitary whenever $\mathrm{b} > 0$ or $|\mathrm{b}| = 1$ while for higher level $N > 1$ the unitarity is only manifest when $|\mathrm{b}| = 1$ while in the case $\mathrm{b} > 1$ the behaviour is more exotic. We will consider

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both situations here and we will use the setting $b > 0$ to present some asymptotic properties in the limit $b \to 0$. The Teichmüller TQFT can be used to define knot invariants starting from triangulations of their complements. In this presentation we update the examples presented in [AK1] to the level N setting together with their asymptotic analysis. For the simplest hyperbolic knot we show the appearance of the Baseilhac–Benedetti invariant from [BB] in such a limit.

It is an interesting challenge how the TQFT's which are reviewed in this paper are related to the Witten-Reshetikhin-Turaev TQFT's [W1, RT1, RT2, BHMV1, BHMV2, B] and in particular how they are related to the geometric construction of these TQFT's [ADW, Hit2, Las, TUY, AU1, AU2, AU3, AU4] and to Witten's proposal for the construction of the complex quantum Chern-Simons theory [W2], which can actually be constructed from a purely mathematical point of view [AG], resulting also in the mathematically well-defined Hitchin-Witten connection in the bundle of quantizations of the moduli space of flat $\mathrm{SL}(n, \mathbb{C})$-connections over Teichmüller space. In the classical case of compact groups, the description of the representations of the mapping class groups via the monodromy of the Hitchin connections turned out to be very useful to prove deep properties about these representations [A1, A2, A3, A4, AH, AHJMMc], some of which also uses the theory of Toeplitz operators [BMS, KS]. Understanding how these kinds of results can be extended to the complex quantum theory discussed in this paper will be very interesting and most likely involve using Higgs bundles techniques [Hit1]. Certainly we have already seen the start of this with the Verlinde formula for Higgs bundle moduli space [AGP].

The paper is organised as follows. In section 2 we recall the definition of the (decorated) Ptolemy groupoid of punctured surfaces, which is the combinatorial foundation over which Quantum Teichmüller Theory is defined. In section 3 we recall the quantum dilogarithm D_b, and we list some of its properties. The function D_b was introduced in [AK3] for the first time, but some of its properties that we list here are not present in the literature. In section 5 we carry out the quantization of the moduli space of $\mathrm{PSL}(2, \mathbb{C})$ flat connections over a four punctured sphere with unipotent holonomies around the punctures. We follow the prescriptions of geometric quantisation, together with a choice of real polarisation, and we connect the resulting algebra of observables to the $\mathrm{L}^2(\mathbb{R} \times \mathbb{Z}/N\mathbb{Z})$ representations of quantum Teichmüller theory of the previous section. Finally in section 6 we construct the the Teichmüller TQFT functor $F_\mathrm{b}^{(N)}$ mirroring the construction in [AK1] and we study some examples and their asymptotic behaviour. It would of course be interesting to go through all the examples treated in [AN] in this volume for the level N theory as well.

Acknowledgements

We would like to thank Rinat Kashaev for many interesting discussions.

2 Ptolemy Groupoid

Let $\Sigma_{g,s}$ be a surface of genus g with s punctures, such that $s > 0$ and $2-2g+s < 0$.

Definition 2.1. An *ideal arc* α is the homotopy class relative endpoints of the embedding of a path in $\Sigma_{g,s}$, such that the endpoints are punctures of the surface. An *ideal triangle* is a triangle with the vertices removed, such that the edges are ideal arcs.
An *ideal triangulation* τ of $\Sigma_{g,s}$ is a collection of disjoint ideal arcs such that $\Sigma_{g,s} \setminus \tau$ is a collection of interiors of ideal triangles.

Given an ideal triangulation τ, $\Delta_j(\tau)$ will denote the set of its j-dimensional cells.

Definition 2.2. A *decorated ideal triangulation* of $\Sigma_{g,s}$ is an ideal triangulation τ up to isotopy relative to the punctures, together with the choice of a distinguished corner in each ideal triangle and a bijective ordering map

$$\overline{\tau} : \{1, \ \ldots, \ s\} \ni j \mapsto \overline{\tau}_j \in \Delta_2(\tau).$$

We denote the set of all decorated ideal triangulation as $\dot{\Delta} = \dot{\Delta}(\Sigma_{g,s})$.

When we say that τ is a decorated ideal triangulation we mean that τ is the set of decorated ideal triangles in the triangulation.

One of the main interests in quantizing moduli spaces is the consequent construction of representations of (central extensions of) the mapping class group of the surfaces [W2, Hit1, Las, A1, AU4, AG]. Quantum Teichmüller theory produce instead representations of a bigger object called the (decorated) *Ptolemy Groupoid* that we are going to introduce now.

Recall that, given a group G acting freely on a set X, we can define an associated groupoid $\mathcal{G}$ as follows. The objects of $\mathcal{G}$ are G-orbits in X while morphisms are G-orbits in $X \times X$ with respect to the diagonal action. Then for any $x \in X$ we can consider the object $[x]$ and for any pair $(x, y) \in X \times X$ we can consider the morphism $[x, y]$. When $[y] = [u]$ there will be a $g \in G$ so that $gu = y$ and we can define the composition $[x, y][u, v] = [x, gv]$. The unit for $[x]$ is given by $[x, x]$. If the action of G is transitive, we would get an actual group. We will abbreviate $[x_1, x_2][x_2, x_3] \cdots [x_{n-1}, x_n]$ with $[x_1, x_2, \ldots, x_n]$.
We define the *decorated Ptolemy groupoid* $\mathcal{G}(\Sigma_{g,s})$ of a punctured surface $\Sigma_{g,s}$ following the above recipe. The set we consider is the set $\dot{\Delta}$ of decorated triangulations τ of $\Sigma_{g,s}$. The free group action is the one of the mapping class group $\mathrm{MCG}_{g,s}$ acting on $\dot{\Delta}$. This action is not transitive, meaning that not all pairs of decorated ideal triangulations can be related by a mapping class group element. However in the language of groupoids, we can still describe generators and relations for the morphism groups. For $\tau \in \dot{\Delta}$ there are three kind of generators $[\tau, \tau^\sigma]$, $[\tau, \rho_i\tau]$ and $[\tau, \omega_{i,j}\tau]$, where τ^σ is obtained by applying the permutation $\sigma \in \mathbb{S}_{|\tau|}$

to the ordering of triangles in τ, $\rho_i\tau$ is obtained by changing the distinguished corner in the triangle $\bar{\tau}_i \in \tau$ as in Figure 1, and $\omega_{i,j}$ is obtained by applying a decorated diagonal exchange to the quadrilateral made of the two decorated ideal triangles $\bar{\tau}_i$ and $\bar{\tau}_j$ as in Figure 2.

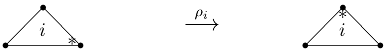


Figure 1: Transformation ρ_i.

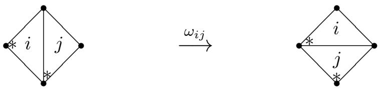


Figure 2: Transformation ω_{ij}.

The relations are usually grouped in two sets, the first being

$$[\tau, \tau^\alpha, (\tau^\alpha)^\beta] = [\tau, \tau^{\alpha\beta}], \quad \alpha, \beta \in \mathbb{S}_\tau, \tag{2.1}$$

$$[\tau, \rho_i\tau, \rho_i\rho_i\tau, \rho_i\rho_i\rho_i\tau] = \mathrm{id}_{[\tau]}, \tag{2.2}$$

$$[\tau, \omega_{i,j}\tau, \omega_{i,k}\omega_{i,j}\tau, \omega_{j,k}\omega_{i,k}\omega_{i,j}\tau] = [\tau, \omega_{j,k}\tau, \omega_{i,j}\omega_{j,k}\tau] \tag{2.3}$$

$$[\tau, \omega_{i,j}\tau, \rho_i\omega_{i,j}\tau, \omega_{j,i}\rho_i\omega_{i,j}\tau] = [\tau, \tau^{(i,j)}, \rho_j\tau^{(i,j)}, \rho_i\rho_j\tau^{(i,j)}] \tag{2.4}$$

The first two relations are obvious, the third is called the Pentagon Relation and the fourth is called the Inversion Relation.

The second set of relations, are commutation relations

$$[\tau, \rho_i\tau, \rho_i\tau^\sigma] = [\tau, \tau^\sigma, \rho_{\sigma^{-1}(i)}\tau^\sigma], \tag{2.5}$$

$$[\tau, \omega_{i,j}\tau, (\omega_{i,j}\tau)^\sigma] = [\tau, \tau^\sigma, \omega_{\sigma^{-1}(i)\sigma^{-1}(j)}\tau^\sigma], \tag{2.6}$$

$$[\tau, \rho_j\tau, \rho_j\rho_i\tau] = [\tau, \rho_i\tau, \rho_i\rho_j\tau], \tag{2.7}$$

$$[\tau, \rho_i\tau, \omega_{j,k}\rho_i\tau] = [\tau, \omega_{j,k}\tau, \rho_i\omega_{j,k}\tau], \ i \notin \{j, k\}, \tag{2.8}$$

$$[\tau, \omega_{i,j}\tau, \omega_{k,l}\omega_{i,j}\tau] = [\tau, \omega_{k,l}\tau, \omega_{i,j}\omega_{k,l}\tau], \ \{i, j\} \cap \{k, l\} = \emptyset, \tag{2.9}$$

To every decorated ideal triangulation $\tau \in \dot{\Delta}$ it is possible to associate a simple symplectic space $\mathcal{R}(\tau)$, called the space of *Ratio Coordinates*. We summarize here its relation with the Ptolemy groupoid and refer to [K3] for a detailed introduction to ratio coordinates and their relation to the Teichmüller space. Let $M \equiv 2g - 2 + s = |\tau|$ be the number of ideal triangles, then $\mathcal{R}(\tau) \equiv (\mathbb{R}_{>0} \times \mathbb{R}_{>0})^M$. Let $x^j \equiv (x_1^j, x_2^j) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, for $j = 1, \ldots, M$ be the coordinates associated to the ideal triangle $\bar{\tau}_j \in \Delta_2(\tau)$. The symplectic form that we consider on $\mathcal{R}(\tau)$ is

$$\omega_\tau \equiv \sum_{j=1}^{M} \frac{\mathrm{d}x_1^j}{x_1^j} \wedge \frac{\mathrm{d}x_2^j}{x_2^j} \tag{2.10}$$

Now we want to describe the action of $\mathcal{G}(\Sigma_{g,s})$ as symplectomorphisms between these spaces. The morphisms $[\tau, \tau^\sigma]$ act by permuting the coordinates in $\mathcal{R}(\tau)$. The morphism $[\tau, \rho_i\tau]$ acts as the identity on any pair $x = (x_1, x_2)$ corresponding to ideal triangles different from $\bar{\tau}_i$ and as $(x_1, x_2) = x \mapsto y = (\frac{x_2}{x_1}, \frac{1}{x_1})$ for the pair of coordinates corresponding to $\bar{\tau}_i$. Finally the action of $[\tau, \omega_{i,j}\tau]$ is the identity on $\bar{\tau}_k$ for $k \neq i,\ j$ while letting $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be the coordinates corresponding to the triangles $\bar{\tau}_i$ and $\bar{\tau}_j$ respectively, and letting $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be the coordinates of the triangles $\overline{\omega_{i,j}\tau}_i$ and $\overline{\omega_{i,j}\tau}_j$, then we have $u = x \bullet y$ and $v = x * y$ where

$$\begin{aligned} x \bullet y &:= (x_1y_1, x_1y_2 + x_2) \\ x * y &:= \left(\frac{y_1x_2}{x_1y_2 + x_2}, \frac{y_2}{x_1y_2 + x_2} \right). \end{aligned} \tag{2.11}$$

Let $\widetilde{\Delta}$ be the set of pairs $(\tau, \mathcal{R}(\tau))$, $\tau \in \dot{\Delta}$. Then the space $\mathcal{R}(\Sigma_{g,s})$ is defined as the quotient of $\widetilde{\Delta}$ by the action of $\mathcal{G}(\Sigma_{g,s})$ as described above. This space of coordinates is now independent of the triangulation. For more details on the (decorated or not) Ptolemy groupoid see [P],[FK],[K6][K4].

3 Quantum Dilogarithm

In this section we recall the quantum dilogarithm $\mathrm{D_b}$ over $\mathbb{A}_N$ and we state some of their properties.

Definition 3.1 (q-Pochammer Symbol)**.** Let $x, q \in \mathbb{C}$, such that $|q| < 1$. Define the *q-Pochammer Symbol* of x as

$$(x;q)_\infty := \prod_{i=0}^{\infty} (1 - xq^i)$$

Theorem 3.2. *Let X, Y satisfying $XY = qYX$. Then the following five-term relation holds true*

$$(Y;q)_\infty\,(X;q)_\infty = (X;q)_\infty\,(-YX;q)_\infty\,(Y;q)_\infty\,. \tag{3.1}$$

Definition 3.3 (Faddeev's Quantum Dilogarithm [F]). Let $z, \mathrm{b} \in \mathbb{C}$ be such that $\mathrm{Re}\,\mathrm{b} \neq 0$, $|\,\mathrm{Im}(z)| < |\,\mathrm{Im}(c_\mathrm{b})|$, where $c_\mathrm{b} := i(\mathrm{b}+\mathrm{b}^{-1})/2$. Let $C \subset \mathbb{C}$, $C = \mathbb{R}+i0$ be a contour equal to the the real line outside a neighborhood of the origin that avoid the singularity in 0 going in the upper half plane. *Faddeev's quantum dilogarithm* is defined to be

$$\Phi_\mathrm{b}(z) = \exp\left(\int_C \frac{e^{-2izw}\mathrm{d}w}{4\sinh(w\mathrm{b})\sinh(w\mathrm{b}^{-1})w}\right). \tag{3.2}$$

It is evident that Φ_b is invariant under the following changes of parameter

$$\mathrm{b} \leftrightarrow \mathrm{b}^{-1} \leftrightarrow -\mathrm{b}, \tag{3.3}$$

so that our choice of b can be restricted to the first quadrant

$$\mathrm{Re}\,\mathrm{b} > 0, \qquad \mathrm{Im}\,\mathrm{b} \geq 0 \tag{3.4}$$

which implies

$$\mathrm{Im}(\mathrm{b}^2) \geq 0. \tag{3.5}$$

Faddeev's quantum dilogarithm has a lot of other interesting properties and applications, see for example [F],[FK],[FKV] and [V].

Let $N \geq 1$ be a positive *odd* integer. Then, following [AK3], we can define a *quantum dilogarithm* over $\mathbb{A}_N$ as follows

$$\mathrm{D}_\mathrm{b}(x,n) := \prod_{j=0}^{N-1} \Phi_\mathrm{b}\left(\frac{x}{\sqrt{N}} + (1-N^{-1})c_\mathrm{b} - i\mathrm{b}^{-1}\frac{j}{N} - i\mathrm{b}\left\{\frac{j+n}{N}\right\}\right) \tag{3.6}$$

where $\{p\}$ is the fractional part of p, and Φ_b is the Faddeev's quantum dilogarithm. Of course for $N = 1$ we have just $\Phi_\mathrm{b}(x)$. The function D_b was introduced in [AK3] only for $|\mathrm{b}| = 1$. It satisfies a series properties that we are going to list.

Lemma 3.4 (Inversion Relation [AK3]).

$$\mathrm{D}_\mathrm{b}(x,n)\mathrm{D}_\mathrm{b}(-x,-n) = e^{\pi i x^2} e^{-\pi i n(n+N)/N} \zeta_{N,\,inv}^{-1},$$

where

$$\zeta_{N,\,inv} = e^{\pi i (N + 2c_\mathrm{b}^2 N^{-1})/6}. \tag{3.7}$$

Unitarity properties are different in the two situations $|\mathrm{b}| = 1$ or $\mathrm{b} \in \mathbb{R}$.

Lemma 3.5 (Unitarity).

$$\overline{\mathrm{D}_\mathrm{b}(x,n)} = \mathrm{D}_\mathrm{b}(\bar{x},n)^{-1} \qquad \textit{if } |\mathrm{b}| = 1, \tag{3.8}$$

$$\overline{\mathrm{D}_\mathrm{b}(x,n)} = \mathrm{D}_\mathrm{b}(\bar{x},-n)^{-1} \qquad \textit{if } \mathrm{b} \in \mathbb{R}_{>0}. \tag{3.9}$$

Remark 3.6. One can see that

$$\mathrm{D}_{\mathrm{b}}(x,-n)=\mathrm{D}_{\mathrm{b}^{-1}}(x,n) \tag{3.10}$$

just by the Definition 3.6 for $\mathrm{D}_{\mathrm{b}^{-1}}$ and carefully substituting $j+n\mapsto j'$. In particular the unitarity for $\mathrm{b}>0$ can be re-expressed as

$$\overline{\mathrm{D}_{\mathrm{b}}(x,n)}=\left(\mathrm{D}_{\mathrm{b}^{-1}}(x,n)\right)^{-1} \tag{3.11}$$

Lemma 3.7 (Faddeev's difference equations)**.** *Let*

$$\chi^{\pm}(x,n)\equiv e^{2\pi\frac{\mathrm{b}^{\pm1}}{\sqrt{N}}x}e^{\pm\frac{2\pi i n}{N}}, \tag{3.12}$$

for every x, $\mathrm{b}\in\mathbb{C}$, $\mathrm{Im}(\mathrm{b})\neq 0$ n, $N\in\mathbb{Z}$ we have

$$\mathrm{D}_{\mathrm{b}}\left(x+i\frac{\mathrm{b}^{\pm1}}{\sqrt{N}},n\pm1\right)=\mathrm{D}_{\mathrm{b}}\left(x,n\right)\left(1+\chi^{\pm}(x,n)e^{-\pi i\frac{N-1}{N}}e^{\pi i\frac{\mathrm{b}^{\pm2}}{N}}\right)^{-1} \tag{3.13}$$

$$\mathrm{D}_{\mathrm{b}}\left(x-i\frac{\mathrm{b}^{\pm1}}{\sqrt{N}},n\mp1\right)=\mathrm{D}_{\mathrm{b}}(x,n)\left(1+\chi^{\pm}(x,n)e^{\pi i\frac{N-1}{N}}e^{-\pi i\frac{\mathrm{b}^{\pm2}}{N}}\right) \tag{3.14}$$

Proposition 3.8. *If* $\mathrm{Im}(\mathrm{b})>0$ *and* $\mathrm{Re}(\mathrm{b})>0$ *we have*

$$\mathrm{D}_{\mathrm{b}}(x,n)=\frac{\left(\chi^{+}(x+\frac{c_{\mathrm{b}}}{\sqrt{N}},n);q^2\omega\right)_\infty}{\left(\chi^{-}(x-\frac{c_{\mathrm{b}}}{\sqrt{N}},n);\tilde{q}^2\overline{\omega}\right)_\infty} \tag{3.15}$$

where $q=e^{i\pi\frac{\mathrm{b}^2}{N}}$, $\tilde{q}=e^{-\pi i\frac{\mathrm{b}^{-2}}{N}}$, $\omega=e^{\frac{2\pi i}{N}}$ *and* $\chi^{\pm}(x,n)=e^{2\pi\frac{\mathrm{b}^{\pm1}}{\sqrt{N}}x}e^{\pm\frac{2\pi i n}{N}}$.

Proposition 3.9. *The quantum dilogarithm* $\mathrm{D}_{\mathrm{b}}(x,n)$*, for* $\mathrm{Im}(\mathrm{b})>0$ *has poles*

$$\begin{cases} x=\frac{c_{\mathrm{b}}}{\sqrt{N}}+i\frac{\mathrm{b}^{-1}}{\sqrt{N}}l+i\frac{\mathrm{b}}{\sqrt{N}}m\\ n=m-l \mod N\end{cases}$$

and zeros

$$\begin{cases} x=-\frac{c_{\mathrm{b}}}{\sqrt{N}}-i\frac{\mathrm{b}^{-1}}{\sqrt{N}}l-i\frac{\mathrm{b}}{\sqrt{N}}m\\ n=l-m \mod N\end{cases}$$

for $l,m\in\mathbb{Z}_{>0}$*. Moreover its residue at* $(x_{l,m},n_{l,m})=\left(\frac{c_{\mathrm{b}}}{\sqrt{N}}+i\frac{\mathrm{b}^{-1}}{\sqrt{N}}l+i\frac{\mathrm{b}}{\sqrt{N}}m,m-l\right)$ *is*

$$\frac{\sqrt{N}}{2\pi\mathrm{b}^{-1}}\frac{\left(q^2\omega;q^2\omega\right)_\infty}{\left(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega}\right)_\infty}\frac{(-\tilde{q}^2\overline{\omega})^l(\tilde{q}^2\overline{\omega})^{l(l-1)/2}}{\left(q^2\omega;q^2\omega\right)_m\left(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega}\right)_l} \tag{3.16}$$

The following Summation Formula can be shown by a residue computation. It is well known for $N=1$, i.e. for Φ_{b}, see [FKV] for example.

Theorem 3.10 (Summation Formula)**.** *Suppose* $\mathrm{Im}(\mathrm{b}) > 0$ *and* N *odd, and let* u, v, $w \in \mathbb{C}$ *and* $a, b, c \in \mathbb{Z}/N\mathbb{Z}$ *satisfy*

$$\text{(3.17)}\quad \mathrm{Im}\left(v + \frac{c_\mathrm{b}}{\sqrt{N}}\right) > 0, \qquad \mathrm{Im}\left(-u + \frac{c_\mathrm{b}}{\sqrt{N}}\right) > 0, \qquad \mathrm{Im}(v-u) < \mathrm{Im}(w) < 0.$$

Define

$$\text{(3.18)}\qquad \Psi(u,v,w,a,b,c) \equiv \int_{\mathbb{A}_N} \frac{\mathrm{D}_\mathrm{b}(x+u,a+d)}{\mathrm{D}_\mathrm{b}(x+v,b+d)} e^{2\pi i w x} e^{-2\pi i \frac{cd}{N}} \mathrm{d}(x,d)$$

Then we have that

$$\begin{aligned}
&\Psi(u,v,w,a,b,c)\\
&\qquad =\zeta_0 \frac{\mathrm{D}_\mathrm{b}\left(v-u-w+\frac{c_\mathrm{b}}{\sqrt{N}}, b-a-c\right)}{\mathrm{D}_\mathrm{b}\left(-w-\frac{c_\mathrm{b}}{\sqrt{N}}, -c\right)\mathrm{D}_\mathrm{b}\left(v-u+\frac{c_\mathrm{b}}{\sqrt{N}}, b-a\right)} e^{2\pi i w\left(\frac{c_\mathrm{b}}{\sqrt{N}}-u\right)} \omega^{ac}\\
&\qquad =\zeta_0^{-1} \frac{\mathrm{D}_\mathrm{b}\left(w+\frac{c_\mathrm{b}}{\sqrt{N}}, c\right)\mathrm{D}_\mathrm{b}\left(-v+u-\frac{c_\mathrm{b}}{\sqrt{N}}, -b+a\right)}{\mathrm{D}_\mathrm{b}\left(-v+u+w-\frac{c_\mathrm{b}}{\sqrt{N}}, -b+a+c\right)} e^{2\pi i w\left(-\frac{c_\mathrm{b}}{\sqrt{N}}-v\right)} \omega^{bc}
\end{aligned}$$

where $\zeta_0 = e^{-\pi i(N-4c_\mathrm{b}^2N^{-1})/12}$.

Remark 3.11. Assumptions (3.17) even though sufficient are not optimal. Indeed they guarantee the theorem to hold true when the integration is performed along the real line, however we can deform the integration contour as long as

$$\text{(3.19)}\quad |\arg(iz)| < \pi - \arg \mathrm{b} \qquad z \text{ being one of } \left\{w, v-u-w, u-v-2\frac{c_\mathrm{b}}{\sqrt{N}}\right\}$$

Using the notation for the Fourier Kernels from (A.13) in Appendix A.2 we have that

Proposition 3.12 (Fourier Transformation Formula, [AK3])**.** *For* N *odd we have that*

$$\begin{aligned}
\int_{\mathbb{A}_N} \mathrm{D}_\mathrm{b}(x,n)\langle (x,n);(w,c)\rangle \mathrm{d}(x,n) &= \frac{e^{2\pi i w \frac{c_\mathrm{b}}{\sqrt{N}}}}{\mathrm{D}_\mathrm{b}\left(-w-\frac{c_\mathrm{b}}{\sqrt{N}}, -k\right)} e^{-\pi i(N-4c_\mathrm{b}^2N^{-1})/12}\\
&= \mathrm{D}_\mathrm{b}\left(w+\frac{c_\mathrm{b}}{\sqrt{N}}, c\right)\overline{\langle (w,c)\rangle} e^{\pi i(N-4c_\mathrm{b}^2N^{-1})/12}\\
\int_{\mathbb{A}_N} (\mathrm{D}_\mathrm{b}(x,n))^{-1}\langle (x,n);(w,c)\rangle \mathrm{d}(x,n) &= \frac{\langle (w,c)\rangle}{\mathrm{D}_\mathrm{b}\left(-w-\frac{c_\mathrm{b}}{\sqrt{N}}, -k\right)} e^{-\pi i(N-4c_\mathrm{b}^2N^{-1})/12}\\
&= \mathrm{D}_\mathrm{b}\left(w+\frac{c_\mathrm{b}}{\sqrt{N}}, c\right) e^{-2\pi i w \frac{c_\mathrm{b}}{\sqrt{N}}} e^{\pi i(N-4c_\mathrm{b}^2N^{-1})/12}
\end{aligned}$$

Proposition 3.13 (Integral Pentagon Relation)**.** *Let* $\widetilde{\mathrm{D}_\mathrm{b}}(x,n) \equiv \mathsf{F}_N \circ \mathcal{F}^{-1}(\mathrm{D}_\mathrm{b})(x,n)$. *We have the following integral relation*

$$\begin{aligned}&\langle (x,n);(y,m)\rangle \widetilde{\mathrm{D}_\mathrm{b}}(x,n)\widetilde{\mathrm{D}_\mathrm{b}}(y,m)\\ &\quad = \int_{\mathbb{A}_N} \widetilde{\mathrm{D}_\mathrm{b}}(x-z,n-k)\widetilde{\mathrm{D}_\mathrm{b}}(z,k)\widetilde{\mathrm{D}_\mathrm{b}}(y-z,m-k)\langle (z,k)\rangle \mathrm{d}(z,k).\end{aligned}$$

Before we look at the asymptotic behavior of Φ_b let us recall the classical dilogarithm function, defined on $|z|<1$ by

$$\mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2} \tag{3.20}$$

and recall that it admits analytic continuation to $\mathbb{C}\setminus[1,\infty]$ through the following integral formula

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u}\mathrm{d}u. \tag{3.21}$$

Proposition 3.14. *We have the following behaviour when* $\mathrm{b}>0$, $\mathrm{b}\to 0$ *and* x, n, N *are fixed*

$$\mathrm{D}_\mathrm{b}(\frac{x}{2\pi\mathrm{b}},n) = \mathrm{Exp}\left(\frac{\mathrm{Li}_2(-e^{x\sqrt{N}})}{2\pi i b^2 N}\right)\phi_x(n)(1+\mathcal{O}(b^2)) \tag{3.22}$$

where $\phi_x(n)$ *is defined by* $\begin{cases}\phi_x(n+1) = \phi_x(n)\frac{(1-e^{x/\sqrt{N}}\overline{\omega}^{n+\frac12})}{(1+e^{x\sqrt{N}})^{1/N}}\\ \phi_x(0) = (1+e^{x\sqrt{N}})^{-\frac{N-1}{2N}}\prod_{j=0}^{N-1}(1-e^{xN^{-\frac12}}\overline{\omega}^{j+\frac12})^{\frac{j}{N}}\end{cases}$
whenever N *is odd.*

Remark 3.15. The function ϕ_x on the finite set $\mathbb{Z}/N\mathbb{Z}$ is a cyclic quantum dilogarithm [FK],[K3], [K1]. Precisely $\frac{1}{\phi_x}$ corresponds to the function Ψ_λ from Proposition 10 in [K3] with $\lambda = e^{x/\sqrt{N}}$.

The Hilbert space $\mathrm{L}^2(\mathbb{A}_N)$ is naturally isomorphic to the tensor product $\mathrm{L}^2(\mathbb{R})\otimes \mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \cong \mathrm{L}^2(\mathbb{R})\otimes\mathbb{C}^N$, see Appendix A.2. Let p and q two self-adjoint operators on $\mathrm{L}^2(\mathbb{R})$ satisfying

$$[\mathsf{p},\mathsf{q}] = \frac{1}{2\pi i} \tag{3.23}$$

and let X and Y unitary operators satisfying

$$YX = e^{2\pi i/N}XY, \quad X^N = Y^N = 1, \tag{3.24}$$

together with the cross relations

$$[\mathsf{p}, X] = [\mathsf{p}, Y] = [\mathsf{q}, X] = [\mathsf{q}, Y] = 0. \tag{3.25}$$

The equations in (3.24) imply that X and Y will have finite and the same spectrum, and this will be a subset of the set $\mathbb{T}_N$ of all N-th complex roots of unity. Let

$$\mathrm{L}_N : \mathbb{T}_N \longrightarrow \mathbb{Z}/N\mathbb{Z}$$

be the natural group isomorphism. We can define $\mathrm{L}_N(A)$, by the spectral theorem, for any operator A of order N, such that it formally satisfies

$$A = e^{2\pi i \mathrm{L}_N(A)/N}.$$

One has that

$$\mathrm{L}_N(-e^{-\pi i/N} XY) = \mathrm{L}_N(X) + \mathrm{L}_N(Y). \tag{3.26}$$

For any function $f : \mathbb{A}_N \longrightarrow \mathbb{C}$ recall the definition of $\tilde{f}$ and the operator function $\not{f}(\mathsf{x}, A) \equiv f(\mathsf{x}, \mathrm{L}_N(A))$ from Appendix A.2. The following Pentagon Identity for D_b was first proved in [AK3], where a projective ambiguity was undetermined and $|\mathrm{b}| = 1$.

Lemma 3.16 (Pentagon Equation)**.** *Let* p,q, *X and Y be as above, then the following five-term relation holds*

$$\not{\mathrm{D}}_\mathrm{b}(\mathsf{p}, X)\not{\mathrm{D}}_\mathrm{b}(\mathsf{q}, Y) = \not{\mathrm{D}}_\mathrm{b}(\mathsf{q}, Y)\not{\mathrm{D}}_\mathrm{b}(\mathsf{p}+\mathsf{q}, -e^{\pi i/N} XY)\not{\mathrm{D}}_\mathrm{b}(\mathsf{p}, X). \tag{3.27}$$

Proof. This is equivalent to the Integral Pentagon equation of Proposition 3.13. To see this we need to use equation (A.14) an all the five terms. Then we compare the coefficients of $e^{2\pi i y \mathsf{q}} Y^m e^{2\pi i x \mathsf{p}} X^{-n}$ and get exactly the integral pentagon equation.
An alternative proof follows from the q-Pochammer presentation of D_b from Proposition 3.8. □

3.1 Charges

We are going to define a *charged* version of the dilogarithm. The charges will assume geometrical meaning in the construction of the partition function, however they already satisfy the purpose of turning all the conditional convergent integral relations of the dilogarithm D_b (e.g. Proposition 3.13 and 3.12) into absolutely convergent integrals.
Let a, b and c be three real positive numbers such that $a+b+c = \frac{1}{\sqrt{N}}$. We define the charged quantum dilogarithm

$$\psi_{a,c}(x, n) := \frac{e^{-2\pi i c_\mathrm{b} a x}}{\mathrm{D}_\mathrm{b}(x - c_\mathrm{b}(a+c), n)}. \tag{3.28}$$

From the Fourier transformation formula, Proposition 3.12, and the inversion formula in Lemma 3.4, we can deduce the following transformation formulas for $\psi_{a,c}$ (recall notation (A.15) for the inverse Fourier transform)

Lemma 3.17. *Suppose* $\mathrm{Im}(\mathrm{b})(1-|\mathrm{b}|)=0$*, then*

$$\tilde{\psi}_{a,c}(x,k)=\psi_{c,b}(x,k)\langle x,k\rangle e^{-\pi i c_{\mathrm{b}}^2 a(a+2c)}\zeta_0 \tag{3.29}$$

$$\overline{\psi_{a,c}(x,k)}=\psi_{c,a}(-x,\epsilon k)\langle x,k\rangle e^{\pi c_{\mathrm{b}}^2(a+c)^2}\zeta_{N,inv} \tag{3.30}$$

$$\overline{\tilde{\psi}_{a,c}(x,k)}=\psi_{b,c}(-x,\epsilon k)e^{-2\pi i c_{\mathrm{b}}^2 ab}\zeta_0 \tag{3.31}$$

where $\zeta_0=e^{-\pi i(N-4c_{\mathrm{b}}^2N^{-1})/12}$ *and* $\zeta_{N,inv}=\zeta_0^2e^{-\pi i c_{\mathrm{b}}^2/N}$ *and* $\epsilon=+1$ *if* $\mathrm{b}>0$ *or* $\epsilon=-1$ *if* $|\mathrm{b}|=1$*.*

Remark 3.18. The hypothesis on positivity of a, b and c assure that the Fourier integral of $\tilde{\psi}_{a,c}$ is absolutely convergent.

Theorem 3.19 (Charged Pentagon Equation)**.** *Let* $a_j,\ c_j>0$ *such that* $\frac{1}{\sqrt{N}}-a_j-c_j>0$ *for* $j=0,\ 1,\ 2,\ 3$ *or* 4*. Define* $\psi_j\equiv\psi_{a_j,b_j}$*. Suppose the following relations hold true*

$$a_1=a_0+a_2 \qquad a_3=a_2+a_4 \qquad c_1=c_0+a_4 \qquad c_3=a_0+c_4 \qquad c_2=c_1+c_3 \tag{3.32}$$

and consider the operators on $\mathrm{L}^2(\mathbb{A}_N)$ *defined to satisfy (3.23 -3.24). We have the following charged pentagon relation*

$$\begin{aligned}\psi_1(\mathsf{q},\mathrm{L}_N(X))\psi_3(\mathsf{p},\mathrm{L}_N(Y))\xi(a,c)&=\\ =\psi_4(\mathsf{p},\mathrm{L}_N(X))\psi_2(\mathsf{p}+\mathsf{q},&\mathrm{L}_N(X)+\mathrm{L}_N(Y))\psi_0(\mathsf{q},\mathrm{L}_N(Y))\end{aligned} \tag{3.33}$$

where $\xi(a,c)=e^{2\pi i c_{\mathrm{b}}^2(a_0a_2+a_0a_4+a_2a_4)}e^{\pi i c_{\mathrm{b}}^2 a_2^2}$*.*

4 Quantum Teichmüller Theory

In this section we are going to quantize the space $\mathcal{R}(\Sigma_{g,s})$ following [K3]. For any fixed τ the quantization of $\mathcal{R}(\tau)$ is just the canonical quantization in exponential coordinates of the space $\mathbb{R}^M_{>0}\times\mathbb{R}^M_{>0}$, where $M=2g-2+s$, with symplectic form $\omega_\tau=\sum_{j=1}^M \mathrm{d}\log u_j\wedge\mathrm{d}\log v_j$. Formally, following the expectations from canonical quantization of $\mathbb{R}^{2M}$, we can quantize $\mathcal{R}(\tau)$ and associate to it an algebra of operator

$$\mathcal{X}(\tau)\text{ generated by }\{\hat{u}_j,\,\hat{v}_j\}\text{, where }0\le j<M\text{, subject to the relations}$$
$$\hat{u}_j\hat{v}_l=q^{\delta(j-l)}\hat{v}_l\hat{u}_j \quad \hat{u}_j\hat{u}_l=\hat{u}_l\hat{u}_j \quad \hat{v}_j\hat{v}_l=\hat{v}_l\hat{v}_j$$

where $q\in\mathbb{C}^*$. The algebra we mean here is the associative algebra of non commutative fractions of non commutative polynomials generated by these generators.

In order to obtain a quantization of $\mathcal{R}(\Sigma_{g,s})$ (i.e. triangulation independent) we have to look at the action of the $\mathcal{G}(\Sigma_{g,s})$ generators on coordinates and translate it into an action on the algebras $\mathcal{X}(\tau)$. Precisely consider the set of the couples $(\tau, \mathcal{X}(\tau))$ and let the generators $[\tau, \tau^\sigma]$, $[\tau, \rho_i\tau]$ and $[\tau, \omega_{i,j}\tau]$ act on them. The action on the algebras is as follows. The elements $[\tau, \tau^\sigma]$ just permutes the indexes of the generators according to the permutation σ. The change of decoration $[\tau, \rho_i\tau]$ acts trivially on the operators $(\hat{u}_j, \hat{v}_j)$ such that $j \neq i$ and as follows on $(\hat{u}_i, \hat{v}_i)$

$$(\hat{u}_i, \hat{v}_i) \mapsto (q^{-1/2}\hat{v}_i\hat{u}_i^{-1}, \hat{u}_i^{-1}). \tag{4.1}$$

The most interesting generator $[\tau, \omega_{i,j}\tau]$, is again trivial in the triangles not involved in the diagonal exchange, but it maps the two couples of operators $(\hat{u}_i, \hat{v}_i)$ and $(\hat{u}_j, \hat{v}_j)$ to the two new couples (following formulas (2.11))

$$(\hat{u}_i, \hat{v}_i) \bullet (\hat{u}_j, \hat{v}_j) \equiv (\hat{u}_i\hat{u}_j, \hat{u}_i\hat{v}_j + \hat{v}_i) \tag{4.2}$$

$$(\hat{u}_i, \hat{v}_i) * (\hat{u}_j, \hat{v}_j) \equiv (\hat{u}_j\hat{v}_i(\hat{u}_i\hat{v}_j + \hat{v}_i)^{-1}, \hat{v}_j(\hat{u}_i\hat{v}_j + \hat{v}_i)^{-1}). \tag{4.3}$$

In order to get an actual quantization we need to provide a representation of $\mathcal{X}(\tau)$ by operators acting on some vector space $\mathcal{H}$. In the original paper [K3], Kashaev proposed representations on the vector spaces $\mathrm{L}^2(\mathbb{R})$ and $\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N$ for N odd. The former was used to construct the Andersen-Kashaev invariants in [AK1], while the latter are related to the colored Jones polynomials ([K1], [MM]) and the Volume Conjecture [K2]. In the more recent work [AK3] a representation on the vector space $\mathrm{L}^2(\mathbb{A}_N) \equiv \mathrm{L}^2(\mathbb{R} \times \mathbb{Z}/N\mathbb{Z}) \simeq \mathrm{L}^2(\mathbb{R}) \otimes \mathbb{C}^N$ was implicitly proposed, or at least all the basics elements to construct it were presented. Here we describe the representations in $\mathrm{L}^2(\mathbb{A}_N)$.

4.1 $\mathrm{L}^2(\mathbb{A}_N)$ Representations

Fix N positive odd integer, $\omega \equiv e^{\frac{2\pi i}{N}}$ and $\mathrm{b} \in \mathbb{C}^*$, $\mathrm{Re}(\mathrm{b}) > 0$. To each decorated ideal triangle $\overline{\tau}_j \in \tau$ we associate the Hilbert space $\mathrm{L}^2(\mathbb{A}_N)$. Then the Hilbert space associated to $\mathcal{R}(\tau)$ will be $\mathcal{H} = \mathrm{L}^2(\mathbb{A}_N)^{\otimes M} \cong \mathrm{L}^2(\mathbb{A}_N^M)$ where $M = 2g-2+s$ is the number of triangles in τ. For conventions and notation on the space $\mathrm{L}^2(\mathbb{A}_N)$ see Appendix A.2. For every $i = 0, \dots, M$ let p_i, q_i be self adjoint operator in $\mathrm{L}^2(\mathbb{R})$ and X_i, Y_i unitary operators in $\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N$ such that

$$[p_i, q_j] = \frac{\delta_{ij}}{2\pi i}, \qquad Y_iX_j = \omega^{\delta_{ij}}X_jY_i, \qquad X_i^N = Y_i^N = 1. \tag{4.4}$$

We can define the operators

$$\mathsf{u}_i = e^{2\pi\frac{\mathrm{b}}{\sqrt{N}}\mathsf{q}_i}Y_i \qquad \mathsf{u}_i^* = e^{2\pi\frac{\mathrm{b}^{-1}}{\sqrt{N}}\mathsf{q}_i}Y_i^{-1} \tag{4.5}$$

$$\mathsf{v}_i = e^{2\pi\frac{\mathrm{b}}{\sqrt{N}}\mathsf{p}_i}X_i \qquad \mathsf{v}_i^* = e^{2\pi\frac{\mathrm{b}^{-1}}{\sqrt{N}}\mathsf{p}_i}X_i^{-1} \tag{4.6}$$

satisfying

$$\mathsf{u}_i\mathsf{v}_j = q^{\delta_{ij}}\mathsf{v}_j\mathsf{u}_i \qquad \mathsf{u}_i^*\mathsf{v}_j^* = \tilde{q}^{\delta_{ij}}\mathsf{v}_j^*\mathsf{u}_i^* \tag{4.7}$$

$$\mathsf{u}_i\mathsf{v}_j^* = \mathsf{v}_j^*\mathsf{u}_i \qquad \mathsf{u}_i^*\mathsf{v}_j = \mathsf{v}_j\mathsf{u}_i^* \tag{4.8}$$

$$q = e^{2\pi i\frac{\mathrm{b}^2}{N}}\omega \qquad \tilde{q} = e^{2\pi i\frac{\mathrm{b}^{-2}}{N}}\omega. \tag{4.9}$$

The Quantum algebra $\mathcal{X}(\tau)$ is generated by the u_j, v_j for $j = 0,\dots M$, and has a $*$–algebra structure when extended to include u_j^* and v_j^*. We remark that the $*$ operator we are using here is the standard hermitian conjugation only if $|\mathrm{b}| = 1$. Explicitly let X_j, Y_j, p_j, q_j , $j = 1,2$ be operators acting on $\mathcal{H} := \mathrm{L}^2(\mathbb{A}_N^2)$ as follow

$$\mathsf{p}_j f(\mathsf{x},\mathsf{m}) = \frac{1}{2\pi i}\frac{\partial}{\partial x_j}f(\mathsf{x},\mathsf{m}), \qquad \mathsf{q}_j f(\mathsf{x},\mathsf{m}) = x_j f(\mathsf{x},\mathsf{m}) \tag{4.10}$$

$$X_1 f(\mathsf{x},\mathsf{m}) = f(\mathsf{x},(m_1+1,m_2)), \quad X_2 f(\mathsf{x},\mathsf{m}) = f(\mathsf{x},(m_1,m_2+1)) \tag{4.11}$$

$$Y_j f(\mathsf{x},\mathsf{m}) = \overline{\omega}^{m_j} f(\mathsf{x},\mathsf{m}), \tag{4.12}$$

where $\mathsf{m} = (m_1,m_2) \in \mathbb{Z}_N^2$ and $\mathsf{x} = (x_1,x_2) \in \mathbb{R}^2$.
These operators satisfy conditions (4.4). Let $\psi_{\mathrm{b}}(x,n) \equiv \frac{1}{\mathrm{D}_{\mathrm{b}}(x,n)}$ and consider the operators

$$\mathsf{D}_{12} \equiv e^{2\pi i\mathsf{q}_2\mathsf{p}_1}\sum_{j,k=0}^{N-1}\omega^{jk}Y_2^jX_1^k \tag{4.13}$$

$$\Psi_{12} \equiv \Psi_{\mathrm{b}}(\mathsf{q}_1+\mathsf{p}_2-\mathsf{q}_2, -e^{-\frac{\pi i}{N}}Y_1X_2\overline{Y_2}) \tag{4.14}$$

$$\mathsf{T}_{12} \equiv \mathsf{D}_{12}\Psi_{12} \tag{4.15}$$

One has

Lemma 4.1 (Tetrahedral Equations)**.**

$$\mathsf{T}_{12}\mathsf{u}_1 = \mathsf{u}_1\mathsf{u}_2\mathsf{T}_{12} \tag{4.16}$$

$$\mathsf{T}_{12}\mathsf{v}_1\mathsf{v}_2 = \mathsf{v}_2\mathsf{T}_{12} \tag{4.17}$$

$$\mathsf{T}_{12}\mathsf{v}_1\mathsf{u}_2 = \mathsf{v}_1\mathsf{u}_2\mathsf{T}_{12} \tag{4.18}$$

$$\mathsf{T}_{12}\mathsf{v}_1 = (\mathsf{u}_1\mathsf{v}_2+\mathsf{v}_1)\mathsf{T}_{12} \tag{4.19}$$

$$\mathsf{T}_{12}\mathsf{T}_{13}\mathsf{T}_{23} = \mathsf{T}_{23}\mathsf{T}_{12} \tag{4.20}$$

Remark 4.2. If we define $\tilde{\mathsf{T}}_{12} = \mathsf{D}_{12}\tilde{\Psi}_{12}$ where

$$\tilde{\Psi} \equiv \Psi_{\mathrm{b}^{-1}}(\mathsf{q}_1+\mathsf{p}_2-\mathsf{q}_2, -e^{-\frac{\pi i}{N}}\overline{Y}_1\overline{X}_2Y_2) \tag{4.21}$$

then $\tilde{\mathsf{T}}$ satisfies equations (4.16 – 4.20) with u_i and v_i substituted by u_i^* and v_i^*. However from Remark 3.6 we know that $\Psi_{\mathrm{b}^{-1}}(x,n) = \Psi_{\mathrm{b}}(x,-n)$, and

$$\left(-e^{-\frac{\pi i}{N}}Y_1X_2\overline{Y_2}\right)^{-1} = -e^{\frac{\pi i}{N}}\overline{Y}_1Y_2\overline{X}_2 = -e^{-\frac{\pi i}{N}}\overline{Y}_1\overline{X}_2Y_2$$

so that

$$\tilde{\mathsf{T}} = \mathsf{T}.$$

From Lemma 4.1 we have the following implementations of equations (4.1 – 4.3).

Proposition 4.3. *Let* $\mathsf{w}_i \equiv (\mathsf{u}_i, \mathsf{v}_i)$ *and* $\mathsf{w}_i^* = (\mathsf{u}_i^*, \mathsf{v}_i^*)$. *Then we have*

$$\mathsf{w}_1 \bullet \mathsf{w}_2 \mathsf{T}_{12} = \mathsf{T}_{12}\mathsf{w}_1, \quad \mathsf{w}_1 * \mathsf{w}_2 \mathsf{T}_{12} = \mathsf{T}_{12}\mathsf{w}_2, \tag{4.22}$$

$$\mathsf{w}_1^* \bullet \mathsf{w}_2^* \mathsf{T}_{12} = \mathsf{T}_{12}\mathsf{w}_1^*, \quad \mathsf{w}_1^* * \mathsf{w}_2^* \mathsf{T}_{12} = \mathsf{T}_{12}\mathsf{w}_2^*. \tag{4.23}$$

Proposition 4.4. *Let*

$$\mathsf{A} \equiv e^{3\pi i \mathsf{q}^2} e^{\pi i (\mathsf{p}+\mathsf{q})^2} \sum_{j=0}^{N-1} \overline{\langle j \rangle}^3 Y^{3j} \sum_{l=0}^{N-1} \overline{\langle l \rangle} (-e^{-\pi i/N} Y X)^l, \tag{4.24}$$

where $\langle n \rangle = e^{-\pi i n(n+N)/N}$ *and* Y *and* X *are as above. Then*

$$\mathsf{A}(\mathsf{u}, \mathsf{v}) = (q^{-1/2}\mathsf{v}\mathsf{u}^{-1}, \, \mathsf{u}^{-1}) \qquad \mathsf{A}(\mathsf{u}^*, \mathsf{v}^*) = (\tilde{q}^{-1/2}\mathsf{v}^*(\mathsf{u}^*)^{-1}, \, (\mathsf{u}^*)^{-1}) \tag{4.25}$$

where q *and* $\tilde{q}$ *are defined by equation* (4.9).

5 Quantization of the Model Space for Complex Chern-Simons Theory

In this Section we want to quantize the space $\mathbb{C}^* \times \mathbb{C}^*$ with the complex symplectic form

$$\omega_{\mathbb{C}} = \frac{\mathrm{d}x \wedge \mathrm{d}y}{xy}.$$

We think of it as a model space for Complex Chern-Simons Theory because it is an open dense of the $\mathrm{PSL}(2, \mathbb{C})$ moduli space of flat connections on a four punctured sphere, with unipotent holonomy around the punctures, [AK3, K5, D, FG]. Tetrahedral operators are supposedly related to states in the quantization of the four punctured sphere. Since we want to construct knot invariants starting from tetrahedral ideal triangulations this is the space we need to quantize. We follow the ideas in Andersen and Kashaev [AK3] using a real polarization with contractible leaves. We will further show that the level N Weil-Gel'fand-Zak Transform relates this quantization with the $\mathrm{L}^2(\mathbb{A}_N)$ representations in Quantum Teichmüller theory. To use this transform to relates the Andersen–Kashaev invariants to complex Chern–Simons Theory was already proposed in [AK3]. However the relation between the two approaches was not as tight as the one present here.

Let $t = N + is \in \mathbb{C}^*$ be the quantization constant, for $N \in \mathbb{R}$ and $s \in \mathbb{R} \sqcup i\mathbb{R}$. Denote also $\tilde{t} = N - is$. Fix $\mathrm{b} \in \mathbb{C}$ such that $s = -iN\frac{1-\mathrm{b}^2}{1+\mathrm{b}^2}$ and $\mathrm{Re}\,\mathrm{b} > 0$. This substitution, for $s \in i\mathbb{R}$, is only possible when $-N < is < N$. Notice that

$$s \in \mathbb{R} \iff |\mathrm{b}| = 1 \text{ and } \mathrm{b} \neq \pm i, \qquad s \in i\mathbb{R} \iff \mathrm{Im}\,\mathrm{b} = 0 \tag{5.1}$$

and notice the following useful expressions

$$t = \frac{2N}{1+\mathrm{b}^2}, \qquad \tilde{t} = \frac{2N}{1+\mathrm{b}^{-2}}. \tag{5.2}$$

Consider the covering maps

$$\begin{aligned} \zeta^{\pm}\colon \mathbb{R}^2 &\longrightarrow \mathbb{C}^* \\ (z,n) &\mapsto \exp\left(2\pi \mathrm{b}^{\pm 1} z \pm 2\pi i n\right) \end{aligned} \tag{5.3}$$

and consequently

$$\pi^{\pm} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{C}^* \times \mathbb{C}^*, \qquad \pi^{\pm} = (\zeta^{\pm}, \zeta^{\pm}) \tag{5.4}$$

such that

$$\begin{aligned} &\mathbb{C}^* \times \mathbb{C}^* \ni (x,y) = \pi^+((z,n),(w,m)), \\ &\mathbb{C}^* \times \mathbb{C}^* \ni (\tilde{x},\tilde{y}) = \pi^-((z,n),(w,m)) \quad \text{for } ((z,n),(w,m)) \in \mathbb{R}^2 \times \mathbb{R}^2. \end{aligned} \tag{5.5}$$

We remark that

$$\overline{\zeta^+(z,n)} = \zeta^-(z,n) \qquad \iff \qquad |\mathrm{b}| = 1 \tag{5.6}$$

in this case $\pi^- = \overline{\pi^+}$ and $\tilde{x} = \overline{x}$, $\tilde{y} = \overline{y}$. In this sense x, y $\tilde{x}$ and $\tilde{y}$ are natural coordinate functions to quantize in $\mathbb{C}^* \times \mathbb{C}^*$. If $\mathrm{b} \in \mathbb{R}$ they are still coordinates functions for the underlying real manifold, but we lose the complex conjugate interpretation. We will first consider the quantization of the covering $\mathbb{R}^2 \times \mathbb{R}^2$. Define the form

$$\omega_t \equiv \frac{t}{4\pi}(\pi^+)^*(\omega_{\mathbb{C}}) + \frac{\tilde{t}}{4\pi}(\pi^-)^*(\omega_{\mathbb{C}}). \tag{5.7}$$

Lemma 5.1.

$$\omega_t = 2\pi N(\mathrm{d}z \wedge \mathrm{d}w - \mathrm{d}n \wedge \mathrm{d}m). \tag{5.8}$$

In particular it is a real symplectic 2 *form on* $\mathbb{R}^2 \times \mathbb{R}^2$*, independent of* b.

Over $\mathbb{R}^2 \times \mathbb{R}^2$ we take the trivial line bundle $\widetilde{\mathcal{L}} = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{C}$. On the N-th tensor power of this line bundle $\widetilde{\mathcal{L}}^N$ we consider the connection

$$\nabla^{(t)} \equiv \mathrm{d} - i\alpha_t \tag{5.9}$$

where

$$\alpha_t \equiv \frac{t}{4\pi}\alpha_{\mathbb{C}}^{+} + \frac{\tilde{t}}{4\pi}\alpha_{\mathbb{C}}^{-}, \tag{5.10}$$

$$\alpha_{\mathbb{C}}^{\pm} \equiv 2\pi^2(\mathrm{b}^{\pm 1}z \pm in)\mathrm{d}(\mathrm{b}^{\pm 1}w \pm im) - 2\pi^2(\mathrm{b}^{\pm 1}w \pm im)\mathrm{d}(\mathrm{b}^{\pm 1}z \pm in) \tag{5.11}$$

In analogy to Lemma 5.1 we have

$$\alpha_t = \pi N(z\mathrm{d}w - w\mathrm{d}z - n\mathrm{d}m + m\mathrm{d}n). \tag{5.12}$$

It is simple to see that

$$\mathrm{d}\alpha_{\mathbb{C}}^{\pm} = (\pi^{\pm})^*(\omega_{\mathbb{C}}), \qquad \text{which implies} \tag{5.13}$$

$$F_{\nabla^{(t)}} = -i\omega_t. \tag{5.14}$$

Further, on $\mathbb{R}^2 \times \mathbb{R}^2$ we have an action of $\mathbb{Z} \times \mathbb{Z}$ compatible with the projection π^+, i.e.

$$(\mathbb{Z} \times \mathbb{Z}) \times \left(\mathbb{R}^2 \times \mathbb{R}^2\right) \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2 \tag{5.15}$$

$$(\lambda_1, \lambda_2) \cdot ((z, n), (w, m)) \mapsto ((z, n + \lambda_1), (w, m + \lambda_2)) \tag{5.16}$$

that satisfies

$$\pi^{\pm}((z, n + \lambda_1), (w, m + \lambda_2)) = \pi^{\pm}((z, n), (w, m)) \tag{5.17}$$

This action can be lifted to an action $\widetilde{\mathcal{L}}^N$ in such a way that the quotient bundle $\mathcal{L}^N \equiv \widetilde{\mathcal{L}}^N\ /(\mathbb{Z})^2 \to \mathbb{R}^4/\mathbb{Z}^2$ has first Chern class $c_1(\mathcal{L}^N) = \frac{1}{2\pi}[\omega_t]$ (ω_t is evidently $\mathbb{Z}^2$–invariant). Such a condition gives the requirement (which is in fact the pre-quantum condition) $\frac{1}{2\pi}[\omega_t] \in H^2((\mathbb{R}^2 \times \mathbb{R}^2)/(\mathbb{Z}^2), \mathbb{Z})$, which boils down to the requirement $N \in \mathbb{Z}$. Explicitly the action of $\mathbb{Z} \times \mathbb{Z}$ on $\widetilde{\mathcal{L}}^N$ is given by the following two multipliers

$$e_{(1,0)} = e^{-\pi N i m}, \qquad e_{(0,1)} = e^{\pi N i n}. \tag{5.18}$$

That means that we consider the space of sections

$$(\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2} \tag{5.19}$$

of $\mathbb{Z}^2$–invariant, smooth sections of $\widetilde{\mathcal{L}}^N$. Explicitly

$$s \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2} \text{ if and only if } s \in \mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N) \text{ and satisfies}$$

$$s((z, n+1), (w, m)) = e^{-\pi i N m} s((z, n), (w, m)), \tag{5.20}$$

$$s((z, n), (w, m+1)) = e^{\pi i N n} s((z, n), (w, m)) \tag{5.21}$$

Lemma 5.2.

$$\nabla^{(t)} s \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2}, \quad \textit{for any } s \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2}$$

The following Hermitian structure on $\widetilde{\mathcal{L}}^N$ is $\mathbb{Z}^2$–invariant and parallel with respect to $\nabla^{(t)}$.

$$s \cdot s'(p) \equiv s(p)\overline{s'(p)}, \quad \text{for any } p \in \mathbb{R}^2 \times \mathbb{R}^2 \tag{5.22}$$

Being parallel here means that

$$\mathrm{d}(s \cdot s') = (\nabla^{(t)} s) \cdot s' + s \cdot (\nabla^{(t)} s'), \tag{5.23}$$

and this is a simple consequence of α_t being a real 1-form. It follows that the following is a well defined inner product in the completion of $\left(\left(\mathrm{L}^2 \cap \mathrm{C}^\infty\right)(\mathbb{R}^4, \widetilde{\mathcal{L}}^N)\right)^{\mathbb{Z}^2}$

$$(s,s') \equiv \int_{\mathbb{R}} \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}w \left(\int_0^1 \mathrm{d}n \int_0^1 \mathrm{d}m \ s \cdot s' \right) \tag{5.24}$$

Lemma 5.3. *We have the following Hamiltonian vector field for the coordinates functions on* $\mathbb{R}^2 \times \mathbb{R}^2$

$$X_z = \frac{1}{2\pi N}\frac{\partial}{\partial w} \qquad X_w = -\frac{1}{2\pi N}\frac{\partial}{\partial z}$$
$$X_n = -\frac{1}{2\pi N}\frac{\partial}{\partial m} \qquad X_m = \frac{1}{2\pi N}\frac{\partial}{\partial n}$$

From the definition of the pre-quantum operator $\hat{f}$ associated to the observable f, we have

$$\hat{f} = -i\nabla_{X_f} + f \tag{5.25}$$

Lemma 5.4 (Pre–Quantum operators)**.** *The following are the pre–quantum operators for the coordinate functions on* $\mathbb{R}^2 \times \mathbb{R}^2$

$$\hat{z} = \frac{-i}{2\pi N}\nabla_w^{(t)} + z \qquad \hat{w} = \frac{i}{2\pi N}\nabla_z^{(t)} + w$$
$$\hat{n} = \frac{i}{2\pi N}\nabla_m^{(t)} + n \qquad \hat{m} = \frac{-i}{2\pi N}\nabla_n^{(t)} + m$$

and they satisfy the following canonical commutation relations

$$[\hat{z},\ \hat{w}] = \frac{1}{2\pi i N} \qquad [\hat{n},\ \hat{m}] = -\frac{1}{2\pi i N} \tag{5.26}$$

$$[\hat{z}, \hat{n}] = [\hat{z}, \hat{m}] = [\hat{w}, \hat{n}] = [\hat{w}, \hat{m}] = 0 \tag{5.27}$$

The Hermitian line bundle $\mathcal{L}^N \to (\mathbb{R}^2 \times \mathbb{R}^2/\mathbb{Z}^2)$ together with the connection $\nabla^{(t)}$ define a pre–Quantization of the theory. In order to finish the quantization program we need to choose a Lagrangian polarization.

Choose the following real Lagrangian polarization

$$\widetilde{\mathcal{P}} \equiv \operatorname{Span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial w} + \frac{\partial}{\partial n}, \frac{\partial}{\partial z} - \frac{\partial}{\partial m} \right\}. \tag{5.28}$$

The leaves of this polarization are all contractible after quotient by the action of $\mathbb{Z}^2$ on $\mathbb{R}^2 \times \mathbb{R}^2$, so we do have polarized global sections. In particular the space $T \subset \mathbb{R}^2 \times \mathbb{R}^2$

$$T \equiv \{z = w = 0\} \tag{5.29}$$

is a transversal for the polarization. For any $\psi \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2}$ polarized by $\widetilde{\mathcal{P}}$, the following two differential equations will determine $\psi \equiv \psi((z,n),(w,m))$ by its value in (n,m)

$$\nabla^{(t)}_w \psi = -\nabla^{(t)}_n \psi \qquad\qquad \nabla^{(t)}_z \psi = \nabla^{(t)}_m \psi. \tag{5.30}$$

The space $T/\mathbb{Z}^2$ is of course $\mathbb{T} \times \mathbb{T}$, and the line bundle $\mathcal{L}^N$ will restrict to a non trivial line bundle over $\mathbb{T} \times \mathbb{T}$ that we shall call $\mathcal{L}^N$ again. The quantum space that we obtain is then

$$\mathcal{H}^{(N)} \equiv \mathrm{C}^\infty \left(\mathbb{T} \times \mathbb{T}, \mathcal{L}^N \right). \tag{5.31}$$

We consider the obvious inner product on $\mathcal{H}^{(N)}$

$$(\psi,\, \phi) = \int_0^1 \int_0^1 \psi \overline{\phi}\; \mathrm{d}n \mathrm{d}m \tag{5.32}$$

that is the standard inner product on the completion $\mathrm{L}^2 \left(\mathbb{T} \times \mathbb{T}, \mathcal{L}^N \right)$. Finally the quantum operators acts on polarized sections as

$$\hat{x} \equiv \exp\left(2\pi \mathrm{b} \hat{z} + 2\pi i \hat{n}\right) = \exp\left(i \frac{\mathrm{b}}{N} \nabla^{(t)}_n - \frac{1}{N} \nabla^{(t)}_m + 2\pi i n \right) \tag{5.33}$$

$$\hat{y} \equiv \exp\left(2\pi \mathrm{b} \hat{w} + 2\pi i \hat{m}\right) = \exp\left(i \frac{\mathrm{b}}{N} \nabla^{(t)}_m + \frac{1}{N} \nabla^{(t)}_n + 2\pi i m \right) \tag{5.34}$$

$$\hat{\tilde{x}} \equiv \exp\left(2\pi \mathrm{b}^{-1} \hat{z} - 2\pi i \hat{n}\right) = \exp\left(i \frac{\mathrm{b}^{-1}}{N} \nabla^{(t)}_n + \frac{1}{N} \nabla^{(t)}_m - 2\pi i n \right) \tag{5.35}$$

$$\hat{\tilde{y}} \equiv \exp\left(2\pi \mathrm{b}^{-1} \hat{w} - 2\pi i \hat{m}\right) = \exp\left(i \frac{\mathrm{b}^{-1}}{N} \nabla^{(t)}_z - \frac{1}{N} \nabla^{(t)}_n - 2\pi i m \right) \tag{5.36}$$

Now we are going to connect the quantization with the Quantum Teichmüller theory. Recall the operators $\mathsf{u} = \mathsf{u}(\mathrm{b})$ and $\mathsf{v} = \mathsf{v}(\mathrm{b})$ from equations (4.4 – 4.12), and recall that they depend on a parameter b. Define the rescaling operator

$$\mathcal{O}_{\sqrt{N}} \colon\ \mathrm{L}^2(\mathbb{A}_N) \longrightarrow \mathrm{L}^2(\mathbb{A}_N)$$

$$\mathcal{O}_{\sqrt{N}}(\mathsf{f})(z,n) = \mathsf{f}\left(\sqrt{N}z, n\right) \tag{5.37}$$

and the following rescaled analogues of the Quantum Teichmüller Theory operators

$$\hat{u} = \mathcal{O}_{\sqrt{N}} \circ \mathsf{u}(\mathrm{b}^{-1}) \circ \mathcal{O}^{-1}_{\sqrt{N}} \qquad \hat{v} = \mathcal{O}_{\sqrt{N}} \circ \mathsf{v}(\mathrm{b}^{-1}) \circ \mathcal{O}^{-1}_{\sqrt{N}} \tag{5.38}$$

$$\hat{u}^* = \mathcal{O}_{\sqrt{N}} \circ \mathsf{u}^*(\mathrm{b}^{-1}) \circ \mathcal{O}^{-1}_{\sqrt{N}} \qquad \hat{v}^* = \mathcal{O}_{\sqrt{N}} \circ \mathsf{v}^*(\mathrm{b}^{-1}) \circ \mathcal{O}^{-1}_{\sqrt{N}} \tag{5.39}$$

which acts on $\mathsf{f} \in \mathrm{L}^2(\mathbb{A}_N)$ as

$$\hat{u}^*\mathsf{f}(z,l) = e^{2\pi \mathrm{b} z} e^{2\pi i l/N}\mathsf{f}(z,l) \qquad \hat{u}\mathsf{f}(z,l) = e^{2\pi \mathrm{b}^{-1} z} e^{-2\pi i l/N}\mathsf{f}(z,l) \tag{5.40}$$

$$\hat{v}^*\mathsf{f}(z,l) = f(z - i\frac{\mathrm{b}}{N}, l-1) \qquad \hat{v}\mathsf{f}(z,l) = f(z - i\frac{\mathrm{b}^{-1}}{N}, l+1) \tag{5.41}$$

We make use of the level-N Weil-Gel'fand-Zak Transform, [AK3].

Theorem 5.5. *Recall the line bundle $\mathcal{L}^N$. The following map $Z^{(N)}\colon \mathcal{S}(\mathbb{A}_N) \longrightarrow \mathrm{C}^\infty(\mathbb{T}\times\mathbb{T}, \mathcal{L}^N)$ is a an isomorphism*

$$Z^{(N)}(\mathsf{f})(n,m) = \frac{1}{\sqrt{N}} e^{\pi i N mn} \sum_{p\in\mathbb{Z}} \sum_{l=0}^{N-1} \mathsf{f}\left(n + \frac{p}{N}, l\right) e^{2\pi i mp} e^{2\pi i l p/N} \tag{5.42}$$

with inverse

$$\overline{Z}^{(N)}(s)(x,j) = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} e^{-2\pi i \frac{lj}{N}} \int_0^1 s\left(x - \frac{l}{N}, v\right) e^{-\pi i N(x+\frac{l}{N})v} \mathrm{d}v.$$

which preserves the inner products $\mathrm{L}^2(\mathbb{A}_N)$ and $(\cdot,\cdot)$, i.e.

$$\left(Z^{(N)}(\mathsf{f}),\, Z^{(N)}(\mathsf{g})\right) = \langle \mathsf{f},\, \mathsf{g}\rangle$$

and so extends to an isometry between $\mathrm{L}^2(\mathbb{A}_N)$ and $\mathrm{L}^2(\mathbb{T}\times\mathbb{T}, \mathcal{L}^N)$.

Proposition 5.6. *We have*

$$Z^{(N)} \circ \hat{u}^* \circ (Z^{(N)})^{-1} = \hat{y}^{-1} \qquad Z^{(N)} \circ \hat{v}^* \hat{\circ} (Z^{(N)})^{-1} = \hat{x}^{-1}$$

$$Z^{(N)} \circ \hat{u} \circ (Z^{(N)})^{-1} = \hat{\tilde{y}}^{-1} \qquad Z^{(N)} \circ \hat{v} \circ (Z^{(N)})^{-1} = \hat{\tilde{x}}^{-1}$$

All together we have showed that the quantization for the model space of complex Chern-Simons theory is equivalent to the $\mathrm{L}^2(\mathbb{A}_N)$ representations of the quantum algebra defined from Quantum Teichmüller Theory. In the following section we will extend the $\mathrm{L}^2(\mathbb{A}_N)$ representations to knots invariants following the recipe given by Andersen and Kashaev in [AK1]. The previous discussion on the different quantizations serves to link such invariants to Complex Quantum Chern–Simons Theory.

6 The Andersen–Kashaev Teichmüller TQFT at Level N

6.1 Angle Structures on 3-Manifolds

We present here shaped triangulated pseudo 3-manifolds, which are the combinatorial data underlying the Andersen-Kashaev construction of their invariant. Following strictly [AK1] we will describe the *categroid* of *admissible* oriented triangulated pseudo 3-manifolds, where the words admissible and categroid go together because admissibility is what will obstruct us to have a full category. See Appendix B for a definition of categroids.

Definition 6.1 (Oriented Triangulated Pseudo 3-manifold)**.** An *Oriented Triangulated Pseudo* 3*-manifold* X is a finite collection of 3-simplices (tetrahedra) with totally ordered vertices together with a collection of *gluing homeomorphisms* between some pairs of codimension 1 faces, so that every face is in, at most, one such pairs. By gluing homeomorphism we mean a vertex order preserving, orientation reversing, affine homeomorphism between the two faces.
The quotient space under the glueing homeomorphisms has the structure of CW-complex with oriented edges.

For $i \in \{0,1,2,3\}$ we denote by $\Delta_i(X)$ the collection of i-dimensional simplices in X and, for $i > j$, we denote

$$\Delta_i^j(X) = \{(a,b) | a \in \Delta_i(X), b \in \Delta_j(a)\}.$$

We have projection maps

$$\phi_{i,j} : \Delta_i^j(X) \longrightarrow \Delta_i(X), \quad \phi^{i,j} : \Delta_i^j(X) \longrightarrow \Delta_j(X),$$

and boundary maps

$$\partial_i : \Delta_j(X) \longrightarrow \Delta_{j-1}(X), \quad \partial_i[v_0, \dots, v_j] \mapsto [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_j]$$

where $[v_0, \dots, v_j]$ is the j-simplex with vertices $v_0, \dots, v_j$ and $i \leq j$.

Definition 6.2 (Shape Structure)**.** Let X be an oriented triangulated pseudo 3-manifold. A *Shape Structure* is a map

$$\alpha_X : \Delta_3^1(X) \longrightarrow \mathbb{R}_{>0},$$

so that, in every tetrahedron, the sum of the values of α_X along three incident edges is π.
The value of the map α_X in an edge e inside a tetrahedron T is called the *dihedral angle* of T at e. If we allow α_X to take values in $\mathbb{R}$ we talk about a *Generalized Shape Structure*.

The set of shape structures supported by X is denoted $\mathrm{S}(X)$. The space of generalized shape structures is denoted by $\widetilde{\mathrm{S}}(X)$. X together with α_X is called *Shaped Pseudo* 3*-manifold.*

Remark 6.3 (Ideal Tetrahedron)**.** A shape structure on a simplicial tetrahedron T as above defines an embedding of $T \setminus \Delta_0(T)$ in the hyperbolic 3–space $\mathbb{H}^3$ which extends to a map of T to $\overline{\mathbb{H}^3}$. In fact we can change a given embedding, so that it send the four vertices (v_0, v_1, v_2, v_3) to the four points $(\infty, 0, 1, z) \in \mathbb{CP}^1 \simeq \partial\mathbb{H}^3$, where

$$z = \frac{\sin \alpha_T([v_0, v_2])}{\sin \alpha_T([v_0, v_3])} \exp\left(i\alpha_T([v_0, v_1])\right).$$

This four points in $\partial\mathbb{H}^3$ extend to a unique ideal tetrahedron in $\mathbb{H}^3$, by taking the geodesic convex hull, that has dihedral angles defined by α_T.

Remark 6.4. In every tetrahedron, its orientation induces a cyclic ordering of all triples of edges meeting in a vertex. Such a cyclic ordering descends to a cyclic ordering of the pairs of opposite edges of the whole tetrahedron. Moreover, it follows from the definition that opposite edges share the same dihedral angle. Hence, we get a well defined cyclic order preserving projection $p : \Delta_3^1(X) \longrightarrow \Delta_3^{1/p}(X)$ which identifies opposite edges. α_X descends to a map from $\Delta_3^{1/p}(X)$ and we can consider the following skew-symmetric functions

$$\varepsilon_{a,b} \in \{0, \pm 1\}, \quad \varepsilon_{a,b} = -\varepsilon_{b,a}, \quad a, b \in \Delta_3^{1/p}(X),$$

defined to be $\varepsilon_{a,b} = 0$ if the underlying tetrahedra are distinct, and $\varepsilon_{a,b} = +1$ if the underlying tetrahedra coincides and b cyclically follows a in the order induced on $\Delta_3^{1/p}(X)$.

Definition 6.5. To any shaped pseudo 3-manifold X, we associate a *Weight* function

$$\omega_X : \Delta_1(X) \longrightarrow \mathbb{R}_{>0}, \quad \omega_X(e) = \sum_{a \in (\phi^{3,1})^{-1}(e)} \alpha_X(a).$$

An edge e in X is called *balanced* if e is internal and $\omega_X(e) = 2\pi$. A shape structure is fully balanced if all its edges are balanced.

The shape structures of closed fully balanced 3-manifolds are called *Angle Structures* in the literature. For more details on them and their geometric admissibility see [Lac] and [LT].

Definition 6.6. A *leveled* (generalised) shaped pseudo 3-manifold is a pair (X, l_X) consisting of a (generalized) shaped pseudo 3-manifold X and a real number $l_X \in \mathbb{R}$, called the *level.* The set of all leveled (generalised) shaped pseudo 3-manifolds is denoted by $\mathrm{LS}(X)$ (respectively $\widetilde{\mathrm{LS}}(X)$).

There is a gauge action of $\mathbb{R}^{\Delta_1(X)}$ on $\widetilde{\mathrm{LS}}(X)$.

Definition 6.7. Let (X, l_X) and (Y, l_Y) be two (generalized) leveled shaped pseudo 3-manifolds. They are said to be *gauge equivalent* if there exists an isomorphism $h : X \longrightarrow Y$ of the underlying cellular structures, and a function $g : \Delta_1(X) \longrightarrow \mathbb{R}$ such that

$$\Delta_1(\partial X) \subset g^{-1}(0),$$
$$\alpha_Y(h(a)) = \alpha_X(a) + \pi \sum_{b \in \Delta_3^1(X)} \varepsilon_{p(a),p(b)} g(\phi^{3,1}(b)), \quad \forall a \in \Delta_3^1(X), \text{ and}$$
$$l_Y = l_X + \sum_{e \in \Delta_1(X)} g(e) \sum_{a \in (\phi^{3,1})^{-1}(e)} (\frac{1}{3} - \frac{\alpha_X(a)}{\pi}).$$

We remark that $\omega_X = \omega_Y \circ h$.

Definition 6.8. Let (α_X, l_X) and $(\alpha_{X'}, l_{X'})$ be two (generalized) leveled shape structures of the oriented pseudo 3-manifold X. They are said *based gauge equivalent* if they are gauge equivalent as in Definition 6.7 if the isomorphism $h : X \longrightarrow X$ is the identity.

Based gauge equivalence is an equivalence relation in the sets $\mathrm{S}(X)$, $\mathrm{LS}(X)$, $\widetilde{\mathrm{S}}(X)$, $\widetilde{\mathrm{LS}}(X)$ and the quotient sets are denoted (resp.) $\mathrm{S}_r(X)$, $\mathrm{LS}_r(X)$, $\widetilde{\mathrm{S}}_r(X)$, $\widetilde{\mathrm{LS}}_r(X)$. We remark that $\mathrm{S}_r(X)$ is an open convex (possibly empty) subset of the space $\widetilde{\mathrm{S}}_r(X)$. We will return to existence of shape structures later. Let us focus on $\widetilde{\mathrm{S}}(X)$ for now. Let

$$\tilde{\Omega}_X : \widetilde{\mathrm{S}}(X) \longrightarrow \mathbb{R}^{\Delta_1(X)}$$

be the map which sends the shape structure α_X to the corresponding weight function ω_X. This map is gauge invariant, so it descends to a map

$$\tilde{\Omega}_{X,r} : \widetilde{\mathrm{S}}_r(X) \longrightarrow \mathbb{R}^{\Delta_1(X)}$$

For fixed $a \in \Delta_3^{1/p}(X)$ we can think of $\alpha_a := \alpha_X(a)$ as an element of $C^\infty\left(\widetilde{\mathrm{S}}(X)\right)$.

Definition 6.9 ([NZ])**.** The Neumann-Zagier symplectic structure on $\widetilde{\mathrm{S}}(X)$ is the unique symplectic structure which induces the Poisson bracket $\{\cdot,\cdot\}$ satisfying

$$\{\alpha_a, \alpha_b\} = \varepsilon_{a,b}$$

for all a, $b \in \Delta_3^{1/p}(X)$.

For a triangulated pseudo 3-manifold we have a symplectic decomposition

$$\widetilde{\mathrm{S}}(X) = \prod_{T \in \Delta_3(X)} \widetilde{\mathrm{S}}(T).$$

Theorem 6.10 ([AK1]). *The gauge action of* $\mathbb{R}^{\Delta_1(X)}$ *on* $\widetilde{\mathrm{S}}(X)$ *is symplectic and* $\tilde{\Omega}_X$ *is a moment map for this action. It follows that* $\widetilde{\mathrm{S}}_r(X) = \widetilde{\mathrm{S}}(X)/\mathbb{R}^{\Delta_1(X)}$ *is a Poisson manifold with symplectic leaves corresponding to the fibers of* $\tilde{\Omega}_{X,r}$.

Let $N_0(X)$ be a sufficiently small neighbourhood of $\Delta_0(X)$, then $\partial N_0(X)$ is a surface which inherits a triangulation from X, with a shape structure, if X has a shape structure. Notice that this surface can have boundary if $\partial X \neq \emptyset$.

Theorem 6.11 ([AK1]). *The map*

$$\tilde{\Omega}_{X,r} : \widetilde{\mathrm{S}}_r(X) \longrightarrow \mathbb{R}^{\Delta_1(X)}$$

is an affine $\mathrm{H}^1(\partial N_0(X), \mathbb{R})$*-bundle. The Poisson structure of* $\widetilde{\mathrm{S}}_r(X)$ *coincide with the one induced by the* $\mathrm{H}^1(\partial N_0(X), \mathbb{R})$*-bundle structure.*
If $h : X \longrightarrow Y$ *is an isomorphisms of cellular structure, the induced morphism* $h^* : \widetilde{\mathrm{S}}_r(Y) \longrightarrow \widetilde{\mathrm{S}}_r(X)$ *is compatible with all this structures, i.e. it is a Poisson affine bundle morphism which fiberwise coincide with the naturally induced group morphism* $h^* : \mathrm{H}^1(\partial N_0(Y), \mathbb{R}) \longrightarrow \mathrm{H}^1(\partial N_0(X), \mathbb{R})$*. Moreover* h^* *maps* $\mathrm{S}_r(Y)$ *to* $\mathrm{S}_r(X)$.

Definition 6.12 (Shaped $3-2$ Pachner moves). Let X be a shaped pseudo 3 manifold and let e be a balanced internal edge in it, shared exactly by three distinct tetrahedra t_1, t_2 and t_3 with dihedral angles at e exactly α_1, α_2 and α_3. Then the triangulated pseudo 3-manifold X_e obtained by removing the edge e, and substituting the three tetrahedra t_1, t_2 and t_3 with other two new tetrahedra t_4 and t_5 glued along one face, is topologically the same space as X. In order to have the same weights of X on X_e, the dihedral angles of t_4 and t_5 are uniquely determined by the ones of t_1, t_2 and t_3 as follows

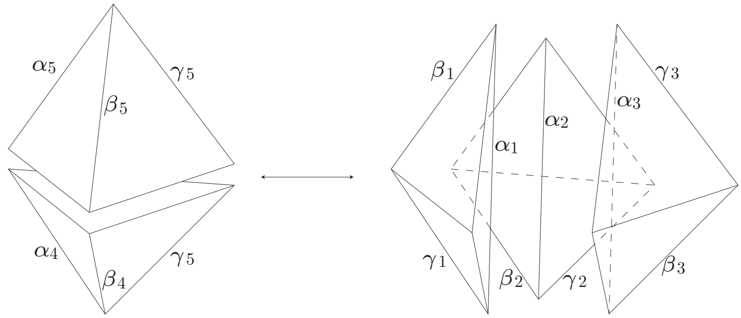


Figure 3: A 3–2 Pachner move.

$$\begin{array}{ll} \alpha_4 = \beta_2 + \gamma_1 & \alpha_5 = \beta_1 + \gamma_2 \\ \beta_4 = \beta_1 + \gamma_3 & \beta_5 = \beta_3 + \gamma_1 \\ \gamma_4 = \beta_3 + \gamma_2 & \gamma_5 = \beta_2 + \gamma_3. \end{array} \tag{6.1}$$

where $(\alpha_i, \beta_i, \gamma_i)$ are the dihedral angles of t_i. In this situation we say that X_e is obtained from X by a *shaped* $3-2$ *Pachner move.*

We remark that the linear system, together with e being balanced, guarantees the positivity of the dihedral angles of t_4 and t_5 provided the positivity for t_1, t_2 and t_3 but it does not provide any guarantees on the converse, i.e. the positivity of a shaped $2-3$ Pachner moves. However, two different solutions for the angles for t_1, t_2 and t_3 from the same starting angles for t_4 and t_5 are always gauge equivalent.
The system (6.1) define a map $P^e : \mathrm{S}(X) \longrightarrow \mathrm{S}(X_e)$, that extends to a map

$$\tilde{P}^e : \widetilde{\mathrm{S}}(X) \longrightarrow \widetilde{\mathrm{S}}(X_e).$$

For a balanced edge e, the latter restricts to the map

$$\tilde{P}_r : \tilde{\Omega}_{X,r}(e)^{-1}(2\pi) \longrightarrow \widetilde{\mathrm{S}}_r(X_e),$$

and it can be noticed that $\tilde{P}_r(\tilde{\Omega}_{X,r}(e)^{-1}(2\pi) \cap \mathrm{S}_r(x)) \subset \mathrm{S}_r(Y)$.

We also say that a leveled shaped pseudo 3-manifold (X, l_X) is obtained from (Y, l_Y) by a *leveled shaped* 3-2 *Pachner move* if, for some balanced $e \in \Delta_1(X)$, $Y = X_e$ as above and

$$l_Y = l_X + \frac{1}{12\pi} \sum_{a \in (\phi^{3,1})^{-1}(e)} \sum_{b \in \Delta_3^1(X)} \varepsilon_{p(a),p(b)} \alpha_X(b).$$

Definition 6.13. A (leveled) shaped pseudo 3-manifold X is called a *Pachner refinement* of a (leveled) shaped pseudo 3-manifold Y if there exists a finite sequence of (leveled) shaped pseudo 3-manifolds

$$X = X_1, X_2, \ldots, X_n = Y$$

such that for any $i \in \{1, \ldots, n-1\}$, X_{i+1} is obtained from X_i by a (leveled) shaped $3-2$ Pachner move. Two (leveled) shaped pseudo 3-manifolds X and Y are called *equivalent* if there exist gauge equivalent (leveled) shaped pseudo 3-manifolds X' and Y' which are respective Pachner refinements of X and Y.

For technical reasons, which we will discuss later, we will restrict the category of triangulated $2+1$ cobordisms, discussed so far, to a certain sub-categroid, as discussed below. This means that we will remove some morphisms as the following definition imposes.

Definition 6.14 (Admissibility)**.** An oriented triangulated pseudo 3-manifold is called *admissible* if

$$\mathrm{S}_r(X) \neq \emptyset,$$

and

$$\mathrm{H}_2(X - \Delta_0(X), \mathbb{Z}) = 0.$$

Definition 6.15. Two (leveled) admissible shaped pseudo 3-manifolds X and Y are said to be *admissibly equivalent* if there exists a gauge equivalence

$$h' : X' \longrightarrow Y'$$

of (leveled) shaped 3-manifolds X' and Y' which are respective Pachner refinements of X and Y such that $\Delta_1(X') = \Delta_1(X) \cup D_X$ and $\Delta_1(Y') = \Delta_1(Y) \cup D_Y$ and the following holds

$$\left[h(\mathrm{S}_r(X) \cap \tilde{\Omega}_{X',r}(D_X)^{-1}(2\pi))\right] \cap \left[\tilde{\Omega}_{Y',r}(D_Y)^{-1}(2\pi)\right] \neq \emptyset.$$

Theorem 6.16 ([AK1])**.** *Suppose two (leveled) shaped pseudo* 3*-manifolds X and Y are equivalent. Then there exist $D \subset \Delta_1(X)$ and $D' \subset \Delta_1(Y)$ and a bijection*

$$i : \Delta_1(X) - D \to \Delta_1(Y) - D'$$

and a Poisson isomorphism

$$R : \tilde{\Omega}_{X,r}(D)^{-1}(2\pi) \to \tilde{\Omega}_{Y,r}(D')^{-1}(2\pi),$$

which is covered by an affine $\mathbb{R}$-bundle isomorphism from $\widetilde{\mathrm{LS}}_r(X)|_{\tilde{\Omega}_{X,r}(D)^{-1}(2\pi)}$ to $\widetilde{\mathrm{LS}}_r(Y)_{\tilde{\Omega}_{X,r}(D')^{-1}(2\pi)}$ and such that we get the following commutative diagram

$$\begin{array}{ccc}
\tilde{\Omega}_{X,r}(D)^{-1}(2\pi) & \xrightarrow{R} & \tilde{\Omega}_{Y,r}(D')^{-1}(2\pi) \\
\Big\downarrow \mathrm{proj} \circ \tilde{\Omega}_{X,r} & & \Big\downarrow \mathrm{proj} \circ \tilde{\Omega}_{Y,r} \\
\mathbb{R}^{\Delta_1(X)-D} & \xrightarrow{i^*} & \mathbb{R}^{\Delta_1(Y)-D'}.
\end{array}$$

Moreover, if X and Y are admissible and admissibly equivalent, the isomorphism R takes an open convex subset U of $S_r(X) \cap \tilde{\Omega}_{X,r}(D)^{-1}(2\pi)$ onto an open convex subset U' of $S_r(Y) \cap \tilde{\Omega}_{Y,r}(D)^{-1}(2\pi)$.

We remark that in the previous notation $D = \Delta_1(X) \cap h^{-1}(D_Y)$ and $D' = \Delta_1(Y) \cap h(D_X)$.

For a tetrahedron $T = [v_0, v_1, v_2, v_3]$ in $\mathbb{R}^3$ with ordered vertices v_0, v_1, v_2, v_3, we define its sign

$$\mathrm{sign}(T) = \mathrm{sign}(\det(v_1 - v_0, v_2 - v_0, v_3 - v_0)),$$

as well as the signs of its faces

$$\operatorname{sign}(\partial_i T) = (-1)^i \operatorname{sign}(T), \text{ for } i \in \{0, \dots, 3\}.$$

For a pseudo 3-manifold X, the signs of faces of the tetrahedra of X induce a sign function on the faces of the boundary of X, $\operatorname{sign}_X : \Delta_2(\partial X) \to \{\pm 1\}$, which permits us to split the components of the boundary of X into two sets, $\partial X = \partial_+ X \cup \partial_- X$, where $\Delta_2(\partial_\pm X) = \operatorname{sign}_X^{-1}(\pm 1)$. Notice that $|\Delta_2(\partial_+ X)| = |\Delta_2(\partial_- X)|$.

Definition 6.17 (Cobordism Categroid)**.** The category $\mathcal{B}$ is the category that has triangulated surfaces as objects,equivalence classes of (leveled) shaped pseudo 3-manifolds X as morphisms (so that $X \in \operatorname{Hom}_{\mathcal{B}}(\partial_- X, \partial_+ X)$) and the composition given by glueing along boundary components, through edge orientation preserving and face orientation reversing CW-homeomorphisms.
The Categroid $\mathcal{B}_a$ is the subcategroid of $\mathcal{B}$ whose morphisms are restricted to be admissible equivalence classes of admissible (leveled) shaped pseudo 3-manifolds. In particular composition is possible only if the gluing gives an (leveled) admissible pseudo 3-manifold.

Remark 6.18. *Admissible Shaped Pseudo* 3*-Manifolds in the real world.*
Even though we will discuss the whole Andersen Kashaev construction of the Teichmüller TQFT functor in the general setting of the above defined cobordism categroid, the the main parts of this construction, we want to put our hands on in this paper, are invariants of links. We interpret Triangulated Pseudo 3-manifolds X as ideal triangulations of the (non closed) manifold $X \setminus \Delta_0(X)$. This interpretation is enlighten in Remark 6.3. We shall ask ourself when a cusped 3-manifold (cusped means non compact with finite volume here) admits a positive fully balanced shape structure. This requirement is weaker than asking for a full geometric structure on the manifold and in our language, this can be expressed by the fact that we did not required a precise gauge to be fixed. The problem of finding positive or generalized angle structures has been studied in [LT], where necessary and sufficient conditions for their existence are given. In the work [HRS] it is proved, among other things, that a particular class of manifolds M supporting positive shape structures are complements in S^3 of hyperbolic links. However the admissibility conditions kicks in here and further restrict us to just complements of hyperbolic knots. So, at the least, we know that the Andersen Kashaev construction will work on complements of hyperbolic knots, and that are the examples we will look a bit closer at below. Now we should clarify the equivalence relation in $\mathcal{B}_a$, in the context of knot complements. Combinatorially speaking, any two ideal triangulations of a knot complement are related by finite sequences of 3–2 or 2–3 Pachner moves. On the other hand it is not known (at least to the authors) if any such sequence can be realised as a sequence of shaped Pachner moves. For sure we know that 3–2 shaped Pachner moves are

well defined in the category $\mathcal{B}_a$ as we remarked when we defined them, and if a shaped Pachner 2–3 move is possible in some particular case, then it is an equivalence in the category $\mathcal{B}_a$. So the knot invariants that we will define starting from $\mathcal{B}_a$ are not guaranteed to be topological invariants. There is however another construction of the Andersen–Kashaev invariant [AK2], that avoid this problem with analytic continuation properties of the partition function. The equivalence of the two constructions is still conjectural though.
In [AK1] another way to define knot invariants is suggested, by taking one vertex Hamiltonian triangulations of knots, that is, one vertex triangulations of S^3 (or a general manifold M) where the knot is represented by a unique edge with a degenerating shape structure, meaning that we take a limit on the shapes, sending all the weights to be balanced except the weight of the knot that is sent to 0. The partition function is actually divergent but a residue can be computed as an invariant. We will show this in a couple of examples in subsection 6.4.

6.2 The target Categroid $\mathcal{D}_N$

W recall all the relevant things regarding tempered distributions and the space $\mathcal{S}(\mathbb{A}_N)$ in Appendix A. As always, here N is an odd positive integer and $\mathrm{b} \in \mathbb{C}$ is fixed to satisfy $\mathrm{Re}(\mathrm{b}) > 0$ and $\mathrm{Im}\,\mathrm{b}(1 - |\mathrm{b}|) = 0$.

Definition 6.19. The categroid $\mathcal{D}_N$ has as objects finite sets and for two finite sets n, m the set of morphisms from n to m is

$$\mathrm{Hom}_{\mathcal{D}_N}(n, m) = \mathcal{S}'(\mathbb{A}_N^{n\sqcup m}) \simeq \mathcal{S}'(\mathbb{R}^{n\sqcup m}) \otimes \mathcal{S}((\mathbb{Z}/N\mathbb{Z})^{n\sqcup m}).$$

Definition 6.20. For $\mathcal{A} \otimes A_N \in \mathrm{Hom}_{\mathcal{D}_N}(n, m)$ and $\mathcal{B} \otimes B_N \in \mathrm{Hom}_{\mathcal{D}_N}(m, l)$, such that $\mathcal{A}$ and $\mathcal{B}$ satisfy condition (A.3) and $\pi^*_{n,m}(\mathcal{A})\pi^*_{m,l}(\mathcal{B})$ continuously extends to $S(\mathbb{R}^{n\sqcup m\sqcup l})_m$, we define

$$(\mathcal{A} \otimes A_N) \circ (\mathcal{B} \otimes B_N) = (\pi_{n,l})_*(\pi^*_{n,m}(\mathcal{A})\pi^*_{m,l}(\mathcal{B})) \otimes A_N B_N \in \mathrm{Hom}_{\mathcal{D}_N}(n, l).$$

Where the product $A_N B_N$ is just the matrix product.
We will frequently use the following notation in what follows. For any $a \in \mathbb{A}_N$, $a = (x, n) \in \mathbb{R} \times \mathbb{Z}/N\mathbb{Z}$ we will consider the b–dependent operator

$$\varepsilon \equiv \varepsilon(\mathrm{b})\colon \mathbb{A}_N \to \mathbb{A}_N$$

defined by

$$\varepsilon(x, n) \equiv \begin{cases} (x, n) & \text{if } |\mathrm{b}| = 1, \\ (x, -n) & \text{if } \mathrm{b} \in \mathbb{R} \end{cases} \tag{6.2}$$

For any $\mathcal{A} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$, we have a unique adjoint $\mathcal{A}^* \in \mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}'(\mathbb{R}^n))$ defined by the formula

$$\mathcal{A}^*(f)(g) = \overline{\bar{f}(\mathcal{A}(\bar{g}))}$$

for all $f \in \mathcal{S}(\mathbb{R}^m)$ and all $g \in \mathcal{S}(\mathbb{R}^n)$.

Definition 6.21 ($\star_{\mathrm{b}}$ structure)**.** Consider $\mathrm{b} \in \mathbb{C}$ fixed as above and $N \in \mathbb{Z}_{>0}$ odd. Let $A_N \in \mathrm{Hom}(\mathcal{S}((\mathbb{Z}/N\mathbb{Z})^m), \mathcal{S}((\mathbb{Z}/N\mathbb{Z})^n)$. Recall the involution ε on $\mathbb{Z}/N\mathbb{Z}$ form equation (6.2). Define $A_N^{\star_{\mathrm{b}}}$ as

$$\langle j_1, \dots j_m | A_N^{\star_{\mathrm{b}}} | p_1, \dots, p_n \rangle = \overline{\langle \varepsilon p_1, \dots, \varepsilon p_n | A_N | \varepsilon j_1, \dots \varepsilon j_m \rangle} \tag{6.3}$$

We can finally define the $*_{\mathrm{b}}$ operator as

$$(\mathcal{A} \otimes A_N)^{*_{\mathrm{b}}} = \mathcal{A}^* \otimes A_N^{\star_{\mathrm{b}}} \tag{6.4}$$

6.3 Tetrahedral Partition Function

Recall the operators from Section 4.1, u_j, v_j, X_j, Y_j, p_j, q_j , $j = 1, 2$ acting on $\mathcal{H} := \mathcal{S}(\mathbb{A}_N^2)$. Define the *Charged Tetrahedral Operator* as follows

Definition 6.22. Let a, b, $c > 0$ such that $a+b+c = \frac{1}{\sqrt{N}}$. Recall the Tetrahedral operator T defined in (4.15). Define the charged tetrahedral operator $\mathsf{T}(a, c)$ as follows

$$\mathsf{T}(a, c) \equiv e^{-\pi i \frac{c_{\mathrm{b}}^2}{\sqrt{N}} \left(2(a-c) + \frac{1}{\sqrt{N}}\right)/6} e^{2\pi i c_{\mathrm{b}}(c\mathsf{q}_2 - a\mathsf{q}_1)} \mathsf{T}_{12} e^{-2\pi i c_{\mathrm{b}}(a\mathsf{p}_2 + c\mathsf{q}_2)} \tag{6.5}$$

Lemma 6.23. *We have that*

$$\mathsf{T}(a, c) = e^{-\pi i \frac{c_{\mathrm{b}}^2}{\sqrt{N}} \left(2(a-c) + \frac{1}{\sqrt{N}}\right)/6} e^{\pi i c_{\mathrm{b}}^2 a(a+c)} \mathsf{D}_{12} \psi_{a,c}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2, -e^{-\frac{\pi i}{N}} Y_1 X_2 \overline{Y_2}) \tag{6.6}$$

were $\psi_{a,c}(x, n)$ is the charged quantum dilogarithm from (3.28)

Extra Notation. Recall the notation for Fourier coefficients and Gaussian exponentials in $\mathbb{A}_N$. For $a = (x, n)$ and $a' = (y, m)$ in $\mathbb{A}_N$ we write

$$\langle a, a' \rangle \equiv e^{2\pi i x y} e^{-2\pi i n m/N} \qquad \langle a \rangle \equiv e^{\pi i x^2} e^{-\pi i n(n+N)/N}$$

For $a = (x, n) \in \mathbb{A}_N$, define $\delta(a) \equiv \delta(x)\delta(n)$ where $\delta(x)$ is Dirac's delta distribution while $\delta(n)$ is the Kronecker delta $\delta_{0,n}$ between 0 and n mod N. Define

$$\varphi_{a,c}(x, n) \equiv \psi_{a,c}(x, -n). \tag{6.7}$$

Denote, for $x, y \in \mathbb{R}$ and $z \in \mathbb{A}_N$

$$\nu(x) \equiv e^{-\pi i \frac{c_{\mathrm{b}}^2}{\sqrt{N}} \left(2x + \frac{1}{\sqrt{N}}\right)/6} \qquad \nu_{x,y} = \nu(x - y) e^{\pi i c_{\mathrm{b}}^2 x(x+y)} \tag{6.8}$$

The equations from Lemma 3.17 can be upgraded to

$$\nu_{a,c} \tilde{\varphi}_{a,c}(z) = \nu_{c,b} \varphi_{c,b}(z) \langle z \rangle e^{-\pi i N/12} \tag{6.9}$$

$$\nu_{a,c} \overline{\varphi_{a,c}}(z) = \nu_{c,a} \varphi_{c,a}(-\varepsilon z) \langle z \rangle e^{-\pi i N/6} \tag{6.10}$$

$$\nu_{a,c} \overline{\tilde{\varphi}_{a,c}}(z) = \nu_{b,c} \varphi_{b,c}(-\varepsilon z) e^{-\pi i N/12} \tag{6.11}$$

Where ε was defined in (6.2). From the Charged Pentagon Equation (3.33) we get the following

Proposition 6.24 (Charged Tetrahedral Pentagon equation)**.** *Let a_j, $c_j > 0$ such that $\frac{1}{\sqrt{N}} - a_j - c_j > 0$ for $j = 0, 1, 2, 3$ and 4, which further satisfies the following relations*

(6.12)
$$a_1 = a_0 + a_2 \qquad a_3 = a_2 + a_4 \qquad c_1 = c_0 + a_4 \qquad c_3 = a_0 + c_4 \qquad c_2 = c_1 + c_3.$$

Then we have that

$$\mathsf{T}_{12}(a_4, c_4)\mathsf{T}_{1,3}(a_2, c_2)\mathsf{T}_{23}(a_0, c_0) = \mu \mathsf{T}_{23}(a_1, c_1)\mathsf{T}_{12}(a_3, c_3) \tag{6.13}$$

where
$$\mu = \exp \pi i \frac{c_{\mathrm{b}}^2}{6\sqrt{N}} \left(2(c_0 + a_2 + c_4) - \frac{1}{\sqrt{N}}\right)$$

We have an integral kernel description for the charged tetrahedral operator. We use the Bra-Ket notation to denote integral kernels, see Appendix A.1.

Proposition 6.25. *Let $\overline{\mathsf{T}}(a, c) \equiv (\mathsf{T}(a, c))^{-1}$.*

$$\begin{aligned}
&\langle a_0,\, a_2|\, \mathsf{T}_{12}(a, c)\, |a_1,\, a_3\rangle \\
&\qquad = \nu(a-c)e^{\pi i c_{\mathrm{b}}^2 a(a+c)}\langle a_3 - a_2,\, a_0\rangle\overline{\langle a_3 - a_2\rangle}\delta(a_0 + a_2 - a_1)\tilde{\varphi}_{a,c}(a_3 - a_2) \\
&\langle a_0,\, a_2|\overline{\mathsf{T}}(a, c)|a_1,\, a_3\rangle \\
&\qquad = \nu(b-c)e^{\pi i c_{\mathrm{b}}^2 b(b+c)}e^{-\pi i N/12}\langle a_3 - a_2,\, a_1\rangle\langle a_3 - a_2\rangle\delta(a_1 + a_3 - a_0)\varphi_{b,c}(a_3 - a_2).
\end{aligned}$$

The appearance of ε is due to the non-unitarity of the theory for $\mathrm{b} > 0$ and $N > 1$.

Let A and B two operators on $\mathrm{L}^2(\mathbb{A}_N)$ defined as bra-ket distributions by

$$\langle a_1, a_2|\mathsf{A}\rangle = \delta(a_1 + a_2)\langle a_1\rangle e^{\pi i N/12} \qquad \langle a_1, a_2|\mathsf{B}\rangle = \langle a_1 - a_2\rangle \tag{6.14}$$
$$\langle\overline{\mathsf{A}}|a_1, a_2\rangle = \overline{\langle\varepsilon a_1, \varepsilon a_2|\mathsf{A}\rangle} \qquad \langle\overline{\mathsf{B}}|a_1, a_2\rangle = \overline{\langle\varepsilon a_1, \varepsilon a_2|\mathsf{B}\rangle}. \tag{6.15}$$

Lemma 6.26 (Fundamental Lemma)**.** *We have the following three relations*

$$\int_{\mathbb{A}_N^2} \langle\overline{\mathsf{A}}|v, s\rangle\langle x, s|\mathsf{T}(a, c)|u, t\rangle\langle t, y|\mathsf{A}\rangle \mathrm{d}s\mathrm{d}t = \langle x, y|\overline{\mathsf{T}}(a, b)\langle u, v\rangle \tag{6.16}$$

$$\int_{\mathbb{A}_N^2} \langle\overline{\mathsf{A}}|u, s\rangle\langle s, x|\mathsf{T}(a, c)|v, t\rangle\langle t, y|\mathsf{B}\rangle \mathrm{d}s\mathrm{d}t = \langle x, y|\overline{\mathsf{T}}(b, c)\langle u, v\rangle \tag{6.17}$$

$$\int_{\mathbb{A}_N^2} \langle\overline{\mathsf{B}}|u, s\rangle\langle s, y|\mathsf{T}(a, c)|t, v\rangle\langle t, x|\mathsf{B}\rangle \mathrm{d}s\mathrm{d}t = \langle x, y|\overline{\mathsf{T}}(a, b)\langle u, v\rangle. \tag{6.18}$$

6.3.1 TQFT Rules, Tetrahedral Symmetries and Gauge Invariance

We consider oriented surfaces with cellular structure such that all 2-cells are either bigons or triangles. Not all the edge orientations will be admitted. We forbid cyclically oriented triangles. For the bigons, we consider only the *essential* ones, the others being contractible to an edge. These essential bigons are precisely the ones with cellular structure isomorphic to the unit disk with vertices $\pm 1 \in \mathbb{C}$ and edges $\{e_1 = e^{\pi i t}; e_2 = -e^{\pi i t}, \text{for } t \in [0,1]\}$ or $\{e_1 = -e^{-\pi i t}; e_2 = e^{-\pi i t}, \text{for } t \in [0,1]\}$. Given such an ideally triangulated surface Σ we will associate a copy of $\mathbb{C}$ to any bigon and a copy of $\mathcal{S}'(\mathbb{A}_N)$ to any triangle. Globally we associate to the surface the space $\mathcal{S}'(\mathbb{A}_N^{\Delta_2(\Sigma)})$. To a shaped tetrahedron T with ordered vertices $\{v_0, v_1, v_2, v_3\}$ we associate the partition function $Z_{\mathrm{b}}^{(N)}(T)$ through the Nuclear Theorem (A.6) as a ket distribution

$$\langle x | \widetilde{Z_{\mathrm{b}}^{(N)}(T)} \rangle = \begin{cases} \langle a_0, a_2 | \mathsf{T}(c(v_0v_1), c(v_0v_3)) | a_1, a_3 \rangle & \text{if } \operatorname{sign}(T) = 1; \\ \langle a_1, a_3 | \overline{\mathsf{T}}(c(v_0v_1), c(v_0v_3)) | a_0, a_2 \rangle & \text{if } \operatorname{sign}(T) = -1. \end{cases} \tag{6.19}$$

where

$$\mathbb{A}_N \ni a_i := a(\partial_i T), \quad i \in \{0,1,2,3\}$$

and

$$c := \frac{1}{\pi\sqrt{N}} \alpha_T : \Delta_1(T) \to \mathbb{R}_{>0}.$$

Having allowed bigons in triangulations of surfaces, we must also allow cones over such as cobordisms. From the 2 classes of bigons described above we have 4 isotopy classes of cellular structures of cones over them, described in the following as embedded in $\mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$. The bigon is identified with the unit disc embedded in $\mathbb{C}$. The apex of the cone will be the point $(0,1) \in \mathbb{C} \times \mathbb{R}$. The 1-cells will be either

$$\{e^1_{0\pm}(t) = (\pm e^{i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm(1-t), t)\}$$

or

$$\{e^1_{0\pm}(t) = (\mp e^{-i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm(1-t), t)\}$$

or

$$\{e^1_{0\pm}(t) = (\pm e^{i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm t, 1-t)\}$$

or

$$\{e^1_{0\pm}(t) = (\mp e^{-i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm t, 1-t)\}.$$

We name these types of cones A_+, A_-, B_+ and B_- respectively. We need TQFT rules for the gluing of these cones. We just need to consider their gluing to a tetrahedra. We assign a partition function to the cones as follows

$$\langle a_1, a_2 | \widetilde{Z_{\mathrm{b}}^{(N)}(A_\pm)} \rangle = \delta(a_1 + a_2) \langle a_1 \rangle^{\pm 1} e^{\pm \pi i N/12}, \quad \langle a_1, a_2 | \widetilde{Z_{\mathrm{b}}^{(N)}(B_\pm)} \rangle = \langle a_1 - a_2 \rangle^{\pm 1}. \tag{6.20}$$

Tetrahedral symmetries are generated by permutation of the ordered vertices. Indeed the group of tetrahedral symmetries is identified with the symmetric group $\mathbb{S}_4$ and is generated by three transpositions. The three equations of the Fundamental Lemma 6.26 gain an interpretation as glueing of cones on the faces of a tetrahedron through definitions (6.20). These three glueing generates all the symmetries of a tetrahedron, and through this interpretation, the Fundamental Lemma assure that the partition function $Z_{\mathrm{b}}^{(N)}$ satisfies all the tetrahedral symmetries. For more details on tetrahedral symmetries and the cone's partition function see [AK1, GKT].

We can now formulate the main Theorem for the Teichmüller TQFT. This theorem was proved by Andersen and Kashaev for the case $N = 1$ in [AK1]. The statement that we have here is for every N odd, and it is strictly speaking not present as such in the literature.

Theorem 6.27 (Level N Teichmüller TQFT, Andersen and Kashaev)**.** *For any* $\mathrm{b} \in \mathbb{C}^*$ *such that* $\operatorname{Im} \mathrm{b}(|\mathrm{b}| - 1) = 0$ *and* $\operatorname{Re} \mathrm{b} > 0$*, and for any* $N \in \mathbb{Z}_{>0}$ *odd there exists a unique* $*_{\mathrm{b}}$*-functor* $F_{\mathrm{b}}^{(N)} \colon \mathcal{B}_a \to \mathcal{D}_N$ *such that* $F_{\mathrm{b}}^{(N)}(A) = \Delta_2(A)$, $\forall A \in \operatorname{Ob} \mathcal{B}_a$*, and for any admissible leveled shaped pseudo 3-manifold* (X, l_X)*, the associated morphism in* $\mathcal{D}_N$ *takes the form*

$$F_{\mathrm{b}}^{(N)}(X, l_X) = Z_{\mathrm{b}}^{(N)}(X) e^{-\pi i \frac{l_X c_{\mathrm{b}}^2}{N}} \in \mathcal{S}'\left(\mathbb{A}_N^{\Delta_2(\partial X)}\right), \tag{6.21}$$

where $Z_{\mathrm{b}}^{(N)}$ *is defined in* (6.19) *for a tetrahedron.*

Here $*_{\mathrm{b}}$-functor means that $F_{\mathrm{b}}^{(N)}(X^*) = F_{\mathrm{b}}^{(N)}(X)^{*_{\mathrm{b}}}$, where X^* is the oppositely oriented pseudo 3-manifold to X.

The discussion so far proves the theorem except for the gauge invariance and the convergence of the partition functions under glueings. We will not discuss the convergence here because it follows directly from the convergence in the case level $N = 1$, which was addressed in [AK1]. We just remark that the hypothesis of admissibility is used to prove the convergence of the partition function.
For the gauge invariance consider the suspension of an n-gone SP_n naturally triangulated into n tetrahedra sharing the only internal edge e. Every gauge transformation can be decomposed in a sequence of gauge transformations involving only one edge e, and every such gauge transformation can be understood in the example of the suspension. Suppose all the tetrahedra to be positive, and having vertex order such that the last two vertices are the endpoints of the internal common edge. After enumerating the tetrahedra in cyclic order, let a_i, c_i be the two shape parameter of T_i, $i = 0, \ldots, n$, and $\mathrm{a} = (a_0, \ldots, a_n)$, $\mathrm{c} = (c_0, \ldots, c_n)$. Notice that $\sqrt{N}\pi a_i$ is the dihedral angle corresponding to the edge e. So a gauge transformation corresponding to e will affect the partition function of SP_N

$$Z_{\mathrm{b}}^{(N)}(SP_N)(\mathrm{a}, \mathrm{c}) := \operatorname{Tr}_0(\mathsf{T}_{01}(a_1, c_1)\mathsf{T}_{02}(a_2, c_2) \cdots \mathsf{T}_{0n}(a_n, c_n))$$

by shifting c by an amount $\lambda = (\lambda, \dots, \lambda)$ say. One can show from the definitions and the discussion above, that

$$\mathsf{T}(a, c+\lambda) = e^{-2\pi i c_{\mathrm{b}} \lambda \mathsf{p}_1} \mathsf{T}(a, c) e^{2\pi i c_{\mathrm{b}} \lambda \mathsf{p}_1} e^{\pi i c_{\mathrm{b}}^2 \left(\frac{1}{\sqrt{N}} - 6a\right) \lambda/3}$$

which, after tracing, leads to the following

Proposition 6.28. *[AK1]*

$$Z_{\mathrm{b}}^{(N)}(SP_N)(\mathrm{a}, \mathrm{c}+\lambda) = Z_{\mathrm{b}}^{(N)}(SP_N)(\mathrm{a}, \mathrm{c}) e^{\pi i c_{\mathrm{b}}^2 \left(\frac{n}{\sqrt{N}} - 6Q_e\right)\lambda/3}$$

where

$$Q_e = a_1 + a_2 + \dots a_n$$

6.4 Knot Invariants: Computations and Conjectures

In this section we update the examples computed in [AK1] to the level $N \geq 1$ setting. Similar results were obtained in [D].
Notation. In the examples we are going to use the following notation for quantum dilogarithms

$$\varphi_{\mathrm{b}}(x, n) \equiv \mathrm{D}_{\mathrm{b}}(x, -n). \tag{6.22}$$

Moreover we will often abuse notation in favor of readability in the following ways. For $z = (x, n) \in \mathbb{A}_N$ we will sometimes write $e^{2\pi i c_{\mathrm{b}} z \alpha}$ in place of $e^{2\pi i c_{\mathrm{b}} x \alpha}$. Moreover sums of the form $z + c_{\mathrm{b}} a$ will always mean $(x + c_{\mathrm{b}} a, n)$.

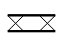
In the following examples we encode an oriented triangulated pseudo 3-manifold X into a diagram where a tetrahedron T is represented by an element

where the vertical segments, ordered from left to right, correspond to the faces $\partial_0 T, \partial_1 T, \partial_2 T, \partial_3 T$ respectively. When we glue tetrahedron along faces, we illustrate this by joining the corresponding vertical segments.

6.4.1 Figure–Eight Knot 4_1

Let X be represented by the diagram

(6.23)



Choosing an orientation, it consists of one positive tetrahedron T_+ and one negative tetrahedron T_- with four identifications

$$\partial_{2i+j} T_+ \simeq \partial_{2-2i+j} T_-, \quad i, j \in \{0, 1\}.$$

Combinatorially, we have $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{e_0, e_1\}$, $\Delta_2(X) = \{f_0, f_1, f_2, f_3\}$, and $\Delta_3(X) = \{T_+, T_-\}$ with the boundary maps

$$f_{2i+j} = \partial_{2i+j}T_+ = \partial_{2-2i+j}T_-, \quad i, j \in \{0,1\},$$

$$\partial_i f_j = \begin{cases} e_0, & \text{if } j - i \in \{0,1\}; \\ e_1, & \text{otherwise,} \end{cases}$$

$$\partial_i e_j = *, \quad i, j \in \{0,1\}.$$

The topological space $X \setminus \{*\}$ is homeomorphic to the complement of the figure–eight knot, and indeed $X \setminus \{*\}$ is an ideal triangulation of such a cuspidal manifold. The set $\Delta_{3,1}(X)$ consists of the elements $(T_\pm, e_{j,k})$ for $0 \le j < k \le 3$. We fix a shape structure

$$\alpha_X \colon \Delta_{3,1}(X) \to \mathbb{R}_{>0}$$

by the formulae

$$\alpha_X(T_\pm, e_{0,1}) = \pi\sqrt{N}a_\pm, \quad \alpha_X(T_\pm, e_{0,2}) = \pi\sqrt{N}b_\pm, \quad \alpha_X(T_\pm, e_{0,3}) = \pi\sqrt{N}c_\pm,$$

where $a_\pm + b_\pm + c_\pm = \frac{1}{\sqrt{N}}$. The weight function

$$\omega_X \colon \Delta_1(X) \to \mathbb{R}_{>0}$$

takes the values

$$\omega_X(e_0) = \sqrt{N}\pi(2a_+ + c_+ + 2b_- + c_-) =: 2\pi w, \quad \omega_X(e_1) = 2\pi(2 - w).$$

As the figure–eight knot is hyperbolic, the completely balanced case $w = 1$ is accessible directly. We can state the balancing condition $w = 1$ as

$$2b_+ + c_+ = 2b_- + c_-. \tag{6.24}$$

The kernel representations for the operators $\mathsf{T}(a_+, c_+)$ and $\mathsf{T}(a_-, c_-)$ are as follows. Let $z_j \in \mathbb{A}_N$, $j = 0, 1, 2, 3,$

$$\begin{aligned} &\langle z_0, z_2 | \mathsf{T}(a_+, c_+) | z_1, z_3 \rangle \\ &\qquad = \nu_{a_+,c_+} \langle z_3 - z_2, z_0 \rangle \overline{\langle z_3 - z_2 \rangle} \delta(z_0 + z_2 - z_1) \tilde{\varphi}_{a_+,c_+}(z_3 - z_2) \end{aligned} \tag{6.25}$$

$$\begin{aligned} &\langle z_3, z_1 | \overline{\mathsf{T}}(a_+, c_-) | z_2, z_0 \rangle = \overline{\langle \varepsilon z_2, \varepsilon z_0 | \mathsf{T}(a_+, c_-) | \varepsilon z_3, \varepsilon z_1 \rangle} \\ &\qquad = \overline{\nu_{a_-,c_-}} \langle z_0 - z_1, z_2 \rangle \langle z_1 - z_0 \rangle \delta(z_0 + z_2 - z_3) \overline{\tilde{\varphi}_{a_-,c_-}(\varepsilon z_1 - \varepsilon z_0)} \end{aligned} \tag{6.26}$$

The Andersen–Kashaev invariant at level N for the complement of the figure–eight knot is then

$$Z_{\mathrm{b}}^{(N)}(X) = \int_{\mathbb{A}_N^4} \langle z_0, z_2 | \mathsf{T}(a_+, c_+) | z_1, z_3 \rangle \langle z_3, z_1 | \overline{\mathsf{T}}(a_+, c_-) | z_2, z_0 \rangle \mathrm{d}z_0 \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3$$

$$
\begin{aligned}
&= \int_{\mathbb{A}_N^4} \nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\varphi_{c_+,b_+}(z_3-z_2)\overline{\varphi_{c_-,b_-}(\varepsilon z_1-\varepsilon z_0)}\delta(z_0+z_2-z_1)\times \\
&\quad \times \delta(z_0+z_2-z_3)\langle z_3-z_2, z_0\rangle\langle z_0-z_1, z_2\rangle \mathrm{d}z_0\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 \\
&= \int_{\mathbb{A}_N^3} \nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\varphi_{c_+,b_+}(z_1-z_2)\overline{\varphi_{c_-,b_-}(\varepsilon z_1-\varepsilon z_0)}\delta(z_0+z_2-z_1)\times \\
&\quad \times \langle z_1-z_2, z_0\rangle\langle z_0-z_1, z_2\rangle \mathrm{d}z_0\mathrm{d}z_1\mathrm{d}z_2 \\
&= \int_{\mathbb{A}_N^2} \nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\varphi_{c_+,b_+}(z_0)\overline{\varphi_{c_-,b_-}(\varepsilon z_2)}\langle z_0, z_0\rangle\langle -z_2, z_2\rangle \mathrm{d}z_0\mathrm{d}z_2 \\
&= \int_{\mathbb{A}_N} \nu_{c_+,b_+}\varphi_{c_+,b_+}(z_0)\langle z_0, z_0\rangle \mathrm{d}z_0 \overline{\int_{\mathbb{A}_N} \nu_{c_-,b_-}\varphi_{c_-,b_-}(\varepsilon z_2)\langle z_2, z_2\rangle \mathrm{d}z_2} \\
&= \sigma_{c_+,b_+}\overline{\sigma_{c_-,b_-}}
\end{aligned}
$$

We can compute

$$
\begin{aligned}
\sigma_{c_\pm,b_\pm} &= \nu_{c_\pm,b_\pm}\int_{\mathbb{A}_N} \frac{e^{-2\pi i c_\mathrm{b} z c_\pm}}{\varphi_\mathrm{b}(z-c_\mathrm{b}(b_\pm+c_\pm))}\langle z\rangle^2 \mathrm{d}z \\
&= \nu'_{c_\pm,b_\pm}\int_{\mathbb{A}_N+di} \frac{e^{4\pi i c_\mathrm{b} z(2b_\pm+c_\pm)}}{\varphi_\mathrm{b}(z)}\langle z\rangle^2 \mathrm{d}z
\end{aligned}
$$

where

$$
\nu'_{c_\pm,b_\pm} = \nu_{c_\pm,b_\pm} e^{4\pi i c_\mathrm{b}^2(c_\pm b_\pm - b_\pm^2)} \tag{6.27}
$$

and the domain of integration $\mathbb{A}_N + di = (\mathbb{R}+di)\times\mathbb{Z}/N\mathbb{Z})$. Note we have shifted the real integral to a contour integral in the complex plane, and $d\in\mathbb{R}$ is such that the integral converges absolutely. We sometimes omit the contour shift in the computations but we state it in the results. Defining

$$
\lambda \equiv 2b_+ + c_+ = 2b_- + c_-
$$

we have

$$
\begin{aligned}
Z_\mathrm{b}^{(N)}(X) &= \nu'_{c_+,b_+}\overline{\nu'_{c_-,b_-}}\int_{\mathbb{A}_N^2} \frac{e^{4\pi i c_\mathrm{b}\lambda(z_0+z_2)}}{\varphi_\mathrm{b}(z_0)\overline{\varphi_\mathrm{b}(\varepsilon z_2)}}\langle z_0\rangle^2\overline{\langle z_2\rangle^2}\mathrm{d}z_0\mathrm{d}z_2 \\
&= \nu'_{c_+,b_+}\overline{\nu'_{c_-,b_-}}\int_{\mathbb{A}_N^2} \frac{\varphi_\mathrm{b}(z_2)}{\varphi_\mathrm{b}(z_0)} e^{4\pi i c_\mathrm{b}\lambda(z_0+z_2)}\langle z_0\rangle^2\overline{\langle z_2\rangle^2}\mathrm{d}z_0\mathrm{d}z_2 \\
&= \nu'_{c_+,b_+}\overline{\nu'_{c_-,b_-}}\int_{\mathbb{A}_N^2} \frac{\varphi_\mathrm{b}(z_2-z_0)}{\varphi_\mathrm{b}(z_0)} e^{4\pi i c_\mathrm{b}\lambda z_2}\langle z_0, z_2\rangle^2\overline{\langle z_2\rangle^2}\mathrm{d}z_0\mathrm{d}z_2
\end{aligned}
$$

that has the structure

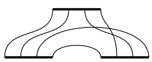
(6.28)
$$Z_{\mathrm{b}}^{(N)}(X) = e^{i\phi} \int_{\mathbb{A}_N + i0} \chi_{4_1}^{(N)}(x,\lambda) \mathrm{d}x,$$

(6.29)
$$\chi_{4_1}^{(N)}(x,\lambda) = \chi_{4_1}^{(N)}(x) e^{4\pi i c_{\mathrm{b}} \lambda x}, \qquad \chi_{4_1}^{(N)}(x) = \int_{\mathbb{A}_N - i0} \frac{\varphi_{\mathrm{b}}(x-y)}{\varphi_{\mathrm{b}}(y)} \langle x,y\rangle^2 \overline{\langle x\rangle^2} \mathrm{d}y$$

where ϕ is some constant quadratic combination of dihedral angles.

6.4.2 The Complement of the Knot 5_2

Let X be the closed S.O.T.P. 3-manifold represented by the diagram



This triangulation has only one vertex $*$ and $X \setminus \{*\}$ is topologically the complement of the knot 5_2. We denote T_1, T_2, T_3 the left, right, and top tetrahedra respectively. We choose the orientation so that all of them are positive. Balancing all the edges correspond to require the following equations to be true

$$2a_3 = a_1 + c_2, \quad b_3 = c_1 + b_2. \tag{6.30}$$

The three integral kernels reads

$$\begin{aligned}
&\langle z, w|\mathsf{T}(a_1, c_1)|u, x\rangle = \\
&\qquad = \nu_{a_1,c_1} \langle x-w, z\rangle \overline{\langle x-w\rangle} \delta(z+w-u) \tilde{\varphi}_{a_1,c_1}(x-w) \\
&\langle x, v|\mathsf{T}(a_2, c_2)|y, w\rangle = \\
&\qquad = \nu_{a_2,c_2} \langle w-v, x\rangle \overline{\langle w-v\rangle} \delta(x+v-y) \tilde{\varphi}_{a_2,c_2}(w-v) \\
&\langle y, u|\mathsf{T}(a_3, c_3)|v, z\rangle = \\
&\qquad = \nu_{a_3,c_3} \langle z-u, y\rangle \overline{\langle z-u\rangle} \delta(y+u-v) \tilde{\varphi}_{a_3,c_3}(z-u)
\end{aligned}$$

Carrying out the computations, defining $\lambda = -c_1 + b_2 - c_2 + a_3$, one gets that

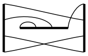
$$Z_{\mathrm{b}}^{(N)}(X) = \int_{\mathbb{A}_N + i0} \chi_{5_2}^{(N)}(x,\lambda) \mathrm{d}x, \qquad \chi_{5_2}^{(N)}(x,\lambda) = \chi_{5_2}^{(N)}(x) e^{2\pi i c_{\mathrm{b}} \lambda x} \tag{6.31}$$

$$\chi_{5_2}^{(N)}(x) = \int_{\mathbb{A}_N - i0} \frac{\overline{\langle x\rangle}\langle z\rangle}{\varphi_{\mathrm{b}}(z+x)\varphi_{\mathrm{b}}(z)\varphi_{\mathrm{b}}(z-x)} \mathrm{d}z \tag{6.32}$$

6.4.3 H-Triangulations

In this section we will look at one vertex H-triangulations of knots.

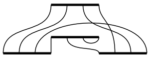
Let X be an H–Triangulation for the figure–eight knot, i.e. let X be given by the diagram



where the figure-eight knot is represented by the edge of the central tetrahedron connecting the maximal and the next to maximal vertices. If we choosing central tetrahedron (T_0) to be positive, the left tetrahedron (T_+) will be positive and the right one (T_-) negative.The shape structure, in the limit $a_0 \to 0$ satisfies $2b_+ + c_+ = 2b_- + c_- =: \lambda$ The partition function satisfies the following limit formula

$$\lim_{a_0 \to 0} \varphi_{\mathrm{b}}(c_{\mathrm{b}} a_0 - c_{\mathrm{b}}/\sqrt{N}) Z_{\mathrm{b}}^{(N)}(X) = \frac{e^{-\pi i N/12}}{\nu(c_0)} \chi_{4_1}^{(N)}(0) \tag{6.33}$$

Similarly let X be represented by the diagram



that is, the H–triangulation for the 5_2 knot. We denote T_0, T_1, T_2, T_3 the central, left, right, and top tetrahedra respectively and we choose the orientation so that the central tetrahedron T_0 is negative then all other tetrahedra are positive. The edge representing the knot 5_2 connects the last two edges of T_0, so that the weight on the knot is given by $2\pi a_0$. In the limit $a_0 \to 0$, all edges, except for the knot, become balanced under the conditions

$$a_1 = c_2 = a_3, \quad b_3 = c_1 + b_2,$$

which in particular imply (6.30). The partition function has the following expression

$$Z_{\mathrm{b}}^{(N)}(X) = \Theta \frac{e^{-\pi i N/12}}{\varphi_{\mathrm{b}}(c_{\mathrm{b}} a - c_{\mathrm{b}}\sqrt{N})} \chi_{5_2}^{(N)}(c_{\mathrm{b}}(a_1 - a_3)) \tag{6.34}$$

For some constant phase factor Θ.

6.4.4 Asymptotics of $\chi_{4_1}^{(N)}(0)$

In this section we want to study the asymptotic behavior of the invariant of the figure–8 knot

$$\begin{aligned}
\chi_{4_1}^{(N)}(0) &= \int_{\mathbb{A}_N} \mathrm{D}_{\mathrm{b}}(-x,-k)\overline{\mathrm{D}_{\mathrm{b}}(x,-k)}\mathrm{d}(x,k) \\
&= \frac{1}{2\pi \mathrm{b}\sqrt{N}} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R}-id} \mathrm{D}_{\mathrm{b}}\left(\frac{-x}{2\pi\mathrm{b}}, -k\right) \overline{\mathrm{D}_{\mathrm{b}}\left(\frac{x}{2\pi\mathrm{b}}, -k\right)} \mathrm{d}x
\end{aligned}$$

when $\mathrm{b} \to 0$. The analysis uses techniques similar to the one presented in [AK1] for $N = 1$, however higher level gives new informations that we will show here. The integration in the complex plane is a contour integral where $d > 0$ so that the integral is absolutely convergent. By means of the asymptotic formula for the quantum dilogarithm (3.22) we have that

$$\begin{aligned}\chi_{4_1}^{(N)}(0) =& \frac{1}{2\pi\mathrm{b}\sqrt{N}} \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R}-id} \mathrm{Exp}\left[\frac{\mathrm{Li}_2(-e^{-\sqrt{N}x}) - \mathrm{Li}_2(-e^{\sqrt{N}x})}{2\pi i\mathrm{b}^2 N}\right] \\ &\times \phi_{-x}(k)\overline{\phi_x(k)}(1+\mathcal{O}(\mathrm{b}^2))\mathrm{d}x\end{aligned}$$

We want to apply the steepest descent method to this integral to get an asymptotic formula for $\mathrm{b} \to 0$. First we show the computation for the exact integral,

$$\frac{1}{2\pi\mathrm{b}\sqrt{N}} \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R}-id} \mathrm{Exp}\left[\frac{\mathrm{Li}_2(-e^{-\sqrt{N}x}) - \mathrm{Li}_2(-e^{\sqrt{N}x})}{2\pi i\mathrm{b}^2 N}\right] \phi_{-x}(k)\overline{\phi_x(k)}\mathrm{d}x \tag{6.35}$$

and then we will argue that the former one can be approximated by the latter when $\mathrm{b} \to 0$.
Let $h(x) := \mathrm{Li}_2(-e^{-\sqrt{N}x}) - \mathrm{Li}_2(-e^{\sqrt{N}x})$. Its critical points are solutions to

$$\begin{cases} h'(x) = 0 \\ h''(x) \neq 0 \end{cases}$$

which are $\mathcal{S} = \left\{\pm\frac{2}{3}\frac{\pi i}{\sqrt{N}} + \frac{2\pi i k}{\sqrt{N}} : k \in \mathbb{Z}\right\}$. We compute the value of $\operatorname{Im} h$ at its critical points to be

$$\operatorname{Im} h\left(\pm\frac{2}{3}\frac{\pi i}{\sqrt{N}} + \frac{2\pi i k}{\sqrt{N}}\right) = \pm 4\Lambda(\frac{\pi}{6}) \tag{6.36}$$

where Λ is the Lobachevsky function

$$\Lambda(\alpha) = -\int_0^\alpha \log|2\sin\varphi|\,\mathrm{d}\varphi \tag{6.37}$$

and we refer the reader to [Kir] for the expressions that relate Lobachevsky function to the classical dilogarithm.

We only remark that $4\Lambda(\frac{\pi}{6}) = \mathrm{Vol}(4_1)$, where by $\mathrm{Vol}(4_1)$ we mean the hyperbolic volume of knot complement $S^3 \setminus (4_1)$.
Fix $\mathbb{C} \ni x_0 = -\frac{2}{3}\frac{\pi i}{\sqrt{N}}$, which is accessible from the original contour without passing through other critical points, and consider the contour

$$\mathcal{C} = \{z \in \mathbb{C} : \mathrm{Re}(h(z)) = \mathrm{Re}(h(x_0)),\ \mathrm{Im}(h(z)) \leq \mathrm{Im}(h(x_0))\}$$

which is asymptotic to $\mathrm{Re}(z)+\mathrm{Im}(z)=0$ for $\mathrm{Re}(z)\to\infty$ and to $\mathrm{Re}(z)-\mathrm{Im}(z)=0$ for $\mathrm{Re}(z)\to-\infty$. Moreover

$$\lim_{\mathrm{Re}(z)\to\pm\infty}\mathrm{Im}(h(z))=\lim_{\mathrm{Re}(z)\to\pm\infty}\pm\mathrm{Re}(z)\,\mathrm{Im}(z)=-\infty \tag{6.38}$$

All together we have found a contour $\mathcal{C}$ along which the integral (6.35) can be computed with the steepest descent method (see [Won]), giving as the following approximation for $\mathrm{b}\to 0$

$$e^{\frac{h(x_0)}{2\pi i \mathrm{b}^2 N}}\frac{g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right)}{\sqrt{iN^{-1}h''(x_0)}}(1+\mathcal{O}(\mathrm{b}^2) \tag{6.39}$$

where

$$g_{4_1}(x):=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\phi_{-x}(k)\overline{\phi}_x(k).$$

We now go back to $\chi_{4_1}^{(N)}(0)$, and we write it as the following integral

$$\chi_{4_1}^{(N)}(0)=\frac{1}{2\pi\mathrm{b}\sqrt{N}}\sum_{k\in\mathbb{Z}/N\mathbb{Z}}\int_{\mathbb{R}-id}f_{\mathrm{b}}(x,k)\mathrm{d}(x,k) \tag{6.40}$$

where

$$f_{\mathrm{b}}(x,k)=\mathrm{D}_{\mathrm{b}}\left(\frac{-x}{2\pi\mathrm{b}},-k\right)\overline{\mathrm{D}_{\mathrm{b}}\left(\frac{x}{2\pi\mathrm{b}},-k\right)}. \tag{6.41}$$

Then consider the contour

$$\mathcal{C}_{\mathrm{b}}=\{z\in\mathbb{C}:\arg f_{\mathrm{b}}(z)=\arg f_{\mathrm{b}}(z_{\mathrm{b}}),\ |f_{\mathrm{b}}(z)|=|f_{\mathrm{b}}(z_{\mathrm{b}})|\} \tag{6.42}$$

where z_{b} is defined as the solution to

$$\frac{\partial}{\partial x}\log f_{\mathrm{b}}(x)=0 \tag{6.43}$$

which minimize the absolute value of f_{b}. Using the asymptotic formula for f_{b} it is simple to show that the contours $\mathcal{C}_{\mathrm{b}}$ approximates $\mathcal{C}$ as $\mathrm{b}\to 0$ and that the points z_{b}'s will converge to x_0. So, in the limit $\mathrm{b}\to 0$, the integral (6.40) is approximated by the integral (6.35), for which we already have an asymptotic formula. We have proved the following

$$\chi_{4_1}^{(N)}(0)=e^{\frac{h(x_0)}{2\pi i \mathrm{b}^2 N}}\frac{g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right)}{\sqrt{iN^{-1}h''(x_0)}}(1+\mathcal{O}(\mathrm{b}^2)), \tag{6.44}$$

As we remarked above $\operatorname{Im} h(x_0) = -\operatorname{Vol}(4_1)$.
Next we look at the number $g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right)$ which is a topological invariant of the knot in the formula above. We have that

$$\begin{aligned}\sqrt{N} g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right) &= \sum_{k=1}^{N} \phi_{\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(k)\overline{\phi}_{-\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(k) \\ &= \left|\prod_{j=1}^{N-1}\left(1-e^{-\frac{\pi i}{3N}}e^{-\frac{2\pi i j}{N}}\right)^{\frac{j}{N}}\right| \sum_{k=0}^{N-1}\prod_{j=1}^{k}\frac{1}{\left|1-e^{\frac{1}{3}\frac{-\pi i}{N}}e^{\frac{2\pi i j}{N}}\right|^2}\end{aligned}$$

The last expression allows us to make the following remark

$$g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right) = \gamma_N \mathcal{H}^0_N(\overline{\rho_{comp}}) \tag{6.45}$$

where $H^0_N(\overline{\rho_{comp}})$ is the Baseilhac–Benedetti invariant for the figure–eight knot found in [BB], computed at the conjugate of the complete hyperbolic structure (meaning that the holonomies of the structure are all complex conjugated) and γ_N is a global rescaling given by

$$\gamma_N = \left|\prod_{j=1}^{N-1}\left(1-e^{-\frac{2\pi i j}{N}}\right)^{\frac{j}{N}}\right| \tag{6.46}$$

Remark 6.29. The very same steps of the previous asymptotic computation for $\chi^{(N)}_{4_1}(0)$ can be applied to $\chi^{(N)}_{5_2}(0)$ up to the point of having an expression

$$\chi^{(N)}_{5_2}(0) = e^{\frac{\phi(x_{5_2})}{2\pi i \mathrm{b}^2 N})}\frac{g_{5_1}(x_{5_2})}{\sqrt{iN^{-1}h''_{5_2}(x_{5_2})}}(1+\mathcal{O}(\mathrm{b}^2)), \tag{6.47}$$

where x_{5_2} is the only critical point in the complex plane that contributes to the steepest descent and

$$g_{5_1}(x) = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\overline{\phi_{-x}}(j)\phi_x(j)\overline{\phi_{-x}}(j). \tag{6.48}$$

The fact that $\operatorname{Im}\phi(x_{5_2}) = -\operatorname{Vol}(5_2)$, can be seen directly, see for example [AK1]. However this situation is already too complicated to allow us to check relations with other theories. The obvious guess is to look for the Baseilhac–Benedetti invariant, but no explicitly computed examples, other then 4_1, are known to the authors.

The following conjecture was originally stated in [AK1] for $N = 1$. Here we restate it in the updated setting.

Conjecture 6.30 ([AK1]). *Let M be a closed oriented compact 3-manifold. For any hyperbolic knot $K \subset M$, there exist a two parameters (b, N) family of smooth functions $J_{M,K}^{(\mathrm{b},N)}(x,j)$ on $\mathbb{R} \times \mathbb{Z}/N\mathbb{Z}$ which has the following properties.*

1. *For any fully balanced shaped ideal triangulation X of the complement of K in M, there exist a gauge invariant real linear combination of dihedral angles λ, a (gauge non-invariant) real quadratic polynomial of dihedral angles ϕ such that*
$$Z_{\mathrm{b}}^{(N)}(X) = e^{ic_{\mathrm{b}}^2\phi}\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\int_{\mathbb{R}} J_{M,K}^{(\mathrm{b},N)}(x,j)e^{ic_{\mathrm{b}}x\lambda}\mathrm{d}x$$

2. *For any one vertex shaped H-triangulation Y of the pair (M,K) there exists a real quadratic polynomial of dihedral angles φ such that*
$$\lim_{\omega_Y\to\tau} \mathrm{D}_{\mathrm{b}}\left(c_{\mathrm{b}}\frac{\omega_Y(K)-\pi}{\pi\sqrt{N}},0\right) Z_{\mathrm{b}}^{(N)}(Y) = e^{ic_{\mathrm{b}}^2\varphi - i\frac{\pi N}{12}} J_{M,K}^{(\mathrm{b},N)}(0,0),$$
where $\tau\colon \Delta_1(Y) \to \mathbb{R}$ takes the value 0 on the knot K and the value 2π on all other edges.

3. *The hyperbolic volume of the complement of K in M is recovered as the following limit*
$$\lim_{\mathrm{b}\to 0} 2\pi\mathrm{b}^2 N \log|J_{M,K}^{(\mathrm{b},N)}(0,0)| = -\operatorname{Vol}(M\setminus K)$$

Remark 6.31. We have proved this extended conjecture for the knots $(S^3, 4_1)$ and $(S^3, 5_2)$, see formulas (6.45), (6.47) and (6.48). Moreover we gave a more explicit expansion, showing the appearance of an extra interesting therm g_K, and a precise relation between g_{4_1} and a known invariant of hyperbolic knots, defined by Baseilhac–Benedetti in [BB], see equation (6.45). We could have been more bold and extend the conjecture declaring the appearance of $g_{(M,K)}$ to be general, and it to be proportional to the Baseilhac–Benedetti invariant. However we feel that there are not enough evidences to state it as general conjecture.

Appendices

A Tempered Distributions

For standard references for the topics of this appendix see e.g. [Hör2, Hör1] and [RS1, RS2].

Definition A.1. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the space of all the functions $\phi \in C^\infty(\mathbb{R}^n, \mathbb{C})$ such that

$$||\phi||_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| < \infty$$

for all multi-indeces α, β.
The space of Tempered Distributions $\mathcal{S}'(\mathbb{R}^n)$ is the space of linear functionals on $\mathcal{S}(\mathbb{R}^n)$ which are continuous with respect to all these semi-norms.

Both these spaces are stable under the action of the Fourier transform $\mathcal{F}$ and we use the notation $\hat{u} = \mathcal{F}(u)$. Let Z_n be the zero section of $T^*(\mathbb{R}^n)$.

Definition A.2. For a temperate distribution $u \in \mathcal{S}'(\mathbb{R}^n)$, we define its *Wave Front Set* to be the following subset of the cotangent bundle of $\mathbb{R}^n$

$$\mathrm{WF}(u) = \{(x, \xi) \in T^*(\mathbb{R}^n) - Z_{\mathbb{R}^n} |\ \xi \in \Sigma_x(u)\}$$

where

$$\Sigma_x(u) = \cap_{\phi \in C_x^\infty(\mathbb{R}^n)} \Sigma(\phi u).$$

Here

$$C_x^\infty(\mathbb{R}^n) = \{\phi \in C_0^\infty(\mathbb{R}^n) | \phi(x) \neq 0\}$$

and $\Sigma(v)$ are all $\eta \in \mathbb{R}^n - \{0\}$ having no conic neighborhood V such that

$$|\hat{v}(\xi)| \leq C_N(1 + |\xi|)^{-N},\ N \in \mathbb{Z}_{>0},\ \xi \in V.$$

Lemma A.3. *Suppose u is a bounded density on a C^∞ sub-manifold Y of $\mathbb{R}^n$, then $u \in \mathcal{S}'(\mathbb{R}^n)$ and*

$$\mathrm{WF}(u) = \{(x, \xi) \in T^*(\mathbb{R}^n) | x \in \operatorname{Supp} u,\ \xi \neq 0 \text{ and } \xi(T_xY) = 0\}.$$

In particular if $\operatorname{Supp} u = Y$, then we see that $\mathrm{WF}(u)$ is the co-normal bundle of Y.

Definition A.4. Let u and v be temperate distributions on $\mathbb{R}^n$. Then we define

$$\mathrm{WF}(u) \oplus \mathrm{WF}(v) = \{(x, \xi_1 + \xi_2) \in T^*(\mathbb{R}^n) | (x, \xi_1) \in \mathrm{WF}(u),\ (x, \xi_2) \in \mathrm{WF}(v)\}.$$

Theorem A.5. *Let u and v be temperate distributions on $\mathbb{R}^n$. If*

$$\mathrm{WF}(u) \oplus \mathrm{WF}(v) \cap Z_n = \emptyset,$$

then the product of u and v exists and $uv \in \mathcal{S}'(\mathbb{R}^n)$.

Definition A.6. We denote by $\mathcal{S}(\mathbb{R}^n)_m$ the set of all $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha(\phi)(x)| < \infty$$

for all multi-indices α and β such that if $\alpha_i = 0$ then $\beta_i = 0$ for $n - m < i \leq n$. We define $\mathcal{S}'(\mathbb{R}^n)_m$ to be the continuous dual of $\mathcal{S}(\mathbb{R}^n)_m$ with respect to these semi-norms.

We observe that if $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-m}$ is the projection onto the first $n-m$ coordinates, then $\pi^*(\mathcal{S}(\mathbb{R}^{n-m})) \subset \mathcal{S}(\mathbb{R}^n)_m$. This means we have a well defined push forward map

$$\pi_* : \mathcal{S}'(\mathbb{R}^n)_m \longrightarrow \mathcal{S}'(\mathbb{R}^{n-m}).$$

Proposition A.7. *Suppose Y is a linear subspace in $\mathbb{R}^n$, u a density on Y with exponential decay in all directions in Y. Suppose $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a projection for some $m < n$. Then $u \in \mathcal{S}'(\mathbb{R}^n)_m$ and $\pi_*(u)$ is a density on $\pi(Y)$ with exponential decay in all directions of the subspace $\pi(Y) \subset \mathbb{R}^m$.*

Tempered distributions can be thought of as functions of growth at most polynomial, thanks to the following

Theorem A.8. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$, then $T = \partial^\beta g$ for some polynomially bounded continuous function g and some multi-index β. That is, for $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$T(f) = \int_{\mathbb{R}^n} (-1)^{|\beta|} g(x)(\partial^\beta f)(x)\mathrm{d}x$$

In particular it is possible to show that $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n) \ni f \mapsto T_f \in \mathcal{S}'(\mathbb{R}^n)$ with $T_f(g) = \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x$.

Denoting by $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$ the space of continuous linear maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^m)$, we remark that we have an isomorphism

$$\tilde{\cdot} : \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m)) \to \mathcal{S}'(\mathbb{R}^{n\sqcup m}) \tag{A.1}$$

determined by the formula

$$\varphi(f)(g) = \tilde{\varphi}(f \otimes g) \tag{A.2}$$

for all $\varphi \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$, $f \in \mathcal{S}(\mathbb{R}^n)$, and $g \in \mathcal{S}(\mathbb{R}^m)$. This is the content of the Nuclear theorem, see e.g. [RS2]. Since we can not freely multiply distributions we end up with a categroid instead of a category. The partially defined composition in this categroid is defined as follows. Let n, m, l be three finite sets and $A \in \mathcal{S}'(\mathbb{R}^{n\sqcup m})$ and $B \in \mathcal{S}'(\mathbb{R}^{m\sqcup l})$. We have pull back maps

$$\pi^*_{n,m} : \mathcal{S}'(\mathbb{R}^{n\sqcup m}) \to \mathcal{S}'(\mathbb{R}^{n\sqcup m\sqcup l}) \text{ and } \pi^*_{m,l} : \mathcal{S}'(\mathbb{R}^{m\sqcup l}) \to \mathcal{S}'(\mathbb{R}^{n\sqcup m\sqcup l}).$$

By what we summarised above, the product

$$\pi^*_{n,m}(A)\pi^*_{m,l}(B) \in \mathcal{S}'(\mathbb{R}^{n\sqcup m\sqcup l})$$

is well defined provided the wave front sets of $\pi^*_{n,m}(A)$ and $\pi^*_{m,l}(B)$ satisfy the condition

$$(\mathrm{WF}(\pi^*_{n,m}(A)) \oplus \mathrm{WF}(\pi^*_{m,l}(B))) \cap Z_{n\sqcup m\sqcup l} = \varnothing \tag{A.3}$$

If we now further assume that $\pi^*_{n,m}(A)\pi^*_{m,l}(B)$ continuously extends to $\mathcal{S}(\mathbb{R}^{n\sqcup m\sqcup l})_m$, then we obtain a well defined element

$$(\pi_{n,l})_*(\pi^*_{n,m}(A)\pi^*_{m,l}(B)) \in \mathcal{S}'(\mathbb{R}^{n\sqcup l}).$$

A.1 Bra-Ket Notation

We often use the Bra-Ket notation to make computations with distributions. For $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ a density and $x \in \mathbb{R}^n$ we will write

$$\langle x|\varphi\rangle := \varphi(x),$$

with distributional meaning

$$\varphi(f) = \int_{\mathbb{R}^n} \langle x|\varphi\rangle f(x)\mathrm{d}x = \int_{\mathbb{R}^n} \varphi(x)f(x)\mathrm{d}x.$$

In particular if $\varphi \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, then

$$\langle x|\varphi\rangle = \varphi(x) = \delta_x(\varphi)$$

The integral kernel of the operator T, if it exists, is a distribution k_{T} such that

$$\mathsf{T}(\psi)(x) = \int_{\mathbb{R}^n} k_{\mathsf{T}}(x,y)\psi(y)\mathrm{d}y \tag{A.4}$$

Working with Schwartz functions, the nuclear theorem expressed by formula (A.2) guarantees that the kernel k_T exists and that it is a tempered distribution. We will usually write the kernel from equation (A.4), in Bra-Ket notation as follows

$$\mathsf{T}(\psi)(x) = \int_{\mathbb{R}^n} \langle x|\mathsf{T}|y\rangle\psi(y)\mathrm{d}y \tag{A.5}$$

and the nuclear theorem morphism (A.2) can be read as

$$\langle x|\mathsf{T}|y\rangle = \langle x,\, y|\widetilde{T}\rangle. \tag{A.6}$$

A.2 $\mathrm{L}^2(\mathbb{A}_N)$ and $\mathcal{S}(\mathbb{A}_N)$

$\mathbb{A}_N \equiv \mathbb{R} \times (\mathbb{Z}/N\mathbb{Z})$ has the structure of a locally compact abelian group, with the normalized Haar measure $\mathrm{d}(x,n)$ defined by

$$\int_{\mathbb{A}_N} f(x,n)\mathrm{d}(x,n) := \frac{1}{\sqrt{N}} \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R}} f(x,n)\mathrm{d}x$$

where $f : \mathbb{A}_N \longrightarrow \mathbb{C}$ is an integrable function. By definition $\mathrm{L}^2(\mathbb{A}_N)$ is the space of functions $\mathsf{f}\colon \mathbb{A}_N \longrightarrow \mathbb{C}$ such that

$$\int_{\mathbb{A}_N} |\mathsf{f}(a)|^2 \,\mathrm{d}a \equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \int_{\mathbb{R}} |\mathsf{f}(x,n)|^2 \,\mathrm{d}x < \infty \tag{A.7}$$

with standard inner product

$$\langle \mathsf{f}, \mathsf{g} \rangle \equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \int_{\mathbb{R}} \mathsf{f}(x,n)\overline{\mathsf{g}(x,n)} \mathrm{d}(x,n) \tag{A.8}$$

Finite square integrable sequences are just a finite dimensional vector space

$$\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N,$$

with a preferred basis given by mod N Kronecker delta functions

$$\delta_j(n) \equiv \begin{cases} 1 & \text{if } j = n \bmod N \\ 0 & \text{otherwise} \end{cases} \tag{A.9}$$

There is a natural isomorphism

$$\mathrm{L}^2(\mathbb{R}) \otimes \mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathrm{L}^2(\mathbb{A}_N) \tag{A.10}$$

defined by

$$f \otimes \delta_j\,(a) = f(x)\delta_j(n), \text{ for } a = (x,n) \in \mathbb{A}_N \tag{A.11}$$

with inverse

$$\mathbb{A}_N \ni \mathsf{f} \mapsto \sum_{j=0}^{N-1} \mathsf{f}(\cdot, j) \otimes \delta_j \in \mathrm{L}^2(\mathbb{R}) \otimes \mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \tag{A.12}$$

Everything just said holds true substituting L^2 with $\mathcal{S}$, with the isomorphism $\mathcal{S}(\mathbb{A}_N) \simeq \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^N$ and further also, the space of tempered distributions on $\mathbb{A}_N$, defined as linear continuous functionals over $\mathcal{S}(\mathbb{A}_N)$, are simply $\mathcal{S}'(\mathbb{R}) \otimes \mathbb{C}^N$. All the Bra-Ket notation extends trivially to $\mathcal{S}(\mathbb{A}_N)$, including the nuclear theorem (A.6), substituting all the integrals over $\mathbb{R}$ with integrals over $\mathbb{A}_N$.
We use a bracket notation for Fourier coefficients and Gaussian exponentials in $\mathbb{A}_N$, following the notation introduced in [AK3]

$$\langle (x,n),\,(y,m) \rangle \equiv e^{2\pi i x y} e^{-2\pi i n m/N} \qquad \langle (x,n) \rangle \equiv e^{\pi i x^2} e^{-\pi i n(n+N)/N} \tag{A.13}$$

For (x,n) and (y,m) in $\mathbb{A}_N$. The Fourier transform then takes the form

$$\mathcal{F}(f)(x,n) = \int_{\mathbb{A}_N} f(y,m) \langle (x,n), (y,m) \rangle \mathrm{d}(y,m).$$

For any operator A of order N, we can define the operator $\mathrm{L}_N(A)$ via the spectral theorem, such that it formally satisfies

$$A = e^{2\pi i\, \mathrm{L}_N(A)/N}.$$

We can define, for any function $f : \mathbb{A}_N \longrightarrow \mathbb{C}$ the operator function $f\!\!\!/(\mathsf{x}, A) \equiv f(\mathsf{x}, \mathrm{L}_N(A))$ for any commuting pair of operators x and A, where the former is self adjoint and the latter is of order N. We have, for x and A as above, that

$$f\!\!\!/(\mathsf{x}, A) = \int_{\mathbb{A}_N} \tilde{f}(y,m) e^{2\pi i y \mathsf{x}} A^{-m} \mathrm{d}(y,m) \tag{A.14}$$

where we use the following notation for the inverse Fourier transforms

$$\tilde{f}(x,n) = \int_{\mathbb{A}_N} f(y,m) \overline{\langle (y,m);(x,n)\rangle} \mathrm{d}(y,m). \tag{A.15}$$

B Categroids

We need a notion which is slightly more general than categories to define the Teichmüller TQFT functor.

Definition B.1. [AK1]
A *Categroid* $\mathcal{C}$ consist of a family of objects $\mathrm{Obj}(\mathcal{C})$ and for any pair of objects A, B from $\mathrm{Obj}(\mathcal{C})$ a set $\mathrm{Mor}_{\mathcal{C}}(A, B)$ such that the following holds

A For any three objects A, B, C there is a subset $K^{\mathcal{C}}_{A,B,C} \subset \mathrm{Mor}_{\mathcal{C}}(A,B) \times \mathrm{Mor}_{\mathcal{C}}(B,C)$, called the composable morphisms and a *composition* map

$$\circ : K^{\mathcal{C}}_{A,B,C} \to \mathrm{Mor}_{\mathcal{C}}(A, C).$$

such that composition of composable morphisms is associative.

B For any object A we have an identity morphism $1_A \in \mathrm{Mor}_{\mathcal{C}}(A, A)$ which is composable with any morphism $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ or $g \in \mathrm{Mor}_{\mathcal{C}}(B, A)$ and we have the equations

$$1_A \circ f = f, \text{ and } g \circ 1_A = g.$$

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Jørgen Ellegaard Andersen and Simone Marzioni,
Center for Quantum Geometry of Moduli Spaces,
Department of Mathematics,
University of Aarhus,
DK-8000, Denmark

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Construction of Modular Functors from Modular Tensor Categories

by Jørgen Ellegaard Andersen and William Elbæk Petersen[1]

Abstract

In this paper we follow the constructions of Turaev's book, "Quantum invariants of knots and 3-manifolds" closely, but with small modifications, to construct a modular functor, in the sense of Kevin Walker, from any modular tensor category. We further show that this modular functor has duality and if the modular tensor category is unitary, then the resulting modular functor is also unitary. We further introduce the notion of a fundamental symplectic character for a modular tensor category. In the cases where such a character exists we show that compatibilities between the structures in a modular functor can be made strict in a certain sense. Finally we establish that the modular tensor categories which arise from quantum groups of simple Lie algebras all have natural fundamental symplectic characters.

1 Introduction

The axioms for a *Topological Quantum Field Theory* (TQFT) was proposed by Atiyah, Segal and Witten in the late 80'ties and further Witten proposed in his seminal paper [32], that quantum Chern-Simons gauge theory should provide examples of TQFT's. This was shortly thereafter demonstrated by Reshetikhin and Turaev in the fundamental papers [24, 25], where they used the representation theory of quantum groups to give a construction of these TQFT's for $U_q(sl(2,\mathbb{C}))$ at a root of unity q, which is now known as the *Witten-Reshetikhin-Turaev* TQFT. This work further identified the needed categorical setup to construct a TQFT, which Turaev presented in his beautiful book [29], namely the notion of a *Modular Tensor Category*. Following these main events it was then shown that all quantum groups of simple Lie algebras at roots of unity gives examples of modular tensor categories (see e.g. the extensive reference list in the second edition of [29]). More topological constructions of these TQFT's, first for the $U_q(sl(2,\mathbb{C}))$-case were given in [17, 18] and then for the $U_q(sl(n,\mathbb{C}))$-case, the corresponding modular

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tensor category and its associated TQFT was given a topological construction in [16].

It was further conjectured by Witten in [32] that these TQFT's could also be constructed by *Conformal Field Theory* techniques. At the time there was a large body of work done on conformal field theory on the physics side and on the mathematical side as well, where we would like to highlight the works of Segal [26] and of Tsuchiya, Ueno and Yamada [27]. As it was later shown in [11, 12], the TUY construction of conformal field theory naturally leads to the construction of a *Modular Functor* in the sense of Kevin Walker [31]. It is well known that a modular functor in the sense of Walker also gives rise to a TQFT and that the TQFT is uniquely determined by the underlying modular functor (see e.g. [20]). The construction of a modular functor from a modular tensor category was provided by Turaev in his book [29]. He works however with similar, but not the same axioms for a modular functor, as Walker does. It is well-known to a broad range of researchers in the TQFT community that one can easily adapt the constructions in Turaev's book so as to provide the construction of a modular functor in the sense of Walker. Indeed, in this paper, we follow the construction from Turaev's book [29] very closely, and provide all details for how any modular tensor category $\mathcal{V}$ gives rise to a modular functor $Z_{\mathcal{V}}$ subject to the axioms formulated by Kevin Walker in [31] and used in [12, 13, 14]. The importance of the isomorphism provided in [11, 12, 13, 14] between the Witten-Reshetikhin-Turaev TQFT and the one coming from conformal field theory [27, 12] is that it provides a geometric construction of the WRT-TQFT's. When one further combines this isomorphism with Laszlo's isomorphism [23] between the covariant constant sections of the TUY connections with that of the Hitchin connection in the context of the geometric quantization of the moduli spaces of flat connections [21, 15, 19, 6, 7], one gets the full picture conjectured by Witten in [32]. This chain of isomorphisms has already been exploited by the first author of this paper, in part with collaborators, to obtain deep results about the asymptotics in terms of the above mentioned root of unity [1, 2, 3, 4, 5, 6, 8, 9, 10].

Let us now briefly review how one adjusts Turaev's construction of a modular functor so as to obtain one in the sense of Walker. A labeled marked surface is a closed oriented surface $\mathbf{\Sigma}$ endowed with a finite set of distinguished points equipped with a direction as well as a label from a finite set Λ. Moreover $\mathbf{\Sigma}$ is equipped with a Lagrangian subspace of its first homology group. A modular functor associates to any labeled marked surface $\mathbf{\Sigma}$ a module called its module of states. See section 4 below where we spell out Walker's axioms for a modular functor in all details.

Given a modular tensor category $(\mathcal{V}, (V_i)_{i\in I})$, Turaev constructs a 2-DMF in [29]. Taking the label set to be $\Lambda = I$, we simply use the modular functor provided by Turaev, and provide natural identifications between certain modules of states to make up for the differences between Turaev's axioms for a 2-DMF and Walker's axioms for a 2-DMF. To do this one needs to fix isomorphisms

$$q_i : V_{i^*} \to (V_i)^* \, . \tag{1.1}$$

Existence of such a family of isomorphisms is guaranteed by the axioms of a modular tensor category as given in [29], but a specified choice is not part of these axioms. Indeed, some of the interesting results in this paper concerning duality are obtained by exploiting that these can be scaled. We first obtain the following result (Theorem 5.8 and 9.2).

Theorem 1.1. *For any choice of the isomorphisms (1.1) we get a modular functor $Z_{\mathcal{V}}$ satisfying Walker's axioms. For any two choices of the isomorphisms (1.1) we get quasi-isomorphic modular functors.*

Here quasi-isomorphism refers to a notation which is exactly like isomorphism of modular functors, except that it allows for scalings of the glueing isomorphisms in a label dependent way, see Definition 9.1. Hence we see that there is a unique quasi-isomorphism class of modular functors associated to every modular tensor category. Two sets of isomorphisms $q_i^{(j)} : V_{i^*} \to (V_i)^*$, $j = 1, 2$ give rise to two strictly isomorphic modular functors if the unique $u_i \in K^*$ determined by $q_i^{(2)} = u_i q_i^{(1)}$ satisfies that $u_{i^*} = u_i$.

Remark 1.2. We stress that for the rest of this paper, the term modular functor generally refers to Walker's axioms.

Theorem 1.3. *For any choice of the isomorphisms (1.1) we get a duality structure on the modular functor $Z_{\mathcal{V}}$. If the modular tensor category is unitary, then we also get a unitary structure compatible with the rest of the structure of the modular functor.*

This is the content of Theorem 11.4 and Theorem 13.1 below. We emphasize that we do not need to choose the same q_i for the glueing maps and for the duality, as discussed in section 14. Further, in the compatibility between glueing and duality, duality with itself and duality with the unitary structure, there are projective factors allowed, as detailed in the Definition 11.1 and Definition 12.1.

The existence of these projective factors in the compatibility between these structures naturally raises the question if one can actually normalise all quantities such that these projective factors disappear. Let us now address this question.

First we establish, that we can normalise the duality pairing and the unitary pairing, such that both are strictly compatible with glueing. This is done in section 14. From this scaling analysis, one sees that the scaling can be separated into a product of two factors, one which only depends on the genus of the surface (see Definition 14.1) and one, which is simply a product of contributions from each of the labels (see equation (14.12)). This provides us with what we call the *canonical symplectic scaling*, where (15.1) in Theorem 15.3 relate the two scalings of the isomorphisms (1.1), which has the effect that the quantum invariant of the flat unknot labeled by i becomes $\dim(V_i)$ (see equation (15.3), which is the corresponding normalisation for the unknot with one negative twist). The multiplicative factor in the compatibility of duality with duality and unitary pairing with duality becomes in this case negative one raised to the number of *symplectic self-dual labels* of a given labeled marked surface (see Definition 15.1 and 15.2 and Theorem 15.3 and 15.5).

In order to analyse if we can find a normalisation such that all projective factors in the compatibility between glueing and duality, duality with itself and duality with unitarity can be made unity, which we call *strict compatibility*, we introduce the *dual fundamental group* $\Pi(\mathcal{V}, I)^*$ of a modular tensor category.

Definition 1.4 ($\Pi(\mathcal{V}, I)^*$). Let $\Pi(\mathcal{V}, I)^*$ consist of the set of functions

$$\tilde{\mu} : I \to K^*$$

that satisfies

$$\tilde{\mu}(i)\tilde{\mu}(i^*) = 1,$$

and such that

$$\tilde{\mu}(i)\tilde{\mu}(j)\tilde{\mu}(k) \neq 1,$$

implies

$$\mathrm{Hom}(\mathbf{1}, V_i \otimes V_j \otimes V_k) = \mathbf{0}.$$

We call it the dual of the fundamental group due to its similarity with the dual of the fundamental group of a simple Lie algebra as spelled out in section 18.

We make the following definition.

Definition 1.5. An element $\tilde{\mu} \in \Pi(\mathcal{V}, I)^*$ with the property that $\tilde{\mu}$ takes on the values ± 1 on the self-dual simple objects, in such way that $\tilde{\mu}$ is -1 on the symplectic simple objects and 1 on the rest of the self-dual simple objects, is called a *fundamental symplectic character*.

We observe that if $\mathcal{V}$ has no symplectic simple objects, then the identity in $\Pi(\mathcal{V}, I)^*$ is a fundamental symplectic character. We ask the question if any modular tensor category has such a fundamental symplectic character.

Theorem 1.6. *If $\mathcal{V}$ has a fundamental symplectic character, then we can arrange that glueing and duality, duality with itself and duality with the unitary paring are strictly compatible.*

This is proven in section 16. In section 17 we provide a fundamental symplectic character for the quantum $\mathrm{SU}(N)$ modular tensor category $H_k^{\mathrm{SU(N)}}$ at the root of unity $q = e^{2\pi i/(k+N)}$ first constructed by Reshetikhin and Turaev for $N = 2$ [24, 25] and by Turaev and Wenzl for general N [28, 30]. See also [17, 18] for a skein theory model of the $N = 2$ case and [16] for the general N. In section 18, we provide a fundamental symplectic character for any modular tensor category associated to the quantum group at a root of unity for any simple Lie algebra. Hence we have established

Theorem 1.7. *Any quantum group at a root of unity gives a modular functor such that glueing and duality, duality with itself and duality with the unitary pairing are strictly compatible.*

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2 Axioms for a modular tensor category

For the axioms of a modular tensor category $(\mathcal{V}, (V_i)_{i\in I})$ we refer to chapter II in [29]. For any modular tensor category, we have an induced involution $^* : I \to I$, determined by

$$(V_i)^* \cong V_{i^*}.$$

Recall that the ground ring is $K = \mathrm{End}(\mathbf{1})$ in the notation of [29]. For an object V we have the important K-linear trace operation $\mathrm{tr} : \mathrm{End}(V) \to K$. We have the following definition $\dim(V) := \mathrm{tr}(\mathrm{id}_V)$ and one gets the following identities for all objects V

$$\dim(V) = \dim(V^*),$$

We simply write $\dim(V_i) = \dim(i)$ and so for all indices $i \in I$

$$\dim(i) = \dim(i^*).$$

3 Labeled marked surfaces, extended surfaces and marked surfaces

3.1 Λ-Labeled marked surfaces

Let Λ be a finite set equipped with an involution $^\dagger : \Lambda \to \Lambda$ and a preferred element $0 \in \Lambda$ with $0^\dagger = 0$. We start by recalling that for a closed connected surface Σ, Poincare duality induces a non-degenerate skewsymmetric pairing

$$(\,\cdot\,,\,\cdot\,) : H_1(\Sigma, \mathbb{Z}) \times H_1(\Sigma, \mathbb{Z}) \longrightarrow \mathbb{Z},$$

called the intersection pairing. For the rest of this paper, $H_1(\Sigma)$ will mean the first integral homology group. We remark that we could just as well have considered $H_1(\Sigma, \mathbb{R})$. For any real vector space W, let $P(W) := (W \setminus \{0\})/\mathbb{R}_+$. We now define the objects of the category of Λ-labeled marked surfaces.

Definition 3.1 (Λ-marked surfaces)**.** A Λ-marked surface is given by the following data: $(\Sigma, P, V, \lambda, L)$. Here Σ is a smooth oriented closed surface. P is a finite subset of Σ. We call the elements of P distinguished points of Σ. V assigns to any p in P an element $v(p) \in P(T_p\Sigma)$. We say that $v(p)$ is the direction at p. λ is an assignment of labels from Λ to the points in P, e.g. it is a map $P \to \Lambda$. We say that $\lambda(p)$ is the label of p. Assume Σ splits into connected components $\{\Sigma_\alpha\}$. L is a Lagrangian subspace of $H_1(\Sigma)$ such that the natural splitting $H_1(\Sigma) \simeq \oplus_\alpha H_1(\Sigma_\alpha)$ induces a splitting $L \simeq \oplus_\alpha L_\alpha$ where $L_\alpha \subset H_1(\Sigma_\alpha)$ is a Lagrangian subspace for each α. By convention the empty set $\emptyset$ is regarded as a Λ-labeled marked surface.

For the sake of brevity, we will refer to a Λ-labeled marked surface as a labeled marked surface, whenever there is no risk of ambiguities. Now we describe the morphisms of this category.

Definition 3.2 (Morphisms)**.** Let $\mathbf{\Sigma}_i, i = 1, 2$ be two (non-empty) Λ-labeled marked surfaces. For $i = 1, 2$, write $\mathbf{\Sigma}_i = (\Sigma_i, P_i, V_i, \lambda_i, L_i)$. A morphism is a pair $\mathbf{f} = (f, s)$, where s is an integer, and f is an equivalence class of orientation preserving diffeomorphisms $\phi : \Sigma_1 \xrightarrow{\sim} \Sigma_2$ that restricts to a bijection of distinguished points $P_1 \xrightarrow{\sim} P_2$ that preserves directions and labels. Two such diffeomorphisms ϕ, ψ are said to be equivalent if they are related by an isotopy of such diffeomorphisms.

For a diffeomorphism such as ϕ, we will write $[\phi]$ for the equivalence class described above. Thus we will sometimes denote a morphism by $([f], s)$ if we want to stress that we are dealing with a pair where the isotopy class is the equivalence class of the diffeomorphism f. Let σ be Wall's signature cocycle for triples of Lagrangian subspaces. We now define composition.

Definition 3.3 (Composition)**.** Assume that we are given two composable morphisms $\mathbf{f}_1 = (f_1, s_1) : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ and $\mathbf{f}_2 = (f_2, s_2) : \mathbf{\Sigma}_2 \to \mathbf{\Sigma}_3$. We then define

$$\mathbf{f}_2 \circ \mathbf{f}_1 := (f_2 \circ f_1, s_2 + s_1 - \sigma((f_2 \circ f_1)_\#(L_1), (f_2)_\#(L_2), L_3)) .$$

Using properties of Wall's signature cocycle we obtain that the composition operation is associative and therefore we obtain the category of Λ-labelled marked surfaces.

Definition 3.4 (The category of Λ-labeled marked surfaces)**.** The category $\mathbf{C}(\Lambda)$ of Λ-labeled marked surfaces has Λ-labeled marked surfaces as objects and morphisms as described in definition 3.2 and composition as described in definition 3.3.

There is an easy way to make this category into a symmetric monoidal category.

Definition 3.5 (The operation of disjoint union)**.** Let $\mathbf{\Sigma}_1, \mathbf{\Sigma}_2$ be two Λ-labeled marked surfaces. For $i = 1, 2$, write $\mathbf{\Sigma}_i = (\Sigma_i, P_i, V_i, \lambda_i, L_i)$. We define their disjoint union $\mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2$ to be

$$(\Sigma_1 \sqcup \Sigma_2, P_1 \sqcup P_2, V_1 \sqcup V_2, \lambda_1 \sqcup \lambda_2, L_1 \oplus L_2).$$

For morphisms $\mathbf{f}_i : \mathbf{\Sigma}_i \to \mathbf{\Sigma}_3$ we define $\mathbf{f_1} \sqcup \mathbf{f_2}$ to be

$$(f_1 \sqcup f_2, s_1 + s_2).$$

We have an obvious natural transformation

$$\mathrm{Perm} : \mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2 \to \mathbf{\Sigma}_2 \sqcup \mathbf{\Sigma}_1.$$

Proposition 3.6 ($\mathbf{C}(\Lambda)$ is a symmetric monoidal category)**.** *The category of Λ-labeled marked surfaces is a symmetric monoidal category with disjoint union as product, the empty surface as unit, and Perm as the braiding.*

We now describe the operation of orientation reversal. For an oriented surface Σ we let $-\Sigma$ be the oriented surface where we reverse the orientation on each component. For a map g with values in Λ we let $g^\dagger$ be the map, with the same domain and codomain, given by $g^\dagger(x) = g(x)^\dagger$.

Definition 3.7 (Orientation reversal)**.** Let $\mathbf{\Sigma} = (\Sigma, P, V, \lambda, L)$ be a Λ-labeled marked surface. Then we define

$$-\mathbf{\Sigma} := (-\Sigma, P, V, \lambda^\dagger, L).$$

We say that $-\mathbf{\Sigma}$ is obtained form $\mathbf{\Sigma}$ by reversal of orientation. For a morphism $\mathbf{f} = (f, s)$ we let

$$-\mathbf{f} := (f, -s).$$

Remark 3.8. We note that we could also have defined the reversal of orientation to also involve changing the sign on the tangent vectors at the marked points. This gives complete equivalent theories, since there is a canonical morphism of labeled marked surfaces, which induces minus the identity at the marked points, and which is the identity on the complement of small disjoint neighbourhoods of the marked points and which locally around each marked point twist half a turn positively according to the surface orientation around the marked point, yet remains the identity near the boundary of the neighbourhood of the marked point.

Finally we describe the factorization procedure, where we obtain a Λ-labeled marked surface by cutting along an oriented simple closed curve γ whose homology class is in the distinguished Lagrangian subspace, and then collapse the resulting two boundary components to points, which get labeled by $(i, i^\dagger)$ as described below.

Definition 3.9 (Factorization data)**.** Factorization data is a triple $(\mathbf{\Sigma}, \gamma, i)$, where $\mathbf{\Sigma}$ is a Λ-labeled marked surface and γ is a smooth, oriented, simple closed curve with a basepoint x_0, such that the homology class of γ lies in L. Further, i is an element of the label set Λ. We also say that the pair (γ, i) is a choice of factorization data for $\mathbf{\Sigma}$.

Definition 3.10 (Factorization)**.** Let $\mathbf{\Sigma} = (\Sigma, P, V, \lambda, L)$ be a Λ-labeled marked surface with factorization data (γ, i). We will define a Λ-labeled marked surface $\mathbf{\Sigma}^i_\gamma$. We denote the underlying smooth surface by Σ_γ. Cutting along γ we get a smooth oriented surface $\tilde{\Sigma}_\gamma$ with two boundary components γ_- and γ_+. The orientation of γ together with the orientation of Σ allows us to define γ_+ to be the component whose induced Stokes orientation agrees with that of γ. The underlying smooth surface is given by $\Sigma_\gamma := \tilde{\Sigma}_\gamma / \sim$ where we collapse γ_- to a point p_- and we collapse γ_+ to a point p_+. We orient this surface such that $\Sigma \setminus \gamma \hookrightarrow \tilde{\Sigma}_\gamma / \sim$ is orientation preserving. The set of distinguished points for Σ_γ is $P \sqcup \{p_-, p_+\}$. Identifying $P(T_{p_\pm}(\tilde{\Sigma}_\gamma))$ with γ, we choose $v(p_\pm)$ to be x_0. We extend the labelling λ by labelling p_+ by i and p_- by $i^\dagger$. There is a topological space X given by identifying p_- and p_+. Clearly this space is naturally homeomorphic to $\Sigma / \sim$, where we collapse γ to a point. Thus we have quotient maps $q : \Sigma \to X$ and $n : \Sigma_\gamma \to X$. Define $L_\gamma := (n_\#)^{-1}(q_\#)(L)$. This yields a Lagrangian subspace of $H_1(\Sigma_\gamma)$ that respects the splitting induced by decomposing Σ_γ into connected components. We say that $\mathbf{\Sigma}^i_\gamma$ is obtained by factorizing $\mathbf{\Sigma}$ along (γ, i).

There is an inverse procedure that we call glueing.

Definition 3.11 (Glueing data)**.** Glueing data consist of a triple $(\mathbf{\Sigma}, (p_0, p_1), c)$. Here $\mathbf{\Sigma} = (\Sigma, P, V, \lambda, L)$ is a Λ-labeled marked surface with $p_0, p_1 \in P$, such that $\lambda(p_0) = \lambda(p_1)^\dagger$ and $c : P(T_{p_0}\Sigma) \xrightarrow{\sim} P(T_{p_+}\Sigma)$ is an orientation reversing projective linear isomorphism mapping $v(p_0)$ to $v(p_1)$. We also say that (p_0, p_1, c) determine glueing data for $\mathbf{\Sigma}$ and that (p_0, p_1) is subject to glueing.

As we are dealing with ordered pairs (p_0, p_1) we will sometimes speak of p_0 as the preferred point.

Definition 3.12 (Glueing)**.** Assume we are given a glueing data $(\mathbf{\Sigma}, (p_0, p_1), c)$. We will define a Λ-labeled marked surface $\mathbf{\Sigma}^{p_0,p_1}_c$. We denote the underlying smooth surface by Σ^{p_0,p_1}_c. Blow up Σ at p_0, p_1 and glue in $P(T_{p_0}\Sigma)$ and $P(T_{p_1}\Sigma)$ to obtain a smooth oriented surface with boundary, that, as a set, can be naturally identified with

$$(\Sigma \setminus \{p_0, p_1\}) \sqcup P(T_{p_0}\Sigma) \sqcup P(T_{p_1}\Sigma).$$

Now identify the two boundary components through $x \sim c(x)$. This yields a smooth oriented surface, that will be the underlying surface of $\mathbf{\Sigma}^{p_0,p_1}_c$. As distinguished points, directions and labels, we simply take those from $\mathbf{\Sigma}$. Let X be the topological space obtained from Σ by identifying p_0 with p_1. We have continuous maps $q : \Sigma \to X$ and $n : \Sigma_c^{p_1,p_2} \to X$. Set $L_{c,p_0,p_1} := (n_\#)^{-1}(q_\#)(L)$. This is a Lagrangian subspace of $H_1(\Sigma_\gamma)$ that respects the splitting induced by decomposing Σ_γ into connected components.

Observe that the homology class of $P(T_{p_0}\Sigma)$ lies in L_{c,p_0,p_1}.

Proposition 3.13 (Consecutive glueing). *Assume that two distinct pairs of points (p_1, p_2, c) and (q_1, q_2, d) are subject to glueing. Then there is a canonical diffeomorphism*

$$s^{p_1,p_2,q_1,q_2} : (\mathbf{\Sigma}_c^{p_1,p_2})_d^{q_1,q_2} \to (\mathbf{\Sigma}_d^{q_1,q_2})_c^{p_1,p_2}.$$

In abuse of notation we will also write s^{p_1,p_2,q_1,q_2} for the induced morphism of labeled marked surfaces given by $([s^{p_1,p_2,q_1,q_2}], 0)$.

We recall that any two orientation reversing self-diffeomorphisms of S^1 fixing a basepoint are isotopic among diffeomorphisms fixing this basepoint. Therefore we wish to detail the independence of the choice of c in the glueing construction.

Proposition 3.14 (Glueing independent of c). *Assume we are given a Λ-labeled marked surface Σ and two pairs of glueing data (p_0, p_1, c_1) and (p_0, p_1, c_2). Then there is an orientation preserving diffeomorphism $f : \Sigma \to \Sigma$ that induces the identity on (P, V, λ, L) and such that $c_1 \circ df = df \circ c_2$. Moreover f can be chosen to induce the identity morphism $(id, 0)$ on $\mathbf{\Sigma}$. Any two such f induces the same morphism of Λ-labeled marked surfaces, and therefore we have a canonical identification morphism $\tilde{f}(c_1, c_2) : \mathbf{\Sigma}_{c_1}^{p_0,p_1} \to \mathbf{\Sigma}_{c_2}^{p_0,p_1}$ given by the pair $([f], 0)$.*

It follows from this that in order to specify glueing, it will suffice to specify an ordered pair (p_0, p_1) with $\lambda(p_0) = \lambda(p_1)^\dagger$.

Proposition 3.15 (Functoriality of glueing). *Let $\mathbf{\Sigma}_i$ for $i = 1, 2$ be Λ-labeled marked surfaces. Assume (p_0^i, p_1^i) are subject to glueing for $i = 1, 2$. Consider any morphism $\mathbf{f} = ([f], s) : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ with $f(p_0^1) = p_0^2$ and $f(p_1^1) = p_1^2$. Let $c : P(T_{p_0^1}\Sigma_1) \to P(T_{p_1^1}\Sigma_1)$ be orientation reversing. Let $c' := df \circ c \circ df^{-1} : P(T_{p_0^2}\Sigma_2) \to P(T_{p_1^2}\Sigma_2)$. This data induces a morphism*

$$\mathbf{f}' = ([f'], s) : (\mathbf{\Sigma}_1)_c^{p_0^1,p_1^2} \longrightarrow (\mathbf{\Sigma}_2)_{c'}^{p_0^2,p_1^2}$$

compatible with $\mathbf{f}$.

3.2 Extended surfaces

We now describe the category of extended surfaces following Turaev [29]. Observe that this is only defined relative to a modular tensor category $(\mathcal{V}, (V_i)_{i\in I})$. We recall that an orientation of a closed topological surface Σ is a choice of fundamental class in $H^2(\Sigma_\alpha, \mathbb{Z})$ for each component Σ_α. A degree 1-homeomorphism between oriented closed surfaces is a homeomorphism that respects this choice. We recall that an arc $\gamma \subset \Sigma$ is a topological embedding of $[0, 1]$.

Definition 3.16 (Extended surfaces)**.** An e-surface Σ is given by the following data $(\Sigma, (\alpha_i), (W_i, \mu_i), L)$. Here Σ is an oriented closed surface, (α_i) is a finite collection of disjoint oriented arcs. To each arc α_i we have an object W_i of $\mathcal{V}$ and a sign $\mu_i \in \{\pm 1\}$. The pair (W_i, μ_i) is called the marking of α_i. Finally, L is a Lagrangian subspace of $H_1(\Sigma, \mathbb{R})$. By convention $\emptyset$ is an e-surface.

We now describe the arrows.

Definition 3.17 (Weak extended homeomorphisms and their composition)**.** Let Σ_1, Σ_2 be two e-surfaces. A weak e-homeomorphism $f : \Sigma_1 \to \Sigma$ is a degree 1-homeomorphism between the underlying topological surfaces $\Sigma_1 \to \Sigma_2$ that induces an orientation and marking preserving bijection between their distinguished arcs. An e-homeomorphism $f : \Sigma_1 \to \Sigma$ is a weak e-homeomorphism that induces an isomorphism of distinguished Lagrangian subspaces $f_\# : L_1 \to L_2$. We observe that the class of weak e-homeomorphisms is closed under compostion, and that this is also the case for e-homeomorphisms.

Thus we have the category of extended surfaces based on $(\mathcal{V}, (V_i)_{i\in I})$.

Definition 3.18 (The category of extended surfaces based on $\mathcal{V}$)**.** The category of extended surfaces based on $\mathcal{V}$ has e-surfaces as objects and weak e-homeomorphisms as morphisms. We denote it by $\mathbf{E}(\mathcal{V})$.

As above we wish to make this into a symmetric monoidal category with an orientation reversal.

Definition 3.19 (Disjoint union of e-surfaces)**.** Let $\Sigma_1 = (\Sigma_1, (\alpha_i), (W_i, \mu_i), L)$ and $\Sigma_2 = (\Sigma_2, (\beta_j), (Z_j, \eta_j), L')$ be two e-surfaces. We define $\Sigma_1 \sqcup \Sigma_2$ to be

$$(\Sigma_1 \sqcup \Sigma_2, (\alpha_i \sqcup \beta_j), (W_i, \mu_i) \sqcup (Z_j, \eta_j), L \oplus L').$$

For a pair of (weak) morphisms $f_i : \Sigma_i \to \Sigma_3$ we observe that $f_1 \sqcup f_2$ is a (weak) morphism. We have an obvious natural transformation

$$\mathrm{Perm} : \Sigma_1 \sqcup \Sigma_2 \to \Sigma_2 \sqcup \Sigma_1.$$

Proposition 3.20 ($\mathbf{E}(\mathcal{V})$ is a symmetric monoidal category)**.** *The category of extended surfaces is a symmetric monoidal category with disjoint union as product, the empty surface as unit, and Perm as the braiding.*

Definition 3.21 (Orientation reversal for e-surfaces)**.** Consider an extended surface $\Sigma = (\Sigma, (\alpha_i), (W_i, \mu_i), L)$. We define $-\Sigma$ to be

$$(-\Sigma, (-\alpha_i), (W_i, -\mu_i), L).$$

That is, we reverse the orientation on each component, reverse the orientation of arcs, keep the labels, multiply all signs by -1, and keep the Lagrangian subspace. We observe that any (weak) e-homeomorphism $f : \Sigma_1 \to \Sigma_2$ yields a (weak) morphism $f : -\Sigma_1 \to -\Sigma_2$.

3.3 Marked surfaces

Finally we describe the category of marked surfaces[2]. This is defined relative to a monoidal class. That is, a class C together with a strictly associative operation $C \times C \to C$ and a unit 1 for this operation. Again we here follow Turaev [29].

Definition 3.22 (Marked surface over C)**.** A marked surface (over C) is a compact oriented surface Σ endowed with a Lagrangian subspace of $H_1(\Sigma, \mathbb{R})$ and such that each connected component X of $\partial\Sigma$ is equipped with a basepoint, a sign δ, and an element V of C called the label. The pair (V, δ) is called the marking of X. By convention $\emptyset$ is an m-surface.

Next we describe the morphisms.

Definition 3.23 (Weak m-homeomorphisms)**.** Let Σ_1, Σ_2 be two marked surfaces. A weak m-homeomorphism $f : \Sigma_1 \to \Sigma_2$ is an orientation preserving homeomorphism f that respects the marks of boundary components. An m-homeomorphism is a weak m-homeomorphism that also preserves the Lagrangian subspaces.

Definition 3.24 (The category of marked surfaces over C)**.** The category of marked surfaces over C has m-surfaces as objects and weak m-homeomorphisms as morphisms. We denote it $\mathbf{M}(C)$.

As above this naturally constitute a symmetric monoidal category with disjoint union as the product.

Definition 3.25 (Disjoint union of marked surfaces)**.** Let Σ_1, Σ_2 be two m-surfaces. Then we define the marked surface $\Sigma_1 \sqcup \Sigma_2$ by declaring that the boundary components naturally inherit basepoints and markings, and equipping it with a Lagrangian subspace of $H_1(\Sigma_1 \sqcup \Sigma_2, \mathbb{R})$, by taking the direct sum of Lagrangian subspaces of Σ_1, Σ_2. If f_1, f_2 are (weak) m-homeomorphisms, then $f_1 \sqcup f_2$ is a (weak) m-homeomorphism. We have a natural transformation

$$\mathrm{Perm} : \Sigma_1 \sqcup \Sigma_2 \to \Sigma_2 \sqcup \Sigma_1.$$

Proposition 3.26 ($\mathbf{M}(C)$ is a symmetric monoidal category)**.** *The category $\mathbf{M}(C)$ of marked surfaces (over C) is a symmetric monoidal category with disjoint union as product, the empty surface as unit, and Perm as the braiding.*

Definition 3.27 (Glueing)**.** Let Σ be an m-surface. Assume that there are two components X, Y with the same label, but with opposite sign. We say that X, Y are subject to glueing. There is a (unique up to isotopy) basepoint preserving orientation-reversing homeomorphism $c : X \to Y$. The quotient $\Sigma' = \Sigma/\sim$ where $x \sim c(x)$ is naturally an oriented compact surface. The quotient map $q : \Sigma \to \Sigma'$ yields a bijection $\partial\Sigma' \sim \partial\Sigma \setminus X \cup Y$. Using this, we equip each component of $\partial\Sigma'$

[2]Not to be confused with Λ-labeled marked surfaces.

with a basepoint and a marking. Finally, equip Σ' with the Lagrangian subspace that is the image of the Lagrangian subspace of Σ under $q_\#$. Denote the resulting m-surface by

$$\Sigma/[X = Y]_c.$$

Proposition 3.28 (Functorial property of glueing of m-surfaces)**.** *Let Σ be an m-surface. Assume $X, Y \subset \partial\Sigma$ are two boundary components subject to glueing. Let $x : X \to Y$ be basepoint preserving and orientation reversing. Let $f : \Sigma \to \Sigma'$ be a (weak) m-homeomorphism. Then $X' = f(X), Y' = f(Y) \subset \partial\Sigma'$ are subject to glueing and the map c' given by $f \circ c \circ f^{-1} : X' \to Y'$ is orientation reversing and basepoint preserving. There is a unique (weak) homeomorphism $f_c : \Sigma/[X = Y]_c \to \Sigma'/[X' = Y']_{c'}$ inducing a commutative diagram:*

$$\begin{tikzcd} \Sigma \arrow[r, "f"] \arrow[d, "q"'] & \Sigma' \arrow[d, "q"] \\ \Sigma/[X = Y]_c \arrow[r, "f_c"] & \Sigma'/[X' = Y']_{c'} \end{tikzcd}$$

Here the vertical maps are the quotient maps.

When dealing with Turaev's 2-DMF, it is convenient to introduce a symmetric monoidal category $\mathcal{M}'(C)$ very similar to $\mathcal{M}(C)$ but with fever morphisms. See remark 3.30 below.

Definition 3.29. Let $\mathcal{M}'(C)$ be the category with the same objects as $\mathbf{M}(C)$, but where morphisms are equivalence classes of weak m-homeomorphisms, where two parallel weak m-homeomorphisms are equivalent if and only if they are isotopic through weak m-homeomorphisms. The braiding and the permutation is defined similarly to those of $\mathcal{M}(C)$.

Remark 3.30. We recall that the 2-DMF $\mathcal{H}_\mathcal{V}$ defined in chapter V of [29] descends to $\mathcal{M}'(C)$ in the sense that if f, g are two equivalent weak m-homeomorphisms, then we have the identity $\mathcal{H}(f) = \mathcal{H}(g)$.

4 Axioms for a modular functor

We now recall Kevin Walker's axioms for a modular functor as they are given and used in [12, 13, 14]. For Turaev's axioms of a modular functor, we refer to chapter V in [29]. We assume familiarity with the notion of symmetric monoidal functors. Roughly speaking, a symmetric monoidal functor between symmetric monoidal categories $(C, \otimes, e) \to (D, \otimes', e')$ is a triple (F, F_2, f) where $F : C \to D$ is a functor, F_2 is a family of morphisms $F_2 : F(a) \otimes' F(b) \to F(a \otimes b)$ and f is a morphism $f : e' \to F(e)$. For the precise formulation of the axioms we refer to [22]. For brevity we will write $F = (F, F_2, f)$. If F_2, f are always isomorphisms, we say that F is a strong monoidal functor.

4.1 The Walker axioms for a modular functor

Let $\Lambda = (\Lambda, ^\dagger, 0)$ be a label set. Let K be a commutative ring (with unit). Let $\mathbf{P}(K) = \mathrm{Proj}(K)$ be the category of finitely generated projective K-modules. We recall that this is a symmetric monoidal category with the tensor product over K as product, and K as unit.

Definition 4.1 (Modular functor V based on Λ and K)**.** A modular functor based on a label set Λ and a commutative ring K is a strong monoidal functor V

$$V : \mathbf{C}(\Lambda) \to \mathbf{P}(K),$$

satisfying the glueing axiom, the one punctured sphere axiom and the twice punctured sphere axiom as these are described below.

The glueing axiom. Assume that (p_1, p_2, c) is a glueing data for a labeled marked surface $\mathbf{\Sigma}$. For any $\lambda \in \Lambda$, let $\mathbf{\Sigma}(\lambda)$ be the labeled marked surface identical to $\mathbf{\Sigma}$ except for the fact that p_1 is labeled with λ and p_2 is labeled with $\lambda^\dagger$. Then (p_1, p_2, c) is a glueing data for $\mathbf{\Sigma}(\lambda)$. We demand that there is a specified isomorphism

$$g : \bigoplus_{\lambda \in \Lambda} V(\mathbf{\Sigma}(\lambda)) \xrightarrow{\sim} V(\mathbf{\Sigma}_c^{p_1,p_2}). \tag{4.1}$$

Let g_λ be the restriction of g to $V(\mathbf{\Sigma}(\lambda))$. If the context is clear, we will simply write g for this restriction, and suppress λ from the notation. If we wish to stress the glueing map c, we will write g^c. The glueing isomorphism is subject to the four axioms below.

(i). The isomorphism should be associative in the following sense. Assume that (q_1, q_2, d) is another pair subject to glueing. For any pair $(\lambda, \mu) \in \Lambda^2$ let $\mathbf{\Sigma}(\lambda, \mu)$ be the labeled marked surface identical to $\mathbf{\Sigma}$ except that p_1 is labeled with λ, p_2 is labeled with $\lambda^\dagger$, q_1 is labeled with μ and q_2 is labeled with $\mu^\dagger$. Then the following diagram is commutative

$$\begin{tikzcd} V(\mathbf{\Sigma}(\lambda,\mu)) \arrow[r, "g_\mu"] \arrow[d, "g_\lambda"'] & V((\mathbf{\Sigma}_d^{q_1,q_2})(\lambda)) \arrow[d, "g_\lambda"] \\ V((\mathbf{\Sigma}_c^{p_1,p_2})(\mu)) \arrow[r, "s'\circ g_\mu"] & V((\mathbf{\Sigma}_d^{q_1,q_2})_c^{p_1,p_2}). \end{tikzcd} \tag{4.2}$$

Here $s' = V(s^{p_1,p_2,q_1,q_2})$, where s^{p_1,p_2,q_1,q_2} is as defined in Prop. 3.13.

(ii). The isomorphism should be compatible with glueing of morphisms in the following sense. Assume that $\mathbf{f} : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ is a morphism such that a pair (p_0, p_1) subject to glueing is taken to the pair (q_0, q_1). Choosing c will induce a morphism $\mathbf{f}' : (\mathbf{\Sigma}_1)_c^{p_0,p_1} \longrightarrow (\mathbf{\Sigma}_2)_{c'}^{q_0,q_1}$ as in Prop 3.15. This should induce a commutative diagram:

$$\begin{array}{ccc} V(\mathbf{\Sigma}_1) & \xrightarrow{g} & V((\mathbf{\Sigma}_1)_c^{p_0,p_1}) \\ {\scriptstyle V(\mathbf{f})}\downarrow & & \downarrow{\scriptstyle V(\mathbf{f}')} \\ V(\mathbf{\Sigma}_2) & \xrightarrow{g} & V((\mathbf{\Sigma}_2)_{c'}^{q_0,q_1}) \end{array} \tag{4.3}$$

(iii). The isomorphism should be compatible with disjoint union in the following way. Assume that (p_0, p_1, c) is a glueing data for $\mathbf{\Sigma}_1$. For any $\mathbf{\Sigma}_2$, we see that (p_0, p_1, c) is also a choice of glueing data for $\mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2$, and that there is a canonical morphism $\iota = (\iota, 0) : (\mathbf{\Sigma}_1)_c^{p_1,p_2} \sqcup \mathbf{\Sigma}_2 \longrightarrow (\mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2)_c^{p_0,p_1}$. This should induce a commutative diagram

$$\begin{array}{ccc} V(\mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2) & \xrightarrow{\quad g \quad} & V((\mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2)_c^{p_0,p_1}) \\ {\scriptstyle V_2}\uparrow & & \uparrow{\scriptstyle V(\iota)\circ V_2} \\ V(\mathbf{\Sigma}_1) \otimes V(\mathbf{\Sigma}_2) & \xrightarrow{g\otimes 1} & V((\mathbf{\Sigma}_1)_c^{p_0,p_1}) \otimes V(\mathbf{\Sigma}_2). \end{array} \tag{4.4}$$

(iv). The isomorphism should be independent of the glueing map c in the following way. Assume a pair of points (p_0, p_1) in $\mathbf{\Sigma}$ is subject to glueing. Assume that $c_1, c_2 : P(T_{p_0}\Sigma) \to P(T_{p_1}\Sigma)$ are two glueing maps. Consider the identification morphism $\tilde{f}(c_1, c_2) : \mathbf{\Sigma}_{c_1}^{p_0,p_1} \to \mathbf{\Sigma}_{c_2}^{p_0,p_2}$ as in Prop 3.14. This should induce a commutative diagram

$$\begin{array}{ccc} V(\mathbf{\Sigma}) & \xrightarrow{g^{c_1}} & V(\mathbf{\Sigma}_{c_1}^{p_0,p_1}) \\ & {\scriptstyle g^{c_2}}\searrow & \downarrow{\scriptstyle V(\tilde{f}(c_1,c_2))} \\ & & V(\mathbf{\Sigma}_{c_2}^{p_0,p_1}). \end{array} \tag{4.5}$$

The once punctured sphere axiom. For any $\lambda \in \Lambda$ consider a sphere with one distinguished point $\mathbf{\Sigma}_\lambda = (S^2, \{p\}, \{v\}, \{\lambda\}, 0)$. We demand that

$$V(\mathbf{\Sigma}_0) \simeq \begin{cases} K \text{ if } \lambda = 0 \\ 0 \text{ if } \lambda \neq 0. \end{cases} \tag{4.6}$$

The twice punctured sphere axiom. For any ordered pair (λ, μ) in Λ, consider a sphere with two distinguished points $\mathbf{\Sigma}_{\lambda,\mu} = (S^2, \{p_1, p_2\}, \{v_1, v_2\}, \{\lambda, \mu\}, 0)$. We demand that

$$V(\mathbf{\Sigma}_{\lambda,\mu}) \simeq \begin{cases} K \text{ if } \mu = \lambda^\dagger \\ 0 \text{ if } \mu \neq \lambda^\dagger. \end{cases} \tag{4.7}$$

Remark 4.2. We stress that the isomorphisms given in (4.6) and (4.7) are not part of the data of a modular functor. Only the existence of such isomorphisms are required. Below we will occasionally denote a modular functor by a pair (V, g) where V is the strong monoidal functor, and g is the glueing isomorphism 4.1.

5 Construction of a modular functor $Z_{\mathcal{V}}$.

5.1 The symmetric monoidal functor

From now on, we consider a modular tensor category $(\mathcal{V}, (V_i)_{i\in I})$ and take $\Lambda = I$ and $^\dagger =^*$. We let K be the commutative ring $\mathrm{End}(\mathbf{1})$, where $\mathbf{1}$ is the unit for the tensor product in $\mathcal{V}$.

Proposition 5.1 (Existence of a strong monoidal functor $\mathbf{C}(I) \to \mathbf{M}'(\mathcal{V})$)**.** *Consider a modular tensor category $(\mathcal{V}, (V_i)_{i\in I})$. Let $\Lambda = I$, $^\dagger =^*$ and let $C = \mathcal{V}$ considered as a monoidal class. There is a strong monoidal functor from the category of I-labeled marked surface into the category $\mathbf{M}'(C)$*

$$\mathcal{G} : \mathbf{C}(I) \to \mathbf{M}'(C).$$

For an I-labeled marked surface $\mathbf{\Sigma} = (\Sigma, P, V, \lambda, L)$ the marked surface $\mathcal{G}(\mathbf{\Sigma})$ is given as follows. For any distinguished point p, blow up Σ at p. That is, the underlying topological surface of $\mathcal{G}(\mathbf{\Sigma})$ is denoted by $\mathcal{G}(\Sigma)$ and is given as follows

$$\left(\Sigma \setminus P \bigsqcup_{p\in P} S^1_p\right) / \sim .$$

Here we glue in the circle S^1_p using smooth coordinates in a neigbourhood of p. The orientation agrees with that on $\mathbf{\Sigma}$. The direction v_p yields a basepoint on S^1, the label $i \in I$ yields a marking $(V_i, 1)$. Collapsing S^1_p to a point at all p yields a surface Σ', that is canonically homeomorphic to Σ. Let η denote the natural homeomorphism $\Sigma' \to \Sigma$. Let q denote the quotient map that collapses any component to a point. The composition $g := \eta \circ q : \mathcal{G}(\Sigma) \to \Sigma$ will be an isomorphism on homology, and this provides us with a Lagrangian subspace $L' := {g_\#}^{-1}(L)$. Given a morphism of labeled marked surfaces $(f, s) : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ any representative of f naturally induces a weak m-homeomorphism $\mathcal{G}(\mathbf{\Sigma_1}) \to \mathcal{G}(\mathbf{\Sigma}_2)$ and we let $\mathcal{G}(f, s)$ be the corresponding equivalence class.

We are now finally ready to define our modular functor. We recall that even though Turaev's axioms for a $2-DMF$ as given in chapter V only requires functoriality with respect to m-homeomorphisms, it is also defined on weak m-homeomorphisms. See section 4.3 in chapter V.

Definition 5.2 (The definition of $Z_{\mathcal{V}}$)**.** Let $\mathcal{H}_{\mathcal{V}}$ be the 2-DMF as defined in chapter V of [29] relative to $(\mathcal{V},(V_i)_{i\in I})$. On the level of objects we define $Z_{\mathcal{V}}$ to be

$$Z_{\mathcal{V}} := \mathcal{H}_{\mathcal{V}} \circ \mathcal{G} : \mathbf{C}(I) \longrightarrow \mathbf{P}(K).$$

For a morphism of labeled marked surfaces $(f,s) : \mathbf{\Sigma} \to \mathbf{\Sigma}'$ we define

$$Z_{\mathcal{V}}(f,s) := (\Delta^{-1}D)^s \mathcal{H}_{\mathcal{V}}(\mathcal{G}(f,s)).$$

Here D,Δ are invertible scalars in K to be introduced in section 7.1 below. We write $Z = Z_{\mathcal{V}}$ and $\mathcal{H} = \mathcal{H}_{\mathcal{V}}$. We need to address the issue of functoriality. That is we must verify that $Z(\mathbf{f}) \circ Z(\mathbf{g}) = Z(\mathbf{f} \circ \mathbf{g})$ for composable morphisms of labeled marked surfaces. Let V, V' be symplectic vector spaces. Recall that Walker's signature cocycle for an ordered triple (L_1, L_2, L_3) of Lagrangian subspaces $L_i \subset V$ coincide with the Maslov index $\mu(L_1, L_2, L_3)$. Recall also that $\mu(L_1, L_2, L_3) = \mu(f(L_1), f(L_2), f(L_3))$ for any symplectomorphism $f : V \to V'$. These facts together with remark 5.4 and lemma 6.3.2 in chapter IV of [29] easily imply functoriality.

We need to define a glueing isomorphism. We start by observing the following proposition.

Proposition 5.3 ($\mathcal{G}$ is compatible with glueing)**.** *Assume $\mathbf{\Sigma}$ is a labeled marked surface. Assume we are given glueing data (p,q,c). Assume p is labeled with i. Consider $\Sigma' = \mathcal{G}(\mathbf{\Sigma})$. If we replace the marking of X_q with $(V_i,-1)$ to obtain a new marked surface Σ'', then $X_p \subset \partial\Sigma''$ and $X_q \subset \partial\Sigma''$ are subject to glueing. We observe*

$$\mathcal{G}(\mathbf{\Sigma}_c^{p,q}) = \Sigma''/[X_p \approx X_q].$$

We now compare the glueing isomorphism axiom of Walker and the splitting axiom of Turaev more closely. Turaev's modular functor is subject to the splitting axiom, which means that the glueing homomorphisms provide an isomorphism

$$g : \bigoplus_{i\in I} \mathcal{H}\left(\mathcal{G}(\Sigma),(V_i,1),(V_i,-1)\right)\right) \xrightarrow{\sim} Z(\mathbf{\Sigma}_c).$$

See chapter V, the splitting axiom on page 246 in [29]. Comparing with Walker's glueing axiom, we see that the summands are not the same, since there we need an isomorphism

$$g : \bigoplus_{i\in I} \mathcal{H}\left(\mathcal{G}(\Sigma),(V_i,1),(V_{i^*},1)\right) \xrightarrow{\sim} Z(\mathbf{\Sigma}_c).$$

Hence we need to provide isomorphisms between modules of states, where we exchange a marking $(V_{i^*}, 1)$ with $(V_i, -1)$. To provide these identifications, we first recall that $\mathcal{H}$ is compatible with the operator invariant τ^e. For an explanation of this see the following remark.

Remark 5.4. Let Σ be a marked surface over $\mathcal{V}$. We recall that $\mathcal{H}(\Sigma)$ is naturally isomorphic to $\mathcal{T}^e(\overline{\Sigma})$, where $\overline{\Sigma}$ is an extended surface obtained from Σ by glueing in discs with preferred diameter, that are taken to be marked arcs. We can therefore use the operator invariant τ^e to obtain morphisms between modules of states. The natural isomorphism $\mathcal{H}(\Sigma) \simeq \mathcal{T}^e(\overline{\Sigma})$ also implies, that for a labeled marked surface $\mathbf{\Sigma}$ we could just as well define $Z(\mathbf{\Sigma})$ as $\mathcal{T}^e(\widetilde{\mathbf{\Sigma}})$, where $\widetilde{\mathbf{\Sigma}}$ is an extended surface naturally obtained from $\mathbf{\Sigma}$. Similarly, we observe that a morphism (f, s) of I-labeled marked surfaces induces an equivalence class f' of weak e-homeomorphisms, and that $\mathcal{H}(\mathcal{G}(f, s)) \sim \mathcal{T}^e(f')$.

Now we provide the needed identifications.

Lemma 5.5 (The natural transformation $\dot{f}$)**.** *Let Σ be an m-surface with a boundary component X_α marked with $(V, 1)$. Assume that Σ' is obtained from Σ by replacing the marking $(V, 1)$ with $(W, 1)$. Assume that $f : V \to W$ is a morphism. There is a K-linear morphism*

$$\dot{f} : \mathcal{H}(\Sigma) \to \mathcal{H}(\Sigma'),$$

where $\dot{f}$ is induced from the extended three manifold $M = \overline{\Sigma} \times I$. Here we think of the bottom as $\overline{\Sigma}$, the top as $\overline{\Sigma'}$, and we provide M with following ribbon graph. For each arc β different from the arc α corresponding to $X_\alpha \subset \partial\Sigma$, we put in the identity strand $\beta \times I$. For α, we put in a coupon colored with f.

Lemma 5.6 (The natural transformation h_α)**.** *Let Σ be an m-surface with a boundary component X_α marked with $(V^*, 1)$. Assume that Σ' is obtained from Σ by replacing the marking $(V^*, 1)$ with the marking $(V, -1)$. There is a K-linear morphism*

$$h_\alpha : \mathcal{H}(\Sigma) \to \mathcal{H}(\Sigma').$$

The morphism h_α is induced from the extended three manifold $M = \overline{\Sigma} \times I$ where we think of the bottom as $\overline{\Sigma}$, the top as $\overline{\Sigma'}$, and we provide M with following ribbon graph. For each arc β different from the arc α corresponding to $X_\alpha \subset \partial\Sigma$, we put in the identity strand $\beta \times I$. For α, we put in a coupon colored with id_{V^*}.

If the relevant boundary component is understood, we will simply write $h_\alpha = h$. In section 8 below we will give all details of how these two lemmas follow directly from similar statements in [29].

5.2 The glueing isomorphism

Let $\mathbf{\Sigma_c}$ be an I-labeled marked surface obtained from $\mathbf{\Sigma}$ by glueing. We must provide an isomorphism

$$\bigoplus_{i\in I} Z(\mathbf{\Sigma}, i, i^\dagger) \xrightarrow{\sim} Z(\mathbf{\Sigma_c}).$$

For each $i \in I$ fix an isomorphism $q_i : V_{i^*} \to V_i^*$. We define the glueing isomorphism as follows. For each i, consider the composition

$$Z(\mathbf{\Sigma}, i, i^*) \xrightarrow{\dot{q}_i} \mathcal{H}\left(\mathcal{G}(\Sigma), (V_i, 1), (V_i^*, 1)\right) \xrightarrow{h} \mathcal{H}\left(\mathcal{G}(\Sigma), (V_i, 1), (V_i, -1)\right). \tag{5.1}$$

Using that $\mathcal{H}$ satisfies the splitting axiom as defined in chapter V, we see that we have an isomorphism

$$g : \bigoplus_{i\in I} \mathcal{H}\left(\mathcal{G}(\Sigma), (V_i, 1), (V_i, -1)\right)) \xrightarrow{\sim} Z(\mathbf{\Sigma}_c).$$

Thus we can define our glueing isomorphism as follows.

Definition 5.7 (The glueing isomorphism)**.** We define

$$\tilde{g}(q) := g \circ (\oplus_{i\in I} h \circ \dot{q}_i) : \bigoplus_{i\in I} Z(\mathbf{\Sigma}, \lambda, i, i^*) \xrightarrow{\sim} Z(\mathbf{\Sigma}_c).$$

We write $\tilde{g}(q)$ to stress that this depends on the choices of isomorphisms q_i.

5.3 Main theorem

We are now ready to state our main theorem.

Theorem 5.8 (Main Theorem)**.** *For any modular tensor category $(\mathcal{V}, (V_i)_{i\in I})$ the symmetric modular functor $Z_\mathcal{V}$ as given in definition 5.2 together with the glueing isomorphism $\tilde{g}(q)$ as given in definition 5.7 satisfies Walker's axioms of a modular functor based on I and K as given in section 4.*

We will sometimes write $Z(q)$ for the modular functor $Z_\mathcal{V}$ equipped with the glueing $\tilde{g}(q)$.

6 Proof of the main theorem

We first state more or less trivial statements about the K-linear morphisms coming from Lemma 5.5 and 5.6. Recalling the setting and notation of Lemma 5.5, it is clear that if $g : \Sigma \to \tilde{\Sigma}$ is a weak m-homeomorphism, then so is $g : \Sigma' \to \tilde{\Sigma}'$, where $\tilde{\Sigma}'$ is obtained from $\tilde{\Sigma}$ by replacing the marking of $g(X_\alpha)$ with $(W, 1)$.

Lemma 6.1 (The natural transformation $\dot{f}$). *For each such g, $\dot{f}$ induces a commutative diagram*

$$\begin{tikzcd}
\mathcal{H}(\Sigma) \arrow[r, "\mathcal{H}(g)"] \arrow[d, "\dot{f}"'] & \mathcal{H}(\tilde{\Sigma}) \arrow[d, "\dot{f}"] \\
\mathcal{H}(\Sigma') \arrow[r, "\mathcal{H}(g)"] & \mathcal{H}(\tilde{\Sigma}').
\end{tikzcd}$$

Moreover, $\dot{f}$ is compatible with disjoint union in the following sense. Assume $\Sigma = \Sigma_1 \sqcup \Sigma_2$, and $X_\alpha \subset \Sigma_2$. Then the following diagram commutes

$$\begin{tikzcd}
\mathcal{H}(\Sigma) \arrow[r] \arrow[d, "\dot{f}"'] & \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2) \arrow[d, "id\otimes\dot{f}"] \\
\mathcal{H}(\Sigma') \arrow[r] & \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2').
\end{tikzcd}$$

Recalling the setting and notation of Lemma 5.6, we observe that if $g : \Sigma \to \tilde{\Sigma}$ is a weak m-morphism, then so is $g : \Sigma' \to \tilde{\Sigma}'$, where $\tilde{\Sigma}'$ is obtained from $\tilde{\Sigma}$ by replacing the marking of $g(X_\alpha)$ with $(V, -1)$.

Lemma 6.2 (The natural transformation h_α). *For each such g, h_α induces a commutative diagram*

$$\begin{tikzcd}
\mathcal{H}(\Sigma) \arrow[r, "\mathcal{H}(g)"] \arrow[d, "h_\alpha"'] & \mathcal{H}(\tilde{\Sigma}) \arrow[d, "h_\alpha"] \\
\mathcal{H}(\Sigma') \arrow[r, "\mathcal{H}(g)"] & \mathcal{H}(\tilde{\Sigma}').
\end{tikzcd}$$

Moreover, h_α is compatible with disjoint union in the following sense. Assume $\Sigma = \Sigma_1 \sqcup \Sigma_2$, and $X_\alpha \subset \Sigma_2$. Then the following diagram commutes

$$\begin{tikzcd}
\mathcal{H}(\Sigma) \arrow[r] \arrow[d, "h_\alpha"'] & \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2) \arrow[d, "id\otimes h_\alpha"] \\
\mathcal{H}(\Sigma') \arrow[r] & \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2').
\end{tikzcd}$$

We need to know how the morphisms $\dot{f}, h$ relate to the glueing homomorphism provided by Turaev, and we need to know what happens if we apply the $\dot{f}, h$ operations consecutively to distinct boundary components. We call a morphism of type $\dot{f}$ or h a coupon morphism.

Lemma 6.3 (Far commutativity)**.** *Assume that an m-surface Σ_2 is obtained from an m-surface Σ_1 by altering the markings of two distinct boundaries X_α, X_β components in one of the two ways described above. Let q be the K-morphism $\mathcal{H}(\Sigma_1) \to \mathcal{H}(\Sigma_2)$ that is obtained from composing the coupon morphism that alters X_α with the coupon morphism that alters X_β. Let p be the K-morphism $\mathcal{H}(\Sigma_1) \to \mathcal{H}(\Sigma_2)$ that is obtained from composing the coupon morphism that alters the labelling of X_β with the coupon morphism that alters the labelling of X_α. Then we have*

$$p = q.$$

Lemma 6.4 (Compatibility of $\dot{f}, h$ with glueing.)**.** *Assume that $\mathbf{\Sigma_c}$ is obtained from $\mathbf{\Sigma}$ by glueing. Consider a component X_β of $\mathbf{\Sigma}$, that is not part of the glueing data. Assume the marking of β is altered either by using a morphism f of objects of $\mathcal{V}$, or by replacing $(V^*, 1)$ with $(V, -1)$. Then this operation applies to $\mathbf{\Sigma}_c$ as well. Let r denote the resulting isomorphisms of modules. Let g denote the glueing homomorphism. Then*

$$r \circ g = g \circ r.$$

These lemmas will be proven in section 8 below.

Proof of the Main Theorem. Since $\mathcal{H}$ is a strong monoidal functor, it is immediate that $Z_\mathcal{V}$ is a symmetric monoidal functor, since it is a composition of strong monoidal functors. Thus it remains to verify the once punctured sphere axiom, the twice punctured sphere axiom, and the glueing axiom. The once punctured sphere axiom follows directly from Turaev's disc axiom, which is axiom 1.5.5 in chapter V. The twice punctured sphere axioms follows directly from the third normalization axiom 1.6.2 in chapter V of [29].

It remains to verify the glueing axiom. If $\mathbf{f} = (f, n)$ is a morphism of labeled marked surfaces, we will abuse notation and write f for $\mathcal{G}(\mathbf{f})$.

(i) In the notation of definition 4.1 we must prove that the following diagram commutes

$$\begin{tikzcd}
Z(\mathbf{\Sigma}(i,j)) \arrow[r, "\tilde{g}_j"] \arrow[d, "\tilde{g}_i"'] & Z((\mathbf{\Sigma}_d^{q_1,q_2})(i)) \arrow[d, "\tilde{g}_i"] \\
Z((\mathbf{\Sigma}_c^{p_1,p_2})(j)) \arrow[r, "s' \circ \tilde{g}_j"] & Z((\mathbf{\Sigma}_d^{q_1,q_2})_c^{p_1,p_2}).
\end{tikzcd}$$

Here $s' = Z(s^{p_1,p_2,q_1,q_2})$, where s^{p_1,p_2,q_1,q_2} is as defined in proposition 3.13, and $i = \lambda$ and $j = \mu$. Let α be the relevant distinguished point labeled with j^*. Let β be the relevant distinguished point labeled with i^*. As above, let g be the glueing homomorphism provided by Turaev in chapter V of [29]. In the following calculation we use that the integer associated to the morphism s^{p_1,p_2,q_1,q_2} is 0. Commutativity of the diagram above can be rewritten as the following equation

$$g_i \circ h_\beta \circ \dot{q}_i \circ g_j \circ h_\alpha \circ \dot{q}_j = s' \circ g_j \circ h_\alpha \circ \dot{q}_j \circ g_i \circ h_\beta \circ \dot{q}_i. \tag{6.1}$$

Using lemma 6.4 and lemma 6.3 we see that

$$g_i \circ h_\beta \circ \dot{q}_i \circ g_j \circ h_\alpha \circ \dot{q}_j = g_i \circ g_j \circ h_\alpha \circ \dot{q}_j \circ h_\beta \circ \dot{q}_i.$$

Using lemma 6.4 we then get that

$$s' \circ g_j \circ h_\alpha \circ \dot{q}_j \circ g_i \circ h_\beta \circ \dot{q}_i = s' \circ g_j \circ g_i \circ h_\alpha \circ \dot{q}_j \circ h_\beta \circ \dot{q}_i.$$

Using axiom 1.5.4(ii) in chapter V of [29] we see that

$$s' \circ g_j \circ g_i = g_i \circ g_j.$$

Therefore we see that equation (6.1) holds.
(ii) In the notation of definition 4.1 we must prove that the following diagram commutes

$$\begin{tikzcd}
Z(\mathbf{\Sigma}_1) \arrow[r, "\tilde{g}"] \arrow[d, "Z(\mathbf{f})"'] & Z((\mathbf{\Sigma}_1)_c^{p_0,p_1}) \arrow[d, "Z(\mathbf{f}')"] \\
Z(\mathbf{\Sigma}_2) \arrow[r, "\tilde{g}"] & Z((\mathbf{\Sigma}_2)_{c'}^{q_0,q_1}).
\end{tikzcd}$$

This amounts to proving

$$\mathcal{H}(f') \circ g \circ h \circ \dot{q} = g \circ h \circ \dot{q} \circ \mathcal{H}(f). \tag{6.2}$$

Here we use that $\mathbf{f}'$ is equipped with the same integer as $\mathbf{f}$. Equation (6.2) follows directly from lemma 5.6, lemma 5.5, and axiom 1.5.4(i) in chapter V of [29]. Here we use that even though the naturality condition is only formulated for m-homeomorphisms in this axiom, Turaev argues in section 4.6 of chapter V that it is also valid for weak m-homeomorphisms.
(iii) In the notation of definition 4.1 we must prove that the following diagram commutes

$$\begin{tikzcd}
Z(\mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2) \arrow[r, "\tilde{g}"] & Z((\mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2)_c^{p_0,p_1}) \\
Z(\mathbf{\Sigma}_1) \otimes Z(\mathbf{\Sigma}_2) \arrow[r, "\tilde{g}\otimes 1"] \arrow[u, "Z_2"] & Z((\mathbf{\Sigma}_1)_c^{p_0,p_1}) \otimes Z(\mathbf{\Sigma}_2). \arrow[u, "Z(\iota)\circ Z_2"']
\end{tikzcd}$$

As the integer associated with the morphism ι is zero, this takes the following equational form

$$g \circ h \circ \dot{q} \circ \mathcal{H}_2 = \mathcal{H}(\iota) \circ \mathcal{H}_2 \circ (g \circ h \circ \dot{q} \otimes 1). \tag{6.3}$$

Rewrite the RHS as $\mathcal{H}(\iota) \circ \mathcal{H}_2 \circ (g \otimes 1) \circ (h \circ \dot{q} \otimes 1)$. Now use lemma 5.5 and lemma 5.6 to rewrite the LHS as $g \circ H_2 \circ (h \circ \dot{q} \otimes 1)$. Now axiom 1.5.4($iii$) entails $g \circ H_2 = \mathcal{H}(\iota) \circ \mathcal{H}_2 \circ (g \otimes 1)$. This implies equation (6.3).
(iv) In the notation of definition 4.1 we must prove that the following diagram commutes

$$\begin{tikzcd} Z(\mathbf{\Sigma}) \arrow[r, "\tilde{g}^{c_1}"] \arrow[dr, "\tilde{g}^{c_2}"'] & Z(\mathbf{\Sigma}_{c_1}^{p_0,p_1}) \arrow[d, "Z(\tilde{f}(c_1,c_2))"] \\ & Z(\mathbf{\Sigma}_{c_2}^{p_0,p_1}). \end{tikzcd}$$

As the integer associated with the morphism $\tilde{f}(c_1, c_2)$ is zero, this takes the form

$$\mathcal{H}(\tilde{f}(c_1, c_2))) \circ g^{c_1} \circ h \circ \dot{q} = g^{c_2} \circ h \circ \dot{q}. \tag{6.4}$$

Lemma 6.5. *The morphisms $\mathcal{H}(\tilde{f}(c_1, c_2))) \circ g^{c_1}$ and g^{c_2} are operator invariants of extended three manifolds that are naturally e-homeomorphic through an e-homeomorphism commuting with boundary parametrizations. In particular they coincide.*

We see that lemma 6.5 implies equation (6.4). The lemma will be proven in section 8. □

7 Review of the TQFT based on extended cobordisms

As observed above, $\mathcal{H}$ is defined as

$$\mathcal{H}(\Sigma) = \mathcal{T}_{\mathcal{V}}^{e}(\overline{\Sigma}),$$

where $\overline{\Sigma}$ is the associated extended surface, and $\mathcal{T}_{\mathcal{V}}^{e}$ is the modular functor based on the category of extended surfaces and the modular tensor category $\mathcal{V}$. In this section we will give a quick review of the TQFT $(\mathcal{T}_{\mathcal{V}}^{e}, \tau_{\mathcal{V}}^{e})$ based on the cobordism theory of extended cobordisms, as defined in chapter IV of [29]. We will assume familiarity with the axioms for a TQFT based on a cobordism theory as defined in chapter III of [29]. We will assume familiarity with the quantum invariant $\tau(M, \Omega)$ of a closed oriented three manifold M containing a ribbon graph Ω with colors in $\mathcal{V}$. This invariant is defined in chapter II of [29]. We will however provide the formula associated to a surgery presentation below, but we will not explain the Reshetikhin-Turaev functor $F_{\mathcal{V}}$ as defined in chapter I of [29].

7.1 Quantum invariants of 3-manifolds

We now recall the construction of $\tau(\tilde{M}, \Omega) \in K$ where $\tilde{M}$ is a closed oriented three manifold with a colored ribbon graph Ω inside. Here $\tau(\tilde{M}, \Omega)$ is called the quantum invariant of $(\tilde{M}, \Omega)$. We may assume $\tilde{M} = \partial W$, where W is a compact oriented four manifold obtained by performing surgery along a framed link $L = \{L_1, ..., L_m\}$ in $S^3 = \partial B^4$. Let $\sigma(L)$ be the signature of the intersection form on $H_2(W, \mathbb{R})$. Let $\mathrm{Col}(L)$ be the set of all colorings of L by colors in $(V_i)_{i\in I}$. For any coloring λ we let $\Gamma(L, \lambda)$ be the associated colored ribbon graph in S^3. Then $\tau(\tilde{M}, \Omega)$ is given by

$$\tau(\tilde{M}, \Omega) = \Delta^{\sigma(L)} D^{-\sigma(L)-m-1} \sum_{\lambda \in \mathrm{Col}(L)} \dim(\lambda) F(\Gamma(L, \lambda) \cup \Omega). \tag{7.1}$$

Here $\dim(\lambda) = \dim(\lambda_1) \cdots \dim(\lambda_m)$ and D is given by

$$D^2 = \sum_{i \in I} \dim(i)^2,$$

and Δ is given by

$$\Delta := \sum_{i \in I} k_i^{-1} (\dim(i))^2 \in K,$$

where the k_i are the standard twists coefficients - Turaev denotes them v_i in [29].

7.2 The TQFT based on decorated cobordisms

7.2.1 Modules of states

Recall the notion of a d-surface and decorated type as defined in section 1.1 of chapter IV in [29]. Recall the notion of a standard d-surface and of a parametrized d-surface as in section 1.2 and section 1.3 of chapter IV in [29]. Assume Σ is a connected parametrized d-surface of topological type t given by $(g; (W_i, \mu_i), ..., (W_m, \mu_m))$. Recall the standard d-surface of type t. This is denoted by Σ_t. These notions can be found in sections $1.1 - 1.3$ of chapter IV. For a decorated type t as above, and for $i \in I^g$, let

$$\Phi(t, i) := W_1^{\mu_1} \otimes \cdots \otimes W_m^{\mu_m} \bigotimes_{s=1}^{g} (V_{i_s} \otimes V_{i_s}^*).$$

Here $W^1 = W$, and $W^{-1} = W^*$. Recall that elements of $\Phi(t, i)$ can be thought of as colorings of the ribbon graph R_t sitting inside Σ_t, as defined in section 1.2 of [29]. Moreover we define $\mathcal{T}(\Sigma) := \Psi(t)$ where

$$\Psi(t) := \bigoplus_{i \in I^g} \mathrm{Hom}(\mathbf{1}, \Phi(t, i)).$$

Finally, if Σ is not connected, then we define $\mathcal{T}(\Sigma)$ to be the unordered tensor product of the modules of states of the components of Σ.

7.2.2 Operator invariants

We now describe the construction of $\tau(M)$, where M is a decorated cobordism. That is, M is a triple $(M, \partial_- M, \partial_+ M)$ where $\partial_\pm M$ are parametrized d-surfaces. For a general decorated type t let U_t be the standard decorated handlebody bounded by Σ_t as in section 1.7 of chapter IV in [29]. Equip it with the RH orientation. For an element $x \in \Phi(t, i)$ consider the three manifold with boundary $H(U_t, R_t, i, x)$. Let U_t^- be the image of U_t under the reflection of $\mathbb{R}^3$ in the plane $\mathbb{R}^2 \times \{1/2\}$. We denote this orientation reversing diffeomorphism by mir : $\mathbb{R}^3 \to \mathbb{R}^3$. Equip U_t^- with the RH orientation. We recall that they contain certain ribbon graphs denoted R_t, R_{-t} respectively. Let $f : \Sigma_{t_0} \to \partial_- M$ be a parametrization of a component of $\partial_- M$. We glue in U_{t_0} by glueing ∂U_{t_0} to $\Sigma_{t_1} \times \{0\}$ through f. We do this for all components of $\partial_- M$. Similarly, for any parametrized component $g : \Sigma_{t_1} \to \partial_+ M$ we glue in $U_{t_1}^-$ by glueing according to -g $\circ$ mir : $\partial(U_{t_1}^-) \to \Sigma_{t_1}$. This produces a closed oriented three manifold $\tilde{M}$ with a ribbon graph inside, such that choosing an element $x \in \mathcal{T}(\partial_- M)$ and an element $y \in \mathcal{T}(\partial_+ M)^*$ will produce a colored ribbon graph $\Omega(x, y) \subset \tilde{M}$. This descends to a K-linear map $\mathcal{T}(\partial_- M) \otimes_K \mathcal{T}(\partial_+ M)^* \longrightarrow K$ given by

$$x \otimes y \longrightarrow \tau(\tilde{M}, \Omega(x, y)),$$

where τ is the quantum invariant defined in chapter II of [29]. This pairing induces a morphism $j : \mathcal{T}(\partial_- M) \to \mathcal{T}(\partial_+ M)$. Finally, composing this with the map $\eta : \mathcal{T}(\partial_+ M) \to \mathcal{T}(\partial_+ M)$ induced by multiplication by $\mathcal{D}^{1-g}\dim(i)$ on $\mathrm{Hom}(\mathbf{1}, \Phi(t_1; i))$, we get the desired K-linear map

$$\tau(M) := \eta \circ j : \mathcal{T}(\partial_- M) \to \mathcal{T}(\partial_+ M).$$

7.3 The TQFT based on extended cobordisms

7.3.1 Module of states

We start by describing the module of states for an e-surface. Start by assuming that Σ is a connected e-surface. Recall the notion of a parametrization of Σ. This is simply a weak e-homeomorphism $\Sigma_t \to \Sigma$. Given two parametrizations $f : \Sigma_{t_0} \to \Sigma$ and $g : \Sigma_{t_1} \to \Sigma$, we wish to define an isomorphism $\varphi(f, g)$ between $\Psi(t_0)$ and $\Psi(t_1)$. We define

$$\varphi(f, g) := (D\Delta^{-1})^{-\mu((f_0)_*(\lambda(t_0)), \lambda(\Sigma), (g_0)_*(\lambda(t_1)))} \mathcal{E}(g^{-1} f) : \Psi(t_0) \to \Psi(t_1).$$

Here μ is the Maslov index for triples of Lagrangian subspaces. $\mathcal{E}$ is the morphism induced by the decorated three manifold $\Sigma_{t_1} \times I$ where the bottom is parametrized by $g^{-1} \circ f : \Sigma_{t_0} \to \Sigma_{t_1}$ and the top is parametrized by the identity. Turaev proves in [29] that

$$\varphi(f_1, f_2) \circ \varphi(f_0, f_1) = \varphi(f_0, f_2).$$

Now $\mathcal{T}^e(\Sigma)$ is defined as the K-module of coherent sequences $(x(t,f))_{(t,f)}$ where we index over all parametrizations. Finally, if Σ is not connected, then we define $\mathcal{T}^e(\Sigma)$ to be the unordered tensor product of the modules of states of the components of Σ.

7.3.2 Operator invariants

Consider an extended 3-manifold $(M, \partial_- M, \partial_+ M)$. Then any two parametrizations $f : \Sigma_- \to \partial_- M$ and $g : \Sigma_+ \to \partial_+ M$ makes M into a decorated cobordism $\tilde{M}$. We now define $\tau^e(M)$ to be composition

$$\mathcal{T}^e(\partial_- M) \to \mathcal{T}(\Sigma_-) \stackrel{\lambda(M)\tau(\tilde{M})}{\longrightarrow} \mathcal{T}(\Sigma_+) \to \mathcal{T}^e(\partial_+ M).$$

Here $\lambda(M)$ is an invertible element of K defined in section 6.5 of chapter IV of [29].

8 Proofs of lemmas

Proposition 8.1 (The cobordism associated to a weak e-homomorphism)**.** *Let $f : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ be a weak e-homomorphism. There is an invertible scalar $c \in K$ such that the operator invariant of the extended cobordism $\Sigma_1 \times I \cup_f \Sigma_2 \times I$ coincide with $c\mathcal{T}^e(f) : \mathcal{T}^e(\Sigma_1) \to \mathcal{T}^e(\Sigma_2)$. The scalar c depends only on the underlying continuous map of f and the Lagrangian subspaces $L_i \subset H_1(\Sigma_i)$.*

Proof. This follows from theorem 7.1 of chapter VII in [29]. □

Here it is understood that the extended three manifold $\Sigma_1 \times I \cup_f \Sigma_2 \times I$ is obtained by glueing the top of $\Sigma_1 \times I$ to the bottom of $\Sigma_2 \times I$ through f.

Proof of Lemmas 5.5 and 6.1. Using 8.1 and theorem 7.1 of chapter VII in [29], we see that both $\mathcal{H}(g) \circ \dot{f}$ and $\dot{f} \circ \mathcal{H}(g)$ are - up to the same scalar - induced by glueing certain extended three manifolds. Let M_1 be the extended three manifold with $\tau(M_1) = c\mathcal{H}(g) \circ \dot{f}$, and let M_2 be the extended three manifold with $\tau^e(M_2) = c\dot{f} \circ \mathcal{H}(g)$ Clearly there is a homeomorphism of extended three manifolds taking M_1 to M_2, commuting with boundary parametrizations. Therefore they induce the same morphism. □

Proof of Lemmas 5.6 and 6.2. The proof is virtually identical to the proof of Lemma 5.5. □

Proof of Lemma 6.3. The proof is virtually identical to the proof of Lemma 5.5. □

Proof of Lemma 6.4. We start by recalling the definition of the glueing homomorphism provided by Turaev in sections 4.4 – 4.6 of chapter V in [29]. Let M_2 be the extended three manifold obtained by attaching handles to $\Sigma \times I$ as in section 4.4. of chapter V of [29]. The attachment uses the glueing data c. The operator invariant $\tau^e(M_2)$ now yields a map $g' : \mathcal{T}^e(\Sigma) \to \mathcal{T}^e(\Sigma_c')$. Here Σ_c' is an e-surface canonically e-homeomorphic to Σ_c. Composing with the associated isomorphism of K-modules, we get the required glueing homomorphism $g : \mathcal{T}^e(\Sigma) \to \mathcal{T}^e(\Sigma_c)$. Similarly, if we let $\widetilde{\Sigma}$ and $\widetilde{\Sigma}_c'$ be the two same e-surfaces with the relevant change of markings, then the glueing g is obtained as the operator invariant of $\widetilde{M_2}$ with the obvious notation. Assume now that $r = \dot{f}$ for some morphism $f : (V, +1) \to (W, +1)$. Recalling the naturality property of $\dot{f}$ we see that it is enough to argue that $\dot{f}$ commute with g'. Let $M_1 = \Sigma \times I$ be the extended cobordism inducing $\dot{f}$. Then $g' \circ \dot{f}$ is a multiple of $\tau^e(\widetilde{M_2} \circ M_1)$, where we glue the top of M_1 to the bottom of $\widetilde{M_2}$ through the identity. To compute the relevant scalar we use theorem 7.1 in Chapter IV of [29]. Since the identity is an e-homeomorphism here, there is only one Maslow index to compute. In the notation of theorem 7.1 we have $\mathrm{id}_\#(N_1)_*(\lambda_-(M_1)) = \mathrm{id}_\#\lambda_+(M_1)$. Thus we see that

$$0 = \mu\left(\mathrm{id}_\#(N_1)_*(\lambda_-(M_1)), \mathrm{id}_\#\lambda_+(M_1), N_2^*(\lambda_+(\widetilde{M_2})\right).$$

See the proof of lemma 6.7.2 in chapter IV of [29]. Thus we get that

$$\tau^e(\widetilde{M_2} \circ M_1) = g' \circ \dot{f}.$$

Clearly $\widetilde{M_2} \circ M_1$ is e-homeomorphic to a cylinder with handles attached on the top, such that the β-band has a coupon colored with the f-coupon, and all other 'vertical' bands are colored with id. The exact same argument will yield a similar description of $\dot{f} \circ g$. Consider $Q := \Sigma_c \times I$ as the extended cobordism inducing $\dot{f} : \mathcal{T}^e(\Sigma_c) \xrightarrow{\sim} \mathcal{T}^e(\widetilde{\Sigma_c})$. Arguing as above, wee see that $\dot{f} \circ \underline{g'}$ is given by the operator invariant $\tau^e(Q \circ M_2)$. But this is e-homeomorphic to $\widetilde{M_2} \circ M_1$. Therefore $g \circ \dot{f} = \dot{f} \circ g'$. Observe that a homomorphism of type h can be dealt with in exactly the same way. □

Proof of Lemma 6.5. This is a consequence of the description of the glueing homomorphism given above, together with the existence of the proclaimed e-morphisms. □

9 Uniqueness up to quasi-isomorphism

We observe that the construction of the glueing $\tilde{g}$ map depended on a choice of isomorphisms $q_i : V_{i^*} \xrightarrow{\sim} V_i^*$. This dependence is not essential.

Definition 9.1 (Quasi-isomorphism)**.** Let (Z, g) and (Z', g') be two modular functors with the same label set Λ. These are said to be quasi-isomorphic if there is a pair (Φ, γ) satisfying the following conditions. Φ is an assignment, which for each labeled marked surface $\mathbf{\Sigma}$ gives an isomorphism

$$\Phi(\mathbf{\Sigma}) : Z(\mathbf{\Sigma}) \xrightarrow{\sim} Z'(\mathbf{\Sigma}).$$

This assignment is required to be natural with respect to morphisms of modules induced by morphisms of labeled marked surfaces. Similarly it is required to preserve the splitting into tensor products induced by disjoint union, as well as the permutation map. γ is an assignment $\gamma : I \to K^*$ such that if $\mathbf{\Sigma}_c$ is obtained from $\mathbf{\Sigma}$ from glueing along an ordered pair (p, q) where p is labeled with λ, then the following diagram is commutative

$$\begin{tikzcd} Z(\mathbf{\Sigma}) \arrow[r, "g"] \arrow[d, "\gamma(\lambda)\Phi(\mathbf{\Sigma})"'] & Z(\mathbf{\Sigma}_c) \arrow[d, "\Phi(\mathbf{\Sigma}_c)"] \\ Z'(\mathbf{\Sigma}) \arrow[r, "g'"] & Z'(\mathbf{\Sigma}_c). \end{tikzcd} \tag{9.1}$$

Moreover we demand that $\gamma(\lambda)\gamma(\lambda^*) = 1$ for all $\lambda \in I$.

This is easily seen to define an equivalence relation on modular functors with the same label set.

Theorem 9.2 (Independence of (q_i) up to quasi-isomorphism)**.** *Let q, q' be two choices of isomorphisms $V_{i^*} \xrightarrow{\sim} V_i^*$. Then the two resulting modular functors $Z(q)$ and $Z(q')$ are quasi-isomorphic.*

Proof. Write $q_i' = f_i q_i$. Then we have $\tilde{g}_j(q') = f_j \tilde{g}_j(q)$. We want to construct a pair (Φ, γ). Consider a labeled marked surface $\mathbf{\Sigma}$ with labels $i_1, ..., i_k$. We want to construct $\Phi(\mathbf{\Sigma})$ to be of the form $(\prod_l^k \alpha_{i_l})\mathrm{Id}_{Z(\mathbf{\Sigma})}$ for some function $\alpha : I \to K^*$.

Assume $\mathbf{\Sigma}_c$ is obtained from $\mathbf{\Sigma}$ by glueing along an ordered pair (p, q) where p is labeled with i. Assume the labels of $\mathbf{\Sigma}$ are $i_1, ..., i_k, i, i^*$. Then equation (9.1) becomes

$$\prod_{l=1}^{k} \alpha(i_l) = \gamma(i) f_i \alpha(i) \alpha(i^*) \prod_{l=1}^{k} \alpha(i_l).$$

Thus we are forced to define

$$\gamma(i) := \frac{1}{\alpha(i)\alpha(i^*) f_i}.$$

We still have to ensure $\gamma(i)\gamma(i^*) = 1$. We see that this will follow for any choice of α with

$$(\alpha(i)\alpha(i^*))^2 f_i f_{i^*} = 1,$$

which is easy to solve (by adjoining the needed square roots to K if needed). $\square$

10 Universal property

In this section we will describe how to apply Z, and how to use it in calculations. Let $\mathbf{\Sigma}$ be a connected labeled marked surface. Recall that a parametrization $f : \Sigma_t \to \overline{\mathcal{G}(\mathbf{\Sigma})}$ is an orientation preserving homeomorphism that preserves all structure of extended surfaces, except possibly the Lagrangian subspaces in homology. These are also called weak e-homeomorphisms. Clearly the set of parametrizations is non-empty. Let f be a parametrization. This will induce an isomorphism

$$\Psi(t) \simeq Z(\mathbf{\Sigma}).$$

For the definition of $\Psi(t)$ see section 7. We now recall the definition of $Z(\mathbf{\Sigma})$ and describe the isomorphism above. For any pair of parametrizations

$$f_i : \Sigma_{t_i} \to \overline{\mathcal{G}(\mathbf{\Sigma})}, i = 1, 2,$$

there is an isomorphism

$$\varphi(f_1, f_2) : \Psi_{t_1} \xrightarrow{\sim} \Psi_{t_2}.$$

See section 7. With the obvious notation these isomorphisms satisfy

$$\varphi(f_1, f_3) = \varphi(f_2, f_3) \circ \varphi(f_1, f_2).$$

The module $Z(\mathbf{\Sigma})$ is the module of coherent sequences. Hence an element of this module is an equivalence class of pairs (x, f) where f is a parametrization with domain Σ_t and x is an element of $\Psi(t)$. We have $(x, f) \sim (y, g)$ if and only if $\varphi(f, g)(x) = y$. The isomorphism $\Psi(t) \simeq Z_{\mathcal{V}}(\mathbf{\Sigma})$ induced from a parametrization is simply $x \mapsto (x, f)$.

Now we describe $Z_{\mathcal{V}}(\mathbf{f})$ when $\mathbf{f} = (f, s) : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ is a morphism of connected labeled marked surfaces. Any representative f' of the equivalence class f will induce $Z_{\mathcal{V}}(\mathbf{f})$ which is given by

$$(x, g) \mapsto ((\Delta^{-1} D)^s x, f' \circ g).$$

Here we also write f' for the induced e-homeomorphism $\overline{\mathcal{G}(\mathbf{\Sigma}_1)} \to \overline{\mathcal{G}(\mathbf{\Sigma}_2)}$.

11 The duality pairing

Consider a modular functor V based on a label set Λ. For a modular functor with duality we would like the operation of orientation reversal to correspond to the operation of taking the dual K-module. That is, we would like a perfect pairing $V(\mathbf{\Sigma}) \otimes V(-\mathbf{\Sigma}) \to K$ that is compatible with the structure of V. Before we formulate the axioms, consider an arbitrary Λ-labeled marked surface $\mathbf{\Sigma}'$. Observe that if $p, q \in \mathbf{\Sigma}'$ are subject to glueing then so are $p, q \in -\mathbf{\Sigma}'$. Observe that if $\mathbf{\Sigma}$ is the result of glueing $\mathbf{\Sigma}$ along p, q then $-\mathbf{\Sigma}$ is the result of glueing $-\mathbf{\Sigma}'$ along the same ordered pair of points.

Definition 11.1 (Duality)**.** Let (V, g) be a modular functor based on Λ and K. A duality for V is a perfect pairing

$$(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}} : V(\mathbf{\Sigma}) \otimes V(-\mathbf{\Sigma}) \to K,$$

subject to the following axioms.

Naturality. Let $\mathbf{f} = (f, s) : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ be a morphism between Λ-labeled marked surfaces. Then

$$(V(\mathbf{f}), V(-\mathbf{f}))_{\mathbf{\Sigma}_2} = (\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}_1}. \tag{11.1}$$

Compatibility with disjoint union. Consider a disjoint union of Λ-labeled marked surfaces $\mathbf{\Sigma} = \mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2$. The modular functor V provides an isomorphism

$$\eta : V(\mathbf{\Sigma}) \otimes V(-\mathbf{\Sigma}) \xrightarrow{\sim} V(\mathbf{\Sigma}_1) \otimes V(-\mathbf{\Sigma}_1) \otimes V(\mathbf{\Sigma}_2) \otimes V(-\mathbf{\Sigma}_2).$$

We demand that with respect to the natural isomorphism $K \otimes K \simeq K$ we have that

$$(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}} = ((\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}_1} \otimes (\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}_2}) \circ \eta. \tag{11.2}$$

Compatibility with glueing. Let $\mathbf{\Sigma}$ be a Λ labeled marked surface obtained from glueing. Consider the glueing isomorphism

$$g : \bigoplus_{\lambda \in \Lambda} V(\mathbf{\Sigma}(\lambda)) \xrightarrow{\sim} V(\mathbf{\Sigma}),$$

as described in definition 4.1. We have that

$$(g, g)_{\mathbf{\Sigma}} = \sum_{\lambda \in \Lambda} \mu_\lambda (\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}(\lambda)}. \tag{11.3}$$

Here $\mu_\lambda \in K$ is invertible and depends only on the isomorphism class of $\mathbf{\Sigma}(\lambda)$ for all λ.

Compatibility with orientation reversal. For a Λ-labeled marked surface $\mathbf{\Sigma}$ we demand that there is an invertible element $\mu \in K^*$ that only depends on $\mathbf{\Sigma}$ such that for all $(v, w) \in V(\mathbf{\Sigma}) \times V(-\mathbf{\Sigma})$ the following equation holds

$$\mu(w, v)_{-\mathbf{\Sigma}} = (v, w)_{\mathbf{\Sigma}}. \tag{11.4}$$

It is worth spelling out how we demand that the duality is compatible with glueing in a little more detail. Observe $-(\mathbf{\Sigma}(\lambda)) = (-\mathbf{\Sigma})(\lambda^\dagger)$. Thus the glueing isomorphism is a splitting

$$g' : \bigoplus_{\lambda \in \Lambda} V(-\mathbf{\Sigma}(\lambda^\dagger)) \xrightarrow{\sim} V(-\mathbf{\Sigma}).$$

This gives a decomposition

$$\bigoplus_{\lambda,\lambda'\in\Lambda} V(\mathbf{\Sigma}(\lambda)) \otimes V(-\mathbf{\Sigma}(\lambda')) \xrightarrow{g\otimes g} V(\mathbf{\Sigma}) \otimes V(-\mathbf{\Sigma}).$$

Then the statement is that $g(V(\mathbf{\Sigma}(\lambda))$ and $g(V(-\mathbf{\Sigma}(\lambda'))$ are orthogonal w.r.t. the duality $(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}}$ unless $\lambda = \lambda'$. In this case we have

$$(g_\lambda, g_{\lambda^\dagger})_{\mathbf{\Sigma}} = \mu_\lambda(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}(\lambda)}.$$

11.1 Review of the duality for Turaev's modular functor based on extended surfaces

Let Σ be an e-surface. Recall the operation of oriental reversal as described in definition 3.21. We can think of $\Sigma \times I$ as a morphism $\Sigma \sqcup (-\Sigma) \to \emptyset$. This induces a perfect pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_\Sigma : \mathcal{T}^e(\Sigma) \otimes \mathcal{T}^e(-\Sigma) \to K.$$

See chapter *III* section 2 in [29]. The pairing is compatible with the action of e-homeomorphisms in the sense that for any e-homeomorphism $f : \Sigma_1 \to \Sigma_2$ we have

$$\langle \mathcal{T}^e(f)(\,\cdot\,), \mathcal{T}^e(-f)(\,\cdot\,)\rangle_{\Sigma_2} = \langle\,\cdot\,,\,\cdot\,\rangle_{\Sigma_1}.$$

It is proven in exercise 7.3 in chapter *IV* in [29] that the pairing is also natural with respect to weak e-homeomorphisms. The pairing is multiplicative with respect to disjoint union. Moreover the pairing is self-dual in the following sense

$$\langle\,\cdot\,,\,\cdot\,\rangle_\Sigma \circ \mathrm{Perm} = \langle\,\cdot\,,\,\cdot\,\rangle_{-\Sigma}.$$

All these properties are stated in axiom 1.2.4 in section 1.2 of chapter *III* in [29].

11.2 Construction of a duality pairing for Z

In order to induce Turaev's duality pairing, we will need an isomorphism

$$Z(-\mathbf{\Sigma}) \xrightarrow{\sim} \mathcal{T}^e(-\overline{\mathcal{G}(\mathbf{\Sigma})}).$$

It is very important to note that our choice made below can be scaled. See remark 11.3, and section 14.

Consider an I-labeled marked surface $\mathbf{\Sigma} = (\Sigma, P, V, (i_p)_{p\in P}, L)$. Write

$$\overline{\mathcal{G}(\mathbf{\Sigma})} = (\widetilde{\Sigma}, (\alpha_p)_{p\in P}, (V_{i_p}, 1)_{p\in P}, L)$$

for the e-surface associated to the m-surface $\mathcal{G}(\mathbf{\Sigma})$. We have that

$$\overline{\mathcal{G}(-\mathbf{\Sigma})} = (\widetilde{\Sigma}, (\alpha_p)_{p\in P}, (V_{i_p^*}, 1)_{p\in P}, L).$$

This is not quite $-\overline{\mathcal{G}(\mathbf{\Sigma})}$. However, let $\dot{q}$ be the isomorphism of states that take all markings $(V_{i^*}, 1)$ to $(V_i^*, 1)$ and let $\tilde{h}$ be the isomorphism of modules of states that exchange all markings $(V_i^*, 1)$ with $(V_i, -1)$. Define

$$*\Sigma := (\widetilde{\Sigma}, (\alpha_p)_{p\in P}, (V_{i_p}, -1)_{p\in P}, L).$$

Now let r be the orientation-preserving diffeomorphism

$$r : *\Sigma \xrightarrow{\sim} -\overline{\mathcal{G}(\mathbf{\Sigma})},$$

that is given by twisting all arcs with a half-twist. This can of course be done in two different ways, the important thing for now is that it is done the same way for all arcs. We return to this choice in the proof of proposition 11.7. Then we have an isomorphism

$$\mathcal{T}^e(r) \circ \tilde{h} \circ \dot{q} : \mathcal{T}^e(\overline{\mathcal{G}(-\mathbf{\Sigma})}) \xrightarrow{\sim} \mathcal{T}^e(-\overline{\mathcal{G}(\mathbf{\Sigma})}).$$

This will allows us to define a perfect pairing. For notational convenience we will simply write $r' = \mathcal{T}^e(r)$. We will write

$$\zeta = \mathcal{T}^e(r) \circ \tilde{h} \circ \dot{q} : Z(-\mathbf{\Sigma}) \xrightarrow{\sim} \mathcal{T}^e(-\overline{\mathcal{G}(\mathbf{\Sigma})}). \tag{11.5}$$

One last notational definition will be convenient. For a decorated type t of the form $(g; (V_{i_l}, \nu_l))$ with $i_l \in I$ for all l, let $t*$ be the decorated type $(g; (V_{i_l^*}, \nu_l))$.

Definition 11.2. Consider an I-labeled marked surface $\mathbf{\Sigma}$. We have a perfect pairing given by the composition

$$(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}} = \langle\,\cdot\,, \zeta(\,\cdot\,)\rangle_{\overline{\mathcal{G}(\mathbf{\Sigma})}}.$$

Remark 11.3. We note that there are choices involved in defining ζ and therefore choices involved in the pairing. In particular we here note that we can choose to scale the q_i used in the definition of ζ, such that we use a different set of isomorphisms in the glueing and in the duality. We will examine this phenomenon in section 14. Moreover we have here resorted to a slight abuse of notation, since technically, $Z(\mathbf{\Sigma})$ is not equal to $\mathcal{T}^e(\overline{\mathcal{G}(\mathbf{\Sigma})})$, but canonically isomorphic to it.

Theorem 11.4 (Duality)**.** *The pairing* $(\cdot, \cdot)_{\mathbf{\Sigma}}$ *is a duality pairing for the modular functor* $Z_{\mathcal{V}}$.

11.3 Description of the glueing homomorphism

For the proof of theorem (11.4) we will use an explicit description of the glueing homomorphism in two cases.

11.3.1 The two points lie on the same component

We will start by assuming that the points subject to glueing lie on the same component. Write $\mathbf{\Sigma}$ for the labeled marked surface resulting from glueing and write $\mathbf{\Sigma}(i)$ for the labeled marked surface with the two points subject to glueing where the preferred point is labeled with i. Due to the multiplicativity of the glueing we will assume that $\mathbf{\Sigma}(i)$ is connected. See equation (4.4). We will start by assuming $\overline{\mathcal{G}(\mathbf{\Sigma}(i))} = \Sigma_t$, where

$$t = (g; (V_{i_1}, 1), ..., (V_{i_k}, 1), (V_i, 1), (V_{i^*}, +1)).$$

Hence $\overline{\mathcal{G}(\mathbf{\Sigma})} = \Sigma_{t'}$ where t' is equal to the topological type $(g+1, (V_{i_1}, 1), ..., (V_{i_k}, 1))$. Moreover let

$$\tilde{t} = (g; (V_{i_1}, 1), ..., (V_{i_k}, 1), (V_i, 1), (V_i, -1)).$$

In Turaev's setup we see that $\Sigma_{\tilde{t}}$ can be glued along the points labeled with $(V_i, 1)$ and $(V_i, -1)$ to obtain $\Sigma_{t'}$. Now the identity parametrizations induces isomorphisms

$$\begin{aligned}
\mathcal{T}^e(\Sigma_{\tilde{t}}) &\simeq \bigoplus_{l \in I^g} \mathrm{Hom}(\mathbf{1}, \Phi(\tilde{t}, l)),\\
Z(\mathbf{\Sigma}) &\simeq \bigoplus_{l \in I^{g+1}} \mathrm{Hom}(\mathbf{1}, \Phi(t', l)),\\
Z(\mathbf{\Sigma}(i)) &\simeq \bigoplus_{l \in I^g} \mathrm{Hom}(\mathbf{1}, \Phi(t, l)).
\end{aligned}$$

With respect to these isomorphisms, we see that our glueing homomorphism is the composition

$$Z(\mathbf{\Sigma}(i)) \overset{h\circ\dot{q}}{\to} \mathcal{T}^e(\Sigma_{\tilde{t}}) \hookrightarrow Z(\mathbf{\Sigma}). \tag{11.6}$$

Here the last map is the natural summandwise inclusion. This is proven in section 5.9 of chapter V of [29]. Thus we only need to describe the first map. Consider a summand in $Z(\mathbf{\Sigma})$ and an element f

$$f \in \mathrm{Hom}\left(\mathbf{1}, (\otimes_{j=1}^k V_{i_j}) \otimes V_i \otimes V_{i^*} \otimes_{r=1}^g (V_{l_r} \otimes V_{l_r}^*)\right).$$

Let $W = (\otimes_{j=1}^k V_{i_j}) \otimes V_i$, $R = \otimes_{r=1}^g (V_{l_r} \otimes V_{l_r}^*)$ and $q_i : V_{i^*} \xrightarrow{\sim} V_i^*$ be the isomorphism used to define the glueing. Post composing f with $(1_W \otimes q_i \otimes 1_R)$ we get an element

$$(1_W \otimes q_i \otimes 1_R) \circ f \in \mathrm{Hom}\left(\mathbf{1}, (\otimes_{j=1}^k V_{i_j}) \otimes V_i \otimes V_i^* \otimes_{r=1}^g (V_{l_r} \otimes V_{l_r}^*)\right).$$

To see that $h \circ \dot{q}(f) = (1_W \otimes q_i \otimes 1_R) \circ f$, one can either go through the construction given in the review in section 7 above and use that the identity cylinder induces the identity, or one can use the techniques of section 2.3 in chapter IV of [29].

Now we describe the glueing in a slightly more general situation. We observe that Σ_r is naturally a labelled marked surface, for any type r where all marks are of type $(V_i, 1)$. Assume that we have a parametrization $f : \Sigma_t \to \overline{\mathcal{G}(\mathbf{\Sigma}(i))}$. There is a natural homeomorphism $\overline{\mathcal{G}(\mathbf{\Sigma}(i))} \simeq \mathbf{\Sigma}(i)$. With respect to this identification we can think of f as a diffeomorphism between labeled marked surface that preserves all the data except possibly the Lagrangian subspaces in homology. There is also a natural homeomorphism of $\Sigma_{t'}$ with the surface obtained from Σ_t by glueing the ordered pair corresponding under f to the relevant ordered pair of $\mathbf{\Sigma}(i)$. We have a natural homeomorphism $\overline{\mathcal{G}(\mathbf{\Sigma})} = \mathbf{\Sigma}$. As in proposition 3.15 we can choose a parametrization diffeomorphism $F = z(f) : \Sigma_{t'} \to \overline{\mathcal{G}(\mathbf{\Sigma})}$ that is compatible with f. Now f, F induces a pair of isomorphisms

$$Z(\mathbf{\Sigma}) \simeq \bigoplus_{l \in I^{g+1}} \mathrm{Hom}(\mathbf{1}, \Phi(t', l)), \tag{11.7}$$

$$Z(\mathbf{\Sigma}(i)) \simeq \bigoplus_{l \in I^{g}} \mathrm{Hom}(\mathbf{1}, \Phi(t, l)). \tag{11.8}$$

With respect to these isomorphisms the description given by (11.6) is valid.

11.3.2 The two points lie on distinct components

We assume that $\mathbf{\Sigma}(i) = \mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-$ where $\mathbf{\Sigma}^+$ and $\mathbf{\Sigma}^-$ are two spheres. Assume that $(p, i) \in \mathbf{\Sigma}^+$ and that $(q, i^*) \in \mathbf{\Sigma}^-$. We will assume $\mathbf{\Sigma}^- = \Sigma_{t_-}$ and $\mathbf{\Sigma}^+ = \Sigma_{t_+}$ where $t_+ = (0; (V_{i_1}, 1), ..., (V_{i_n}, 1), (V_i, 1))$ and $t_- = (0; (V_{i^*}, 1), (V_{i_{n+1}}, 1), ..., (V_{i_m}, 1))$. Thus we get $\mathbf{\Sigma} = \Sigma_t$ where

$$t = (0, (V_{i_1}, 1), ..., (V_{i_m}, 1)).$$

We get isomorphisms

$$\begin{aligned} Z(\mathbf{\Sigma}(i)) &\simeq \mathrm{Hom}(\mathbf{1}, \Phi(t_+)) \otimes \mathrm{Hom}(\mathbf{1}, \Phi(t_-)), \\ Z(\mathbf{\Sigma}) &\simeq \mathrm{Hom}(\mathbf{1}, \Phi(t)). \end{aligned}$$

Let $V_1 = V_{i_1} \otimes \cdots \otimes V_{i_n}$ and $V_2 = V_{i_{n+1}} \otimes \cdots \otimes V_{i_m}$. With respect to these isomorphisms the glueing homomorphism is given by

$$Z(\mathbf{\Sigma}(i)) \ni x \otimes y \mapsto (1_{V_1} \otimes \overline{d_{V_i}} \otimes 1_{V_2}) \circ (x \otimes q_i \circ y) \in Z(\mathbf{\Sigma}). \tag{11.9}$$

Here $\overline{d_V}$ is given by $F(\cap_V^-)$ where F is the Reshetikhin-Turaev functor and $\cap_V^-$ is defined in Figure 2.6 in section 2.3 of chapter I in [29]. This formula is verified by using the description of $h \circ \dot{q}$ given above and by arguing very similar to the reasoning in section 5.10 of chapter V in [29]. In the general case, where $\mathbf{\Sigma}^+, \mathbf{\Sigma}^-$ are homeomorphic to spheres, we start with a parametrization of each component $\overline{\mathcal{G}(\mathbf{\Sigma}^\pm)}$ and then we glue these two together to obtain a parametrization of $\overline{\mathcal{G}(\mathbf{\Sigma})}$. These will induce isomorphisms with respect to which the glueing is given by (11.9). If we start with parametrizations f, g we will write $z'(f \otimes g)$ for the resulting parametrization.

11.4 Proof of theorem 11.4

Proposition 11.5. *The pairing $(\cdot,\cdot)_{\mathbf{\Sigma}}$ is functorial and is compatible with disjoint union.*

Proof. This easily follows from the properties of $\langle\cdot,\cdot\rangle$ and the functorial properties of $\dot{q}, h$ and r. Observe that even though the axioms in section 1 of chapter II of [29] only ensure that $\langle\,\cdot\,,\,\cdot\,\rangle$ is natural with respect to e-homeomorphisms, it is proven in exercise 7.3 in chapter IV that the pairing is also natural with respect to weak e-homeomorphisms. $\square$

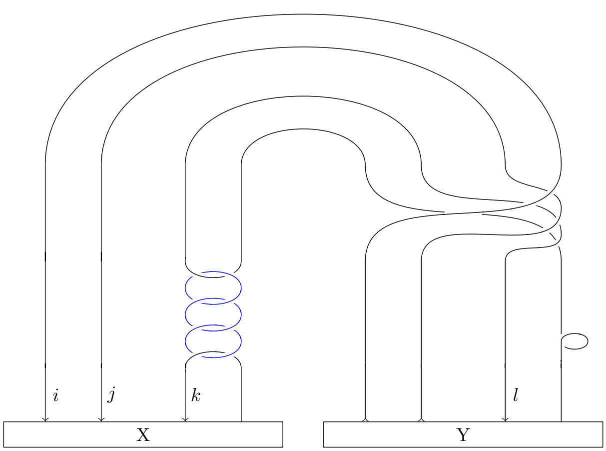
Thus it remains to prove that it is compatible with glueing, and that it is self-dual. The proof of the main propositions needed for these results is based on an explicit ribbon graph presentation of $\langle\,\cdot\,,\,\cdot\,\rangle_\Sigma : \mathcal{T}^e(\Sigma)\otimes\mathcal{T}^e(-\Sigma)\to K$.

All of the proofs in this section are modifications of material appearing in section 10.4. in chapter IV of [29].

Proposition 11.6 (Surgery presentation of $\langle\,\cdot\,,\,\cdot\,\rangle_\Sigma$)**.** *Assume Σ is a connected e-surface. For any parametrization $f:\Sigma_t\to\Sigma$ there is an induced parametrization $y(f):\Sigma_{-t}\to-\Sigma$ such that with respect to the two induced isomorphisms*

$$\Psi(t)\simeq\mathcal{T}^e(\Sigma),$$
$$\Psi(-t)\simeq\mathcal{T}^e(-\Sigma),$$

we have the following surgery presentation of $\langle\,\cdot\,,\,\cdot\,\rangle_\Sigma$.



Observe that in this proposition, orientation reversal is with respect to extended e-surfaces. Here the blue unknot's are the surgery link components. We depict here the genus 1 case. It is obvious how to generalize to higher genus.

We stress that the tangle is a presentation of a perfect pairing

$$\Psi(t) \times \Psi(-t) \to K, \tag{11.10}$$

and that we can only use it as a presentation of the duality pairing with respect to certain pairs of parametrizations $f : \Sigma_t \to \Sigma$ and $y(f) : \Sigma_{-t} \to -\Sigma$. This will be explained in the proof. We will denote the pairing from (11.10) by $\langle \cdot , \cdot \rangle_t$.

Proof. This proof is a slight modification of the proof of theorem 10.4.1 in chapter IV of [29]. Observe that in this proof $-\Sigma$ is the result of using the operation of orientation reversal of extended surfaces on Σ.

Choose a parametrization $f : \Sigma_t \to \Sigma$. This will induce a weak e-homeomorphism $-f : -\Sigma_t \to -\Sigma$. Consider the weak e-homeomorphism $s : \Sigma_{-t} \to -\Sigma_t$ given by a reflection in $y = 0$ followed by counter clockwise half twists in the plane at the distinguished arcs - with respect to the usual identification $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$. This yields a parametrization $y(f) := (-f) \circ s : \Sigma_{-t} \to -\Sigma$. These two parametrizations provides isomorphisms

$$\Psi(t) \simeq \mathcal{T}^e(\Sigma),$$
$$\Psi(-t) \simeq \mathcal{T}^e(-\Sigma).$$

Now let

$$x \in \mathrm{Hom}(\mathbf{1}, \Phi(t, i)) \subset \mathcal{T}^e(\Sigma), \quad y \in \mathrm{Hom}(\mathbf{1}, \Phi(-t, j)) \subset \mathcal{T}^e(-\Sigma).$$

Consider the standard handlebodies denoted by $P(x) = H(U_t, R_t, i, x)$ and $Q'(y) = H(U_{-t}, R_t, j, y)$. We recall that $\langle x, y\rangle$ is given by $\tau(W(x, y))$, where $W(x, y)$ is the closed three manifold with a colored ribbon graph inside it, that is obtained by glueing $P \sqcup Q'$ to $\Sigma \times I$ through the orientation reversing homeomorphism

$$\partial(P(x) \sqcup Q'(y)) = \Sigma_t \sqcup \Sigma_{-t} \xrightarrow{f \sqcup y(f)} \Sigma \times \{0\} \sqcup \Sigma \times \{1\} = \partial(\Sigma \times I).$$

We now observe that the parametrization $s : \Sigma_{-t} \to -\Sigma$ extends to an e-homeomorphism of three manifolds

$$Q'(y) \to Q(y),$$

where $Q(y)$ is the same handlebody with the LH-orientation and the induced colored ribbon graph. We have a homeomorphism of extended three manifolds with colored ribbon graphs

$$P(x) \cup_{\mathrm{id}} Q(y) \xrightarrow{\sim} W(x, y).$$

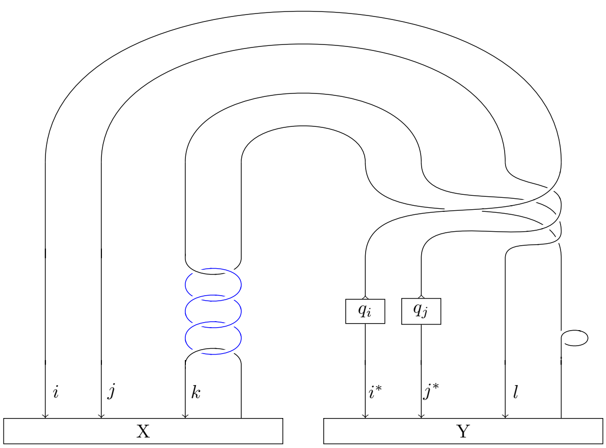
Comparing $P(x) \cup_{\mathrm{id}} Q(y)$ with the three manifold $M \sqcup_{\mathrm{id}} -N$ as considered in the proof of theorem 10.4.1 in [29], we obtain the desired presentation. $\square$

Now we want to use this to provide a presentation of the induced duality pairing on $Z_{\mathcal{V}}$. Again, it should be stressed that this presentation is only valid with respect to certain parametrizations.

Proposition 11.7 (Surgery presentation of $(\cdot,\cdot)_{\mathbf{\Sigma}}$). *Let $\mathbf{\Sigma}$ be a connected I-labeled marked surface. For any parametrization $f : \Sigma_t \to \overline{\mathcal{G}(\mathbf{\Sigma})}$ there is a parametrization $u(f) : \Sigma_{t*} \to \overline{\mathcal{G}(-\mathbf{\Sigma})}$ such that with respect to the induced isomorphisms*

$$\Psi(t) \simeq Z(\mathbf{\Sigma}),$$
$$\Psi(t*) \simeq Z(-\mathbf{\Sigma}),$$

we have the following presentation of $(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}}$.



Proof. Choose a parametrization $f : \Sigma_t \to \overline{\mathcal{G}(\mathbf{\Sigma})}$. Consider the induced parametrization $y(f) = (-f) \circ s : \Sigma_{-t} \to -\overline{\mathcal{G}(\mathbf{\Sigma})}$ as in the previous proof. This provides isomorphisms

$$\oplus_{i\in I^g}\mathrm{Hom}(\mathbf{1}, \Phi(t,i)) \simeq \mathcal{T}^e(\overline{\mathcal{G}(\mathbf{\Sigma})}), \tag{11.11}$$

$$\oplus_{i\in I^g}\mathrm{Hom}(\mathbf{1}, \Phi(-t,i)) \simeq \mathcal{T}^e(-\overline{\mathcal{G}(\mathbf{\Sigma})}). \tag{11.12}$$

Recall the isomorphism

$$\mathcal{T}^e(r) \circ \tilde{h} \circ \dot{q} : \mathcal{T}^e(\overline{\mathcal{G}(-\mathbf{\Sigma})}) \xrightarrow{\sim} \mathcal{T}^e(-\overline{\mathcal{G}(\mathbf{\Sigma})}). \tag{11.13}$$

We can define r such that the half-twists cancel with those of s. More precisely there is a choice of convention such that the following holds. Let $\tilde{s}$ be the same as

s but without the twists at the arcs. That is, $\tilde{s}$ is only the reflection in the $y = 0$ plane. Observe that f induces a parametrization

$$(-f) \circ \tilde{s} : \Sigma_{t*} \to \overline{\mathcal{G}(-\mathbf{\Sigma})}.$$

Take $u(f) = (-f) \circ \tilde{s}$. We see that with respect to the parametrizations $u(f)$ and $y(f)$, the isomorphism

$$\zeta : \Psi(t*) \xrightarrow{\sim} \Psi(-t)$$

is given by post composing with q. Now combine the description of the pairing $\Psi(t) \otimes \Psi(-t) \to K$ given in the previous proposition with the description of $\tilde{h} \circ \dot{q}$ as post composition with q in each factor to obtain the desired presentation. □

We show next that the pairing is compatible with glueing. That is, we prove that the formula holds and we explicitly calculate the μ'_λs. This calculation will depend on whether or not the two points subject to glueing are on the same connected component or not.

Proposition 11.8. *Let $\mathbf{\Sigma}$ be a connected I-labeled marked surface obtained from glueing two points subject to glueing that lie on the same component. Consider the glueing isomorphism*

$$\tilde{g} : \bigoplus_{i \in I} Z(\mathbf{\Sigma}(i)) \xrightarrow{\sim} Z(\mathbf{\Sigma}),$$

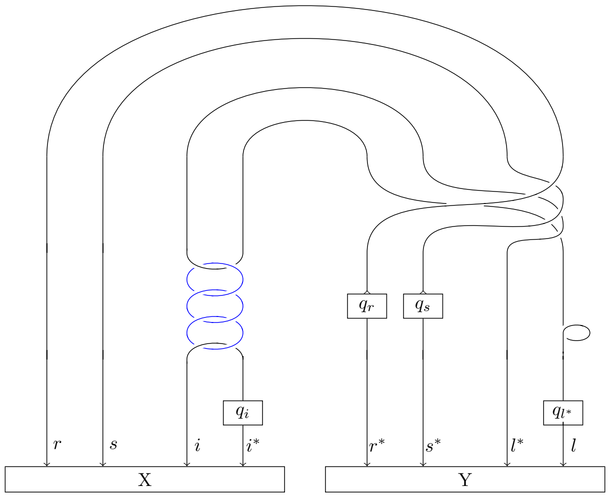
as described in definition 4.1. We have

$$(\tilde{g}, \tilde{g})_{\mathbf{\Sigma}} = \sum_{i \in I} D^4 dim(i)^{-1} (\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}(i)}.$$

Proof. We will use the description of the glueing homomorphism given in section 11.3 above. To do this we need to compare two parametrizations of $-\overline{\mathcal{G}(\mathbf{\Sigma})}$. Recall that to use (11.6) in general, we start with a paramtrization $f : \Sigma_t \to \overline{\mathcal{G}(\mathbf{\Sigma}(i))}$ and provide a parametrization $F : \Sigma_{t'} \to \overline{\mathcal{G}(\mathbf{\Sigma})}$ that agrees with f away from the points subject to glueing. Then (11.6) holds with respect to the isomorphisms (11.7),(11.8). We will write $F = z(f)$. Recall that (11.6) is only valid with respect to a pair of parametrizations $(r, y(r))$. In order to use (11.6) for the pairing on $\overline{\mathcal{G}(\mathbf{\Sigma})}$, we use the pair of parametrizations $(z(f), y(z(f)))$. For the pairing on $\overline{\mathcal{G}(\mathbf{\Sigma}(i))}$ we use the pair $(f, y(f))$. Since (11.6) only holds with respect to isomorphisms induced by compatible parametrizations, we need to compare $z(y(f))$ with $y(z(f))$. We see that $z(y(f))$ is $y(z(f))$ followed by a Dehn twists at the attached handle. This implies that

$$(k_l Y', z(y(f))) = (Y', y(z(f))), \tag{11.14}$$

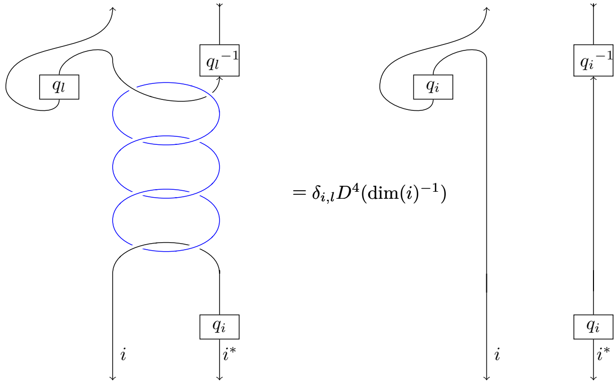
for all $Y' \in \mathrm{Hom}(\mathbf{1}, \Phi(-t', l)) \subset \Psi(-t')$, where l is such that the cap corresponding to the points subject to glueing is colored with V_l or V_{l^*}. To verify (11.14) recall the description of how to pass from one parametrization to another given in section 7, and use that a Dehn twists followed by a reflection in $y = 0$ is the same as the same reflection followed by the reverse Dehn twist. Using proposition 11.7, equation (11.14) and equation (11.6), we see that with respect to the isomorphisms $\Psi(t) \simeq Z(\mathbf{\Sigma}(i))$ and $\Psi(-t) \simeq \mathcal{T}^e(-\overline{\mathcal{G}(\mathbf{\Sigma}(i))})$ induced by the pair of parametrizations $(f, y(f))$, we have the following presentation of $(\tilde{g}_i, k_l \tilde{g}_{l^*})$



Here the blue unknot's are the surgery link components. For the sake of notational simplicity we have assumed that

$$t = (0; (V_r, 1), (V_s, 1), (V_i, 1), (V_{i^*}, 1).$$

The proof in the general case is easily obtained from this, as it relies on a local argument involving the surgery links. As in the proof of theorem 10.4.1, we get the following local equality



Thus this pairing is zero unless $l^* = i^*$. If so, we conclude that the claimed equation holds. $\square$

Proposition 11.9. *Let $\mathbf{\Sigma}$ be a connected I-labeled marked surface obtained from glueing two points subject to glueing that lie on two distinct components. Consider the glueing isomorphism*

$$\tilde{g} : \bigoplus_{i \in I} Z(\mathbf{\Sigma}(i)) \xrightarrow{\sim} Z(\mathbf{\Sigma}),$$

as described in definition 4.1. We have that

$$(\tilde{g}, \tilde{g})_{\mathbf{\Sigma}} = \sum_{i \in I} dim(i)^{-1} (\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}(i)}.$$

Proof. Let $\mathbf{\Sigma}_1$, be the component containing the first point and let $\mathbf{\Sigma}_2$ be the component containing the second point. We may assume that both of these components are homeomorphic to spheres. To see this, let $x \in Z(\mathbf{\Sigma}_1)$ and let $y \in Z(\mathbf{\Sigma}_2)$. We want to compare $\langle x, y \rangle$ with $\langle \tilde{g}(x), \tilde{g}(y) \rangle$. We can reduce the genus by 1 on one of the components by factorization. That is, assume $\mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2$ is obtained from $\tilde{\mathbf{\Sigma}}$ by glueing, where the glueing increase the genus. That is, the points subject to glueing lie on the same component. Then we may assume $x = h(\tilde{x})$ and $y = h(\tilde{y})$, where h is the glueing homomorphism. Now the task is to identify a scalar λ such that

$$(h(\tilde{x}), h(\tilde{y})) = \lambda(\tilde{g} \circ h(\tilde{x}), \tilde{g} \circ h(\tilde{y})).$$

But we already know from the previous proposition, that there is a $C \in K^*$ such that

$$(h(\tilde{x}), h(\tilde{y})) = C(\tilde{x}, \tilde{y})$$

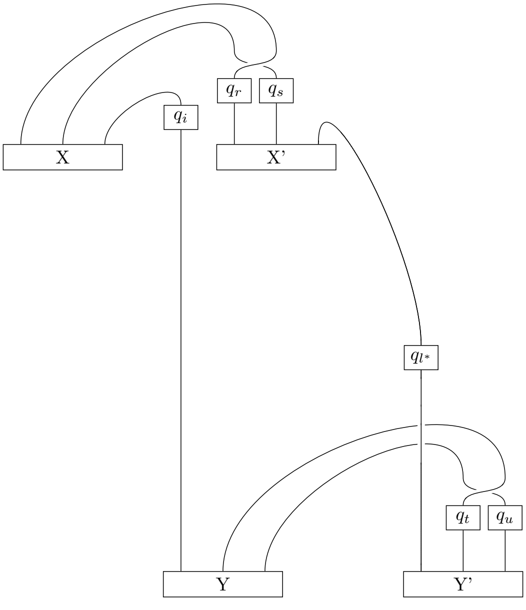
and

$$(\tilde{g} \circ h(\tilde{x}), \tilde{g} \circ h(\tilde{y})) = (h \circ \tilde{g}(\tilde{x}), h \circ \tilde{g}(\tilde{y})) = C(\tilde{g}(\tilde{x}), \tilde{g}(\tilde{y})).$$

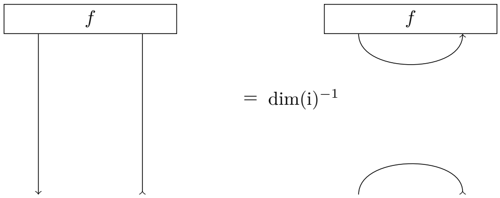
In the last equation we use that the pairing is compatible with morphisms and that glueing is associative. We now see that it will suffice to find λ such that

$$\lambda(\tilde{g}(\tilde{x}), \tilde{g}(\tilde{y})) = (\tilde{x}, \tilde{y}).$$

We have reduced the genus by 1 and can proceed inductively. Thus we can assume that we deal with spheres. Thus we can use the description of the glueing given above in section 11.3. For the sake of notational simplicity, we illustrate the case, where we have two spheres with three marked points. As in the previous proposition one starts by observing $y(z'(f \otimes g))$ followed by a Dehn twists is $z'(y(f) \otimes y(g))$. This will allow us to adopt the same strategy. We consider $\langle \tilde{g}_i(X \otimes Y), k_l \tilde{g}_{l^*}(X' \otimes Y')\rangle_{\mathbf{\Sigma}}$. The following presentations shows that the given presentation above naturally factors as a composition $P(X, X') \circ Q(Y, Y')$ where $P(X, X')$ is an element of $\mathrm{Hom}(V_i \otimes V_l^*, \mathbf{1})$ and $Q(Y, Y') \in \mathrm{Hom}(\mathbf{1}, V_i \otimes V_l^*)$.



Now the orthogonality follows from the fact that $\mathrm{Hom}(V_i \otimes V_l^*, \mathbf{1})$ is 0 if $l \neq i$ and isomorphic to K otherwise. For $l = i$ we note the following equation that holds for all $f \in \mathrm{Hom}(V_i \otimes V_l^*, \mathbf{1})$



Applying this to $P(X, X')$ and taking into account the twist that occurs when applying the isotopy to pull q_i to the right of $q_{s,}$, we see that the claim holds. $\square$

Corollary 11.10 (Compatibility of the pairing with glueing)**.** *The pairing* $(\cdot, \cdot)_{\mathbf{\Sigma}}$ *is compatible with glueing. It can be rescaled to a pairing* $\langle \cdot \mid \cdot \rangle_{\mathbf{\Sigma}}$ *according to topological types, such that*

$$\langle \tilde{g} \mid \tilde{g} \rangle_{\mathbf{\Sigma}} = \sum_{i \in I} \langle \cdot \mid \cdot \rangle_{\mathbf{\Sigma}(i)}.$$

It is easily verified that the following normalization has the given property. Since the pairing is multiplicative with respect to disjoint union, it is enough to specify the normalization on connected $\mathbf{\Sigma}$. Assume therefore that $\mathbf{\Sigma}$ is of genus g with labels $i_1, ..., i_k$. Then the normalization is given by

$$\langle \cdot \mid \cdot \rangle_{\mathbf{\Sigma}} = \left(D^{-4g} \prod_{l=1}^{k} \sqrt{\dim(i_l)} \right) (\cdot, \cdot)_{\mathbf{\Sigma}}. \tag{11.15}$$

It only remains to prove that the pairing is compatible with orientation reversal.

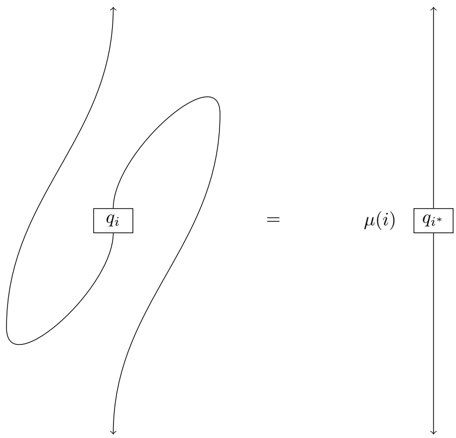
Proposition 11.11 (Compatibility with orientation reversal)**.** *The two pairings* $\langle \cdot \mid \cdot \rangle$ *and* $(\cdot, \cdot)$ *are both compatible with orientation reversal.*

Proof. Since the normalization factor is the same for $\mathbf{\Sigma}$ and $-\mathbf{\Sigma}$, we see, that it is enough to consider $(\cdot, \cdot)$. For $v \in Z(\mathbf{\Sigma})$ and $w \in Z(-\mathbf{\Sigma})$ we want to find a scalar μ such that $\mu(v, w)_{\mathbf{\Sigma}} = (w, v)_{-\mathbf{\Sigma}}$. For the moment let Σ' be an extended surface. Recall that in order to use the presentation of the pairing as given in proposition (11.6), we choose a parametrization f of Σ' and then we constructed a parametrization $y(f) := (-f) \circ s$. These give isomorphisms $\mathcal{T}^e(\Sigma') \simeq \Psi(t_0)$ and $\mathcal{T}^e(-\Sigma') \simeq \Psi(-t_0)$. With respect to these parametrizations we can use the presentation of (11.6). For $x \in \Psi(t_0)$ and $y \in \Psi(-t_0)$ it is an easy exercise to verify

$$\langle x, y \rangle_{t_0} = \langle y, x \rangle_{-t_0}.$$

This identity is also necessary for self-duality, because if we take $y(f)$ as the parametrization $\Sigma_{-t_0} \to \Sigma'$ then we see that $y(y(f)) = f$. This follows from the fact that $s^2 = \mathrm{id}$, which can be seen from the fact that counter clockwise half twists at the arcs, followed by a reflection in the $y = 0$ plane is the same as a reflection in the $y = 0$ plane followed by clockwise half twists at the arcs. Now choose a parametrization $f : \Sigma_t \to \overline{\mathcal{G}(\mathbf{\Sigma})}$. This induces a parametrization $u(f) = (-f) \circ \tilde{s} : \Sigma_{t*} \to \overline{\mathcal{G}(-\mathbf{\Sigma})}$. Here $\tilde{s}$ is simply the reflection in the y-plane. With respect to these isomorphisms we see, that $\langle \cdot, \cdot \rangle_{\mathbf{\Sigma}}$ is given as $\langle \cdot, \dot{q} \rangle_t$, where $\dot{q}$ is given by post composing suitably in each factor of the tensor product. Now choose $g = u(f) : \Sigma_{t*} \to \overline{\mathcal{G}(-\mathbf{\Sigma})}$. Observe $u(g) = f$. Thus for $(v, w) \in \Psi(t) \times \Psi(t*)$ we simply need to compare $\langle v, \dot{q}(w) \rangle_t$ with $\langle w, \dot{q}(v) \rangle_{t*}$.

Assume the labeled marked points of $\mathbf{\Sigma}$ are $i_1, ..., i_k$. Let $\mu(i) \in K^*$ be defined by the following equation



Let $\mu = \mu_{i_1} \cdots \mu_{i_k}$. Recall the fact that $\langle v, \dot{q}(w) \rangle_t = \langle \dot{q}(w), v \rangle_{-t}$. Now use the surgery presentation given in proposition (11.6). In the presentation of $\langle \dot{q}(w), v \rangle_{-t}$ pull over the coupons colored with q_{i_l} from left to right to obtain

$$(v, w)_{\mathbf{\Sigma}} = \mu(w, v)_{-\mathbf{\Sigma}},$$

which finishes the proof. $\square$

Remark 11.12. We observe that if $i \neq i^*$, it is possible to consistently choose q_i and q_{i^*}, such that $\mu(i)$ and $\mu(i^*)$ takes any values, as long as $\mu(i)\mu(i^*) = 1$. This follows from the fact that turning a coupon upside down, and then turning the resulting morphism upside down will yield the original morphism. Call this operation F. Then the equation above reads $F(q_i) = \mu(i) q_{i^*}$. Similarly, it can be seen that if $i^* = i$, then we must have $\mu(i)^2 = 1$. Using the axioms for the unit object of a modular tensor category, it is also easily seen that $\mu(0) = 1$.

We note the following result, that allow us to define μ on the self-dual objects independently of q.

Proposition 11.13 (μ is well defined on self-dual objects)**.** *Assume that $i \in I$ satisfies $i = i^*$. Then $\mu(i)$ is independent of q_i.*

The fact that $\mu(i)$ might be -1 for $i = i^*$ leads us to consider the strict self-duality question in section 16, where we introduce a new algebraic concept associated to a modular tensor category. As will be clear below, this will in many cases produce a very interesting normalization of the duality pairing, that will be strictly self-dual.

12 Unitarity

Consider a complex vector space W with scalar multiplication $(\lambda, w) \mapsto \lambda.w$ Let $\overline{W}$ be the complex vector space with the same underlying Abelian group and scalar multiplication given by $(\lambda, w) \mapsto \overline{\lambda}.w$. Here $\overline{\lambda}$ is the complex conjugate of λ.

Definition 12.1 (Unitarity)**.** Let (V, g) be a modular functor based on Λ and $\mathbb{C}$. A unitary structure on V is a positive definite hermitian form

$$(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}} : V(\mathbf{\Sigma}) \otimes \overline{V(\mathbf{\Sigma})} \to \mathbb{C},$$

subject to the following axioms.

Naturality. Let $\mathbf{f} = (f, s) : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ be a morphism between Λ-labeled marked surfaces. Then

$$(V(f), V(f))_{\mathbf{\Sigma}_2} = (\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}_1}. \tag{12.1}$$

Compatibility with disjoint union. Consider a disjoint union of Λ-labeled marked surface $\mathbf{\Sigma} = \mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2$. Composing with the permutation of the factors, the modular functor V provides an isomorphism

$$\eta : V(\mathbf{\Sigma}) \otimes \overline{V(\mathbf{\Sigma})} \xrightarrow{\sim} V(\mathbf{\Sigma}_1) \otimes \overline{V(\mathbf{\Sigma}_1)} \otimes V(\mathbf{\Sigma}_2) \otimes \overline{V(\mathbf{\Sigma}_2)}.$$

We demand that with respect to the natural isomorphism $\mathbb{C} \otimes \mathbb{C} \simeq \mathbb{C}$ we have that

$$(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}} = ((\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}_1} \otimes (\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}_2}) \circ \eta. \tag{12.2}$$

Compatibility with glueing. Let $\mathbf{\Sigma}$ be a Λ labeled marked surface obtained from glueing. Consider the glueing isomorphism

$$g : \bigoplus_{\lambda \in \Lambda} V(\mathbf{\Sigma}(\lambda)) \xrightarrow{\sim} V(\mathbf{\Sigma}),$$

as described in the definition of a modular functor. Clearly g also induces an isomorphism

$$g : \bigoplus_{\lambda\in\Lambda} \overline{V(\mathbf{\Sigma}(\lambda))} \xrightarrow{\sim} \overline{V(\mathbf{\Sigma})}.$$

We have

$$(g,g)_{\mathbf{\Sigma}} = \sum_{\lambda\in\Lambda} \mu_\lambda(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}(\lambda)}, \tag{12.3}$$

where $\mu_\lambda \in \mathbb{R}_{>0}$ for all λ. We allow the μ_λ to depend on the isomorphism class of $(\mathbf{\Sigma}, (p,q))$.

If the modular functor (V, g) also has a duality pairing we demand the unitary structure and the duality is compatible in the following sense.

Compatibility with duality. For all labeled marked surfaces $\mathbf{\Sigma}$, we demand that the following diagram is commutative up to a scalar $\rho(\mathbf{\Sigma})$ depending only on the isomorphism class of $\mathbf{\Sigma}$

$$\begin{tikzcd} V(\mathbf{\Sigma}) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & V(-\mathbf{\Sigma})^* \arrow[d, "\simeq"] \\ \overline{V(\mathbf{\Sigma})^*} \arrow[r, "\simeq"] & \overline{V(-\mathbf{\Sigma})}. \end{tikzcd} \tag{12.4}$$

Here, the horizontal isomorphisms are induced by the duality pairing, whereas the vertical isomorphisms are induced by the unitary structure.

We now make explicit what the isomorphisms of the diagram (12.4) are. We start with the composition

$$\omega : V(\mathbf{\Sigma}) \xrightarrow{\simeq} V(-\mathbf{\Sigma})^* \xrightarrow{\simeq} \overline{V(-\mathbf{\Sigma})}.$$

Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be the duality pairing and $(\,\cdot\,,\,\cdot\,)$ be the Hermitian form. The first map is given by

$$V(\mathbf{\Sigma}) \ni f \mapsto \langle\,\cdot\,,f\rangle_{-\mathbf{\Sigma}} : V(-\mathbf{\Sigma}) \to \mathbb{C}.$$

The second map is the inverse of the linear isomorphism $\overline{V(-\mathbf{\Sigma})} \xrightarrow{\simeq} V(-\mathbf{\Sigma})^*$ given by

$$\overline{V(-\mathbf{\Sigma})} \ni u \mapsto (\,\cdot\,,u)_{-\mathbf{\Sigma}} : V(-\mathbf{\Sigma}) \to \mathbb{C}.$$

Thus $\omega(f)$ is defined by

$$\langle x, f\rangle_{-\mathbf{\Sigma}} = (x, \omega(f))_{-\mathbf{\Sigma}}, \tag{12.5}$$

for all x in $V(-\mathbf{\Sigma})$. We now consider the composition

$$\phi : V(\mathbf{\Sigma}) \xrightarrow{\simeq} \overline{V(\mathbf{\Sigma})^*} \xrightarrow{\simeq} \overline{V(-\mathbf{\Sigma})}.$$

The first is the linear map

$$V(\mathbf{\Sigma}) \ni f \mapsto (\,\cdot\,, f)_{-\mathbf{\Sigma}} : V(\mathbf{\Sigma}) \to \mathbb{C}.$$

The second map is the inverse of the linear isomorphism $\overline{V(-\mathbf{\Sigma})} \xrightarrow{\simeq} \overline{V(\mathbf{\Sigma})^*}$ given by

$$\overline{V(-\mathbf{\Sigma})} \ni u \mapsto \langle\,\cdot\,, u\rangle_{-\mathbf{\Sigma}} : V(\mathbf{\Sigma}) \to \mathbb{C}.$$

Thus $\phi(f)$ is defined by

$$\langle y, \phi(f)\rangle_{\mathbf{\Sigma}} = (y, f)_{\mathbf{\Sigma}}, \tag{12.6}$$

for all y in $V(\mathbf{\Sigma})$.

Projective commutativity of (12.4) can be reformulated as the existence of $\rho(\mathbf{\Sigma})$ in $\mathbb{C}$ with

$$\phi = \rho(\mathbf{\Sigma})\omega. \tag{12.7}$$

13 Unitary structure from a unitary MTC

Recall the definition of a unitary modular tensor category $(\mathcal{V}, (V_i)_{i\in I})$ with conjugation $f \mapsto \overline{f}$ as defined in section 5.5. of chapter V in [29]. Recall that $K = \mathbb{C}$ in this case. Assume we are given a unitary modular tensor category. For an e-surface Σ let $(\,\cdot\,,\,\cdot\,)_\Sigma$ be the Hermitian form on $\mathcal{T}^e(\Sigma)$ as defined in section 10 of chapter IV in [29].

Theorem 13.1 (Unitarity)**.** *Let $(\mathcal{V}, (V_i)_{i\in I})$ be a unitary modular tensor category. Let $\mathbf{\Sigma}$ be an I-labeled marked surface. Consider the positive definite Hermitian form*

$$(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}} = (\,\cdot\,,\,\cdot\,)_{\overline{\mathcal{G}(\mathbf{\Sigma})}}.$$

This defines a unitary structure on $Z_{\mathcal{V}}$ compatible with duality.

Proof. It is proven by Turaev, that the induced Hermitian form is natural with respect to weak e-morphisms, and that it is multiplicative with respect to disjoint union. As Turaev also proves that $\overline{\Delta^{-1}D} = (\Delta^{-1}D)^{-1}$ these two properties carry over. All of this is proven in section 10 of chapter IV in [29].

Let us now prove that it is compatible with glueing. We first consider the case where the two points lies on the same component. Since the glueing as well as the Hermitian form is multiplicative with respect to disjoint union, as well as natural with respect to morphisms, we may assume that we are in the situation described in section 11.3. We adopt the the notation form the first subsection of this section. It follows directly from theorem 10.4.1 in section 10.4 of chapter

IV in [29] that if $i \neq j$ then $(\tilde{g}_i, \tilde{g}_j)_{\mathbf{\Sigma}} = 0$. Let $x, y \in \mathrm{Hom}(\mathbf{1}, \Phi(t, j)) \subset Z(\mathbf{\Sigma}(i))$. Using theorem 10.4.1, linearity of tr and $\mathbb{C} \simeq \mathrm{End}(V_{i^*})$, we get that

$$(\tilde{g}_i(x), \tilde{g}_i(y))_{\mathbf{\Sigma}} = D^g \left(\dim(i) \prod_{c=1}^{g} \dim(j_c) \right)^{-1} \mathrm{tr}(\tilde{g}_i(x) \circ \overline{\tilde{g}_i(y)}).$$

Here g is the genus of $\mathbf{\Sigma}$. Unwinding the glueing formula and using properties of the conjugation as well as of the trace we get that

$$\begin{aligned} &D^g((\dim(i)\dim(j))^{-1} \mathrm{tr}((1_W \otimes q_i \otimes 1_R) \circ x \circ \overline{y} \circ (1_W \otimes \overline{q_i} \otimes 1_R)) \\ &= D^g((\dim(i)\dim(j))^{-1} \mathrm{tr}((1_W \otimes \overline{q_i} \circ q_i \otimes 1_R) \circ x \circ \overline{y}) \\ &= D \dim(i)^{-1} \lambda_i(x, y)_{\mathbf{\Sigma(i)}}. \end{aligned}$$

Here $\lambda_i \in \mathbb{C}$ is defined by

$$\lambda_i 1_{V_{i^*}} = \overline{q_i} \circ q_i. \tag{13.1}$$

Thus we get that

$$(\tilde{g}, \tilde{g})_{\mathbf{\Sigma}} = \sum_{i \in I} D \dim(i)^{-1} \lambda_i(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}(i)}. \tag{13.2}$$

We now consider the case where the two points subject to glueing lie on distinct components. Using the result above, we may assume that these are homeomorphic to spheres. This can be argued as in the proof of Proposition (11.9). Using naturality and multiplicativity of both the glueing and the Hermitian form, we may assume that we are in the situation of the second subsection in section 11.3. An argument based on the surgery presentation of the form given in the proof of theorem 10.4.1 and based on the ideas of section 10.6 will show that in this case, we have that

$$(\tilde{g}, \tilde{g})_{\mathbf{\Sigma}} = \sum_{i \in I} \dim(i)^{-1} \lambda_i(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}(i)}. \tag{13.3}$$

Finally we prove that the unitary structure is compatible with duality. This is done by considering a surgery presentations of the equations (12.5) and (12.6). Let $f \in Z(\mathbf{\Sigma})$. We may assume $\mathbf{\Sigma}$ is connected. Equation (12.5) is presented as an equation involving $\overline{\omega(f)}$ and equation (12.6) is presented as an equation involving $\phi(f)$. Conjugating the surgery presentation of (12.5) we see that $\phi(f) = \frac{1}{\sigma(i_1)\cdots\sigma(i_m)} \omega(f)$ where $\sigma(i)$ is defined as $\lambda_{i^*}\mu(i)$. Thus, if $\mathbf{\Sigma}$ is an I-labeled marked surface (not necessarily connected) with labels $i_1, ..., i_m$ we see that

$$\rho(\mathbf{\Sigma}) = \prod_{l=1}^{m} \sigma(i_l)^{-1}, \tag{13.4}$$

which concludes the proof. $\square$

14 Scaling of the duality, unitarity and glueing

14.1 A choice of isomorphisms q_i

For the remainder of section 14,15, and 16 we fix a choice $q := (q_i)_{i\in I}$ where $q_i : V_{i^*} \to V_i^*$ is an isomorphism. Recall the definition of μ c.f. proposition 11.13. We can and will assume that $\mu(i) = 1$ for all i with $i \neq i^*$. In addition, if $\mathcal{V}$ is unitary, we can and will assume that $\overline{q_i} \circ q_i = \mathrm{id}_{V_i^*}$. Note however that all results from section 14.2 are true independently of these two extra assumptions. Let k_i be the twist coefficients of the modular tensor category. For the remainder of this article we fix for all i a choice of $\sqrt{\dim(i)}$ and a choice of $\sqrt{k_i}$. We make these choices invariant under $i \mapsto i^*$. If $K = \mathbb{C}$ and $\dim(i)$ is positive, we of course choose the positive square root. This will be the case if $\mathcal{V}$ is assumed to be unitary. We recall that if this is the case then $k_i \in S^1$.

The choice q gives us a modular functor with duality $(Z, \tilde{g}, \langle\,\cdot\,,\,\cdot\,\rangle)$. This is the content of theorem 5.8 and theorem 11.4. It serves as a set of reference isomorphisms, when dealing with scalings.

14.2 The scaling analysis

As mentioned above, both the glueing $\tilde{g}$ and the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ depends on our choice of isomorphisms $V_{i^*} \xrightarrow{\sim} V_i^*, i \in I$. As we are considering isomorphisms between simple objects, any other choice $\hat{q}_i$ will be of the form $\hat{q}_i = u_i q_i$ for some $u_i \in K^*$, where K^* denotes the group of units in K. We now investigate how the different compatibilities of the modular functor with duality are affected by scaling. For $u \in K^{*I}$ let $q(u) := (u_i q_i)_{i\in I}$. Let $\tilde{g}_u$ be the glueing defined using $q(u)$. To be precise, this means that in equation (5.1) we use $u_i q_i$ instead of q_i. Let $\langle\,\cdot\,,\,\cdot\,\rangle^u$ be the pairing defined using $q(u)$. To be precise, this means that in the equation (11.5) we use $u_i q_i$ instead of q_i. We observe that if $\mathbf{\Sigma}$ has labels $i_1, ..., i_m$ then we have that

$$\langle\,\cdot\,,\,\cdot\,\rangle^u_{\mathbf{\Sigma}} = \left(\prod_{l=1}^m u_{i_l}\right)\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{\Sigma}}. \tag{14.1}$$

We will write

$$\langle\,\cdot\,,\,\cdot\,\rangle^u_{\mathbf{\Sigma}} = u(\mathbf{\Sigma})\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{\Sigma}}.$$

Let $u, w \in K^{*I}$. Below we will consider what happens, if we use $q(u)$ to define the glueing, and $q(w)$ to define the pairing.

Definition 14.1 (Genus normalized pairing)**.** Let $w \in K^{*I}$. For a surface of genus g we consider the following normalization

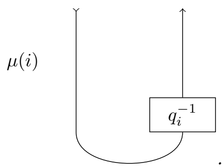
$$\langle\,\cdot\,,\,\cdot\,\rangle^w_{*,\mathbf{\Sigma}} := D^{-4g}\langle\,\cdot\,,\,\cdot\,\rangle^w.$$

Consider a general modular functor V with label set I. Assume V has a duality pairing $\langle \cdot, \cdot \rangle$. Consider S^2 equipped with the Stokes orientation, where B^3 is given the RHS orientation. Let (S^2, i, j) have the northpole colored with i and the southpole colored with j. There is a natural isotopy to the standard decorated surface of type $(0; (V_i, 1), (V_j, 1))$. This induces an isomorphism $Z(S^2, i, j) \simeq \mathrm{Hom}(1, V_i \otimes V_j)$. Let $\omega(i)$ be the unique vector in $Z(S^2, i, i^*)$ that solves $\tilde{g}(\omega(i) \otimes \omega(i)) = \omega(i)$. Let $\zeta(i) \in Z(-(S^2, i, i^*))$ be the unique vector that solves the analogous glueing problem. We define

$$Z(i) := \langle \omega(i), \zeta(i) \rangle_{(\mathbf{S}^2, i, i^*)}. \tag{14.2}$$

We now return to $Z_{\mathcal{V}}$.

Proposition 14.2. *Under the isomorphism $Hom(\mathbf{1}, V_i \otimes V_{i^*}) \simeq Z(S^2, i, i^*)$ induced by the identity parametrization we see that $\omega(i)$ is given as*



Proposition 14.3. *We have that*

$$\langle \omega(i), \zeta(i) \rangle = k_i^{-1} dim(i). \tag{14.3}$$

We start by observing that if $\mathbf{\Sigma}$ has labels $i_1, ..., i_m$ then we have

$$\langle \cdot\, , \cdot \rangle^w_{\mathbf{\Sigma}} = \left(\prod_{l=1}^{m} w_{i_l} \right) \langle \cdot\, , \cdot \rangle_{\mathbf{\Sigma}}. \tag{14.4}$$

We will write

$$\langle \cdot\, , \cdot \rangle^w_{\mathbf{\Sigma}} = w(\mathbf{\Sigma}) \langle \cdot\, , \cdot \rangle_{\mathbf{\Sigma}}.$$

Proposition 14.4. *Assume i, i^* label points on the same component. Then we have that*

$$\langle \tilde{g}^i_u, \tilde{g}^{i^*}_u \rangle^w_{\mathbf{\Sigma}} = \frac{u_i u_{i^*}}{w_i w_{i^*}} D^4 dim(i)^{-1} \langle \cdot\, , \cdot \rangle^w_{\mathbf{\Sigma}(i)}. \tag{14.5}$$

Assume i, i^* label points on distinct components. Then we have that

$$\langle \tilde{g}^i_u, \tilde{g}^{i^*}_u \rangle^w_{\mathbf{\Sigma}} = \frac{u_i u_{i^*}}{w_i w_{i^*}} dim(i)^{-1} \langle \cdot\, , \cdot \rangle^w_{\mathbf{\Sigma}(i)}. \tag{14.6}$$

We want to know how scaling affects the self-duality scalar μ.

Proposition 14.5. *We have that*

$$\mu(i,w) = \frac{w_i}{w_{i^*}}\mu(i). \tag{14.7}$$

Let $\omega(i,u) \in Z(S^2, i, i^*)$ be the unique vector that solves the analogous equation $\tilde{g}_u(\omega(i,u) \otimes \omega(i,u)) = \omega(i,u)$. Then

$$\omega(i,u) = u_i^{-1}\omega(i). \tag{14.8}$$

Let $\zeta(i) \in Z(-(S^2, i, i^*)) = Z(-S^2, i^*, i)$ be the image of $\omega(i,u)$ under $Z(r)$. Again we have that

$$\zeta(i,u) = u_i^{-1}\zeta(i). \tag{14.9}$$

We now investigate how this affect the compatibility of the unitary structure with glueing.

Proposition 14.6. *Let $\mathbf{\Sigma}$ be a connected I-labeled marked surface obtained from glueing two points subject to glueing that lie on one and the same component. Consider the glueing isomorphism*

$$\tilde{g}_u : \bigoplus_{i\in I} Z(\mathbf{\Sigma}(i)) \xrightarrow{\sim} Z(\mathbf{\Sigma}).$$

We have that

$$(\tilde{g}_u, \tilde{g}_u)_{\mathbf{\Sigma}} = \sum_{i\in I} D^4 dim(i)^{-1}\lambda_i u_i \overline{u_i}(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}(i)}. \tag{14.10}$$

Proposition 14.7. *Let $\mathbf{\Sigma}$ be a connected I-labeled marked surface obtained from glueing two points subject to glueing that lie on two distinct components. Consider the glueing isomorphism*

$$\tilde{g}_u : \bigoplus_{i\in I} Z(\mathbf{\Sigma}(i)) \xrightarrow{\sim} Z(\mathbf{\Sigma}),$$

We have that

$$(\tilde{g}_u, \tilde{g}_u)_{\mathbf{\Sigma}} = \sum_{i\in I} dim(i)^{-1}\lambda_i u_i \overline{u_i}(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}(i)}. \tag{14.11}$$

14.3 Normalizations

For $u, w \in K^{*I}$ we now consider the compatibility of the glueing $\tilde{g}_u$ with the pairing $\langle\,\cdot\,,\,\cdot\,\rangle^w$.

Let $\langle\,\cdot\mid\cdot\,\rangle^{u,w}$ be the normalized pairing given by

$$\langle\,\cdot\mid\cdot\,\rangle^{u,w}_{\Sigma_g,i_1,\dots,i_k} = \left(D^{-4g}\prod_{l=1}^{k} s^{u,w}_{i_l}\right)\langle\,\cdot\,,\,\cdot\,\rangle^{w}_{\Sigma_g,i_1,\dots,i_k}, \tag{14.12}$$

where

$$s^{u,w}_i := \frac{\sqrt{w_i w_{i^*}}\sqrt{\dim(i)}}{\sqrt{u_i u_{i^*}}}. \tag{14.13}$$

From 14.4 we immediately get the following

Corollary 14.8. *The pairing $\langle\,\cdot\mid\cdot\,\rangle^{u,w}$ is strictly compatible with the glueing $\tilde{g}_u$. That is*

$$\langle \tilde{g}^i_u \mid \tilde{g}^{i^*}_u\rangle^{u,w}_{\mathbf{\Sigma}} = \langle\,\cdot\mid\cdot\,\rangle^{u,w}_{\mathbf{\Sigma(i)}}. \tag{14.14}$$

We will write

$$\langle\,\cdot\mid\cdot\,\rangle^{u,w}_{\Sigma_g,i_1,\dots,i_k} := s(\mathbf{\Sigma},u,w)\langle\,\cdot\,,\,\cdot\,\rangle.$$

If we want to stress the choice of square roots chosen for $w_i w_i^*$ and $u_i u_i^*$ we will write

$$\langle\,\cdot\mid\cdot\,\rangle^{u,w,S}.$$

Observe that

$$s(\mathbf{\Sigma},u,w) = \left(D^{-4g}\prod_{l=1}^{k} s^{u,w}_{i_l} w_{i_l}\right),$$

when $\mathbf{\Sigma} = (\Sigma_g, i_1, \dots, i_k)$. We now consider a normalization of the Hermitian form. This normalized Hermitian form will be strictly compatible with the glueing $\tilde{g}_u$. First we note that

$$(\,\cdot\mid\cdot\,)^u_{\mathbf{\Sigma}} = \left(D^{-4g}\prod_{l=1}^{k} r^u_{i_l}\right)(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}}, \tag{14.15}$$

where

$$r^u_i = \frac{\sqrt{\dim(i)}}{\sqrt{\lambda_i u_i \overline{u_i}}}. \tag{14.16}$$

Corollary 14.9. *The Hermitian form $(\,\cdot\mid\cdot\,)^u$ is strictly compatible with the glueing $\tilde{g}_u$. That is*

$$(\tilde{g}^i_u \mid \tilde{g}^i_u)^u_{\mathbf{\Sigma}} = (\,\cdot\mid\cdot\,)^u_{\mathbf{\Sigma(i)}}. \tag{14.17}$$

Recall the convention that the square root of any positive number is assumed to be chosen positive. Thus there is no ambiguity in choosing the r_i. Of course we have to choose them positively, if we want the pairing to remain positive definite. We will write

$$(\,\cdot\mid\cdot\,)^u_{\mathbf{\Sigma}} = r(\mathbf{\Sigma}, u)(\,\cdot\,,\,\cdot\,)_{\mathbf{\Sigma}}$$

Proposition 14.10. *Assume $\mathbf{\Sigma}$ is an I-labeled marked surface (not necessarily connected) with labels $i_1, ..., i_m$. With respect to the normalized duality $\langle\,\cdot\mid\cdot\,\rangle^{u,w}$ and the normalized Hermitian form $(\,\cdot\mid\cdot\,)^u$ we have the following equation*

$$\rho^{u,w}_N(\mathbf{\Sigma}) = \left(r(\mathbf{\Sigma}, u)\overline{r(-\mathbf{\Sigma}, u)}\right)\left(s(\mathbf{\Sigma}, u, w)\overline{s(-\mathbf{\Sigma}, u, w)}\right)^{-1}\prod_{l=1}^{m}\sigma(i_l)^{-1}. \tag{14.18}$$

With respect to the genus normalized pairing $\langle\cdot,\cdot\rangle^w_*$ we have the following equation

$$\rho^{u,w}_{g,N}(\mathbf{\Sigma}) = \prod_{l=1}^{m}\left(r^u_{i_l} r^u_{i_l^*}\right)\left(\sigma(i_l) w_{i_l}\overline{w_{i_l^*}}\right)^{-1}. \tag{14.19}$$

Observe that since r is always real (and positive) we have $\overline{r(-\mathbf{\Sigma}, u)} = r(-\mathbf{\Sigma})$.

15 The canonical symplectic rescaling

Assume in the following that K is an integral domain. Recall that in accordance with the assumptions made in section 14.1 we have $\mu(i) = 1$ for all i with $i \neq i^*$. In this section, we only consider scalings $u : I \to K^*$ that satisfies $u_i = u_{i^*}$. Recall the definition of the normalization coefficients

$$s^{u,w}_i := \frac{\sqrt{w_i w_{i^*}}\sqrt{\dim(i)}}{\sqrt{u_i u_{i^*}}}.$$

If u, w are invariant under $i \mapsto i^*$ we see that we have canonical square roots given by $\sqrt{X^2} = X$. In the following, the normalization coefficients shall be interpreted according to this.

Definition 15.1 (Symplectic labels)**.** Let $i \in I$ satisfy $i = i^*$. We say that i is symplectic if $\mu(i) = -1$.

Definition 15.2 (Symplectic multiplicity)**.** Let $\mathbf{\Sigma}$ be a labeled marked surface. Let $\nu(\mathbf{\Sigma})$ denote the number of marked points on $\mathbf{\Sigma}$ labeled with symplectic labels. We call this number the symplectic multiplicity.

Theorem 15.3 (Canonical symplectic scaling)**.** *Choose $u, w \in (K^*)^I$ that solves*

$$u_i = s^{u,w}_i w_i \tag{15.1}$$

for all i. Then the modular functor $Z_{\mathcal{V}}$ with glueing $\tilde{g}(u)$ and genus normalized duality $\langle\,\cdot\,,\,\cdot\,\rangle_*^u$ satisfies that glueing and duality are strictly compatible and the duality is self-dual up to a sign which is given by the symplectic multiplicity

$$\mu = (-1)^{\nu}. \tag{15.2}$$

We have that

$$Z(i) = \frac{dim(i)}{k_i}. \tag{15.3}$$

Moreover, any two solutions (u, w) and (u', w') results in modular functors with duality that are isomorphic through an isomorphism that preserves the duality pairing.

Remark 15.4. We emphasize that equation (15.1) means that one uses the same scaling for the glueing isomorphism as one uses in the duality paring.

Before commencing the proof, we observe that up to a sign there is a preferred solution given by choosing $w_i = 1$ for all i and solving

$$\frac{\sqrt{\dim(i)}}{u_i} = u_i.$$

If there is no such u_i we may formally add it. If (u, w) is a solution, we will write Z^u for the resulting modular functor with duality $(Z, \tilde{g}(u), \langle\,\cdot\,,\,\cdot\,\rangle_*^u)$.

Proof. Let (u, w) be a solution. The fact that this is a solution to (15.1) implies that

$$\langle\,\cdot\,,\,\cdot\,\rangle_*^u = \langle\,\cdot\,\mid\,\cdot\,\rangle^{u,w}.$$

Since the bracket on the left is strictly compatible with $\tilde{g}(u)$ by corollary 14.8 the first claim follows. Equation (15.2) is an easy consequence of $w_i = w_{i^*}$, proposition 14.5 and the proof of proposition 11.11. Equation (15.3) follows from the fact that a proof of proposition 14.3 only depends on the fact that the same set of isomorphisms $V_{i^*} \xrightarrow{\sim} V_i^*$ is used for the duality as well as for the glueing, and that we always have $\mu(i, w)\mu(i^*, w) = 1$ for all $w : I \to K^*$. The proof of proposition 14.3 is a straightforward calculation of the surgery presentation given above.

Finally we prove that if we have two solutions, then they are isomorphic as modular functors through an isomorphism that preserves the duality. Consider a function $\alpha : I \to K^*$. Let $\mathbf{\Sigma}$ be a labeled marked surface with labels $i_1, ..., i_k$. Then α induces an automorphism $\Phi_\alpha = \Phi$,

$$\Phi(\mathbf{\Sigma}) : Z(\mathbf{\Sigma}) \xrightarrow{\sim} Z(\mathbf{\Sigma}),$$

given by $\Phi(\mathbf{\Sigma}) = \left(\prod_{l=1}^{k} \alpha(i_l)\right) \mathrm{id}_{Z(\mathbf{\Sigma})}$. Since this is multiplicative with respect to labels, it is easily seen that Φ is compatible with the action induced by morphisms of labeled marked surfaces, disjoint union and the permutation.

We will think of Φ as a morphism of modular functors $Z^u \to Z^{u'}$. We now identify sufficient conditions for Φ to be compatible with glueing and with the duality pairings. We start with glueing. Let $\mathbf{\Sigma}(\lambda)$ be obtained by glueing $\mathbf{\Sigma}(\lambda, i, i^*)$. Compatibility with glueing is equivalent to the following equation

$$\tilde{g}^i_{u'} \circ \Phi(\lambda, i, i^*) = \Phi(\lambda) \circ \tilde{g}^i_u.$$

This is equivalent to

$$\alpha(i)\alpha(i^*) = \frac{u_i}{u'_i}. \tag{15.4}$$

For Φ to be compatible with duality, we must have that

$$\langle\,\cdot\,,\,\cdot\,\rangle_{*,u} = \langle \Phi(\mathbf{\Sigma})(\,\cdot\,), \Phi(-\mathbf{\Sigma})(\,\cdot\,)\rangle_{*,u'}.$$

For this equation to be satisfied, we see that equation (15.4) is sufficient. If this is so, then $\beta = \alpha^{-1} : I \to K^*$ will satisfy

$$\beta(i)\beta(i^*) = \frac{u'_i}{u_i}.$$

Therefore Φ_β will be an inverse morphism that preserves the duality. Thus we can choose any function $\alpha : I \to K^*$ that satisfies (15.4), and then $\Phi = \Phi_\alpha$ will be an isomorphism $Z^u \xrightarrow{\sim} Z^{u'}$ that preserves the duality pairing. That such a function exists follows from the fact that u, u' are both invariant under $i \mapsto i^*$. $\square$

15.1 Unitarity

Assume now that $K = \mathbb{C}$ and that $\mathcal{V}$ comes equipped with a unitary structure $\mathrm{Hom}(V, W) \ni f \mapsto \overline{f} \in \mathrm{Hom}(W, V)$. Recall that by section 14.1 we have $\overline{q_i} \circ q_i = \mathrm{id}_{V_i^*}$. Thus $\lambda_i = 1$ for all i.

Theorem 15.5. *Assume (u, w) is a solution to (15.1) with $|w_i| = 1$ for all i. Then the following holds. Up to a sign the genus normalized duality pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{*,u}$ is compatible with the normalized Hermitian form $(\,\cdot \mid \cdot\,)^u$. This sign is given by the parity of the symplectic multiplicity*

$$\rho = (-1)^\nu.$$

Moreover any two solutions (u, w) and (u', w') to (15.1) yields modular functors $Z^u, Z^{u'}$ that are isomorphic through an isomorphism that respects the duality pairing as well as the Hermitian form.

Proof. Consider a labeled marked surface $\mathbf{\Sigma}$ with labels $i_1, ..., i_k$. Recall the following formula from proposition 14.10

$$\rho^{u,u}_{g,N}(\mathbf{\Sigma}) = \prod_{l=1}^{m} \left(r^u_{i_l} r^u_{i_l^*}\right) \left(\sigma(i_l) u_{i_l} \overline{u_{i_l^*}}\right)^{-1}.$$

Recall that $\sigma(i) = \lambda_{i^*}\mu(i) = \mu(i)$. So in our situation we see that the product of the σ_i's is equal to $\mu(\mathbf{\Sigma})$, which we already know is given by $(-1)^{\nu(\mathbf{\Sigma})}$. Since $u_i = u_i^*$ and $\lambda_i = 1$ for all i we get

$$r^u_{i_l} r^u_{i_l^*} = \frac{\dim(i)}{|u_i|^2}.$$

Thus ρ/μ is seen to be a product of factors of the form

$$\frac{\dim(i)}{|u_i|^4}.$$

The equation

$$u_i^2 = w_i^2 \sqrt{\dim(i)},$$

implies that all of these factors are 1. Here we use that $|w_i| = 1$ for all i.

Assume now that (u', w') is another solution. We recall that the isomorphism $Z^u \xrightarrow{\sim} Z^{u'}$ from theorem 15.3 can be constructed by choosing a suitable function $\alpha : I \to \mathbb{C}$ with $\alpha(i)\alpha(i^*) = u_i/u_i'$ for all i. For any labeled marked surface $\mathbf{\Sigma}$ the isomorphism

$$\Phi_\alpha : Z^u(\mathbf{\Sigma}) \xrightarrow{\sim} Z^{u'}(\mathbf{\Sigma}),$$

will be multiplication by $\alpha(i_1)\cdots\alpha(i_k)$ where $i_1, ..., i_k$ are the labels of $\mathbf{\Sigma}$. However, since $|w_i| = |w_i'| = 1$ for all i we see that $|u_i| = |u_i'|$ for all i. This implies the following two things. First $r^u_i = r^{u'}_i$ for all i. Second it implies that we can choose $\alpha(i) = \alpha(i^*)$ to be a square root of u_i/u_i', which lies on the unit circle. Therefore Φ_α will be a Hermitian isomorphism. □

16 The dual of the fundamental group of a modular tensor category

Recall the definition of the dual of the fundamental group and a fundamental symplectic character of a modular tensor category given in the introduction.

Theorem 16.1. *Assume that a modular tensor category $(\mathcal{V}, I)$ has a fundamental symplectic character. Then there exists $u : I \to K^*$ such that the genus normalized duality pairing $\langle \cdot, \cdot \rangle^u_*$ is strictly self-dual, strictly compatible with glueing and we have that*

$$Z(u, i) = \frac{dim(i)}{k_i}.$$

Moreover if $\mathcal{V}$ is unitary and the image of $\tilde{\mu}$ above is a subset of

$$S(K) = \{z \in K | z\bar{z} = 1\},$$

then we can choose u, such that $\langle \cdot , \cdot \rangle_*^u$ is strictly compatible with the Hermitian form $(\cdot \mid \cdot)^u$.

Before commencing the proof, we remark that if $\mu(i) = 1$ for all self-dual objects i, then the neutral element $e \in \Pi(\mathcal{V}, I)^*$ is such an extension.

Proof. Partition $I = I_{SD} \sqcup I_{NSD}$ such that $i \in I_{SD}$ if and only if $i^* = i$. We further have the natural splitting

$$I_{SD} = I_{SD}^+ \sqcup I_{SD}^-,$$

where $i \in I_{SD}^+$ if and only if $\mu(i) = 1$. Hence we have of course that $i \in_{SD}^-$ if and only if $\mu(i) = -1$.

Let us now pick a splitting

$$I_{NSD} = I_{NSD}^1 \sqcup I_{NSD}^2,$$

such that $i \in I_{NSD}^1$ if and only if $i^* \in I_{NSD}^2$.

We start by describing the normalization. Recall that the choice made in section 14.1 implies that $\mu(i) = \mu(i^*) = 1$ whenever $i \neq i^*$. Let $w : I \to K^*$. We consider the genus normalized pairing $\langle \cdot , \cdot \rangle_*^w$. We have that

$$\mu(i, w) = \frac{w_i}{w_i^*}\mu(i).$$

Since $\tilde{\mu}(i) = \tilde{\mu}(i^*)^{-1}$ this implies that we can consistently choose w_i, w_i^* such that $\mu(i, w) = \tilde{\mu}(i)$ whenever i is not self-dual. Since $\tilde{\mu}$ is assumed to extend the μ on the self-dual objects we conclude that we can normalize such that $\mu(i, w) = \tilde{\mu}(i)$. Assume that we can choose w such that $\langle \cdot \mid \cdot \rangle_*^w$ is also strictly compatible with glueing. We want to argue that in this case, we have $\mu(\mathbf{\Sigma}) = 1$ unless $Z(\mathbf{\Sigma}) = \mathbf{0}$.

Let $\mathbf{\Sigma}$ be a labeled marked surface. We recall that proving strict self-duality is the same as proving that for all $(x, y) \in Z(\mathbf{\Sigma}) \times Z(-\mathbf{\Sigma})$ we have

$$\langle x, y \rangle_{\mathbf{\Sigma}} = \langle y, x \rangle_{-\mathbf{\Sigma}}.$$

Let C be a collection of simple closed curves on $\mathbf{\Sigma}$, whose homology classes are contained in the Lagrangian subspace of $\mathbf{\Sigma}$ and such that factorization along all of these will produce a disjoint union of spheres with one, two or three marked points. For the existence of such a collection see [31]. Let $\lambda \in I^C$ and let $\mathbf{\Sigma}_C(\lambda)$ be the labeled marked surface obtained from factorization in C. Thus $\mathbf{\Sigma}_C(\lambda)$ is a disjoint union of labeled marked surfaces of genus zero with one, two or three labels. Write $\mathbf{\Sigma}_C(\lambda) = \sqcup_{l=1}^k \mathbf{S}_l(\lambda)$. Let $P_\lambda : Z(\Sigma) \to Z(\mathbf{\Sigma}_C(\lambda))$ be the projection resulting from the factorization isomorphism. Let $x \in Z(\mathbf{\Sigma})$ and let $y \in Z(-\mathbf{\Sigma})$. We can write $P_\lambda(x)$ as a finite sum

$$\sum_{\alpha \in (x,\lambda)} x(\alpha,\lambda)^{(1)} \otimes \cdots \otimes x(\alpha,\lambda)^{(l)},$$

with $x(\alpha,\lambda)^{(i)} \in Z(\mathbf{S}_i(\lambda))$. Here (x,λ) is a finite index set depending only on x and λ. In a similar way we write $P_{\lambda^*}(y)$ as a finite sum of the form

$$\sum_{\beta \in (y,\lambda^*)} y(\beta,\lambda^*)^{(1)} \otimes \cdots \otimes y(\beta,\lambda^*)^{(l)}.$$

Recalling the following identity $-(\mathbf{\Sigma}(\lambda)) = (-\mathbf{\Sigma})(\lambda^*)$, we have that

$$\begin{aligned}\langle x,y\rangle_{\mathbf{\Sigma}} &= \sum_{\lambda \in I^C} \langle P_\lambda(x), P_{\lambda^*}(y)\rangle_{\mathbf{\Sigma}_C(\lambda)} \\ &= \sum_{\lambda \in I^C} \sum_{\alpha \in (x,\lambda), \beta \in (y,\lambda^*)} \prod_{l=1}^{k} \langle x(\alpha,\lambda)^{(l)}, y(\beta,\lambda^*)^{(l)}\rangle_{\mathbf{S}_l(\lambda)}.\end{aligned}$$

Similarly we see that

$$\langle y,x\rangle_{-\mathbf{\Sigma}} = \sum_{\lambda^* \in I^C} \sum_{\alpha \in (x,\lambda), \beta \in (y,\lambda^*)} \prod_{l=1}^{k} \langle y(\beta,\lambda^*)^{(l)}, x(\alpha,\lambda)^{(l)}\rangle_{(-\mathbf{S}_l)(\lambda^*)}.$$

Therefore we see that it reduces to the case of spheres marked with one, two or three points. If a sphere is marked with one point its module of states is zero unless the point is labeled with 0, but we already saw that $\mu(0,w) = 1$. If it is marked with two points then we use that its associated module of states is zero unless its labels are i, i^*. If this is the case then the desired equality follows from $\tilde{\mu}(i)\tilde{\mu}(i^*) = 1$. Finally, for a sphere with three points labeled by i, j, k, we recall that the associated module of states is isomorphic to $\mathrm{Hom}(\mathbf{1}, V_i \otimes V_j \otimes V_k)$ which is zero unless $\tilde{\mu}(i)\tilde{\mu}(j)\tilde{\mu}(k) = 1$.

Therefore it amounts to choosing (u,w) such that

$$u_i = \frac{\sqrt{w_i w_i^*}\sqrt{\dim(i)}}{\sqrt{u_i u_{i^*}}} w_i, \tag{16.1}$$

and

$$\mu(i)\frac{u_i}{u_{i^*}} = \tilde{\mu}(i). \tag{16.2}$$

Choose a square root of $\tilde{\mu}(i)$ and a square root of $\mu(i)$ for each i. This can be done consistently such that $\sqrt{\tilde{\mu}(i^*)} = 1/\sqrt{\tilde{\mu}(i)}$, for $i \neq i^*$. Now define $\eta : I \to K^*$ by

$$\eta_i := \sqrt{\tilde{\mu}(i)}\sqrt{\mu(i)}.$$

With such a choice we have for all $i \in I$ that

$$\eta_i \eta_{i^*} = 1.$$

This implies the important equation

$$\frac{\eta_i}{\eta_{i^*}} = \tilde{\mu}(i)\mu(i). \tag{16.3}$$

Take $w_i = \eta_i$ for all i. Consider $i \in I^1_{NSD}$. Fix a choice $\sqrt{\tilde{\mu}(i^*)}$ and then solve

$$u_i^2 = \frac{\sqrt{\dim(i)}}{\sqrt{\tilde{\mu}(i^*)}}\eta_i.$$

Therefore, if we define $u_{i^*} = u_i \tilde{\mu}(i^*)$ then equation (16.1) is true for i, since we may choose $\sqrt{u_i u_{i^*}} = u_i\sqrt{\tilde{\mu}(i^*)}$ in this case. We now need to check that equation (16.1) is also true for i^*. We can choose $\sqrt{u_i u_{i^*}} = u_{i^*}\sqrt{\tilde{\mu}(i)}$, with $\sqrt{\tilde{\mu}(i)} = 1/\sqrt{\tilde{\mu}(i^*)}$. Then we must check that

$$u_{i^*}^2 = \frac{\sqrt{\dim(i)}}{\sqrt{\tilde{\mu}(i)}}\eta_{i^*}$$

We compute that

$$\begin{aligned} u_{i^*}^2 &= u_i^2 \tilde{\mu}(i^*)^2 \\ &= \tilde{\mu}(i^*) u_i^2 \tilde{\mu}(i)^{-1} \\ &= \frac{\tilde{\mu}(i^*)}{\sqrt{\tilde{\mu}(i^*)}}\sqrt{\dim(i)}\eta_i \tilde{\mu}(i)^{-1} \\ &= \sqrt{\tilde{\mu}(i^*)}\sqrt{\dim(i)}\eta_{i^*} \\ &= \frac{1}{\sqrt{\tilde{\mu}(i)}}\sqrt{\dim(i)}\eta_{i^*}. \end{aligned}$$

Thus (16.1) holds for all $j \in I_{NSD}$. For $i \in I_{SD}$ we have $\eta_i = 1$ and it is easy to choose u_i satisfying (16.1). That (16.2) holds is an easy consequence of equation (16.3). That

$$Z(i) = \frac{\dim(i)}{k_i},$$

follow as in the proof of theorem 15.3.

Finally, assume that $(\mathcal{V}, I)$ is unitary. The choice made in section 14.1 ensures $\lambda_i = 1$ for all i. Observe that $\eta_i \in S^1$ for all i. Therefore $|u_i| = |u_{i^*}| = \frac{1}{\dim(i)^4}$. According to proposition 14.10 we know ρ is given by

$$\rho^{u,u}_{g,N}(\mathbf{\Sigma}) = \prod_{l=1}^{m}\left(r^u_{i_l} r^u_{i_l^*}\right)\left(\sigma(i_l) u_{i_l}\overline{u_{i_l^*}}\right)^{-1}.$$

We have

$$r_i^u r_{i^*}^u = \frac{\dim(i)}{|u_i||u_{i^*}|}.$$

Using $\sigma(i) = \mu(i)$ we see that

$$\sigma(i)u_i\overline{u_{i^*}} = \mu(i)\frac{u_i}{u_{i^*}}u_{i^*}\overline{u_{i^*}} = \tilde{\mu}(i)|u_{i^*}|^2.$$

Thus we get that

$$\rho_{g,N}^{u,u}(\mathbf{\Sigma}) = \prod_{l=1}^{m} \tilde{\mu}(i_l^*).$$

The argument given above proves that this is 1 unless $Z(\mathbf{\Sigma}) = 0$. $\square$

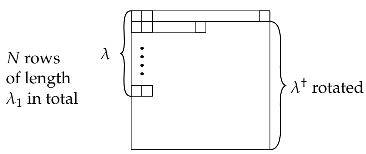
Corollary 16.2. *Assume $\tilde{\mu}$ is as above. Assume a labeled marked $\mathbf{\Sigma}$ surface has labels $i_1, ..., i_k$. We see that $\prod_{l=1}^{l} \tilde{\mu}(i_l) \neq 1$ implies $Z(\mathbf{\Sigma}) = \mathbf{0}$.*

17 The quantum SU(N) modular tensor categories

We refer to the papers [28], [30] and [16] (which uses the skein theory model for the SU(2) case build in [17], [18]) for the complete construction of the quantum SU(N) modular tensor category $H_k^{\mathrm{SU(N)}}$ at the root of unity $q = e^{2\pi i/(k+N)}$. For a short review see also [14]. The simple objects of this category are indexed by the following set of young diagrams

$$\Gamma_{N,k} = \{(\lambda_1, \ldots, \lambda_p) \mid \lambda_1 \leq k, p < N\}.$$

The involution $\dagger : \Gamma_{N,k} \to \Gamma_{N,K}$ is defined as follows. For a Young diagram λ in $\Gamma_{N,k}$ we define $\lambda^\dagger \in \Gamma_{N,k}$ to be the Young diagram obtained from the skew-diagram $(\lambda_1^N)/\lambda$ by rotation as indicated in the following figure.



Let $\mu = e^{2\pi i/N}$ and ζ_N be the set of N'th roots of 1 in $\mathbb{C}$. We then consider the following map

$$\tilde{\mu} : \Gamma_{N,k} \to \zeta_N$$

given by

$$\tilde{\mu}(\lambda) = \mu^{|\lambda|},$$

where $|\lambda| = \lambda_1 + \ldots + \lambda_p$.

Proposition 17.1. *We have that*

$$\tilde{\mu} \in \Pi(H_k^{\mathrm{SU(N)}}, \Gamma_{N,k})^*.$$

Proof. We have

$$\tilde{\mu}(\lambda)\tilde{\mu}(\lambda^\dagger) = 1,$$

since $|\lambda| + |\lambda^\dagger| = N\lambda_1$ by construction. Now consider $\lambda, \mu, \nu \in \Gamma_{N,k}$. By the very definition $H_k^{\mathrm{SU(N)}}(0, \lambda \otimes \mu \otimes \nu) = 0$ if

$$|\lambda| + |\mu| + |\nu| \notin N\mathbb{Z},$$

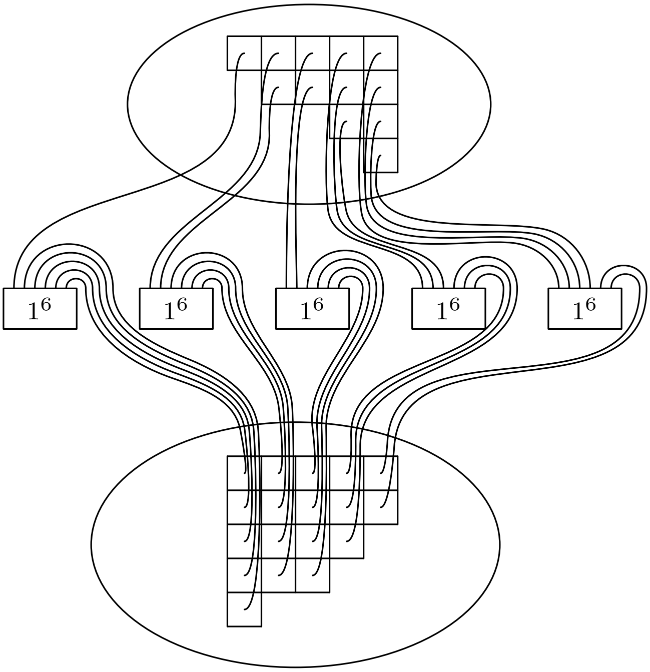
since the $|\lambda| + |\mu| + |\nu|$ ingoing strands at the top of the cylinder over the disc can only disappear into coupons N at the time inside the cylinder, since we have the empty diagram at be bottom determined by the label 0.

□

Using the notation in [14], we will now fix isomorphisms

$$q_\lambda \in H^{\mathrm{SU(N)}}(\lambda^\dagger, \lambda^*),$$

as indicated in the figure below (illustrated for some particular element $\lambda \in \Gamma_{6,k}$ for $k \geq 5$)



This gives us $\mu : \Gamma_{N,k} \to \mathbb{C}^*$ such that

$$F(q_\lambda) = \mu(q) q_{\lambda^\dagger}.$$

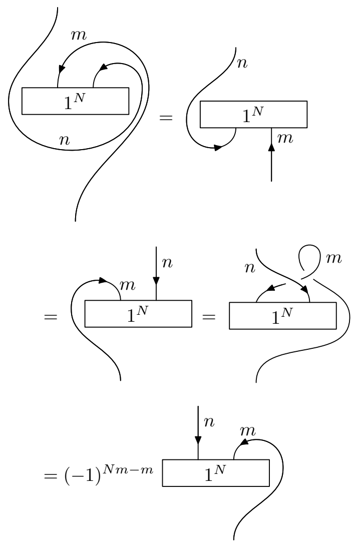
Proposition 17.2. *For N odd and any $\lambda \in \Gamma_{N,k}$, we have that*

$$\mu(\lambda) = 1.$$

For N even and any $\lambda \in \Gamma_{N,k}$ we have that

$$\mu(\lambda) = (-1)^{|\lambda|}.$$

Proof. We observe that if we apply F to q_λ, top and bottom can by a half rotation be brought into the right position for comparison with $q_{\lambda^\dagger}$ and the relevant computation for each coupons in between is the following



Here the last sign is a result of the following calculation in the notation of [16], using that

$$a = q^{-\frac{1}{2N}}, v = q^{-\frac{N}{2}}, s = q^{\frac{1}{2}},$$

namely, the braiding and the twist on top of the coupon contributes

$$(-a^{-1}s)^{nm+m(m-1)}(a^{-1}v)^m = (-1)^{Nm-m}$$

times the coupon with the strands in the original position again. □

From this proposition, we observe that if N is odd or N is divisible by 4, then there are no self-dual symplectic objects in $H_k^{\mathrm{SU(N)}}$. If however, N is even, but $N/2$ is odd, then the self-dual objects are symplectic if and only if they have an odd number of boxes. Moreover, we observe that on these self-dual objects

$$\tilde{\mu}(\lambda) = -1.$$

In all cases, we see that $\tilde{\mu}$ is a fundamental symplectic character.

18 The general quantum group modular tensor categories

We will now fix a simple Lie algebra $\mathfrak{g}$ and we will consider the corresponding quantum group at the root of unity $q = e^{2\pi i/(k+h)}$. The associated modular tensor category will be denoted $(H_k^{\mathfrak{g}}, \Lambda_k^{\mathfrak{g}})$. Let $\mathcal{W}$ be the weight lattice and $\mathcal{R}$ the root lattice for $\mathfrak{g}$. We recall that the fundamental group of $\mathfrak{g}$ is $\Pi(\mathfrak{g}) = \mathcal{W}/\mathcal{R}$. We have three general facts about $\Pi(\mathfrak{g})$. The first one is that

$$\lambda + \lambda^\dagger = 0 \bmod \mathcal{R},$$

for all dominant weights $\lambda \in \mathcal{W}_+$ and $\lambda^\dagger = -w_0(\lambda)$, where w_0 is the longest element of the Weyl group, e.g. $\lambda^\dagger$ is the highest weight vector of the dual of the irreducible representation V_λ, corresponding to λ. We further observe that if V_λ is self-dual, then 2λ will be in $\mathcal{R}$.

The second fact is that if we know that for $\lambda, \mu, \nu \in \mathcal{W}_+$

$$\mathrm{Hom}_{\mathrm{G}}(0, \mathrm{V}_\lambda \otimes \mathrm{V}_\mu \otimes \mathrm{V}_\nu) \neq 0,$$

then

$$\lambda + \mu + \nu \neq 0 \bmod \mathcal{R}.$$

We recall that

$$\mathrm{Hom}_{\mathrm{G}}(0, \mathrm{V}_\lambda \otimes \mathrm{V}_\mu \otimes \mathrm{V}_\nu) = 0,$$

implies that

$$H_k^{\mathfrak{g}}(0, V_\lambda \otimes V_\mu \otimes V_\nu) = 0.$$

The corresponding property for the modular functor coming from Conformal Field Theory (see [12]) is clear by construction.

We now recall that $\Pi(\mathfrak{g})$ is cyclic unless $\mathfrak{g} = D_n$, where $\Pi(\mathfrak{g}) = \mathbb{Z}_2 \times \mathbb{Z}_2$. In the last case, one knows that $(1,0)$ and $(0,1)$ are symplectic, but $(1,1)$ is not. From this we conclude that in all cases, we see that there exist some even N and a homomorphism

$$\tilde{\mu}' : \Pi(\mathfrak{g}) \to \zeta_N,$$

such that $\tilde{\mu}'(\lambda) = -1$ if and only if λ is symplectic. But then we define

$$\tilde{\mu} : \Lambda_k^{\mathfrak{g}} \to \zeta_N$$

to be the composite of the projection from $\Lambda_k^{\mathfrak{g}}$ to $\Pi(\mathfrak{g})$ followed by $\tilde{\mu}'$.

We then see that $\tilde{\mu} \in \Pi(H_k^{\mathfrak{g}})^*$ and indeed it is a fundamental symplectic character.

We remark that the result of this section applied to $\mathfrak{g} = \mathfrak{sl}(N)$ gives a second proof for the existence of a fundamental symplectic character in that case.

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Jørgen Ellegaard Andersen
Center for Quantum Geometry of Moduli Spaces
University of Aarhus
DK-8000, Denmark
andersen@qgm.au.dk

William Elbæk Petersen
Center for Quantum Geometry of Moduli Spaces
University of Aarhus
DK-8000, Denmark
william@qgm.au.dk

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The boundary length and point spectrum enumeration of partial chord diagrams using cut and join recursion

by Jørgen Ellegaard Andersen, Hiroyuki Fuji,
Robert C. Penner, and Christian M. Reidys[1]

Abstract

We introduce the boundary length and point spectrum, as a joint generalization of the boundary length spectrum and boundary point spectrum introduced by Alexeev, Andersen, Penner and Zograf. We establish by cut-and-join methods that the number of partial chord diagrams filtered by the boundary length and point spectrum satisfies a recursion relation, which combined with an initial condition determines these numbers uniquely. This recursion relation is equivalent to a second order, non-linear, algebraic partial differential equation for the generating function of the numbers of partial chord diagrams filtered by the boundary length and point spectrum.

1 Introduction

A *partial chord diagram*, is a special kind of graph, which can be specified as follows. The graph consists of a number of line segments (which we will also call backbones) arranged along the real line (hence they come with an ordering) with a number of vertices on each. A number of semi-circles (called chords) arranged in the upper half plan are attached at a subset of the vertices of the line segments, in such a way that no two chords have endpoints on the line segments in common. The vertices which are not attached to chord ends are called the marked points. A *chord diagram* is by definition a partial chord diagram with no marked points. Partial chord diagrams occur in many branches of mathematics, including topology [12, 15], geometry [8, 9, 2] and representation theory [13].

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Furthermore, they play a very prominent role in macro molecular biology. Please see the introduction of [6] for a short review of these applications.

As documented in [17, 25, 24, 11, 10, 7, 3, 4, 1, 23, 22, 5, 6], the notion of a *fatgraph* [18, 19, 20, 21] is a useful concept when studying partial chord diagrams[2]. A fatgraph is a graph together with a cyclic ordering on each collection of half-edges incident on a common vertex. A partial linear chord diagram c has a natural fatgraph structure induced from its presentation in the plane. The fatgraph c has canonically a two dimensional surface with boundary Σ_c associated to it (e.g. see Figure 1).

Figure 1: The partial chord diagram c and the surface Σ_c associated to the fatgraph with marked points. This partial chord diagram has the type $\{g, k, l; \{b_i\}; \{n_i\}; \{\ell_i\}\} = \{1, 6, 2; \{b_6 = 1, b_8 = 1\}; \{n_0 = 2, n_1 = 2\}; \{\ell_1 = 1, \ell_2 = 2, \ell_9 = 1\}\}$. The boundary length-point spectra are $\{m_{(1)} = 1, m_{(0,0)} = 2, m_{(0,0,0,0,0,1,0,0,0)} = 1\}$.

We now recall the basic definitions from [1] for a partial chord diagram c.

- The number of chords, the number of marked points, and the number of backbones of c are denoted k, l, and b respectively.

- The Euler characteristic and the genus of Σ_c, are denoted χ and g respectively. If n is the number of boundary components of Σ_c, we have that

  $$\chi = 2 - 2g, \tag{1.1}$$

  and g obeys Euler's relation

  $$2 - 2g = b - k + n. \tag{1.2}$$

- The *backbone spectrum* $\boldsymbol{b} = (b_0, b_1, b_2, \ldots)$ are assigned to c, if it has b_i backbones with precisely $i \geq 0$ vertices (of degree either two or three);

- The *boundary point spectrum* $\boldsymbol{n} = (n_0, n_1, \ldots)$ is assigned to c, if its boundary contains n_i connected components with i marked points;

[2] In [16, 14], the Schwinger-Dyson approach to the enumeration of chord diagrams is also discussed.

- The *boundary length spectrum* $\boldsymbol{\ell} = (\ell_1, \ell_2, \ldots)$ is assigned to c, if the boundary cycles of the diagram consist of ℓ_K edge-paths of length $K \geq 1$, where the *length* of a boundary cycle is the number of chords it traverses counted with multiplicity (as usual on the graph obtained from the diagram by collapsing each backbone to a distinct point) *plus* the number of backbone undersides it traverses (or in other words, the number of traversed connected components obtained by removing all the chord endpoints from all the backbones).

We now introduce the combination of the boundary length spectrum and the boundary point spectrum, namely our new boundary length and point spectrum.

- The *boundary length and point spectrum* $\boldsymbol{m} = (m_{(d_1,\ldots,d_K)})$ is assigned to c, if its boundary contains $m_{(d_1,\ldots,d_K)}$ connected components of length K with marked point spectrum $(d_1, \ldots, d_K)$, meaning that there cyclically around the boundary components are d_1 marked points, then a chord or a backbone underside, then d_2 marked points, then a chord or a backbone underside, and so on all the way around the boundary component. In fact we will not need to distinguish which way around the boundary we go. Hence it is only the cyclic ordered tuple of the numbers $d_1, \ldots, d_K$, which we need and which we denote as $\boldsymbol{d}_K = (d_1, \ldots, d_K)$. We remark that some of the d_I $(1 \leq I \leq K)$ might be zero.

We have the following relations

$$
\begin{aligned}
b &= \sum_{i\geq 0} b_i, \ n = \sum_{i\geq 0} n_i = \sum_{K\geq 1} \ell_K = \sum_{K\geq 1}\sum_{\boldsymbol{d}_K} m_{\boldsymbol{d}_K},\\
2k + l &= \sum_{i>0} i b_i, \ l = \sum_{i>0} i n_i = \sum_{K\geq 1}\sum_{\boldsymbol{d}_K} |\boldsymbol{d}_K| m_{\boldsymbol{d}_K}\\
2k + b &= \sum_{K\geq 1} K\ell_K = \sum_{K\geq 1}\sum_{\boldsymbol{d}_K} K m_{\boldsymbol{d}_K},
\end{aligned}
$$

where $|\boldsymbol{d}_K| = \sum_{I=1}^{K} d_I$. For all K and i, we also have that

$$
\ell_K = \sum_{\boldsymbol{d}_K} m_{\boldsymbol{d}_K}, \quad n_i = \sum_{K\geq 1}\sum_{i=|\boldsymbol{d}_K|} m_{\boldsymbol{d}_K}.
$$

We define $\mathcal{M}_{g,k,l}(\boldsymbol{b}, \boldsymbol{m})$ to be the number of connected partial chord diagrams of type $\{g, k, l; \boldsymbol{b}; \boldsymbol{m}\}$ taken to be zero if there is none of the specified type. In [1], $\mathcal{N}_{g,k,l}(\boldsymbol{b}, \boldsymbol{n}, \boldsymbol{p})$ is defined as the number of distinct connected partial chord diagrams of type $\{g, k, l; \boldsymbol{b}; \boldsymbol{n}; \boldsymbol{p}\}$. We find the relation between these numbers by the following formula

$$
\mathcal{N}_{g,k,l}(\boldsymbol{b}, \boldsymbol{\ell}, \boldsymbol{n}) = \sum_{\boldsymbol{m}\in M(\boldsymbol{\ell},\boldsymbol{n})} \mathcal{M}_{g,k,l}(\boldsymbol{b}, \boldsymbol{m}),
$$

where

$$M(\boldsymbol{\ell},\boldsymbol{n}) = \{\boldsymbol{m} \mid \ell_K = \sum_{\boldsymbol{d}_K} m_{\boldsymbol{d}_K},\, n_i = \sum_{K\geq 1} \sum_{i=|\boldsymbol{d}_K|} m_{\boldsymbol{d}_K}\}.$$

In particular, the numbers $\mathcal{N}_{g,k,l}(\boldsymbol{b},\boldsymbol{n})$ and $\mathcal{N}_{g,k,b}(\boldsymbol{\ell})$ are given by

$$\mathcal{N}_{g,k,l}(\boldsymbol{b},\boldsymbol{n}) = \sum_{\boldsymbol{\ell}} \mathcal{N}_{g,k,l}(\boldsymbol{b},\boldsymbol{\ell},\boldsymbol{n}), \quad \mathcal{N}_{g,k,b}(\boldsymbol{\ell}) = \sum_{\boldsymbol{n}} \sum_{\sum b_i = b} \mathcal{N}_{g,k,l=0}(\boldsymbol{b},\boldsymbol{\ell},\boldsymbol{n}),$$

For the index $\boldsymbol{b} = (b_i)$, we consider the variable $\boldsymbol{t} = (t_i)$ and denote

$$\boldsymbol{t}^{\boldsymbol{b}} = \prod_{i\geq 0} t_i^{b_i}.$$

And for the index $\boldsymbol{d} = (\boldsymbol{d}_K)$, we consider the variable $\boldsymbol{u} = (u_{\boldsymbol{d}_K})$ and denote

$$\boldsymbol{u}^{\boldsymbol{m}} = \prod_{K\geq 1} \prod_{\boldsymbol{d}_K} u_{\boldsymbol{d}_K}^{m_{\boldsymbol{d}_K}}$$

for any $\boldsymbol{m} = (\boldsymbol{m}_{\boldsymbol{d}_K})$. We define the orientable, multi-backbone, boundary length and point spectrum generating function $H(x,y;\boldsymbol{t};\boldsymbol{u}) = \sum_{b\geq 0} \mathcal{F}_b(x,y;\boldsymbol{t};\boldsymbol{u})$, where

$$H_b(x,y;\boldsymbol{t};\boldsymbol{u}) = \frac{1}{b!} \sum_{g=0}^{\infty} \sum_{k=2g+b-1}^{\infty} \sum_{\substack{\sum_K \sum_{\boldsymbol{d}_K} m_{\boldsymbol{d}_K} \\ = k-2g-b+2}} \sum_{\sum b_i = b} \mathcal{M}_{g,k,l}(\boldsymbol{b},\boldsymbol{m}) x^{2g} y^k \boldsymbol{t}^{\boldsymbol{b}} \boldsymbol{u}^{\boldsymbol{m}}, \tag{1.3}$$

For an element $\boldsymbol{p} = (p_{(d_1,\ldots d_K)})$, where each $p_{(d_1,\ldots d_K)} \in \mathbb{Z}$, we write

$$\boldsymbol{p} = \boldsymbol{p}^+ - \boldsymbol{p}^-,$$

where $\boldsymbol{p}^+$ contains all the positive entries and $\boldsymbol{p}^-$ the absolute value of all the negative ones, which we assume to both be finite. We define the differential operator

$$D_{\boldsymbol{p}} = \prod_{\boldsymbol{d}} u_{\boldsymbol{d}}^{\boldsymbol{p}_{\boldsymbol{d}}^-} \prod_{\boldsymbol{d}} \left(\frac{\partial}{\partial u_{\boldsymbol{d}}}\right)^{\boldsymbol{p}_{\boldsymbol{d}}^+}.$$

We now define $s_{I,J,\ell,m}(\boldsymbol{d}_K)$, $s_{I,\ell,m}(\boldsymbol{d}_K)$ and $q_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)$ to be strings like $\boldsymbol{p}$ given by the following formulae

$$\begin{aligned}
&s_{I,J,\ell,m}(\boldsymbol{d}_K) = \boldsymbol{e}_{\boldsymbol{d}_K} - \boldsymbol{e}_{(d_1,\ldots,d_{I-1},d_I-\ell-1,m,d_{J+1},\ldots,d_K)} - \boldsymbol{e}_{(\ell,d_{I+1},\ldots,d_{J-1},d_J-m-1)},\\
&s_{I,\ell,m}(\boldsymbol{d}_K) = \boldsymbol{e}_{\boldsymbol{d}_K} - \boldsymbol{e}_{(d_1,\ldots,d_{I-1},\ell,m,d_{I+1},\ldots,d_K)} - \boldsymbol{e}_{(d_I-\ell-m-2)},\\
&q_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)\\
&= \boldsymbol{e}_{\boldsymbol{d}_K} + \boldsymbol{e}_{\boldsymbol{f}_L} - \boldsymbol{e}_{(d_1,\ldots,d_{I-1},d_I-\ell-1,m,f_{J+1},\ldots,f_L,f_1,\ldots,f_{J-1},f_J-m-1,\ell,d_{I+1},\ldots,d_K)}.
\end{aligned}$$

where $\boldsymbol{e}_{\boldsymbol{d}_K}$ denotes the sequence $(0,\ldots,0,1,0,\ldots)$ where the component 1 appears only at the entry indexed by $\boldsymbol{d}_K$. We further define the index $c_{I,J,\ell,h}(\boldsymbol{d}_K,\boldsymbol{f}_M)$ by the formula

$$\begin{aligned} &c_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)\\ &=(d_1,\ldots,d_{I-1},d_I-\ell-1,m,f_{J+1},\ldots,f_L,f_1,\ldots,f_{J-1},f_J-m-1,\ell,d_{I+1},\ldots,d_K), \end{aligned}$$

which is identical to the index on the last term of the above assignments.

Theorem 1.1 (Enumeration of partial chord diagrams filtered by their boundary length and point spectrum)**.**

Define the first and second order linear differential operators

$$M_0=\sum_{K\geq 1}\sum_{\boldsymbol{d}_K}\left(\sum_{1\leq J<I\leq K}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{d_J-1}D_{s_{I,J,\ell,m}(\boldsymbol{d}_K)}+\sum_{I=1}^{K}\sum_{\ell,m=0}^{d_I-1}D_{s_{I,\ell,m}(\boldsymbol{d}_K)}\right), \tag{1.4}$$

$$M_2=\frac{1}{2}\sum_{K,L\geq 1}\sum_{\boldsymbol{d}_K,\boldsymbol{f}_L}\sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{d_J-1}D_{q_{I,J,\ell,h}(\boldsymbol{d}_K)}, \tag{1.5}$$

and the quadratic differential operator

$$S(H)=\frac{1}{2}\sum_{K,L\geq 1}\sum_{\boldsymbol{d}_K,\boldsymbol{f}_L}\sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{f_L-1}u_{c_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)}D_{\boldsymbol{d}_K}(H)D_{\boldsymbol{f}_L}(H)\ . \tag{1.6}$$

Then the following partial differential equations hold

$$\frac{\partial H_1}{\partial y}=(M_0+x^2M_2)H_1,\quad \frac{\partial H}{\partial y}=(M_0+x^2M_2+S)H. \tag{1.7}$$

Together with the initial conditions

$$H_1(x,y=0;\boldsymbol{t}=(t_1);\boldsymbol{u})=u_{(0)}t_1,\quad H(x,y=0;\boldsymbol{t};\boldsymbol{u})=\sum_{i\geq 1}u_{(i)}t_i, \tag{1.8}$$

they determine the functions H_1 and H uniquely.

In this article, we also consider the non-oriented analogue of partial chord diagrams. The generalization of the above analysis is straightforward, as we will now explain. A *non-oriented* partial chord diagrams, is a partial chord diagram together with a decoration of a binary variable at each chord, which indicates if the chord is *twisted* or not. When associating the surface Σ_c, to a non-oriented partial chord diagram, a twisted band is associated along twisted chords as indicated in Figure 2. By this construction, 2^k orientable and non-orientable surfaces are obtained from one partial chord diagram with k chords, when we vary over all assignments of twisting or not to the k chords. In the non-oriented case, we have the following definition of the Euler characteristic.

- Euler characteristic χ.
  The Euler characteristic of the two dimensional surface Σ_c is defined by the formula

  $$\chi = 2 - h, \tag{1.9}$$

  where h is the number of cross-caps and we have Euler's relation

  $$2 - h = b - k + n. \tag{1.10}$$

With this set-up, the enumeration of the non-oriented partial chord diagrams is considered in parallel to the oriented case discussed above with a small change for the boundary length and point spectrum $\boldsymbol{m}$. In this non-oriented case, there are now induced orientation on the boundaries of Σ_c and hence for an index $\boldsymbol{d}_K = (d_1, \ldots, d_K)$ corresponding some boundary component of Σ_c, we not only need to consider this tuple up to cyclic permutation of the tuple, but also reversal of the order

$$\boldsymbol{d}_K = (d_1, d_2, \ldots, d_K) = (d_K, \ldots d_2, d_1).$$

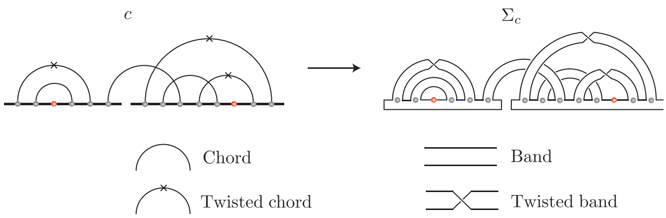

Figure 2: The non-oriented surface constructed out of untwisted and twisted chords.

Let $\widetilde{\mathcal{M}}_{h,k,l}(\boldsymbol{b}, \boldsymbol{m})$ be the number of non-oriented partial chord diagrams of type $\{h, k, l; \boldsymbol{b}; \boldsymbol{m}\}$. In [1], $\widetilde{\mathcal{N}}_{h,k,l}(\boldsymbol{b}, \boldsymbol{\ell}, \boldsymbol{n})$ is defined as the number of non-oriented connected partial chord diagrams of type $\{h, k, l; \boldsymbol{b}; \boldsymbol{\ell}; \boldsymbol{n}\}$. These numbers are related by the following formula

$$\widetilde{\mathcal{N}}_{h,k,l}(\boldsymbol{b}, \boldsymbol{\ell}, \boldsymbol{n}) = \sum_{\boldsymbol{m} \in M(\boldsymbol{\ell}, \boldsymbol{n})} \widetilde{\mathcal{M}}_{h,k,l}(\boldsymbol{b}, \boldsymbol{m}),$$

and the numbers $\widetilde{\mathcal{N}}_{h,k,l}(\boldsymbol{b}, \boldsymbol{n})$ and $\widetilde{\mathcal{N}}_{h,k,b}(\boldsymbol{\ell})$ are given by

$$\widetilde{\mathcal{N}}_{h,k,l}(\boldsymbol{b}, \boldsymbol{n}) = \sum_{\boldsymbol{\ell}} \widetilde{\mathcal{N}}_{h,k,l}(\boldsymbol{b}, \boldsymbol{\ell}, \boldsymbol{n}), \qquad \widetilde{\mathcal{N}}_{h,k,b}(\boldsymbol{\ell}) = \sum_{\boldsymbol{n}} \sum_{\sum b_i = b} \widetilde{\mathcal{N}}_{h,k,l=0}(\boldsymbol{b}, \boldsymbol{\ell}, \boldsymbol{n}).$$

We define the non-oriented generating function $\widetilde{H}(x,y;\boldsymbol{t};\boldsymbol{u}) = \sum_{b\geq 1}\widetilde{H}_b(x,y;\boldsymbol{t};\boldsymbol{u})$ to be given by

$$\widetilde{H}_b(x,y;\boldsymbol{t};\boldsymbol{u}) = \frac{1}{b!}\sum_{h=0}^{\infty}\sum_{k=h+b-1}^{\infty}\sum_{\substack{\sum_K\sum_{\boldsymbol{d}_K} m_{\boldsymbol{d}_K}\\ =k-h-b+2}}\sum_{\sum b_i=b}\widetilde{\mathcal{M}}_{h,k,l}(\boldsymbol{b},\boldsymbol{m})x^h y^k \boldsymbol{t}^{\boldsymbol{b}}\boldsymbol{u}^{\boldsymbol{m}}. \tag{1.11}$$

We define $s^{\times}_{I,J,\ell,h}(\boldsymbol{d}_K)$, $s^{\times}_{I,\ell,h}(\boldsymbol{d}_K)$ and $q^{\times}_{I,J,\ell,h}(\boldsymbol{d}_K,\boldsymbol{f}_L)$ to be by

$$\begin{aligned}
&s^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K) = \boldsymbol{e}_{\boldsymbol{d}_K} - \boldsymbol{e}_{(d_1,\dots,d_{I-1},\ell,m,d_{J-1},\dots,d_{I+1},d_J-\ell-1,d_J-m-1,d_{J+1},\dots,d_K)},\\
&s^{\times}_{I,\ell,m}(\boldsymbol{d}_K) = \boldsymbol{e}_{\boldsymbol{d}_K} - \boldsymbol{e}_{(d_1,\dots,d_{I-1},\ell,d_I-\ell-m-2,m,d_{I+1},\dots,d_K)},\\
&q^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)\\
&\ = \boldsymbol{e}_{\boldsymbol{d}_K} + \boldsymbol{e}_{\boldsymbol{f}_M} - \boldsymbol{e}_{(f_1,\dots,f_{J-1},f_J-m-1,\ell,d_{I-1},\dots,d_1,d_K,\dots,d_{I+1},d_I-\ell-1,m,f_{J+1},\dots,f_L)}.
\end{aligned}$$

And we also define indices $c^{\times}_{I,J,\ell,h}(\boldsymbol{d}_K,\boldsymbol{f}_M)$ by the formula

$$\begin{aligned}
&c^{\times}_{I,J,\ell,h}(\boldsymbol{d}_K,\boldsymbol{f}_L)\\
&= (f_1,\dots,f_{J-1},f_J-m-1,\ell,d_{I-1},\dots,d_1,d_K,\dots,d_{I+1},d_I-\ell-1,m,f_{J+1},\dots,f_L),
\end{aligned}$$

which again, we note is identical to the index on the last term of the above assignments.

Theorem 1.2 (Enumeration of non-oriented partial chord diagrams filtered by their boundary length and point spectrum)**.**

Define the first and second order linear differential operators

$$M_1^{\times} = \sum_{K\geq 1}\sum_{\boldsymbol{d}_K}\left(\sum_{1\leq J<I\leq K}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{d_J-1}D_{s^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K)} + \sum_{I=1}^{K}\sum_{\ell,m=0}^{d_I-1}D_{s^{\times}_{I,\ell,m}(\boldsymbol{d}_K)}\right), \tag{1.12}$$

$$M_2^{\times} = \frac{1}{2}\sum_{K,L\geq 1}\sum_{\boldsymbol{d}_K,\boldsymbol{f}_L}\sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{d_J-1}D_{q^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K)}, \tag{1.13}$$

and the quadratic differential operator

$$S^{\times}(H) = \frac{1}{2}\sum_{K,L\geq 1}\sum_{\boldsymbol{d}_K,\boldsymbol{f}_L}\sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{f_L-1}u_{c^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)}D_{\boldsymbol{d}_K}(H)D_{\boldsymbol{f}_L}(H)\,. \tag{1.14}$$

Then the following partial differential equations hold

$$\begin{aligned}
\frac{\partial\widetilde{H}_1}{\partial y} &= (M_0 + xM_1^{\times} + x^2(M_2+M_2^{\times}))\widetilde{H}_1,\\
\frac{\partial\widetilde{H}}{\partial y} &= (M_0 + xM_1^{\times} + x^2(M_2+M_2^{\times}) + S + S^{\times})\widetilde{H}.
\end{aligned} \tag{1.15}$$

Together with the following initial conditions

$$\widetilde{H}_1(x, y=0; \boldsymbol{t}=(t_1); \boldsymbol{u}) = u_{(0)}t_1, \quad \widetilde{H}(x, y=0; \boldsymbol{t}; \boldsymbol{u}) = \sum_{i\geq 1} u_{(i)}t_i, \tag{1.16}$$

determines $\widetilde{H}_1$ *and* $\widetilde{H}$ *uniquely.*

This paper is organized as follows. Section 2 contains basic combinatorial results on the boundary length and point spectra of partial chord diagrams and derives the recursion relation of the number of diagrams (Proposition 2.1), by the cut-and-join method. This cut-and-join equation is rewritten as a second order, non-linear, algebraic partial differential equation for generating function of the number of partial chord diagrams filtered by the boundary length and point spectrum (Proposition 2.2). Section 3 extends these results to include the non-oriented analogues of the partial chord diagrams. The cut-and-join equation is extended to provide a recursion on the number of non-oriented partial chord diagrams (Proposition 3.1), and is also rewritten as partial differential equation (Proposition 3.2).

2 Combinatorial proof of the cut-and-join equation

In this section, we devote to prove Theorem 1.1. The partial differential equation (1.7) is equivalent to the following recursion relation for the numbers of connected partial chord diagrams.

Proposition 2.1. *The numbers* $\mathcal{M}_{g,k,l}(\boldsymbol{b}, \boldsymbol{m})$ *enumerating connected partial chord diagrams of type* $\{g, k, l; \boldsymbol{b}, \boldsymbol{m}\}$ *obey the following recursion relation*

$$\begin{aligned}
&k\mathcal{M}_{g,k,l}(\boldsymbol{b}, \boldsymbol{m}) \\
&= \sum_{K\geq 1}\sum_{\boldsymbol{d}_K}(m_{\boldsymbol{d}_K}+1)\Bigg[\sum_{1\leq I<J\leq K}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{d_J-1}\mathcal{M}_{g,k-1,l+2}\left(\boldsymbol{b}, \boldsymbol{m}+s_{I,J,\ell,m}(\boldsymbol{d}_K)\right) \\
&\qquad\qquad + \sum_{I=1}^{K}\sum_{\substack{\ell,m=0 \\ \ell+m\leq d_I-2}}^{d_I-1}\mathcal{M}_{g,k-1,l+2}\left(\boldsymbol{b}, \boldsymbol{m}+s_{I,\ell,m}(\boldsymbol{d}_K)\right)\Bigg] \\
&\quad + \frac{1}{2}\sum_{K\geq 1}\sum_{L\geq 1}\sum_{\boldsymbol{d}_K}\sum_{\boldsymbol{f}_L}(m_{\boldsymbol{d}_K}+1)(m_{\boldsymbol{f}_L}+1-\delta_{\boldsymbol{d}_K,\boldsymbol{f}_L}) \\
&\quad\times\sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{f_J-1}\mathcal{M}_{g-1,k-1,l+2}\left(\boldsymbol{b}, \boldsymbol{m}+q_{I,J,\ell,m}(\boldsymbol{d}_K, \boldsymbol{f}_L)\right)
\end{aligned}$$

$$+\frac{1}{2}\sum_{K\geq 1}\sum_{L\geq 1}\sum_{\boldsymbol{d}_K}\sum_{\boldsymbol{f}_L}\sum_{g_1+g_2=g}\sum_{k_1+k_2=k-1}\sum_{b^{(1)}+b^{(2)}=b}$$
$$\times\sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{f_J-1}\sum_{\substack{\boldsymbol{m}^{(1)}+\boldsymbol{m}^{(2)}\\=\boldsymbol{m}+q_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)}}$$
$$\times\, m^{(1)}_{\boldsymbol{d}_K} m^{(2)}_{\boldsymbol{f}_L}\frac{b!}{b^{(1)}!b^{(2)}!}\mathcal{M}_{g_1,k_1,l_1}\big(\boldsymbol{b}^{(1)},\boldsymbol{m}^{(1)}\big)\mathcal{M}_{g_2,k_2,l_2}\big(\boldsymbol{b}^{(2)},\boldsymbol{m}^{(2)}\big). \tag{2.1}$$

This recursion relation is referred to as the *cut-and-join equation*, since it follows from a cut-and-join argument, which we shall now provide.

Proof. When one removes one chord from a partial chord diagram, there are essentially three distinct possible outcomes. First of all the diagram can stay connected and then there are two cases to consider. In the first one, the chord that is removed is adjacent to two different boundary components and in the second one it is adjacent to just one. The third case is when the chord diagram becomes disconnected.

In the first case, the genus of the partial chord diagram is not changed, but two boundary components join into one component. On the other hand, in the second case, the genus decreases by one, and one boundary component splits into two components.

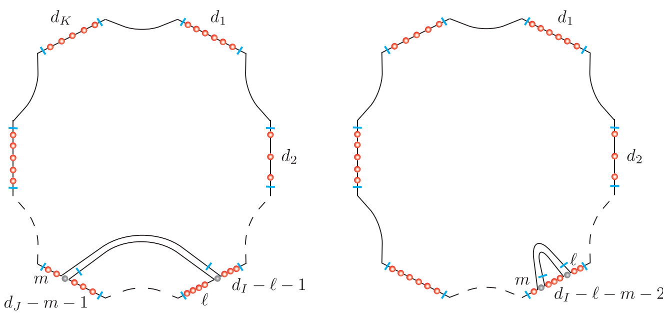

Figure 3: Removal of a chord in case one. The chord is depicted as a band. After the removal of this chord, two boundary components join into one component. Left: The clusters of marked points $(d_I-\ell-1,m)$ and (d_J-m-1,ℓ) join into two clusters d_I and d_J Right: The clusters of marked points (ℓ,m) and (d_J-m-1) join into one cluster d_I.

In the first case, and let us say that after removing this chord, the two adjacent boundary components join into one component with the marked point spectrum

$\boldsymbol{d}_K = (d_1, \ldots, d_K)$. (See Figure 3.) Under this elimination, the numbers k and n change to $k-1$ and $n-1$, the genus g is not changed (c.f. Euler's relation $2-2g = b-k+n$). The number of marked points l changes to $l+2$, because the chord ends of the chord which is removed become new marked points. There are two distinct possible sub cases, namely either the chord ends belong to two distinct clusters of marked points d_I and d_J in the resulting chord diagram, or chord ends belong to the same cluster of marked points d_I.

We will consider the former kind of chord, and assume $I < J$ without loss of generality. Before we remove the chord, the two boundaries adjacent to the chord needs to have the following two marked point spectra

$$\begin{aligned}
&(d_1, \ldots, d_{I-1}, d_I - \ell - 1, m, d_{J+1}, \ldots, d_K), \quad \text{and} \quad (\ell, d_{I+1}, \ldots, d_{J-1}, d_J - m - 1), \\
&0 \le \ell \le d_I - 1, \quad 0 \le m \le d_J - 1,
\end{aligned}$$

When removing the chord, we connect the clusters of marked points $(d_I - \ell - 1, m)$ and $(d_J - m - 1, \ell)$. If the original partial chord diagram has the boundary length-point spectrum $\boldsymbol{m}$, the resulting diagram has

$$\begin{aligned}
&\boldsymbol{m} - \boldsymbol{e}_{(d_1,\ldots,d_{I-1},d_I-\ell-1,m,d_{J+1},\ldots,d_K)} - \boldsymbol{e}_{(\ell,d_{I+1},\ldots,d_{J-1},d_J-m-1)} + \boldsymbol{e}_{\boldsymbol{d}_K} \\
&\quad = \boldsymbol{m} + s_{I,J,\ell,m}(\boldsymbol{d}_K).
\end{aligned}$$

For the latter kind, we must have two boundary components with the marked point spectra

$$\begin{aligned}
&(d_1, \ldots, d_{I-1}, \ell, m, d_{I+1}, \ldots, d_K), \quad \text{and} \quad (d_I - \ell - m - 2). \\
&0 \le \ell, m \le d_I - 1, \quad 0 \le \ell + m \le d_I - 2,
\end{aligned}$$

and removing the chord connects the clusters of marked points (ℓ, m) and $(d_J - m - 1)$. This manipulation changes the boundary length and point spectrum $\boldsymbol{m}$ into

$$\boldsymbol{m} - \boldsymbol{e}_{(d_1,\ldots,d_{I-1},\ell,m,d_{I+1},\ldots,d_K)} - \boldsymbol{e}_{(d_I-\ell-m-2)} + \boldsymbol{e}_{\boldsymbol{d}_K} = \boldsymbol{m} + s_{I,\ell,m}(\boldsymbol{d}_K).$$

For both of these two kinds of removal, there are $m_{\boldsymbol{d}_K}+1$ possibilities to choose the boundary components in the partial chord diagram. Therefore, the number of possibilities for the first way of removal is

$$\begin{aligned}
&\sum_{K\ge 1}\sum_{\boldsymbol{d}_K}(m_{\boldsymbol{d}_K}+1)\Bigg[\sum_{1\le I<J\le K}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{d_J-1}\mathcal{M}_{g,k-1,l+2}\left(\boldsymbol{b}, \boldsymbol{m} + s_{I,J,\ell,m}(\boldsymbol{d}_K)\right) \\
&\qquad + \sum_{I=1}^{K}\sum_{\substack{\ell,m=0 \\ \ell+m\le d_I-2}}^{d_I-1}\mathcal{M}_{g,k-1,l+2}\left(\boldsymbol{b}, \boldsymbol{m} + s_{I,\ell,m}(\boldsymbol{d}_K)\right)\Bigg].
\end{aligned} \tag{2.2}$$

In the second case (see Figure 4), the removal changes the numbers k and n to $k-1$ and $n+1$ and the genus of the partial chord diagram decreases by one. For partial chord diagram with a boundary with marked point spectrum

$$(d_1,\dots,d_{I-1},d_I-\ell-1,m,f_{J+1},\dots,f_L,f_1,\dots,f_{J-1},f_J-m-1,\ell,d_{I+1},\dots,d_K),$$
$$0\le \ell\le d_I-1,\quad 0\le m\le f_J-1,$$

we remove the chord which connects the two clusters (f_J-m-1,ℓ) and $(d_I-\ell-1,m)$ of marked points. The boundary component then splits into two boundary components with marked point spectra $\boldsymbol{d}_K=(d_1,\dots,d_K)$ and $\boldsymbol{f}_L=(f_1,\dots,f_L)$. If the original partial chord diagram has the boundary length and point spectrum $\boldsymbol{m}$, after removal of this chord, we find that

$$\begin{aligned}&\boldsymbol{m}-\boldsymbol{e}_{(d_1,\dots,d_{I-1},d_I-\ell-1,m,f_{J+1},\dots,f_L,f_1,\dots,f_{J-1},f_J-m-1,\ell,d_{I+1},\dots,d_K)}+\boldsymbol{e}_{\boldsymbol{d}_K}+\boldsymbol{e}_{\boldsymbol{f}_L}\\ &\quad=\boldsymbol{m}+q_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L).\end{aligned}$$

The number of possibilities of this removal is $(m_{\boldsymbol{d}_K}+1)(m_{\boldsymbol{f}_L}+1)$ for $\boldsymbol{d}_K\neq\boldsymbol{f}_L$. If $\boldsymbol{d}_K=\boldsymbol{f}_L$, the number of possibilities becomes $m_{\boldsymbol{d}_K}(m_{\boldsymbol{d}_K}+1)/2$. In total, the number of possibilities for the second way of elimination is

$$\begin{aligned}&\frac{1}{2}\sum_{K=1}^{\infty}\sum_{L=1}^{\infty}\sum_{\boldsymbol{d}_K}\sum_{\boldsymbol{f}_L}(m_{\boldsymbol{d}_K}+1)(m_{\boldsymbol{f}_L}+1-\delta_{\boldsymbol{d}_K,\boldsymbol{f}_L})\\ &\qquad\times\sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{d_I-1}\sum_{h=0}^{f_J-1}\mathcal{M}_{g-1,k-1,l+2}\left(\boldsymbol{b},\boldsymbol{m}+q_{I,J,\ell,h}(\boldsymbol{d}_K,\boldsymbol{f}_L)\right).\end{aligned}\tag{2.3}$$

The factor $1/2$ in front of the sum takes care of the over counting in the cases $\boldsymbol{d}_K\neq\boldsymbol{f}_L$.

In the third case, the partial chord diagram split into two connected components. We consider the case that the original diagram has the type $\{g,k,l;\boldsymbol{b},\boldsymbol{m}\}$ and the resulting two connected components have types $\{g_1,k_1,l_1;\boldsymbol{b}^{(1)},\boldsymbol{m}^{(1)}\}$ and $\{g_2,k_2,l_2;\boldsymbol{b}^{(2)},\boldsymbol{m}^{(2)}\}$. These types are related such that

$$g=g_1+g_2,\quad k-1=k_1+k_2,\quad \boldsymbol{b}=\boldsymbol{b}^{(1)}+\boldsymbol{b}^{(2)}.$$

Since a boundary component also split into two components, the boundary length and point spectrum changes in the same manner as in the second case.

$$\boldsymbol{m}+q_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)=\boldsymbol{m}^{(1)}+\boldsymbol{m}^{(2)}.$$

There are $m^{(1)}_{\boldsymbol{d}_K}m^{(2)}_{\boldsymbol{f}_L}$ ways to choose the boundary components which are to be fused under the inverse operation of chord removal. And the number of different ordered splittings of a b-backbone diagram is $\frac{b!}{b^{(1)}!b^{(2)}!}$ where $b^{(a)}=\sum_i b_i^{(a)}$ $(a=1,2)$. Therefore, the total number of possibilities of this case is

$$\frac{1}{2}\sum_{I=1}^{\infty}\sum_{J=1}^{\infty}\sum_{\boldsymbol{d}_K}\sum_{\boldsymbol{f}_L}\sum_{g_1+g_2=g}\quad\sum_{k_1+k_2=k-1}\quad\sum_{b^{(1)}+b^{(2)}=b}$$

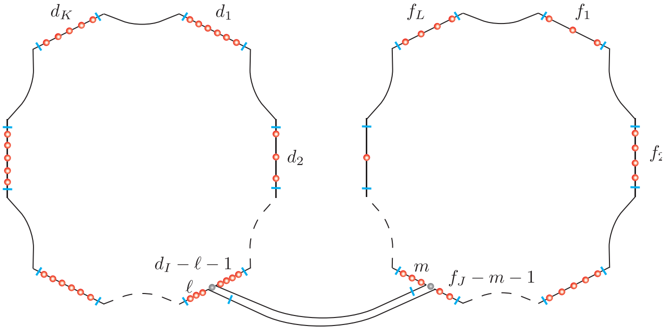

Figure 4: The second and third way of elimination of a chord. After the elimination of this chord, a boundary component split into two different boundary components.

$$\times \sum_{I=1}^{K} \sum_{J=1}^{L} \sum_{\ell=0}^{d_I-1} \sum_{m=0}^{f_J-1} \sum_{\substack{\boldsymbol{m}^{(1)}+\boldsymbol{m}^{(2)} \\ =\boldsymbol{m}+q_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)}}$$

$$\times m^{(1)}_{\boldsymbol{d}_K} m^{(2)}_{\boldsymbol{f}_L} \frac{b!}{b^{(1)}!b^{(2)}!} \mathcal{M}_{g_1,k_1,l_1}\big(\boldsymbol{b}^{(1)}, \boldsymbol{m}^{(1)}\big) \mathcal{M}_{g_2,k_2,l_2}\big(\boldsymbol{b}^{(2)}, \boldsymbol{m}^{(2)}\big). \tag{2.4}$$

The factor $1/2$ corrects for the over counting due to the ordering of the two connected components.

The sum of the contributions (2.2), (2.3), and (2.4) from the three different cases of chord removals equals $k\mathcal{M}_{g,k,l}(\boldsymbol{b}, \boldsymbol{m})$, because there are k possibilities for the choice of the chord to be removed. This gives the cut-and-join equation (2.1). □

Proposition 2.2. *The generating function $H(x, y; \boldsymbol{t}, ; \boldsymbol{u})$ is uniquely determined by the differential equation*

$$\frac{\partial H}{\partial y} = (M + S)H,$$

where $M = M_0 + x^2 M_2$. The generating function $Z(x, y; \boldsymbol{t}, ; \boldsymbol{u}) = \exp[H]$ of the number of connected and disconnected partial chord diagrams satisfies

$$\frac{\partial Z}{\partial y} = MZ, \tag{2.5}$$

and is as such determined by the initial conditions

$$H(x, y = 0; \boldsymbol{t}; \boldsymbol{u}) = \sum_{i \geq 1} t_i u_{(i)}, \quad Z(x, y = 0; \boldsymbol{t}, ; \boldsymbol{u}) = \mathrm{e}^{\sum_{i\geq 1} t_i u_{(i)}}.$$

Proof. It is straightforward to check that the differential equation $\frac{\partial H}{\partial y} = (M+S)H$ is equivalent to the cut-and-join equation (2.1). The actions in the quadratic differential S on H can be rewritten by following relation

$$D_{\boldsymbol{d}_K}(H)D_{\boldsymbol{f}_L}(H) + D_{\boldsymbol{d}_K}D_{\boldsymbol{f}_L}H = \frac{1}{Z}D_{\boldsymbol{d}_K}D_{\boldsymbol{f}_L}Z.$$

The derivatives on the right hand side are contained in M_2, and the differential equation $\frac{\partial Z}{\partial y} = MZ$ follows from that of H.

On the initial condition, every partial chord diagram of type $\{g,k,l;\boldsymbol{b};\boldsymbol{m}\}$ can be obtained from the disjoint collection of type $\{0,0,i;\boldsymbol{e}_i,\boldsymbol{e}_{(i)}\}$ with multiplicity b_i by connecting them with k chords. This implies $H(x,y=0;\boldsymbol{t};\boldsymbol{u}) = \sum_{i\geq 1} t_i u_{(i)}$. Since this is the first order differential equation of y, the coefficient of y^k is determined uniquely using this initial condition.

□

3 Non-oriented analogue of the cut-and-join equation

In this section, we will prove Theorem 1.2. We first establish the following proposition.

Proposition 3.1. *The number $\widetilde{\mathcal{M}}_{g,k,l}(\boldsymbol{b},\boldsymbol{m})$ of connected non-oriented partial chord diagrams of type $\{g,k,l;\boldsymbol{b},\boldsymbol{m}\}$ obeys the following recursion relation*

$$\begin{aligned}
&k\widetilde{\mathcal{M}}_{g,k,l}(\boldsymbol{b},\boldsymbol{m})\\
&=\sum_{K\geq 1}\sum_{\boldsymbol{d}_K}(m_{\boldsymbol{d}_K}+1)\\
&\quad\times\Bigg[\sum_{I<J}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{d_J-1}\Big\{\widetilde{\mathcal{M}}_{h,k-1,l+2}\left(\boldsymbol{b},\boldsymbol{m}+s_{I,J,\ell,m}(\boldsymbol{m})\right)+\widetilde{\mathcal{M}}_{h-1,k-1,l+2}\big(\boldsymbol{b},\boldsymbol{m}+s^{\times}_{I,J,\ell,m}(\boldsymbol{m})\big)\Big\}\\
&\quad+\sum_{I=1}^{K}\sum_{\ell+m\leq d_I-2}\Big\{\widetilde{\mathcal{M}}_{h,k-1,l+2}\left(\boldsymbol{b},\boldsymbol{m}+s_{I,\ell,m}(\boldsymbol{m})\right)+\widetilde{\mathcal{M}}_{h-1,k-1,l+2}\big(\boldsymbol{b},\boldsymbol{m}+s^{\times}_{I,\ell,m}(\boldsymbol{m})\big)\Big\}\Bigg]\\
&\quad+\frac{1}{2}\sum_{K\geq 1}\sum_{L\geq 1}\sum_{\boldsymbol{d}_K}\sum_{\boldsymbol{f}_L}(m_{\boldsymbol{d}_K}+1)(m_{\boldsymbol{f}_L}+1-\delta_{\boldsymbol{d}_K,\boldsymbol{f}_L})\\
&\quad\times\sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{f_J-1}\Big\{\widetilde{\mathcal{M}}_{h-2,k-1,l+2}\left(\boldsymbol{b},\boldsymbol{m}+q_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)\right)\\
&\qquad\qquad+\widetilde{\mathcal{M}}_{h-2,k-1,l+2}\big(\boldsymbol{b},\boldsymbol{m}+q^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)\big)\Big\}\\
&\quad+\frac{1}{2}\sum_{K\geq 1}\sum_{L\geq 1}\sum_{\boldsymbol{d}_K}\sum_{\boldsymbol{f}_L}\sum_{h_1+h_2=h}\sum_{k_1+k_2=k-1}\sum_{b^{(1)}+b^{(2)}=b}
\end{aligned}$$

$$\times \sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{f_J-1}\left(\sum_{\substack{\boldsymbol{m}^{(1)}+\boldsymbol{m}^{(2)}\\=\boldsymbol{m}+q_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)}} + \sum_{\substack{\boldsymbol{m}^{(1)}+\boldsymbol{m}^{(2)}\\=\boldsymbol{m}+q^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)}}\right) m^{(1)}_{\boldsymbol{d}_K} m^{(2)}_{\boldsymbol{f}_L}$$

(3.1)

$$\times \frac{b!}{b^{(1)}!b^{(2)}!}\widetilde{\mathcal{M}}_{h_1,k_1,l_1}\big(\boldsymbol{b}^{(1)},\boldsymbol{m}^{(1)}\big)\widetilde{\mathcal{M}}_{h_2,k_2,l_2}\big(\boldsymbol{b}^{(2)},\boldsymbol{m}^{(2)}\big).$$

Proof. If we remove a non-twisted chord, then we find the same recursive structure as for the numbers (2.2), (2.3), and (2.4) for $\widetilde{\mathcal{M}}_{h,k,l}\big(\boldsymbol{b},\boldsymbol{m}\big)$ in the oriented case. As we did in the proof of proposition 2.1, we also consider three cases, organised the same way, when removing a twisted chord.

In the first case (see Figure 5), there are again two possibilities, namely the twisted chord ends belong to two different or the same clusters of marked points on the boundary component in the resulting diagram after removal. Contrary to the case of non-twisted chords, the boundary cycle does not split, but the marked point spectrum changes due to the recombination of the boundary component. For both of these two cases, the numbers k and n change to $k-1$ and n, and the cross-cap number h decreases by one under this elimination (c.f. Euler's relation $2-h=b-k+n$). The chord ends become marked points and l changes to $l+2$.

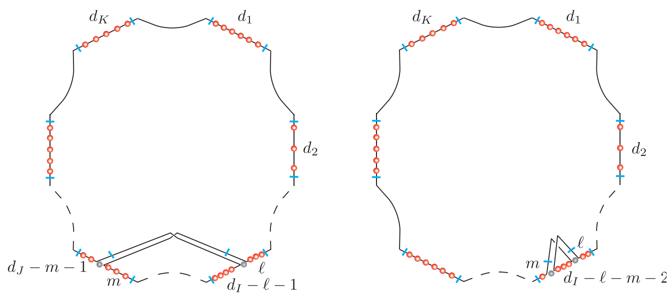

Figure 5: Removal of a twisted chord from a non-oriented partial chord diagram. The chord is depicted as a twisted band. After the elimination of this chord, the boundary component is reconnected into one component with different marked point spectrum. Left: The clusters of marked points $(d_I-\ell-1,m)$ and (d_J-m-1,ℓ) join into two clusters d_I and d_J. Right: The clusters of marked points (ℓ,m) and (d_J-m-1) join into one cluster d_I.

In the former situation, we must have a boundary component with the marked point spectrum

$$(d_1,\ldots,d_{I-1},\ell,m,d_{J-1},d_{J-2}\ldots,d_{I+1},d_I-\ell-1,d_J-m-1,d_{J+1},\ldots,d_K)$$

$$I < J, \quad 0 \le \ell \le d_I - 1, \quad 0 \le m \le d_J - 1,$$

from which we remove one twisted chord with one end between the two clusters (ℓ, m) and the other between $(d_I - \ell - 1, d_J - m - 1)$. Then the removal will result in a boundary component with the marked point spectrum $\boldsymbol{d}_K$ and the boundary length and point spectrum $\boldsymbol{m}$ is changed as follows

$$\begin{aligned}&\boldsymbol{m} - \boldsymbol{e}_{(d_1,\dots,d_{I-1},\ell,m,d_{J-1},\dots,d_{J+1},d_J-\ell-1,d_J-m-1,d_{J+1},\dots,d_K)} + \boldsymbol{e}_{\boldsymbol{d}_K}\\ &= \boldsymbol{m} + s^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K).\end{aligned}$$

The possible number of choices for this kind of removal is $m_{\boldsymbol{d}_K} + 1$, and the total number of diagrams which can be obtained in this way is

$$\sum_{K\ge1}\sum_{\boldsymbol{d}_K}(m_{\boldsymbol{d}_K}+1)\sum_{I<J}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{d_J-1}\widetilde{\mathcal{M}}_{h-1,k-1,l+2}\big(\boldsymbol{b},\boldsymbol{m}+s^{\times}_{I,J,\ell,m}(\boldsymbol{m})\big). \tag{3.2}$$

For the removal of the latter kind of twisted chords, we must start with a diagram with a boundary component with the marked point spectrum

$$\begin{aligned}&(d_1,\dots,d_{I-1},\ell,d_I-\ell-m-2,m,d_{I+1},\dots,d_K),\\ &0\le \ell,m\le d_I, \quad \ell+m\le d_I-2.\end{aligned}$$

from which we remove one twisted chords with one end between the two clusters $(\ell, d_I - \ell - m - 2)$ and the other one between the two clusters $(d_I - \ell - m - 2, m)$. After removal, we obtain a boundary component with the marked point spectrum $\boldsymbol{d}_K$. Thus, the boundary length and point spectrum $\boldsymbol{m}$ is changed to

$$\boldsymbol{m} - \boldsymbol{e}_{(d_1,\dots,d_{I-1},\ell,d_I-\ell-m-2,m,d_{I+1},\dots,d_K)} + \boldsymbol{e}_{\boldsymbol{d}_K} = \boldsymbol{m} + s^{\times}_{I,\ell,m}(\boldsymbol{d}_K).$$

The number of such chords to be removed is $m_{\boldsymbol{d}_K} + 1$, and the total number of partial chord diagrams obtained in this way is

$$\sum_{K\ge1}\sum_{\boldsymbol{d}_K}(m_{\boldsymbol{d}_K}+1)\sum_{I=1}^{K}\sum_{\ell+m\le d_I-2}\widetilde{\mathcal{M}}_{h-1,k-1,l+2}\big(\boldsymbol{b},\boldsymbol{m}+s^{\times}_{I,\ell,m}(\boldsymbol{m})\big). \tag{3.3}$$

Next, we consider the second case (see Figure 6), where we must start with a non-oriented partial chord diagram with a boundary component with the marked point spectrum

$$\begin{aligned}&(f_1,\dots,f_{J-1},f_J-m-1,\ell,d_{I-1},\dots,d_1,d_K,\dots,d_{I+1},d_I-\ell-1,m,f_{J+1},\dots,f_L),\\ &0\le \ell\le d_I-1, \quad 0\le m\le f_J-1,\end{aligned}$$

from which we remove a twisted chord with one end between the two clusters $(f_J - m - 1, \ell)$ and the other end between the two clusters $(d_I - \ell - 1, m)$. After removal of this chord, the boundary component has been split into two components with spectra $\boldsymbol{d}_K$ and $\boldsymbol{f}_L$, and the cross-cap number h decreases by two. Then, the boundary length and point spectrum $\boldsymbol{m}$ changes to

$$\begin{aligned}&\boldsymbol{m} - \boldsymbol{e}_{(f_1,\dots,f_{J-1},f_J-m-1,\ell,d_{I-1},\dots,d_1,d_K,\dots,d_{I+1},d_I-\ell-1,m,f_{J+1},\dots,f_L)} + \boldsymbol{e}_{\boldsymbol{d}_K} + \boldsymbol{e}_{\boldsymbol{f}_L}\\ &= \boldsymbol{m} + q^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K, \boldsymbol{f}_L).\end{aligned}$$

The number of choices for the chord to be removed is $(m_{\boldsymbol{d}_K}+1)(m_{\boldsymbol{f}_L}+1)$ for $\boldsymbol{d}_K \neq \boldsymbol{f}_L$ and $m_{\boldsymbol{d}_K}(m_{\boldsymbol{d}_K}+1)/2$ for $\boldsymbol{d}_K = \boldsymbol{f}_L$, and the total number of partial chord diagrams obtained this way is

$$\begin{aligned}&\frac{1}{2}\sum_{K\geq 1}\sum_{L\geq 1}\sum_{\boldsymbol{d}_K}\sum_{\boldsymbol{f}_L}(m_{\boldsymbol{d}_K}+1)(m_{\boldsymbol{f}_L}+1-\delta_{\boldsymbol{d}_K,\boldsymbol{f}_L})\\ &\times \sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{f_J-1}\widetilde{\mathcal{M}}_{h-2,k-1,l+2}\big(\boldsymbol{b}, \boldsymbol{m} + q^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K, \boldsymbol{f}_L)\big).\end{aligned} \tag{3.4}$$

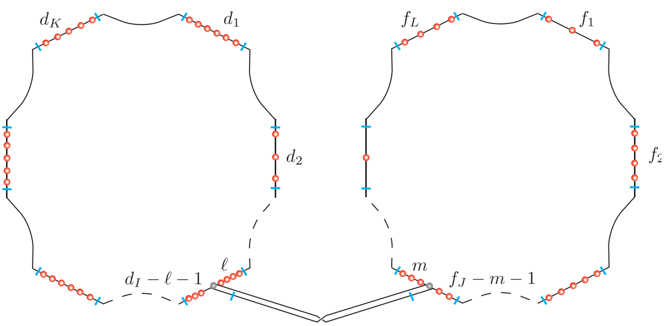

Figure 6: The second case of a twisted chord removal. After the removal of this chord, the boundary component split into two distinct boundary components.

In case three partial chord diagram split into two connected components when we remove the chord. Assume that the original diagram has the type $\{h,k,l;\boldsymbol{b},\boldsymbol{m}\}$ and the resulting two connected components have types $\{h_1,k_1,l_1;\boldsymbol{b}^{(1)},\boldsymbol{m}^{(1)}\}$ and $\{h_2,k_2,l_2;\boldsymbol{b}^{(2)},\boldsymbol{m}^{(2)}\}$. Then these types are related by

$$h = h_1 + h_2, \quad k-1 = k_1 + k_2, \quad \boldsymbol{b} = \boldsymbol{b}^{(1)} + \boldsymbol{b}^{(2)}.$$

The marked point spectrum changes in the same way as the second case

$$\boldsymbol{m} + q^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K, \boldsymbol{f}_L) = \boldsymbol{m}^{(1)} + \boldsymbol{m}^{(2)}.$$

The total number of resulting diagrams is

$$\begin{aligned}&\frac{1}{2}\sum_{K\geq 1}\sum_{L\geq 1}\sum_{\boldsymbol{d}_K}\sum_{\boldsymbol{f}_L}\sum_{h_1+h_2=h}\sum_{k_1+k_2=k-1}\sum_{b^{(1)}+b^{(2)}=b}\\&\times\sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{d_I-1}\sum_{m=0}^{f_J-1}\sum_{\substack{\boldsymbol{m}^{(1)}+\boldsymbol{m}^{(2)}\\=\boldsymbol{m}+q^{\times}_{I,J,\ell,m}(\boldsymbol{d}_K,\boldsymbol{f}_L)}}\\&\times m^{(1)}_{\boldsymbol{d}_K}m^{(2)}_{\boldsymbol{f}_L}\frac{b!}{b^{(1)}!b^{(2)}!}\widetilde{\mathcal{M}}_{h_1,k_1,l_1}\big(\boldsymbol{b}^{(1)},\boldsymbol{m}^{(1)}\big)\widetilde{\mathcal{M}}_{h_2,k_2,l_2}\big(\boldsymbol{b}^{(2)},\boldsymbol{m}^{(2)}\big).\end{aligned}\tag{3.5}$$

Therefore, in total, the number of possible partial chord diagrams obtained by removing a twisted or a non-twisted chord is the sum of (3.2) – (3.5) and of (2.2) – (2.4) for $\widetilde{\mathcal{M}}_{h,k,l}\big(\boldsymbol{b},\boldsymbol{m}\big)$. This number gives the right hand side of equation (3.1), which we have just argued also gives the left side of equation (3.1).

□

Along the same line of arguments as the ones which proved Proposition 2.2, we obtain the proposition below.

Proposition 3.2. *The generating function $\widetilde{H}(x,y;\boldsymbol{t},;\boldsymbol{u})$ is uniquely determined by the differential equation*

$$\frac{\partial\widetilde{H}}{\partial y}=(\widetilde{M}+\widetilde{S})\widetilde{H},$$

where $\widetilde{M}=M_0+xM_1^{\times}+x^2(M_2+M_2^{\times})$ and $\widetilde{S}=S+S^{\times}$. The generating function $\widetilde{Z}(x,y;\boldsymbol{t},;\boldsymbol{u})=\exp[\widetilde{H}]$ of the number of connected and disconnected partial chord diagrams filtered by the boundary length and point spectrum satisfies

$$\frac{\partial\widetilde{Z}}{\partial y}=\widetilde{M}\widetilde{Z}.\tag{3.6}$$

As such they are uniquely determined by the initial conditions

$$\widetilde{H}(x,y=0;\boldsymbol{t};\boldsymbol{u})=\sum_{i\geq 1}t_iu_{(i)},\quad \widetilde{Z}(x,y=0;\boldsymbol{t},;\boldsymbol{u})=\mathrm{e}^{\sum_{i\geq 1}t_iu_{(i)}}.$$

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Jørgen Ellegaard Andersen
Centre for Quantum Geometry of Moduli Spaces,
Department of Mathematics, Aarhus University,
DK-8000 Aarhus C Denmark
jea.qgm@gmail.com

Hiroyuki Fuji
Faculty of Education, Kagawa University,
Takamatsu 760-8522 Japan
fuji@ed.kagawa-u.ac.jp

Robert C. Penner
Institut des Hautes Études Scientifiques,
35 route de Chartres, 91440 Burs-sur-Yvette, France;
Division of Physics, Mathematics and Astronomy,
California Institute of Technology, Pasadena, CA 91125, USA
rpenner@caltech.edu, rpenner@ihes.fr

Christian M. Reidys
Biocomplexity Institute of Virginia Tech
Blacksburg, VA 24061, USA
duckcr@vbi.vt.edu

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Partial Chord Diagrams and Matrix Models

by Jørgen Ellegaard Andersen, Hiroyuki Fuji, Masahide Manabe, Robert C. Penner, and Piotr Sułkowski[1]

Abstract

In this article, the enumeration of partial chord diagrams is discussed via matrix model techniques. In addition to the basic data such as the number of backbones and chords, we also consider the Euler characteristic, the backbone spectrum, the boundary point spectrum, and the boundary length spectrum. Furthermore, we consider the boundary length and point spectrum that unifies the last two types of spectra. We introduce matrix models that encode generating functions of partial chord diagrams filtered by each of these spectra. Using these matrix models, we derive partial differential equations – obtained independently by cut-and-join arguments in an earlier work – for the corresponding generating functions.

1 Introduction

A *partial chord diagram* is a special kind of graph, which is specified as follows. The graph consists of a number of line segments (which are called *backbones*) arranged along the real line (hence they come with an ordering), with a number of vertices on each. A number of semi-circles (called *chords*) arranged in the upper half plane is attached at a subset of the vertices of the line segments, in such a way that no two chords have endpoints at the same vertex. The vertices which are not attached to chord ends are called the marked points. A *chord diagram* is by

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definition a partial chord diagram with no marked points. Partial chord diagrams occur in many branches of mathematics, including topology [14, 30], geometry [9, 10, 3] and representation theory [16].

To each partial chord diagram c one can associate canonically a two dimensional surface with boundary Σ_c, see Figure 1. Moreover, as discussed in [56, 12, 2, 7], the notion of a *fatgraph* [42, 43, 44, 45] is a useful concept when studying partial chord diagrams. A fatgraph is a graph together with a cyclic ordering on each collection of half-edges incident on a common vertex. A partial linear chord diagram c has a natural fatgraph structure induced from its presentation in the plane.

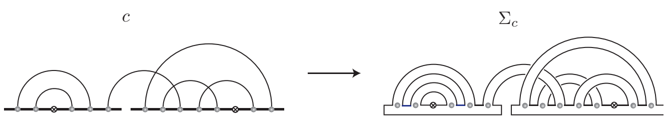


Figure 1: The partial chord diagram (with marked points) c and the corresponding surface Σ_c. The type of this partial chord diagram reads $\{g,k,l;\{b_i\};\{n_i\};\{p_i\}\} = \{1,6,2;\{b_6 = 1, b_8 = 1\};\{n_0 = 2, n_1 = 2\};\{p_1 = 1, p_2 = 2, p_9 = 1\}\}$. The boundary length and point spectrum is $\{n_{(1)} = 1, n_{(0,0)} = 2, n_{(0,0,0,0,0,1,0,0,0)} = 1\}$.

The partial chord diagram c is characterized by various topological data, and we will consider the following five types of data, introduced in [2] and [7].

- The number of chords k in c and the number of backbones b in c.

- Euler characteristic χ and genus g.
  Let χ and g denote respectively the Euler characteristic and genus of Σ_c, which are related as follows
  $$\chi = 2 - 2g.$$
  Denoting by n the number of boundary components of Σ_c, the Euler relation can be written as
  $$2 - 2g = b - k + n. \tag{1.1}$$

- Backbone spectrum $(b_0, b_1, \ldots)$.
  Let b_i denote the number of backbones with i trivalent (i.e. chord ends) or bivalent (i.e. marked points) vertices. The total number of backbones b is then
  $$b = \sum_{i\geq 0} b_i, \tag{1.2}$$

and the total number m of trivalent (i.e. chord ends) and bivalent (i.e. marked points) vertices of the partial chord diagram c is

$$m = \sum_{i \geq 1} i b_i. \tag{1.3}$$

- Boundary point spectrum $(n_0, n_1, \ldots)$.
  Let n_i denote the number of boundary components containing $i \geq 0$ marked points of Σ_c. The total number n of boundary components is

$$n = \sum_{i \geq 0} n_i, \tag{1.4}$$

  and the total number l of marked points is

$$l = \sum_{i \geq 1} i n_i. \tag{1.5}$$

  These three numbers m, k and l satisfies

$$m = 2k + l. \tag{1.6}$$

- Boundary length spectrum $(p_1, p_2, \ldots)$.
  Define the *length* of a boundary component to be the sum of the number of chords and the number of backbone undersides traversed by the boundary cycle. Let p_i be the number of boundary cycles with length $i \geq 1$. By definition, the following two relations hold

$$n = \sum_{i \geq 1} p_i, \tag{1.7}$$

$$2k + b = \sum_{i \geq 1} i p_i. \tag{1.8}$$

The data $\{g, k, l; \{b_i\}; \{n_i\}; \{p_i\}\}$ is called the *type* of a partial chord diagram c.

As a unification of the boundary length spectrum and the boundary point spectrum, we consider the *boundary length and point spectrum* introduced in [7]. Let us here recall its definition.

- Boundary length and point spectrum.
  We associate a K-tuple of numbers $\boldsymbol{i} = (i_1, \ldots, i_K)$ with a boundary component of length K, where i_L $(L = 1, \ldots, K)$ is the number of marked points between the L'th and $(L+1)$'th (taken modulo K) either chord or underpass of a backbone component (in either order) along the boundary.

Let $n_{\boldsymbol{i}}$ be the number of boundary components labeled in this way by $\boldsymbol{i}$. The total number l of marked points is

$$l = \sum_{K\geq 1} \sum_{\boldsymbol{i}} \sum_{L=1}^{K} i_L n_{(i_1,\ldots,i_K)}, \tag{1.9}$$

and the total number n of boundary cycles is

$$n = \sum_{\boldsymbol{i}} n_{\boldsymbol{i}}. \tag{1.10}$$

The data $\{g, k, l; \{b_i\}, \{n_{\boldsymbol{i}}\}\}$ stores more detailed information on the distribution of marked points on each boundary component. One can determine the previous two kinds of spectra from the boundary length and point spectrum by forgetting the partitions of marked points on the boundary cycles.

It is known that the enumeration of chord diagrams is intimately related to matrix models and cut-and-join equations [4, 5, 6, 20, 38]. In this paper, the enumeration of partial chord diagrams labeled by the boundary length and point spectrum with the genus filtration is studied using matrix model techniques. Let $\mathcal{N}_{g,k,l}(\{b_i\}, \{n_{\boldsymbol{i}}\})$ denote the number of connected chord diagrams labeled by the set of parameters $(g, k, l; \{b_i\}; \{n_{\boldsymbol{i}}\})$. We define the generating function of these numbers

(1.11)

$$\mathcal{F}(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\}) = \sum_{b\geq 1} \mathcal{F}_b(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\}),$$

$$\mathcal{F}_b(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\}) = \frac{1}{b!} \sum_{\sum_i b_i = b} \sum_{\{n_{\boldsymbol{i}}\}} \mathcal{N}_{g,k,l}(\{b_i\}, \{n_{\boldsymbol{i}}\}) x^{2g-2} y^k \prod_{i\geq 0} s_i^{b_i} \prod_{K\geq 1} \prod_{\{i_L\}_{L=1}^K} u_{\boldsymbol{i}}^{n_{\boldsymbol{i}}}.$$

Generating functions of disconnected and connected diagrams are related via the exponential relation

$$\mathcal{Z}(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\}) = \exp\left[\mathcal{F}(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\})\right]. \tag{1.12}$$

To analyze this enumeration further, we write the above generating function as a certain Hermitian matrix integral. Let $\mathcal{Z}_N(y; \{s_i\}; \{u_{\boldsymbol{i}}\})$ be the matrix integral over rank N Hermitian matrices $\mathcal{H}_N$

$$\begin{aligned} &\mathcal{Z}_N(y; \{s_i\}; \{u_{\boldsymbol{i}}\}) = \\ &= \frac{1}{\mathrm{Vol}_N} \int_{\mathcal{H}_N} dM \, \exp\left[- N \mathrm{Tr}\left(\frac{M^2}{2} - \sum_{i\geq 0} s_i (y^{1/2} \Lambda_{\mathrm{L}}^{-1} M + \Lambda_{\mathrm{P}})^i \Lambda_{\mathrm{L}}^{-1} \right) \right], \end{aligned} \tag{1.13}$$

where Λ_{P} and Λ_{L} are *external matrices* [29] of rank N, and the normalization factor Vol_N is defined in (2.4). In this matrix integral representation, the counting parameter $u_{(i_1,\dots,i_K)}$ is identified with the trace of the corresponding product of external matrices

$$u_{(i_1,\dots,i_K)} = \frac{1}{N}\mathrm{Tr}\left(\Lambda_{\mathrm{P}}^{i_1}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{i_2}\Lambda_{\mathrm{L}}^{-1}\cdots\Lambda_{\mathrm{P}}^{i_K}\Lambda_{\mathrm{L}}^{-1}\right). \tag{1.14}$$

In Theorem 2.13 we show that

$$\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \mathcal{Z}(N^{-1},y;\{s_i\};\{u_{\boldsymbol{i}}\}). \tag{1.15}$$

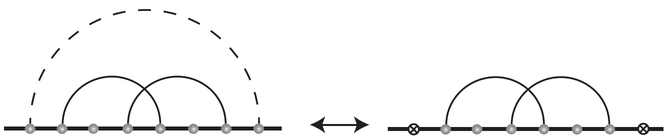


Figure 2: The cut-and-join manipulations on chord diagrams.

This matrix integral representation provides a new, matrix model proof of the cut-and-join equation found by combinatorial means in [7]. The cut-and-join equation can be written as

$$\frac{\partial}{\partial y}\mathcal{Z}(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \mathcal{M}\mathcal{Z}_N(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}), \tag{1.16}$$

where $\mathcal{M}$ is the second order partial differential operator in variables $u_{\boldsymbol{i}}$ (see Theorem 3.11 for details). This cut-and-join equation can be regarded as the evolution equation in the variable y, and its formal solution reads

$$\begin{aligned}\mathcal{Z}(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}) &= \mathrm{e}^{y\mathcal{M}}\mathcal{Z}(x,0;\{s_i\};\{u_{\boldsymbol{i}}\}),\\ \mathcal{Z}(x,0;\{s_i\};\{u_{\boldsymbol{i}}\}) &= \mathrm{e}^{N^2\sum_{i\geq 0}s_iu_{(i)}}.\end{aligned} \tag{1.17}$$

Expanding the operator $\mathrm{e}^{y\mathcal{M}}$ around $y=0$, one determines the number of connected partial chord diagrams $\mathcal{N}_{g,k,l}(\{b_i\},\{n_{\boldsymbol{i}}\})$ iteratively from this formal solution. The cut-and-join equation is a powerful method to systematically count partial chord diagrams of a given length and point spectrum.

In this work we also generalize the above analysis to non-oriented analogues of partial chord diagrams. By non-oriented partial chord diagrams we mean diagrams with all chords decorated by a binary variable, which indicates if they are *twisted* or not. When associating the surface Σ_c to a non-oriented partial chord diagram, twisted bands are associated along the twisted chords as indicated in Figure 3. This construction leads to 2^k orientable or non-orientable surfaces associated to one particular partial chord diagram with k chords, if we consider all possible assignments of twisting or untwisting of k bands. In the non-oriented case the Euler characteristic is defined as follows.

- Euler characteristic χ.
  The Euler characteristic of the two dimensional surface Σ_c is defined by the formula
$$\chi = 2 - h,$$
  where h is the number of cross-caps. The Euler relation holds
$$2 - h = b - k + n. \tag{1.18}$$

With this setup, the enumeration of non-oriented partial chord diagrams can be considered analogously to the orientable case.

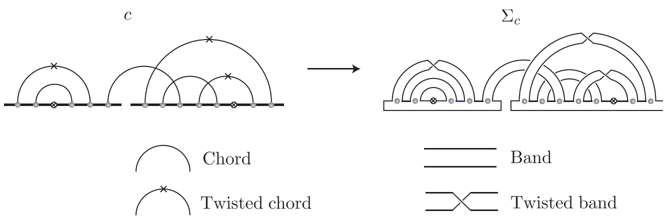


Figure 3: A non-oriented surface Σ_c associated to a non-oriented partial chord diagram c.

Let $\widetilde{\mathcal{N}}_{h,k,l}(\{b_i\},\{n_{\boldsymbol{i}}\})$ denote the number of connected non-oriented partial chord diagrams with the cross-cap number h, k chords, the backbone spectrum $\{b_i\}$, l marked points, and the boundary length and point spectrum $n_{\boldsymbol{i}}$. The generating function $\widetilde{\mathcal{F}}(x,y;\{s_i\};\{u_{\boldsymbol{i}}\})$ is defined by

(1.19)
$$\widetilde{\mathcal{F}}(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \sum_{b\geq 1} \widetilde{\mathcal{F}}_b(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}),$$
$$\widetilde{\mathcal{F}}_b(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \frac{1}{b!} \sum_{\sum_i b_i = b} \sum_{\{n_{\boldsymbol{i}}\}} \widetilde{\mathcal{N}}_{h,k,l}(\{b_i\},\{n_{\boldsymbol{i}}\}) x^{h-2} y^k \prod_{i\geq 0} s_i^{b_i} \prod_{K\geq 1} \prod_{\{i_L\}_{L=1}^K} u_{\boldsymbol{i}}^{n_{\boldsymbol{i}}}.$$

We also define the generating function of the numbers of connected and disconnected non-oriented partial chord diagrams
$$\widetilde{\mathcal{Z}}(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \exp\left[\widetilde{\mathcal{F}}(x,y;\{s_i\};\{u_{\boldsymbol{i}}\})\right]. \tag{1.20}$$

In Theorem 4.5 we show that this generating function can be expressed as a real symmetric matrix integral with two external symmetric matrices Ω_{P} and Ω_{L}

(1.21)
$$\widetilde{\mathcal{Z}}(N^{-1},y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}),$$

$$\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \tag{1.22}$$
$$= \frac{1}{\mathrm{Vol}_N(\mathbb{R})}\int_{\mathcal{H}_N(\mathbb{R})} dM \ \exp\left[-N\mathrm{Tr}\left(\frac{M^2}{4} - \sum_{i\geq 0} s_i(y^{1/2}\Omega_{\mathrm{L}}^{-1}M + \Omega_{\mathrm{P}})^i\Omega_{\mathrm{L}}^{-1}\right)\right],$$

where the normalization factor $\mathrm{Vol}_N(\mathbb{R})$ is defined in (4.8), and $\mathcal{H}_N(\mathbb{R})$ is the space of real symmetric matrices of rank N. The parameter $u_{(i_1,\ldots,i_K)}$ is identified with a trace of the external matrices via the formula

$$u_{(i_1,\ldots,i_K)} = \frac{1}{N}\mathrm{Tr}\left(\Omega_{\mathrm{P}}^{i_1}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{i_2}\Omega_{\mathrm{L}}^{-1}\cdots\Omega_{\mathrm{P}}^{i_K}\Omega_{\mathrm{L}}^{-1}\right). \tag{1.23}$$

Using this matrix integral representation of the generating function, one can again prove the cut-and-join equation, established independently by combinatorial arguments in [7]

$$\frac{\partial}{\partial y}\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \widetilde{\mathcal{M}}\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}), \tag{1.24}$$

where $\widetilde{\mathcal{M}}$ is a second order partial differential operator in the variables $u_{\boldsymbol{i}}$. The details of the differential operator $\widetilde{\mathcal{M}}$ and the matrix model derivation of the cut-and-join equation are presented in Theorem 4.10.

1.1 Motivation: RNA chains

One important motivation to study partial chord diagrams in this and the preceding work [2, 7] is a complicated problem of RNA structure prediction in molecular biology, which we now shortly review.

An RNA molecule is a linear polymer, referred to as the backbone, that consists of four types of nucleotides: adenine, cytosine, guanine, and uracil, denoted respectively **A**, **C**, **G**, and **U**. The backbone is endowed with an orientation from 5'-end to 3'-end, and the primary sequence is the sequence of nucleotides read with respect to this orientation. Between nucleotides hydrogen bonds are formed, resulting in the so-called Watson-Click pairs involving $\mathbf{A}-\mathbf{U}$ or $\mathbf{G}-\mathbf{C}$ nucleotides; in addition Wobble pairs $\mathbf{U}-\mathbf{G}$ can be formed. The set of base pairs formed by such hydrogen bonds is referred to as the secondary structure.[2] Prediction of the secondary structure from the primary sequence is an outstanding problem that was initiated by the pioneering work of Michael Waterman [57] and has been studied intensively for last three decades.

Topologically, we can represent the base pairings for a given RNA structure by a partial chord diagram as follows. The backbone is represented as a disjoint union of horizontal straight line segments (arranged along the real line in the

[2]There are other types of interactions in RNA secondary structure, which are however less common and we ignore them in this discussion.

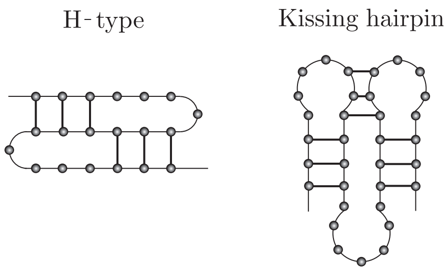


Figure 4: Pseudoknot structures in RNA. The long curved line, blobs (i.e. marked points), and short lines represent the backbone, nucleotides, and base pairs, respectively.

plane), one for each backbone component, and each nucleotide is represented as a marked point on this union of line segments. The base pairs are represented by chords in the upper-half plane attached at two marked points corresponding to the bonded pair of nucleotides.

Note that a partial chord diagram has genus zero if no two of its chords cross each other. If however such crossings exist, then the structure is referred to as a pseudoknot, and its genus is non-zero. Considerable number of pseudoknot structures have been observed, e.g. tRNAs, RNAseP [31], telomerase RNA [53] and ribosomal RNAs [28]. According to the online database "RNA-strand" half of the known structures form pseudoknots [13]. There are various kinds of pseudoknots classified by the topology of the RNA [12], referred to as e.g. H-type [1], kissing hairpin [17, 51], etc.

In recent years, a combinatorial description of RNA structures in terms of linear chord diagrams has been developed in a series of works [41, 56, 55, 12, 11, 8, 4, 5, 2, 49, 46]. However, a large class of reasonable energy-based models that predict the secondary structure including pseudoknots are NP complete [32, 1], and a fully satisfactory energy model for RNA, including pseudoknot structures, has not been established yet.

In the search of a realistic energy function for RNA structures with pseudoknots, the boundary length and point spectrum should provide a useful tool that includes more detailed information about the location of marked points. In standard algorithms developed by Waterman [58], Nussinov et al. [40], Zucker and Stiegler [61], etc., dynamic programming (DP) has been used to predict most likely secondary structures. Indeed, in famous algorithms such as [60, 25], the (loop-based) energy in each configuration of RNA is considered. In these algorithms, the most probable secondary structure is determined as the minimum free energy configuration, and to make them more efficient the statistical mechanical

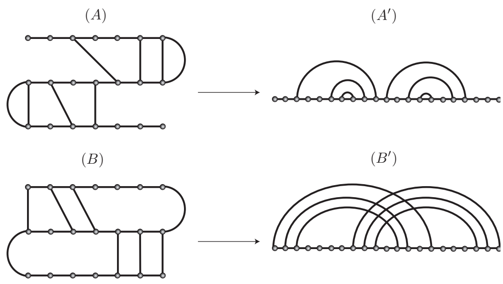


Figure 5: Partial chord diagrams unveil the difference in the topological structure of RNA molecules.

ensemble (i.e. the partition function algorithm) is implemented [34]. The application of these algorithms, which include pseudoknot structures stratified by γ *structures*, was studied in [50, 49]. Most of the energy functions essentially respect the boundary point and length spectra independently. In order to improve the energy model for RNA structure prediction with pseudoknots, it would be useful to explore energy parameters for more realistic and efficient energy function on the basis of the boundary length and point spectrum.

1.2 Plan of the paper

This paper is organized as follows. In Section 2 we construct Hermitian matrix models with external matrices, which encode generating functions of orientable partial chord diagrams labeled by the boundary point spectrum (in Subsection 2.1), the boundary length spectrum (in Subsection 2.2), and the boundary length and point spectrum (in Subsection 2.3). All these constructions are established by the correspondence between chord diagrams and Wick contractions via the Wick theorem. The matrix model encoding the boundary length and point spectrum is given in Theorem 2.13. In Section 3 we derive partial differential equations for matrix integrals found in Section 2. These partial differential equations coincide with the cut-and-join equations found combinatorially in [2, 7]. The cut-and-join equation for partial chord diagrams labeled by the boundary length and point spectrum is determined in Theorem 3.11. Section 4 is devoted to the analysis of non-oriented analogues of the results obtained in Section 2 and 3. In Subsection 4.1 we find real symmetric matrix models with external matrices, that encode generating functions of both orientable and non-orientable partial chord diagrams. The non-oriented analogue of the matrix integral from Theorem 2.13 is given in Theorem 4.5. Non-oriented analogues of cut-and-join equations from Section 3 are determined in Theorem 4.10. In Appendix A we derive a partial differential equation from Proposition 4.7 for a real symmetric matrix integral with external matrices. In Appendix B we prove Lemma 4.9.

2 Enumerating partial linear chord diagrams via matrix models

The enumeration problem of partial chord diagrams with respect to the genus filtration has been reformulated in terms of matrix integrals. Matrix model techniques for enumeration of the RNA structures with pseudoknots have been developed in a series of papers [41, 56, 55], and independently in [4, 5, 6]. Subsequently the analysis involving boundary point and length spectra of partial linear chord diagrams has been conducted in [2, 7]. In this section we develop a new perspective on this problem and construct a matrix model that enumerates partial chord diagrams labeled by the boundary length and point spectrum.

2.1 A matrix model enumerating partial chord diagrams

In the first step we construct a matrix model that counts partial chord diagrams labeled by the boundary point spectrum $\{n_i\}$.

Definition 2.1. Let $\mathcal{N}_{g,k,l}(\{b_i\},\{n_i\},\{p_i\})$ denote the number of connected partial chord diagrams of type $\{g,k,l;\{b_i\};\{n_i\};\{p_i\}\}$. In particular, focusing on the boundary point spectrum we define the following number of partial chord diagrams characterized by the data $\{g,k,l;\{b_i\},\{n_i\}\}$,

$$\mathcal{N}_{g,k,l}(\{b_i\},\{n_i\}) = \sum_{\{p_i\}} \mathcal{N}_{g,k,l}(\{b_i\},\{n_i\},\{p_i\}).$$

We introduce the generating function[3] for the numbers $\mathcal{N}_{g,k,l}(\{b_i\},\{n_i\})$

$$\begin{aligned} F(x,y;\{s_i\};\{t_i\}) &= \sum_{b\geq 1} F_b(x,y;\{s_i\};\{t_i\}), \\ F_b(x,y;\{s_i\};\{t_i\}) &= \frac{1}{b!}\sum_{\sum_i b_i=b}\sum_{\{n_i\}} \mathcal{N}_{g,k,l}(\{b_i\},\{n_i\}) x^{2g-2} y^k \prod_{i\geq 0} s_i^{b_i} t_i^{n_i}. \end{aligned} \tag{2.1}$$

The generating function for the numbers $\widehat{\mathcal{N}}_{k,b,l}(\{b_i\},\{n_i\})$ of connected and disconnected partial chord diagrams arises in the usual way from the exponent

$$\begin{aligned} Z^{\mathrm{P}}(x,y;\{s_i\};\{t_i\}) &= \exp\left[F(x,y;\{s_i\};\{t_i\})\right] \\ &= \sum_{\{b_i\}}\sum_{\{n_i\}} \widehat{\mathcal{N}}_{k,b,l}(\{b_i\},\{n_i\}) x^{-b+k-n} y^k \prod_{i\geq 0} s_i^{b_i} t_i^{n_i}. \end{aligned} \tag{2.2}$$

[3]The parameters s_i and t_i in this article and in [2] are related by $s_i \leftrightarrow t_i$.

In the following we rewrite the generating function $Z^{\mathrm{P}}(x,y;\{s_i\};\{t_i\})$ as a Hermitian matrix integral. To this end, we consider first Gaussian averages over Hermitian matrices.

Definition 2.2. Let $\mathcal{O}(M)$ be a function of a rank N Hermitian matrix M. The Gaussian average $\langle\mathcal{O}(M)\rangle_N^{\mathrm{G}}$ is defined by the integral over the space $\mathcal{H}_N$ of rank N Hermitian matrices with respect to the Haar measure dM with the Gaussian weight $\mathrm{e}^{-N\mathrm{Tr}\frac{M^2}{2}}$,

$$\langle\mathcal{O}(M)\rangle_N^{\mathrm{G}} = \frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N} dM\ \mathcal{O}(M)\,\mathrm{e}^{-N\mathrm{Tr}\frac{M^2}{2}}, \tag{2.3}$$

where the normalization factor Vol_N takes form

$$\mathrm{Vol}_N = \int_{\mathcal{H}_N} dM\ \mathrm{e}^{-N\mathrm{Tr}\frac{M^2}{2}} = N^{N(N+1)/2}\mathrm{Vol}(\mathcal{H}_N). \tag{2.4}$$

In particular for $\mathcal{O}(M) = M_{\alpha\beta}M_{\gamma\epsilon}$ $(\alpha,\beta,\gamma,\epsilon = 1,\dots,N)$, the Gaussian average is

$$\overbracket{M_{\alpha\beta}M}_{\gamma\delta} := \langle M_{\alpha\beta}M_{\gamma\epsilon}\rangle_N^{\mathrm{G}} = \frac{1}{N}\delta_{\alpha\epsilon}\delta_{\beta\gamma}. \tag{2.5}$$

This quantity is called the *Wick contraction.* By definition, a multiple Wick contraction is a product of the Gaussian average of each Wick contracted pair.

It follows from the definition (2.3) that Gaussian averages of an odd number of matrix elements vanish. On the other hand, Gaussian averages of an even number of matrix elements are non-zero, and can be computed using the *Wick theorem* [15, 43, 37], as we now recall. Consider an ordered sequence

$$M_{\alpha_1\beta_1}M_{\alpha_2\beta_2}\cdots M_{\alpha_{2k}\beta_{2k}}$$

of $2k$ matrix elements $M_{\alpha_n\beta_n}$ $(n = 1,\dots,2k)$.

Let P_k denote a set of matchings by k Wick contractions among the $2k$ matrix elements in the above sequence. P_k is isomorphic to the following quotient of groups

$$P_k \simeq G_H/G_E, \quad G_H = S_{2k}, \quad G_E = S_k \rtimes (S_2)^k.$$

Here the elements of the permutation group S_{2k} permute $2k$ matrix elements. The factors S_k of G_E act by permuting k Wick contractions and $(S_2)^k$ swaps matrix elements in each Wick contracted pair. The Wick theorem implies the following result.

Theorem 2.3. *The Gaussian average of* $2k$ *matrix elements* $M_{\alpha_n\beta_n}$ $(n = 1,\dots,k)$ *equals*

$$\begin{aligned}\langle M_{\alpha_1\beta_1}M_{\alpha_2\beta_2}\cdots M_{\alpha_{2k}\beta_{2k}}\rangle_N^{\mathrm{G}} &= \sum_{\sigma\in P_k}\prod_{i=1}^{k}\overbracket{M_{\alpha_{\sigma(2i-1)}\beta_{\sigma(2i-1)}}M}_{\alpha_{\sigma(2i)}\beta_{\sigma(2i)}} \\ &= \frac{1}{N^k}\sum_{\sigma\in P_k}\prod_{i=1}^{k}\delta_{\alpha_{\sigma(2i-1)}\beta_{\sigma(2i)}}\delta_{\alpha_{\sigma(2i)}\beta_{\sigma(2i-1)}}.\end{aligned} \tag{2.6}$$

2.1.1 Chord diagrams and Wick contractions

Let c be a chord diagram. We now recall the explicit relation between a surface Σ_c associated to a chord diagram c and k-matchings or Wick contractions in the Gaussian average. To illustrate this correspondence we depict chord ends on backbones in Σ_c as trivalent vertices that consist of upright and horizontal line segments, see Figure 6. This correspondence is specified by the following four points **C1**–**C4**.

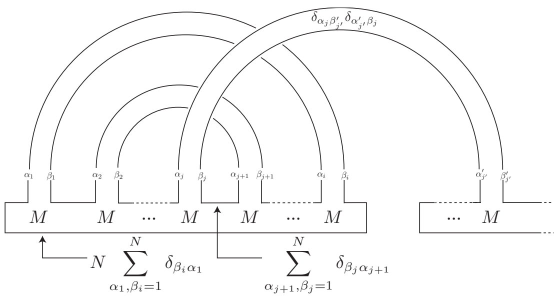


Figure 6: Bijective correspondence between chord diagrams and Wick contractions.

C1 A matrix element $M_{\alpha\beta}$ corresponds to a chord end on a backbone. Indices $\alpha, \beta (= 1, \ldots, N)$ are assigned to two upright line segments on the upper edge of the backbone.

C2 If two matrix elements $M_{\alpha_j\beta_j}M_{\alpha_{j+1}\beta_{j+1}}$ correspond to two adjacent chord ends on the same backbone, then the following quantity is assigned to the horizontal segment between these two chord ends on the upper edge of the backbone

$$\sum_{\alpha_{j+1},\beta_j=1}^{N} \delta_{\beta_j\alpha_{j+1}}.$$

This assignment encodes matrix multiplication of matrix elements corresponding to adjacent chord ends on the backbone.

C3 For the product of i matrix elements M

$$\sum_{\alpha_2,\ldots,\alpha_i=1}^{N} \sum_{\beta_1,\ldots,b_{i-1}=1}^{N} M_{\alpha_1\beta_1}\delta_{\beta_1\alpha_2}M_{\alpha_2\beta_2}\ldots\delta_{\beta_{i-1}\alpha_i}M_{\alpha_i\beta_i} = (M^i)_{\alpha_1\beta_i},$$

which corresponds to a backbone with i chord ends, the following quantity is assigned to the bottom edge of the backbone

$$N \sum_{\alpha_1,\beta_i=1}^{N} \delta_{\beta_i\alpha_1}.$$

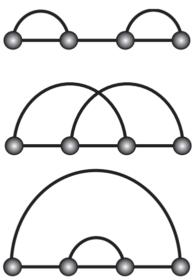
Thus, a backbone with i chord ends corresponds to a single trace of the i'th power of M, namely $N\mathrm{Tr}M^i$.

C4 The Wick contraction between $M_{\alpha_j\beta_j}$ and $M_{\alpha'_{j'}\beta'_{j'}}$ corresponds to a band connecting two chord ends. Each Wick contraction imposes a constraint $\delta_{\alpha_j\beta'_{j'}}\delta_{\alpha'_{j'}\beta_j}$ on matrix indices assigned to the edges of the chord ends matched by the Wick contraction.

The above rules imply the following bijective correspondence

$$W_N^{\mathrm{C}}(\{b_i\}) = \Big\langle \prod_i \left(N\mathrm{Tr}M^i\right)^{b_i} \Big\rangle_N^{\mathrm{G}}, \qquad \sum_i ib_i = 2k, \tag{2.7}$$

between matchings by k Wick contractions in the Gaussian average on one hand, and chord diagrams that consist of b_i backbones with i chord ends on the other hand, see Figure 7.



$$N \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4=1}^{N} M_{\alpha_1\alpha_2}M_{\alpha_2\alpha_3}M_{\alpha_3\alpha_4}M_{\alpha_4\alpha_1}$$

$$N \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4=1}^{N} M_{\alpha_1\alpha_2}M_{\alpha_2\alpha_3}M_{\alpha_3\alpha_4}M_{\alpha_4\alpha_1}$$

$$N \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4=1}^{N} M_{\alpha_1\alpha_2}M_{\alpha_2\alpha_3}M_{\alpha_3\alpha_4}M_{\alpha_4\alpha_1}$$

Figure 7: Chord diagrams and Wick contractions for $\langle N\mathrm{Tr}M^4\rangle_{\mathrm{N}}^{\mathrm{G}}$.

The Wick contractions (2.6) in $W_N^{\mathrm{C}}(\{b_i\})$ replace all matrix elements M's by products of δ's, and summing over matrix indices along a boundary cycle one finds a factor of N corresponding to each boundary cycle in a chord diagram. Therefore the overall N dependence following from the above rules amounts to assigning N^{b-k+n} factor to the term $W_N^{\mathrm{C}}(\{b_i\})$, corresponding to a chord diagram with backbone spectrum $\{b_i\}$ and n boundary cycles. Combing the Wick theorem and this bijective correspondence between matchings by k Wick contractions in the Gaussian average $W_N^{\mathrm{C}}(\{b_i\})$ and the set of chord diagrams with backbone spectrum $\{b_i\}$, the following proposition follows.

Proposition 2.4. *The Gaussian average $W_N^{\mathrm{C}}(\{b_i\})$ in equation (2.7) agrees with the generating function of chord diagrams with backbone spectrum $\{b_i\}$*

$$W_N^{\mathrm{C}}(\{b_i\}) = \sum_{n\geq 0} \widehat{\mathcal{N}}_{k,b,n}(\{b_i\}) N^{b-k+n}. \tag{2.8}$$

Here $\widehat{\mathcal{N}}_{k,b,n}(\{b_i\})$ is the number of chord diagrams that consist of b_i backbones with i trivalent vertices

$$\widehat{\mathcal{N}}_{k,b,n}(\{b_i\}) = \sum_{\{p_i\}} \widehat{\mathcal{N}}_{k,b,l=0}(\{b_i\}, n_0 = n, \{n_i = 0\}_{i\geq 1}, \{p_i\}). \tag{2.9}$$

2.1.2 Partial chord diagrams and Wick contractions

We now generalize the above bijective correspondence to partial chord diagrams. Let c be a partial chord diagram. On the boundary cycles of the surface Σ_c we add additional marked points, which correspond to those marked points on c which are not chord ends. These marked points are represented by *external matrices* Λ_{P} of rank N in the Gaussian average. The rules **P1**–**P5** below provide the correspondence between partial chord diagrams with backbone spectrum $\{b_i\}$ and matchings with k Wick contractions in the Gaussian average.

P1 A matrix element $M_{\alpha\beta}$ corresponds to a chord end on a backbone. The graphical rule is the same as the rule **C1**.

P2 A matrix element $\Lambda_{\mathrm{P}\alpha\beta}$ corresponds to a marked point on a backbone in Σ_c. Indices $\alpha, \beta (= 1, \ldots, N)$ are assigned to two upright line segments at each marked point on the upper edge of the backbone, see Figure 8.

P3 To a line segment (on the upper edge of the backbone) between adjacent chord ends or marked points (located on the same backbone), corresponding to matrix elements $U_{\alpha_j\beta_j}$ and $V_{\alpha_{j+1}\beta_{j+1}}$ (for $U, V = M$ or Λ_{P}), we assign

$$\sum_{\beta_j, \alpha_{j+1}=1}^{N} \delta_{\beta_j \alpha_{j+1}}, \tag{2.10}$$

just as in **C2**.

P4 Let $v_j, w_j \in \mathbb{Z}_{\geq 0}$ $(j = 1, \ldots, i)$ with $\sum_{j=1}^{i}(v_j + w_j) = i$. For an ordered matrix product

$$(M^{v_1}\Lambda_{\mathrm{P}}^{w_1} M^{v_2}\Lambda_{\mathrm{P}}^{w_2} \cdots M^{v_i}\Lambda_{\mathrm{P}}^{w_i})_{\alpha_1\beta_i}, \tag{2.11}$$

corresponding to a backbone which is an ordered sequence of v_j chord ends and w_j marked points, we assign

$$N \sum_{\alpha_1, \beta_i = 1}^{N} \delta_{\beta_i \alpha_1}$$

to the bottom edge of this backbone. It follows that the trace

$$N\mathrm{Tr}(M^{v_1}\Lambda_{\mathrm{P}}^{w_1}M^{v_2}\Lambda_{\mathrm{P}}^{w_2}\cdots M^{v_i}\Lambda_{\mathrm{P}}^{w_i}) \tag{2.12}$$

is assigned to this backbone.

P5 The Wick contraction between $M_{\alpha_j\beta_j}$ and $M_{\alpha'_{j'},\beta'_{j'}}$ corresponds to a band connecting two chord ends, and it is represented in the same way as specified in **C4**.

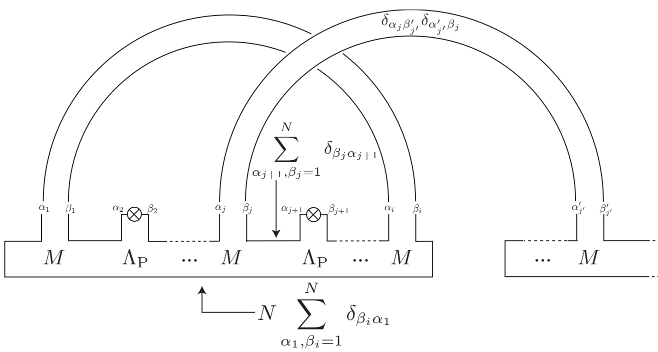


Figure 8: Bijective correspondence between partial chord diagrams and matchings of Wick contractions

For a fixed backbone spectrum $\{b_i\}$, all possible sequences $\{\alpha_j,\beta_j\}$ in the expression (2.12) are generated by the following product of traces

$$\prod_{i\geq 0}\left(N\mathrm{Tr}(M+\Lambda_{\mathrm{P}})^i\right)^{b_i}. \tag{2.13}$$

Hence, by the above rules, all partial chord diagrams with the backbone spectrum $\{b_i\}$ correspond bijectively to all matchings by Wick contractions among the M's in the expansion of the Gaussian average

$$W_N^{\mathrm{P}}(\{b_i\},\{r_i\}) = \left\langle\prod_{i\geq 0}\left(N\mathrm{Tr}(M+\Lambda_{\mathrm{P}})^i\right)^{b_i}\right\rangle_N^{\mathrm{G}}, \tag{2.14}$$

where we introduced the reverse Miwa times

$$r_i = \frac{1}{N}\mathrm{Tr}\Lambda_{\mathrm{P}}^i. \tag{2.15}$$

If there are n_i boundary components containing i marked points, then one finds a trace factor $(\mathrm{Tr}\Lambda_{\mathrm{P}}^{i})^{n_i}$ in the corresponding term in the Gaussian average (2.14), see Figure 9. Therefore, for partial chord diagrams with the backbone spectrum $\{b_i\}$ and the boundary point spectrum $\{n_i\}$, the corresponding term in $W_N^{\mathrm{P}}(\{b_i\},\{r_i\})$ contributes the factor

$$N^{b-k+n}\prod_{i\geq 0} r_i^{n_i}.$$

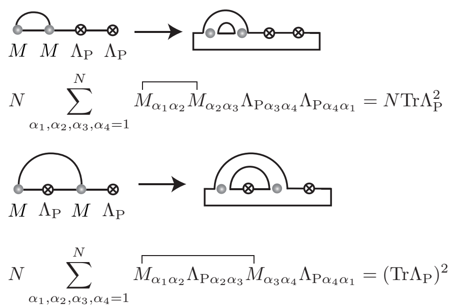


Figure 9: Partial chord diagrams of types $\{g=0,k=1,l=2;b_4=1;n_0=1,n_2=1\}$ and $\{g=0,k=1,l=2;b_4=1;n_1=2\}$, and the corresponding Wick contractions.

Therefore, from Wick theorem and the above bijective correspondence between partial chord diagrams and matchings by Wick contractions, one finds the following proposition.

Proposition 2.5. *The Gaussian average (2.14) is the generating function for the numbers $\widehat{\mathcal{N}}_{k,b,l}(\{b_i\},\{n_i\})$ of partial chord diagrams with the backbone spectrum $\{b_i\}$ and the boundary point spectrum $\{n_i\}$*

$$W_N^{\mathrm{P}}(\{b_i\},\{r_i\}) = \sum_{\{n_i\}}\widehat{\mathcal{N}}_{k,b,l}(\{b_i\},\{n_i\})N^{b-k+n}\prod_{i\geq 0} r_i^{n_i}, \tag{2.16}$$

where the summation is constrained by $\sum in_i = \sum ib_i - 2k$.

Using this proposition, we consider the full generating function $Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\})$ for the numbers $\widehat{\mathcal{N}}_{k,b,l}(\{b_i\},\{n_i\})$ of partial chord diagrams weighted by

$$N^{b-k+n}y^k\prod_{i\geq 0} s_i^{b_i}r_i^{n_i}.$$

Since the contribution from a partial chord diagram is invariant under permutations of its backbones, the full generating function

$$Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}) = \sum_{\{b_i\}}\sum_{\{n_i\}} \widehat{\mathcal{N}}_{k,b,l}(\{b_i\},\{n_i\}) N^{b-k+n} y^k \prod_{i\geq 0} s_i^{b_i} r_i^{n_i}$$

can be rewritten as a sum over all backbone spectra $\{b_i\}$ of the terms

$$y^{\sum_i ib_i/2} W_N^{\mathrm{P}}(\{b_i\},\{y^{-i/2}r_i\}) \prod_i \frac{s_i^{b_i}}{b_i!}.$$

It follows that

$$Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}) = \sum_{\{b_i\}}\prod_{i\geq 0} \frac{s_i^{b_i} y^{ib_i/2}}{b_i!} \Big\langle \Big(N\mathrm{Tr}(M + y^{-1/2}\Lambda_{\mathrm{P}})^i\Big)^{b_i}\Big\rangle_N^{\mathrm{G}}.$$

Performing the summation over b_i's, one finds that the full generating function is given by the matrix integral

$$Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}) = \tag{2.17}$$
$$= \frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N} dM\ \exp\Big[- N\mathrm{Tr}\Big(\frac{M^2}{2} - \sum_{i\geq 0} s_i (y^{1/2}M + \Lambda_{\mathrm{P}})^i\Big)\Big].$$

This matrix integral and $Z^{\mathrm{P}}(x,y;\{s_i\};\{t_i\})$ in equation (2.2) are identified by a change of variables. Since the reverse Miwa time for $i=0$ yields $r_0 = 1$ automatically, we need to introduce the parameter t_0 by the following change of variables

$$N \to t_0 N, \quad y \to t_0 y, \quad s_i \to t_0^{-1} s_i, \quad r_i \to t_0^{-1} t_i.$$

As a result, we find the main theorem in this subsection.

Theorem 2.6. *The generating function (2.2) is given by the matrix integral (2.17),*

$$Z^{\mathrm{P}}(N^{-1}, y;\{s_i\};\{t_i\}) = Z_{t_0 N}^{\mathrm{P}}(t_0 y;\{t_0^{-1}s_i\};\{t_0^{-1}t_i\}). \tag{2.18}$$

2.2 A matrix model for the enumeration of chord diagrams

Next we turn to the enumeration of chord diagrams labeled by the backbone spectrum $\{b_i\}$ and the boundary length spectrum $\{p_i\}$. The number $\mathcal{N}_{g,k}(\{b_i\},\{p_i\})$ of connected chord diagrams is given by

$$\mathcal{N}_{g,k}(\{b_i\},\{p_i\}) = \sum_{\{n_i\}} \mathcal{N}_{g,k,0}(\{b_i\},\{n_i\},\{p_i\}).$$

We introduce the following generating function of these numbers[4]

[4]The parameters q_i's in our paper correspond to s_i's in [2].

Definition 2.7. Let $G(x,y;\{s_i\};\{q_i\})$ denote the generating function of chord diagrams labeled by the boundary length spectrum

$$
\begin{aligned}
G(x,y;\{s_i\};\{q_i\}) &= \sum_{b\geq 1} G_b(x,y;\{s_i\};\{q_i\}),\\
G_b(x,y;\{s_i\};\{q_i\}) &= \frac{1}{b!}\sum_{\sum b_i=b}\sum_{\{p_i\}}\mathcal{N}_{g,k}(\{b_i\},\{p_i\})x^{2g-2}y^k\prod_{i\geq 0}s_i^{b_i}\prod_{i\geq 1}q_i^{p_i}.
\end{aligned}
\tag{2.19}
$$

In the same way as the generating function $Z^{\mathrm{P}}(x,y;\{s_i\};\{t_i\})$ in (2.2), the generating function for the numbers $\widehat{\mathcal{N}}_{k,b}(\{b_i\},\{p_i\})$ of connected and disconnected chord takes form

$$
\begin{aligned}
Z^{\mathrm{L}}(x,y;\{s_i\};\{q_i\}) &= \exp\left[G(x,y;\{s_i\};\{q_i\})\right]\\
&= \sum_{\{b_i\}}\sum_{\{p_i\}}\widehat{\mathcal{N}}_{k,b}(\{b_i\};\{p_i\})x^{-b+k-n}y^k\prod_{i\geq 0}s_i^{b_i}\prod_{i\geq 1}q_i^{p_i}.
\end{aligned}
\tag{2.20}
$$

2.2.1 A matrix model for the boundary length spectrum

Let c be a chord diagram. The boundary length spectrum filters chord diagrams according to combinatorial length of each boundary cycle, i.e. the sum of the number of chords and backbone underpasses. This length can be determined by counting marked points of a new type, which we now introduce. We introduce marked points of a new type between all chord ends and backbone ends, see the left diagram in Figure 10. For chord diagram decorated in this way, we get new marked points on the boundaries of the surface Σ_c by sliding each new marked point along the boundary of Σ_c until it reaches the first chord or backbone underside midpoint, as indicated in the right hand side of Figure 10.

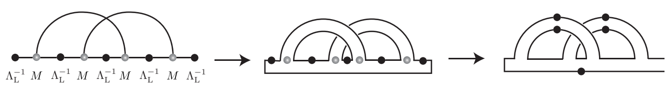


Figure 10: Decorating a chord diagram with new marked points for partitions.

In order to construct a Gaussian matrix integral which counts this type of chord diagrams we introduce another external matrix Λ_{L}, which is an invertible rank N matrix that keeps track of new marked points. We introduce a new model model based on the following rules **L1**–**L5**, in which Wick contractions in the Gaussian average correspond bijectively to decorated chord diagrams.

L1 A matrix element $M_{\alpha\beta}$ corresponds to a chord end on a backbone. This graphical rule is the same as the rule **C1**.

L2 A matrix element $(\Lambda_{\rm L}^{-1})_{\alpha_j\beta_j}$ is adjacent to a matrix element $M_{\alpha_{j+1}\beta_{j+1}}$ on an upper edge of a backbone in Σ_c. Without loss of generality, we can put $\Lambda_{\rm P}^{-1}$'s on the left hand side of the M's. Indices $\alpha_j, \beta_j (= 1, \ldots, N)$ are assigned to two upright line segments nipping a marked point in the upper edge of the backbone, see Figure 11.

L3 If two matrix elements $U_{\alpha_j\beta_j}$ and $V_{\alpha_{j+1}\beta_{j+1}}$ ($U, V = M$ or $\Lambda_{\rm L}^{-1}$) on the same backbone are adjacent, we form a matrix product $(UV)_{\alpha_j\beta_{j+1}}$. This graphical rule is the same as the rule **C2**.

L4 If a matrix product

$$(\Lambda_{\rm L}^{-1}M)^i_{\alpha_1\beta_i}$$

corresponds to a backbone with a marked point, we assign the expression

$$N\sum_{\alpha_1,\beta_i=1}^{N}(\Lambda_{\rm L}^{-1})_{\alpha_1\beta_i}$$

to the bottom edge of this backbone. This gives the contribution

$$N{\rm Tr}((\Lambda_{\rm L}^{-1}M)^i\Lambda_{\rm L}^{-1})$$

with i chord ends and therefore $i+1$ new marked points.

L5 The Wick contraction between $M_{\alpha_j\beta_j}$ and $M_{\alpha'_{j'}\beta'_{j'}}$ corresponds to a band connecting two chord ends. This graphical rule is the same as the rule **C4**.

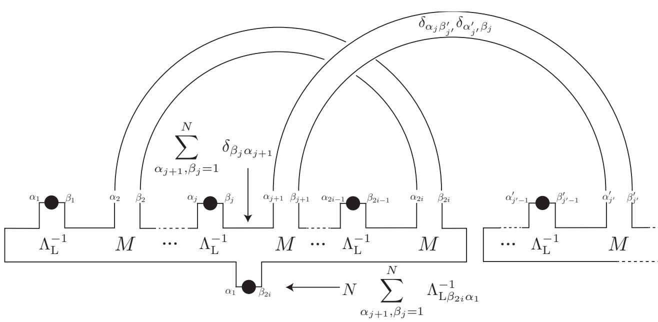


Figure 11: Bijective correspondence between decorated chord diagrams and matchings of Wick contractions.

Repeating the same discussions as in the previous subsection, one finds that every chord diagram with the backbone spectrum $\{b_i\}$ corresponds to matchings with $k = \sum ib_i/2$ Wick contractions, which arise from the following Gaussian average

$$W_N^{\mathrm{L}}(\{b_i\};\{q_i\}) = \Big\langle \prod_{i\geq 0} \left(N\mathrm{Tr}(\Lambda_{\mathrm{L}}^{-1}M)^i\Lambda_{\mathrm{L}}^{-1}\right)^{b_i} \Big\rangle_N^{\mathrm{G}}, \tag{2.21}$$

where we introduced Miwa times

$$q_i = \frac{1}{N}\mathrm{Tr}\Lambda_{\mathrm{L}}^{-i}. \tag{2.22}$$

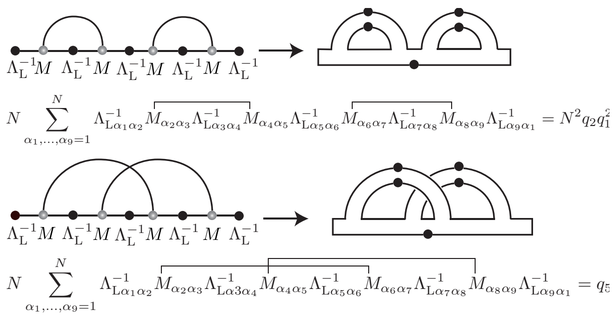


Figure 12: Chord diagrams of types $\{g, k; \{b_i\}; \{p_i\}\} = \{0, 2; b_5 = 1; p_1 = 2, p_3 = 1\}$ and $\{g, k; \{b_i\}; \{p_i\}\} = \{1, 2; b_5 = 1; p_5 = 1\}$.

It follows from the rules **L1**–**L5** that i $\Lambda_{\mathrm{L}}^{-1}$'s are aligned along the boundary cycle with length i. Therefore, for chord diagrams with the backbone spectrum $\{b_i\}$ and the boundary length spectrum $\{p_i\}$, the corresponding Wick contractions in $W_N^{\mathrm{L}}(\{b_i\};\{q_i\})$ involve the factor

$$N^{b-k+n}\prod_{i\geq 1} q_i^{p_i},$$

see Figure 12. The key proposition of this subsection follows.

Proposition 2.8. *The Gaussian average $W_N^{\mathrm{L}}(\{b_i\};\{q_i\})$ in eq.(2.21) is the generating function of the numbers $\widehat{N}_{k,b}(\{b_i\},\{p_i\})$ of chord diagrams with the backbone spectrum $\{b_i\}$*

$$W_N^{\mathrm{L}}(\{b_i\};\{q_i\}) = \sum_{\{p_i\}} \widehat{N}_{k,b}(\{b_i\},\{p_i\}) N^{b-k+n}\prod_{i\geq 1} q_i^{p_i}. \tag{2.23}$$

We also consider the full generating function for the numbers $\widehat{N}_{k,b}(\{b_i\},\{p_i\})$ of chord diagrams

$$Z_N^{\mathrm{L}}(y;\{s_i\};\{q_i\}) = \sum_{\{b_i\}}\sum_{\{p_i\}} \widehat{N}_{k,b}(\{b_i\},\{p_i\}) N^{b-k+n} y^k \prod_{i\geq 0} s_i^{b_i} \prod_{i\geq 1} q_i^{p_i}.$$

This full generating function is given by the sum of Gaussian averages (2.21), and in consequence by the following Hermitian matrix integral

$$\begin{aligned} Z_N^{\mathrm{L}}(y;\{s_i\};\{q_i\}) &= \sum_{\{b_i\}} \frac{1}{\prod_i b_i!} y^k W_N^{\mathrm{L}}(\{b_i\},\{y^{-ib_i/2}q_i\}) \prod_i s_i^{b_i} \\ &= \frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N} dM\ \exp\left[-N\mathrm{Tr}\left(\frac{M^2}{2} - \sum_{i\geq 0} s_i y^{i/2}\left(\Lambda_{\mathrm{L}}^{-1}M\right)^i \Lambda_{\mathrm{L}}^{-1}\right)\right]. \end{aligned} \tag{2.24}$$

Comparing this matrix integral and the generating function $Z_N^{\mathrm{L}}(y;\{s_i\};\{q_i\})$ in equation (2.20), we arrive at the main theorem of this subsection.

Theorem 2.9. *The matrix integral (2.24) agrees with the generating function (2.20)*

$$Z_N^{\mathrm{L}}(y;\{s_i\};\{q_i\}) = Z^{\mathrm{L}}(N^{-1},y;\{s_i\};\{q_i\}). \tag{2.25}$$

Specialization of the model

The cut-and-join equation for the numbers of chord diagrams is discussed in Subsection 3.2. For technical reasons, the partial differential equation for the generating function (2.20) with general parameter $\{s_i\}$ cannot be written in a simple form. Therefore we consider the specialization of the generating function (2.20) defined by[5]

$$s_i = s.$$

Under this specialization, the matrix integral (2.24) reduces to

$$\begin{aligned} Z_N^{\mathrm{L}}(y;s;\{q_i\}) &= Z_N^{\mathrm{L}}(y;\{s_i=s\};\{q_i\}) \\ &= \frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N} dM\ \exp\left[-N\mathrm{Tr}\left(\frac{M^2}{2} - \frac{s}{1-y^{1/2}\Lambda_{\mathrm{L}}^{-1}M}\Lambda_{\mathrm{L}}^{-1}\right)\right]. \end{aligned} \tag{2.26}$$

For $Z^{\mathrm{L}}(x,y;s;\{q_i\}) = Z^{\mathrm{L}}(x,y;\{s_i=s\};\{q_i\})$, we find

$$Z_N^{\mathrm{L}}(y;s;\{q_i\}) = Z^{\mathrm{L}}(N^{-1},y;s;\{q_i\}). \tag{2.27}$$

In Subsection 3.2 we derive the cut-and-join equation for this specialized model, and show the agreement with the cut-and-join equation found by combinatorial means in [2].

[5] In [2], the length spectrum generating function $G_b(x,y;\{s_i\})$ is the same as in this specialized model.

2.3 The boundary length and point spectrum and the unified model

So far we have discussed separately the enumeration of chord diagrams and partial chord diagrams labeled by the boundary point spectrum and the boundary length spectrum. In this subsection we consider a unification of these two kinds of spectra, which is referred to as the boundary length and point spectrum. This unified spectrum was introduced and analyzed by cut-and-join methods in [7]. In what follows we construct a matrix model that encodes this new spectrum, and in Subsection 3.3 we show how the cut-and-join equation found in [7] follows from this matrix model.

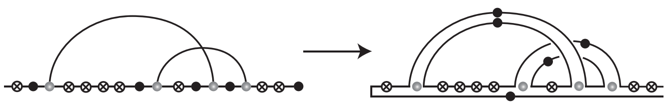


Figure 13: Decorating a partial chord diagram with the boundary label $\boldsymbol{i} = (1, 0, 1, 4, 2)$ with marked points for partitions.

The boundary length and point spectrum $\{n_{\boldsymbol{i}}\}$ is defined as follows [7].

Definition 2.10. Let c be a partial chord diagram. We associate the K tuple of numbers $\boldsymbol{i} = (i_1, i_2, \ldots, i_K)$ to a boundary component of Σ_c, if we find the tuple $\boldsymbol{i}$ of marked points around this boundary component, once we record different numbers of marked points in between chord ends and backbone underpasses along the boundary in the cyclic order induced from the orientation of Σ_c. The boundary length and point spectrum $\{n_{\boldsymbol{i}}\}$ counts the number of boundary cycles indexed by $\boldsymbol{i}$ for the partial chord diagram c.

To enumerate the number of partial chord diagrams labeled by $\{g, k, l; \{b_i\}; \{n_{\boldsymbol{i}}\}\}$, we consider the generating functions introduced in [7].

Definition 2.11. Let $\mathcal{N}_{g,k,l}(\{b_i\}, \{n_{\boldsymbol{i}}\})$ denote the number of connected chord diagrams labeled by the set of parameters $(g, k, l; \{b_i\}; \{n_{\boldsymbol{i}}\})$ in the boundary length and point spectrum. The generating function for these numbers is defined as

(2.28)
$$\mathcal{F}(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\}) = \sum_{b\geq 1} \mathcal{F}_b(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\}),$$
$$\mathcal{F}_b(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\}) = \frac{1}{b!} \sum_{\sum_i b_i = b} \sum_{\{n_{\boldsymbol{i}}\}} \mathcal{N}_{g,k,l}(\{b_i\}, \{n_{\boldsymbol{i}}\}) x^{2g-2} y^k \prod_{i\geq 0} s_i^{b_i} \prod_{K\geq 1} \prod_{\{i_L\}_{L=1}^K} u_{\boldsymbol{i}}^{n_{\boldsymbol{i}}}.$$

Exponentiating this generating function, one obtains the full generating function for the numbers $\widehat{\mathcal{N}}_{k,b,l}(\{b_i\},\{n_{\boldsymbol{i}}\})$ of partial chord diagrams

$$\begin{aligned} \mathcal{Z}(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}) &= \exp\left[\mathcal{F}(x,y;\{s_i\};\{u_{\boldsymbol{i}}\})\right] \\ &= \sum_{\{b_i\}}\sum_{\{n_{\boldsymbol{i}}\}} \widehat{\mathcal{N}}_{k,b,l}(\{b_i\},\{n_{\boldsymbol{i}}\}) x^{-b+k-n} y^k \prod_{i\geq 0} s_i^{b_i} \prod_{K\geq 1}\prod_{\{i_L\}_{L=1}^K} u_{\boldsymbol{i}}^{n_{\boldsymbol{i}}}, \end{aligned} \tag{2.29}$$

where l, k, and b obey

$$l = \sum_{K\geq 1}\sum_{\{i_L\}_{L=1}^K}\sum_{L=1}^{K} i_L n_{(i_1,\dots,i_K)}, \qquad 2k+l = \sum_{i\geq 1} i b_i, \qquad b = \sum_{i\geq 0} b_i.$$

The enumeration of partial chord diagrams decorated by the boundary length and point spectrum can also be expressed in terms of Gaussian averages over Hermitian matrices. To this end we again make use of extra marked points, just as in the previous section (concerning the length spectrum to mark the separation between marked points on the backbone, counted by the index $\boldsymbol{i}$), see Figure 13. Indeed, the boundary length and point spectrum also encodes the length spectrum, simply as the number K of partitions of marked points on boundary cycles.

To represent the boundary length and point spectrum, we introduce two external matrices Λ_{P} and Λ_{L}. In order to faithfully represent the ordering between marked points and partitions on each boundary cycle, we assume that these two external matrices do not commute

$$[\Lambda_{\mathrm{P}}, \Lambda_{\mathrm{L}}] \neq 0.$$

The correspondence between partial chord diagrams with the backbone spectrum $\{b_i\}$ and matchings by Wick contractions in a Gaussian average is given by a combination of the previous rules **C1**, **C2**, **P2**, **L2**, **P4**, **L4**, and **L5**. We summarize this correspondence in Table 1.

Table 1: The correspondence between partial chord diagrams with the backbone spectrum $\{b_i\}$ and matchings by Wick contractions in the Gaussian average.

A partial chord diagram	Gaussian average
A chord end on a backbone	$\Lambda_{\mathrm{L}}^{-1}M$
A marked point on a backbone	Λ_{P}
An underside of a backbone	$N\Lambda_{\mathrm{L}}^{-1}$
A backbone	$N\mathrm{Tr}\left(\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{\alpha_1}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{\alpha_2}\Lambda_{\mathrm{L}}^{-1}\cdots\Lambda_{\mathrm{P}}^{\alpha_K}\Lambda_{\mathrm{L}}^{-1}\right)$
A chord	Wick contraction $\overbrace{MM}$

Based on these rules, one finds a bijective correspondence between partial chord diagrams with the backbone spectrum $\{b_i\}$ and matchings by Wick contractions in the Gaussian average

$$\mathcal{W}_N(\{b_i\};\{u_{\boldsymbol{i}}\}) = \Big\langle \prod_{i\geq 0} \left(N\mathrm{Tr}(\Lambda_{\mathrm{L}}^{-1}M+\Lambda_{\mathrm{P}})^i\Lambda_{\mathrm{L}}^{-1}\right)^{b_i}\Big\rangle_N^{\mathrm{G}}, \tag{2.30}$$

where in order to represent trace factors Λ_{P} and Λ_{L} we introduced the *generalized Miwa times*

$$u_{(i_1,\ldots,i_K)} = \frac{1}{N}\mathrm{Tr}\left(\Lambda_{\mathrm{P}}^{i_1}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{i_2}\Lambda_{\mathrm{L}}^{-1}\cdots\Lambda_{\mathrm{P}}^{i_K}\Lambda_{\mathrm{L}}^{-1}\right). \tag{2.31}$$

If a partial chord diagram c contains a boundary cycle labeled by $\boldsymbol{i}=(i_1,\ldots,i_K)$, one finds the following trace factor in the corresponding Wick contractions in $\mathcal{W}_N(\{b_i\};\{u_{\boldsymbol{i}}\})$

$$\mathrm{Tr}\left(\Lambda_{\mathrm{P}}^{i_1}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{i_2}\Lambda_{\mathrm{L}}^{-1}\cdots\Lambda_{\mathrm{P}}^{i_K}\Lambda_{\mathrm{L}}^{-1}\right).$$

Finally, combining Propositions 2.5 and 2.8, we obtain the key proposition.

Proposition 2.12. *The Gaussian average $\mathcal{W}_N(\{b_i\};\{u_{\boldsymbol{i}}\})$ in the equation (2.30) is the generating function for the numbers $\widehat{\mathcal{N}}_{k,b,l}(\{b_i\},\{n_{\boldsymbol{i}}\})$ of partial chord diagrams*

$$\mathcal{W}_N(\{b_i\};\{u_{\boldsymbol{i}}\}) = \sum_{\{n_{\boldsymbol{i}}\}}\widehat{\mathcal{N}}_{k,b,l}(\{b_i\},\{n_{\boldsymbol{i}}\})N^{b-k+n}\prod_{K\geq 1}\prod_{\{i_L\}_{L=1}^K} u_{\boldsymbol{i}}^{n_{\boldsymbol{i}}}. \tag{2.32}$$

Repeating the same combinatorics as in the previous subsections, we find the main theorem of this section.

Theorem 2.13. *The Hermitian matrix integral*

$$\begin{aligned}&\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) =\\ &= \frac{1}{\mathrm{Vol}_N}\int dM\ \exp\left[-N\mathrm{Tr}\left(\frac{M^2}{2}-\sum_{i\geq 0}s_i(y^{1/2}\Lambda_{\mathrm{L}}^{-1}M+\Lambda_{\mathrm{P}})^i\Lambda_{\mathrm{L}}^{-1}\right)\right]\end{aligned} \tag{2.33}$$

agrees with the generating function (2.29)

$$\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \mathcal{Z}(N^{-1},y;\{s_i\};\{u_{\boldsymbol{i}}\}). \tag{2.34}$$

3 Cut-and-join equations via matrix models

In Section 2 we discussed matrix models that enumerate partial chord diagrams filtered by the boundary point spectrum, the boundary length spectrum, and the boundary length and point spectrum. In this section we derive partial differential equations for these matrix models, and show that they agree with the cut-and-join equations found in [2, 7]. To derive these differential equations, it is useful to introduce the following matrix integral.

Definition 3.1. Let A and B denote invertible matrices of rank N. We define a formal matrix integral with parameters y, $\{g_i\}_{i=-\infty}^{+\infty}$, and matrices A and B, as follows

$$\begin{aligned} &Z_N(y;\{g_i\};A;B) = \\ &= \frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N} dM\ \exp\left[-N\mathrm{Tr}\left(\frac{1}{2}M^2 - \sum_{i\in\mathbb{Z}} g_i(y^{1/2}B^{-1}M+A)^i B^{-1}\right)\right]. \end{aligned} \tag{3.1}$$

By the following specializations of this matrix integral one finds matrix integrals discussed in Section 2

$$\begin{aligned} &Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}): \quad g_{i<0}=0, \quad g_{i\geq 0}=s_i, \quad A=\Lambda_{\mathrm{P}}, \quad B=I_N, \\ &Z_N^{\mathrm{L}}(y;\{s_i\};\{q_i\}): \quad g_{i<0}=0, \quad g_{i\geq 0}=s_i, \quad A=0, \quad B=\Lambda_{\mathrm{L}}, \\ &Z_N^{\mathrm{L}}(y;s;\{q_i\}): \quad g_{i\neq -1}=0, \quad g_{-1}=-s, \quad A=\Lambda_{\mathrm{L}}, \quad B=-I_N, \\ &\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}): \quad g_{i<0}=0, \quad g_{i\geq 0}=s_i, \quad A=\Lambda_{\mathrm{P}}, \quad B=\Lambda_{\mathrm{L}}, \end{aligned}$$

where I_N is the rank N identity matrix.

The matrix integral (3.1) satisfies the following partial differential equation.

Proposition 3.2. *The matrix integral $Z_N(y;\{g_i\};A;B)$ obeys a partial differential equation*

$$\left[\frac{\partial}{\partial y} - \frac{1}{2N}\mathrm{Tr}(B^{-1})^{\mathrm{T}}\frac{\partial}{\partial A}(B^{-1})^{\mathrm{T}}\frac{\partial}{\partial A}\right] Z_N(y;\{g_i\};A;B) = 0, \tag{3.2}$$

where the trace in the second term is defined, for rank N matrices X and Y, as

$$\mathrm{Tr}X\frac{\partial}{\partial Y} = \sum_{\alpha,\beta=1}^{N} X_{\alpha\beta}\frac{\partial}{\partial Y_{\beta\alpha}}.$$

Proof. By a shift $M = X - y^{-1/2}BA$, the matrix integral (3.1) can be rewritten as

$$\begin{aligned} &Z_N(y;\{g_i\};A;B) = \\ &= \frac{1}{\mathrm{Vol}_N}\int_{\widetilde{\mathcal{H}}_N} dX\ \exp\left[-N\mathrm{Tr}\left(\frac{1}{2}(X-y^{-1/2}BA)^2 - \sum_{i\in\mathbb{Z}} y^{i/2}g_i(B^{-1}X)^i B^{-1}\right)\right]. \end{aligned} \tag{3.3}$$

Here $\widetilde{\mathcal{H}}_N$ is the space of shifted matrices $X = M + y^{-1/2}BA$ with $M\in\mathcal{H}_N$. The invariance of this matrix integral under the infinitesimal scaling $X_{\alpha\beta}\to(1+\epsilon)X_{\alpha\beta}$ leads to a constraint equation

$$\left\langle N^2 - N\mathrm{Tr}X^2 + y^{-1/2}N\mathrm{Tr}BAX + N\sum_{i\in\mathbb{Z}} i y^{i/2} g_i \mathrm{Tr}(B^{-1}X)^i B^{-1}\right\rangle = 0, \tag{3.4}$$

where the first term N^2 comes from the measure factor, as $dX \to (1+N^2\epsilon)dX$. Here we have defined the unnormalized average for an observable $\mathcal{O}(X)$

$$\langle \mathcal{O}(X)\rangle = \int_{\widetilde{\mathcal{H}}_N} dX\; \mathcal{O}(X) \exp\left[-N\mathrm{Tr}\left(\frac{1}{2}(X-y^{-1/2}BA)^2 - \sum_{i\in\mathbb{Z}} y^{i/2} g_i (B^{-1}X)^i B^{-1}\right)\right].$$

Using

$$\frac{1}{N} y^{1/2} \sum_{\gamma=1}^{N} (B^{-1})^{\mathrm{T}}_{\beta\gamma} \frac{\partial}{\partial A_{\gamma\alpha}} Z_N(y;\{g_i\};A;B) = \left\langle X_{\alpha\beta} - y^{-1/2} N \sum_{\gamma=1}^{N} B_{\alpha\gamma} A_{\gamma\beta} \right\rangle,$$
$$\frac{1}{N}\frac{\partial}{\partial g_i} Z_N(y;\{g_i\};A;B) = y^{i/2}\left\langle \mathrm{Tr}(B^{-1}X)^i B^{-1}\right\rangle,$$

one finds that the constraint equation (3.4) yields

$$\left[-\frac{1}{N} y \mathrm{Tr}(B^{-1})^{\mathrm{T}} \frac{\partial}{\partial A} (B^{-1})^{\mathrm{T}} \frac{\partial}{\partial A} - \mathrm{Tr} A^{\mathrm{T}} \frac{\partial}{\partial A} + \sum_{i\in\mathbb{Z}} i g_i \frac{\partial}{\partial g_i}\right] Z_N(y;\{g_i\};A;B) = 0.$$

It follows from (3.3) that the last two derivatives in the expression above can be replaced by $2y\partial/\partial y$, so that the partial differential equation (3.2) is obtained. □

Remark 3.3. In the above proof of the constraint equation (3.4) we considered the infinitesimal scaling $X_{\alpha\beta} \to (1+\epsilon)X_{\alpha\beta}$. More generally, matrix integral (3.3) is invariant under infinitesimal shifts

$$X_{\alpha\beta} \longrightarrow X_{\alpha\beta} + \epsilon (X^{n+1})_{\alpha\beta}, \qquad n = -1, 0, 1, \ldots.$$

It is known that for the matrix integral without external matrices A and B this symmetry yields the Virasoro symmetry, and in particular the scaling $X_{\alpha\beta} \to (1+\epsilon)X_{\alpha\beta}$ is related to the Virasoro generator L_0^{Vir} [22, 19].[6]

3.1 The boundary point spectrum

In Subsection 2.1 we showed that the matrix integral $Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\})$ in (2.17)

$$Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}) =$$
$$= \frac{1}{\mathrm{Vol}_N} \int_{\mathcal{H}_N} dM\; \exp\left[-N\mathrm{Tr}\left(\frac{M^2}{2} - \sum_{i\geq 0} s_i y^{i/2} (M + y^{-1/2}\Lambda_{\mathrm{P}})^i\right)\right],$$

enumerates partial chord diagrams labeled by the boundary point spectrum. By the specialization

$$g_{i<0} = 0, \qquad g_{i\geq 0} = s_i, \qquad A = \Lambda_{\mathrm{P}}, \qquad B = I_N = \text{identity matrix},$$

[6] In [33, 18], the Schwinger-Dyson approach to the enumeration of chord diagrams is also discussed.

of the matrix integral $Z_N(y;\{g_i\};A;B)$ in (3.1) we see that

$$Z_N(y;s_{i<0}=0,\{s_i\}_{i\geq 0};\Lambda_{\mathrm{P}};I_N)=Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}), \tag{3.5}$$

where the reverse Miwa times r_i are defined in (2.15). From (3.2) we obtain the partial differential equation satisfied by $Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\})$.

Corollary 3.4. *The matrix integral $Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\})$ obeys the partial differential equation*

$$\left[\frac{\partial}{\partial y}-\frac{1}{2N}\mathrm{Tr}\frac{\partial^2}{\partial\Lambda_{\mathrm{P}}^2}\right]Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\})=0. \tag{3.6}$$

This corollary implies the following theorem.

Theorem 3.5. *Let L_0 and L_2 be the differential operators*[7]

$$\begin{aligned}L_0&=\frac{1}{2}\sum_{i\geq 2}\sum_{j=0}^{i-2} i r_j r_{i-j-2}\frac{\partial}{\partial r_i},\\ L_2&=\frac{1}{2}\sum_{i\geq 2}\sum_{j=1}^{i-1} j(i-j)r_{i-2}\frac{\partial^2}{\partial r_i\partial r_{i-j}}.\end{aligned} \tag{3.7}$$

The matrix integral $Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\})$ obeys the cut-and-join equation

$$\frac{\partial}{\partial y}Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\})=\mathcal{L}Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}), \tag{3.8}$$

where

$$\mathcal{L}=L_0+\frac{1}{N^2}L_2.$$

The formal solution of this cut-and-join equation, which gives the matrix integral $Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\})$, is iteratively determined from the initial condition at $y=0$,

$$Z_N^{\mathrm{P}}(y;\{s_i\};\{r_i\})=\mathrm{e}^{y\mathcal{L}}Z_N^{\mathrm{P}}(0;\{s_i\};\{r_i\})=\mathrm{e}^{y\mathcal{L}}\mathrm{e}^{N^2\sum_{i\geq 0}s_ir_i}. \tag{3.9}$$

This theorem follows from the lemma below by rewriting the derivative $\mathrm{Tr}\partial^2/\partial\Lambda_{\mathrm{P}}^2$ in the partial differential equation (3.6).

Lemma 3.6. *For a function $f(\{r_i\})$ of the reverse Miwa times r_i, the derivative $\mathrm{Tr}\partial^2/\partial\Lambda_{\mathrm{P}}^2$ can be rewritten as*

$$\frac{1}{2N}\mathrm{Tr}\frac{\partial^2}{\partial\Lambda_{\mathrm{P}}^2}f(\{r_i\})=\left(L_0+\frac{1}{N^2}L_2\right)f(\{r_i\}). \tag{3.10}$$

[7] In [36] the differential operators L_0 and L_2 were denoted by $W^{(3)}$.

Proof. Consider the derivative $\partial/\partial\Lambda_{\mathrm{P}\beta\alpha}$ of r_i,

$$\frac{\partial r_i}{\partial \Lambda_{\mathrm{P}\beta\alpha}} = \frac{i}{N}\Lambda^{i-1}_{\mathrm{P}\alpha\beta}, \qquad \mathrm{Tr}\frac{\partial^2 r_i}{\partial \Lambda_{\mathrm{P}}^2} = iN\sum_{j=0}^{i-2} r_j r_{i-j-2}.$$

Then the derivative $\mathrm{Tr}\partial^2/\partial\Lambda_{\mathrm{P}}^2$ of the function $f(\{r_i\})$ is re-expressed as

$$\begin{aligned}\frac{1}{2N}\mathrm{Tr}\frac{\partial^2}{\partial\Lambda_{\mathrm{P}}^2}f(\{r_i\}) &= \frac{1}{2N}\sum_{i\geq 0}\mathrm{Tr}\frac{\partial^2 r_i}{\partial\Lambda_{\mathrm{P}}^2}\frac{\partial f(\{r_i\})}{\partial r_i} + \frac{1}{2N}\sum_{i,j\geq 0}\sum_{\alpha,\beta=1}^{N}\frac{\partial r_i}{\partial\Lambda_{\mathrm{P}\beta\alpha}}\frac{\partial r_j}{\partial\Lambda_{\mathrm{P}\alpha\beta}}\frac{\partial^2 f(\{r_i\})}{\partial r_i\partial r_j}\\ &= \frac{1}{2}\sum_{i\geq 2}\sum_{j=0}^{i-2} i r_j r_{i-j-2}\frac{\partial f(\{r_i\})}{\partial r_i} + \frac{1}{2N^2}\sum_{i,j\geq 1} ij r_{i+j-2}\frac{\partial^2 f(\{r_i\})}{\partial r_i\partial r_j}.\end{aligned}$$

This coincides with the right hand side of (3.10). □

The cut-and-join equation for the rescaled matrix integral (2.18) yields

$$\frac{\partial}{\partial y}Z^{\mathrm{P}}_{t_0N}(t_0y;\{t_0^{-1}s_i\};\{t_0^{-1}t_i\}) = \mathcal{L}Z^{\mathrm{P}}_{t_0N}(t_0y;\{t_0^{-1}s_i\};\{t_0^{-1}t_i\}), \tag{3.11}$$

where $\mathcal{L}$ is given by

$$\begin{aligned}&\mathcal{L} = L_0 + x^2 L_2, \quad x = N^{-1},\\ &L_0 = \frac{1}{2}\sum_{i\geq 2}\sum_{j=0}^{i-2} i t_j t_{i-j-2}\frac{\partial}{\partial t_i}, \quad L_2 = \frac{1}{2}\sum_{i\geq 2}\sum_{j=1}^{i-1} j(i-j)t_{i-2}\frac{\partial^2}{\partial t_i\partial t_{i-j}}.\end{aligned} \tag{3.12}$$

This cut-and-join equation agrees with the partial differential equation in Theorem 1 of [2], where it was proven combinatorially by the recursion relation for the number of partial chord diagrams.[8] This completes the proof of Theorem 2.6.

3.2 The boundary length spectrum

In Subsection 2.2 we showed that the matrix integral $Z^{\mathrm{L}}_N(y;\{s_i\};\{q_i\})$ in (2.24) enumerates chord diagrams labeled by the boundary length spectrum. By the specialization

$$g_{i<0} = 0, \quad g_{i\geq 0} = s_i, \quad A = 0, \quad B = \Lambda_{\mathrm{L}},$$

of the matrix integral $Z_N(y;\{g_i\};A;B)$ in (3.1) we see that

$$Z_N(y; s_{i<0} = 0, \{s_i\}_{i\geq 0}; 0; \Lambda_{\mathrm{L}}) = Z^{\mathrm{L}}_N(y;\{s_i\};\{q_i\}), \tag{3.13}$$

where the Miwa times q_i are defined in equation (2.22).

[8] For the Grothendieck's dessin counting, a similar cut-and-join equation was found in [27].

Obviously, for $A = 0$ the partial differential equation (3.2) does not hold. Instead we consider the matrix integral (2.26) obtained by the specialization $s_i = s$

$$Z_N^{\mathrm{L}}(y; s; \{q_i\}) = \frac{1}{\mathrm{Vol}_N} \int_{\mathcal{H}_N} dM \ \exp\left[- N\mathrm{Tr}\left(\frac{M^2}{2} + \frac{s}{y^{1/2}M - \Lambda_{\mathrm{L}}} \right) \right].$$

The same matrix integral can be obtained by the specialization

$$g_{i \neq -1} = 0, \qquad g_{-1} = -s, \qquad A = \Lambda_{\mathrm{L}}, \qquad B = -I_N,$$

and thus

$$Z_N(y; s_i = -\delta_{i,-1}; \Lambda_{\mathrm{L}}; B = -I_N) = Z_N^{\mathrm{L}}(y; s; \{q_i\}). \tag{3.14}$$

Then from (3.2) we obtain a partial differential equation for $Z_N^{\mathrm{L}}(y; s; \{q_i\})$.

Corollary 3.7. *The matrix integral $Z_N^{\mathrm{L}}(y; s; \{q_i\})$ obeys the partial differential equation*

$$\left[\frac{\partial}{\partial y} - \frac{1}{2N} \mathrm{Tr} \frac{\partial^2}{\partial \Lambda_{\mathrm{L}}^2} \right] Z_N^{\mathrm{L}}(y; s; \{r_i\}) = 0. \tag{3.15}$$

This corollary implies the following theorem.

Theorem 3.8. *Let K_0 and K_2 be the differential operators*

$$\begin{aligned} K_0 &= \frac{1}{2} \sum_{i \geq 3} \sum_{j=1}^{i-1} (i-2) q_j q_{i-j} \frac{\partial}{\partial q_{i-2}}, \\ K_2 &= \frac{1}{2} \sum_{i \geq 2} \sum_{j=1}^{i-1} j(i-j) q_{i+2} \frac{\partial^2}{\partial q_i \partial q_{i-j}}. \end{aligned} \tag{3.16}$$

The matrix integral $Z_N^{\mathrm{L}}(y; s; \{q_i\})$ obeys the cut-and-join equation

$$\frac{\partial}{\partial y} Z_N^{\mathrm{L}}(y; s; \{q_i\}) = \mathcal{K} Z_N^{\mathrm{L}}(y; s; \{q_i\}), \tag{3.17}$$

where

$$\mathcal{K} = K_0 + \frac{1}{N^2} K_2.$$

The formal solution of this cut-and-join equation, which gives the matrix integral $Z_N^{\mathrm{L}}(y; s; \{q_i\})$, is iteratively determined from the initial condition at $y = 0$,

$$Z_N^{\mathrm{L}}(y; s; \{q_i\}) = \mathrm{e}^{y\mathcal{K}} Z_N^{\mathrm{L}}(y = 0; s; \{q_i\}) = \mathrm{e}^{y\mathcal{K}} \mathrm{e}^{N^2 s q_1}. \tag{3.18}$$

The cut-and-join equation (3.17) was combinatorially proven in Theorem 2 of [2] for the generating function $Z^{\mathrm{L}}(x, y; s; \{q_i\})$ in (2.19), and thus Theorem 2.9 for $s_i = s$ is reproved.

The claim of Theorem 3.8 is proven by rewriting the derivative $\mathrm{Tr}\partial^2/\partial\Lambda_{\mathrm{L}}^2$ in the partial differential equation (3.15) using the following lemma.

Lemma 3.9. *For a function $g(\{q_i\})$ of the Miwa times $\{q_i\}$, the derivative $\mathrm{Tr}\partial^2/\partial\Lambda_{\mathrm{L}}^2$ can be rewritten as follows*

$$\frac{1}{2N}\mathrm{Tr}\frac{\partial^2}{\partial\Lambda_{\mathrm{L}}^2}g(\{q_i\}) = \left(K_0 + \frac{1}{N^2}K_2\right)g(\{q_i\}). \tag{3.19}$$

Proof. By acting $\partial/\partial\Lambda_{\mathrm{L}}$ on the Miwa time q_i one obtains

$$\frac{\partial q_i}{\partial\Lambda_{\mathrm{L}\alpha\beta}} = -\frac{i}{N}\Lambda_{\mathrm{L}\beta\alpha}^{-i-1}, \qquad \mathrm{Tr}\frac{\partial^2 q_i}{\partial\Lambda_{\mathrm{L}}^2} = iN\sum_{j=1}^{i+1} q_j q_{i-j+2}.$$

Adopting this relation via the chain rule applied to the Λ_{L} derivatives, one finds that

$$\begin{aligned}\frac{1}{2N}\mathrm{Tr}\frac{\partial^2 g(\{q_i\})}{\partial\Lambda_{\mathrm{L}}^2} &= \frac{1}{2N}\sum_{i\geq 0}\mathrm{Tr}\frac{\partial^2 q_i}{\partial\Lambda_{\mathrm{L}}^2}\frac{\partial g(\{q_i\})}{\partial q_i} + \frac{1}{2N}\sum_{i,j\geq 0}\sum_{\alpha,\beta=1}^{N}\frac{\partial q_i}{\partial\Lambda_{\mathrm{L}\alpha\beta}}\frac{\partial q_j}{\partial\Lambda_{\mathrm{L}\beta\alpha}}\frac{\partial^2 g(\{q_i\})}{\partial q_i\partial q_j}\\ &= \frac{1}{2}\sum_{i\geq 1} iq_jq_{i-j+2}\frac{\partial g(\{q_i\})}{\partial q_i} + \frac{1}{2N^2}\sum_{i,j\geq 1} ijq_{i+j+2}\frac{\partial^2 g(\{q_i\})}{\partial q_i\partial q_j}.\end{aligned}$$

This coincides with the right hand side of (3.19). □

3.3 The boundary length and point spectrum

In Subsection 2.3 we showed that the matrix integral $\mathcal{Z}_N(y; \{s_i\}; \{u_{\boldsymbol{i}}\})$ in (2.29)

$$\begin{aligned}&\mathcal{Z}_N(y; \{s_i\}; \{u_{\boldsymbol{i}}\}) =\\ &= \frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N} dM\ \exp\left[-N\mathrm{Tr}\left(\frac{M^2}{2} - \sum_{i\geq 0} s_i(y^{1/2}\Lambda_{\mathrm{L}}^{-1}M + \Lambda_{\mathrm{P}})^i\Lambda_{\mathrm{L}}^{-1}\right)\right]\end{aligned}$$

enumerates partial chord diagrams labeled by the boundary length and point spectrum. By the specialization

$$g_{i<0} = 0, \qquad g_{i\geq 0} = s_i, \qquad A = \Lambda_{\mathrm{P}}, \qquad B = \Lambda_{\mathrm{L}},$$

of the matrix integral $Z_N(y; \{g_i\}; A; B)$ in (3.1) we see that

$$Z_N(y; s_{i<0} = 0, \{s_i\}_{i\geq 0}; \Lambda_{\mathrm{P}}; \Lambda_{\mathrm{L}}) = \mathcal{Z}_N(y; \{s_i\}; \{u_{\boldsymbol{i}}\}), \tag{3.20}$$

where the generalized Miwa times $u_{(i_1,\ldots,i_K)}$ are defined in (2.31)

$$u_{(i_1,\ldots,i_K)} = \frac{1}{N}\mathrm{Tr}\left(\Lambda_{\mathrm{P}}^{i_1}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{i_2}\Lambda_{\mathrm{L}}^{-1}\cdots\Lambda_{\mathrm{P}}^{i_K}\Lambda_{\mathrm{L}}^{-1}\right).$$

From (3.2) we obtain a partial differential equation for $\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\})$.

Corollary 3.10. *The matrix integral $\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\})$ obeys the partial differential equation*

$$\left[\frac{\partial}{\partial y} - \frac{1}{2N}\mathrm{Tr}(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}\right]\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = 0. \tag{3.21}$$

This corollary implies the following main theorem of this section.

Theorem 3.11. *Let M_0 and M_2 be the following differential operators with respect to parameters $u_{\boldsymbol{i}}$*

$$\begin{aligned} M_0 = \frac{1}{2}&\sum_{K\geq 1}\sum_{\{i_1,\ldots,i_K\}}\sum_{1\leq I\neq M\leq K}\sum_{\ell=0}^{i_I-1}\sum_{m=0}^{i_M-1} \\ &u_{(i_I-\ell-1,i_{I+1},\ldots,i_{M-1},m)}u_{(i_M-m-1,i_{M+1},\ldots,i_{I-1},\ell)}\frac{\partial}{\partial u_{(i_1,\ldots,i_K)}} \\ &+\sum_{K\geq 1}\sum_{\{i_1,\ldots,i_K\}}\sum_{I=0}^{K}\sum_{\ell+m\leq i_I-2}u_{(\ell,m,i_{I+1},\ldots,i_{I-1})}u_{(i_I-\ell-m-2)}\frac{\partial}{\partial u_{(i_1,\ldots,i_K)}}, \\ M_2 = \frac{1}{2}&\sum_{K,L\geq 0}\sum_{\{i_1,\ldots,i_K\}}\sum_{\{j_1,\ldots,j_L\}}\sum_{I=0}^{K}\sum_{J=0}^{L}\sum_{\ell=0}^{i_I-1}\sum_{m=0}^{j_J-1} \\ &u_{(i_I-\ell-1,i_{I+1},\ldots,i_{I-1},\ell,j_J-m-1,j_{J+1},\ldots,j_{J-1},m)}\frac{\partial^2}{\partial u_{(i_1,\ldots,i_K)}\partial u_{(j_1,\ldots,j_L)}}, \end{aligned} \tag{3.22}$$

where labels I, M's are defined modulo K, and the label J is defined modulo L. The matrix integral $\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\})$ obeys the cut-and-join equation

$$\frac{\partial}{\partial y}\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \mathcal{M}\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}), \tag{3.23}$$

where

$$\mathcal{M} = M_0 + \frac{1}{N^2}M_2.$$

The formal solution of this cut-and-join equation, which gives the matrix integral $\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\})$, is iteratively determined from the initial condition at $y=0$,

$$\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \mathrm{e}^{y\mathcal{M}}\mathcal{Z}_N(y=0;\{s_i\};\{u_{\boldsymbol{i}}\}) = \mathrm{e}^{y\mathcal{M}}\mathrm{e}^{N^2\sum_{i\geq 0}s_i u_{(i)}}. \tag{3.24}$$

The partial differential equation (3.23) agrees with the cut-and-equation obtained combinatorially in Theorem 1.1 of [7]. Here we prove this theorem by rewriting the derivative in the second term of the partial differential equation (3.21), taking advantage of the following lemma.

Lemma 3.12. *For a function $h(\{u_{\boldsymbol{i}}\})$ of the generalized Miwa times $u_{\boldsymbol{i}}$, the derivative in the second term of the partial differential equation (3.21) can be rewritten as follows*

$$\frac{1}{2N}\mathrm{Tr}\left[(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}\right]h(\{u_{\boldsymbol{i}}\}) = \left(M_0+\frac{1}{N^2}M_2\right)h(\{u_{\boldsymbol{i}}\}). \tag{3.25}$$

Proof. By the chain rule, the derivative on the left hand side of (3.25) is rewritten as follows

$$\begin{aligned}
&\mathrm{Tr}\left[(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}\right]h(\{u_{\boldsymbol{i}}\}) = \\
&= \sum_{K\geq 0}\sum_{(i_1,\dots,i_K)}\mathrm{Tr}\left[(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}u_{(i_1,\dots,i_K)}\right]\frac{\partial}{\partial u_{(i_1,\dots,i_K)}}h(\{u_{\boldsymbol{i}}\}) \\
&\quad + \sum_{K,L\geq 0}\sum_{(i_1,\dots,i_K)}\sum_{(j_1,\dots,j_L)}\mathrm{Tr}\left[(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}u_{(i_1,\dots,i_K)}(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}u_{(j_1,\dots,j_L)}\right] \\
&\qquad\qquad \times\frac{\partial^2}{\partial u_{(i_1,\dots,i_K)}\partial u_{(j_1,\dots,j_L)}}h(\{u_{\boldsymbol{i}}\}).
\end{aligned}$$

Each of the coefficients yields

$$\begin{aligned}
&\mathrm{Tr}\left[(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}u_{(i_1,\dots,i_K)}\right] = \\
&= \sum_{1\leq I\neq M\leq K}\sum_{\ell=0}^{i_I-1}\sum_{m=0}^{i_M-1}\frac{1}{N}\mathrm{Tr}(\Lambda_{\mathrm{P}}^{i_I-\ell-1}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{i_{I+1}}\Lambda_{\mathrm{L}}^{-1}\cdots\Lambda_{\mathrm{P}}^{i_{M-1}}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{m}\Lambda_{\mathrm{L}}^{-1}) \\
&\qquad\qquad \times\mathrm{Tr}(\Lambda_{\mathrm{P}}^{i_M-m-1}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{i_{M+1}}\Lambda_{\mathrm{L}}^{-1}\cdots\Lambda_{\mathrm{P}}^{i_{I-1}}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{\ell}\Lambda_{\mathrm{L}}^{-1}) \\
&\quad + 2\sum_{L=0}^{K}\sum_{\ell+m\leq i_I-2}\frac{1}{N}\mathrm{Tr}(\Lambda_{\mathrm{P}}^{\ell}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{m}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{i_{I+1}}\Lambda_{\mathrm{L}}^{-1}\cdots\Lambda_{\mathrm{P}}^{i_{L-1}}\Lambda_{\mathrm{L}}^{-1})\mathrm{Tr}(\Lambda_{\mathrm{P}}^{i_I-\ell-m-2}\Lambda_{\mathrm{L}}^{-1}) \\
&= N\sum_{1\leq I\neq M\leq K}\sum_{\ell=0}^{i_I-1}\sum_{m=0}^{i_M-1}u_{(i_I-\ell-1,i_{I+1},\dots,i_{M-1},m)}u_{(i_M-m-1,i_{M+1},\dots,i_{I-1},\ell)} \\
&\quad + 2N\sum_{L=0}^{K}\sum_{\ell+m\leq i_I-2}u_{(\ell,m,i_{I+1},\dots,i_{I-1})}u_{(i_I-\ell-m-2)},
\end{aligned}$$

and

$$\mathrm{Tr}\left[(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}u_{(i_1,\dots,i_K)}(\Lambda_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}u_{(j_1,\dots,j_L)}\right] =$$

$$
\begin{aligned}
&= \sum_{I=1}^{K} \sum_{J=1}^{L} \sum_{\ell=0}^{i_I-1} \sum_{m=0}^{j_J-1} \frac{1}{N^2} \mathrm{Tr}(\Lambda_{\mathrm{P}}^{i_I-\ell-1}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{i_{I+1}}\Lambda_{\mathrm{L}}^{-1}\cdots\Lambda^{i_{I-1}}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{\ell}\Lambda_{\mathrm{L}}^{-1} \\
&\qquad\qquad\qquad\qquad \cdot \Lambda_{\mathrm{P}}^{j_J-m-1}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{j_{J+1}}\Lambda_{\mathrm{L}}^{-1}\cdots\Lambda_{\mathrm{P}}^{j_{J-1}}\Lambda_{\mathrm{L}}^{-1}\Lambda_{\mathrm{P}}^{m}\Lambda_{\mathrm{L}}^{-1}) \\
&= \frac{1}{N} \sum_{I=1}^{K} \sum_{J=1}^{L} \sum_{\ell=0}^{i_I-1} \sum_{m=0}^{j_J-1} u_{(i_I-\ell-1, i_{I+1},\ldots,i_{I-1},\ell,j_J-m-1,j_{J+1},\ldots,j_{J-1},m)}.
\end{aligned}
$$

In this way one obtains the right hand side of (3.25). □

As a corollary of Theorem 3.11, one finds the cut-and-join equation for the 1-backbone generating function.[9]

Corollary 3.13. *The 1-backbone generating function $\mathcal{F}_1(x,y;\{s_i\};\{u_{\boldsymbol{i}}\})$ obtained by picking up the $\mathcal{O}(s_i^1)$ terms in $\mathcal{Z}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\})$ as follows*

$$
\begin{aligned}
&\mathcal{F}_1(N^{-1},y;\{s_i\};\{u_{\boldsymbol{i}}\}) \\
&= \frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N} dM\, \mathrm{e}^{-N\mathrm{Tr}\frac{M^2}{2}} N \sum_{i\geq 0} s_i \mathrm{Tr}(y^{1/2}\Lambda_{\mathrm{L}}^{-1}M+\Lambda_{\mathrm{P}})^i\Lambda_{\mathrm{L}}^{-1},
\end{aligned} \tag{3.26}
$$

obeys the cut-and-join equation

$$
\frac{\partial}{\partial y}\mathcal{F}_1(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \mathcal{M}\mathcal{F}_1(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}), \tag{3.27}
$$

where $\mathcal{M} = M_0 + x^2 M_2$. The solution is iteratively determined by

$$
\mathcal{F}_1(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \mathrm{e}^{y\mathcal{M}}\mathcal{F}_1(x,y=0;\{s_i\};\{u_{\boldsymbol{i}}\}) = \mathrm{e}^{y\mathcal{M}}\left(x^{-2}\sum_{i\geq 0} s_i u_{(i)}\right). \tag{3.28}
$$

4 Non-oriented analogues

In this section we consider the enumeration of both orientable and non-orientable (jointly called non-oriented) partial chord diagrams [2, 7]. To this end we generalize the matrix models introduced in Section 2. In Subsection 4.1, matrix models for the boundary point spectrum, the boundary length spectrum, and the boundary length and point spectrum are introduced, based on the corresponding Gaussian matrix integrals over the space of rank N real symmetric matrices. Subsequently, in Subsection 4.2, we derive cut-and-join equations for the generating functions of non-oriented partial chord diagrams, using analogous methods as those discussed in Section 3.

[9] For $\Lambda_{\mathrm{L}} = I_N$ (or $s_i = s$ and $\Lambda_{\mathrm{P}} = 0$) the cut-and-join equation for the 1-backbone generating function labeled by the boundary point spectrum (or boundary length spectrum) was proven combinatorially in [2].

4.1 Non-oriented analogues of the matrix models

In this subsection we generalize matrix models found in Section 2, in order to enumerate both orientable and non-orientable partial chord diagrams [2, 7].

Definition 4.1. Let $\widetilde{\mathcal{N}}_{h,k,l}(\{b_i\},\{n_i\},\{p_i\})$ denote the number of connected non-oriented partial chord diagrams of type $\{h,k,l;\{b_i\};\{n_i\};\{p_i\}\}$. Analogously as in the orientable case, we define

$$\widetilde{\mathcal{N}}_{h,k,l}(\{b_i\},\{n_i\}) = \sum_{\{p_i\}} \widetilde{\mathcal{N}}_{h,k,l}(\{b_i\},\{n_i\},\{p_i\}),$$
$$\widetilde{\mathcal{N}}_{h,k}(\{b_i\},\{p_i\}) = \sum_{\{n_i\}} \widetilde{\mathcal{N}}_{h,k,l=0}(\{b_i\},\{n_i\},\{p_i\}),$$

and introduce generating functions

$$\begin{aligned}\widetilde{F}(x,y;\{s_i\};\{t_i\}) &= \sum_{b\geq 1} \widetilde{F}_b(x,y;\{s_i\};\{t_i\}),\\ \widetilde{F}_b(x,y;\{s_i\};\{t_i\}) &= \frac{1}{b!}\sum_{\sum_i b_i=b}\sum_{\{n_i\}} \widetilde{\mathcal{N}}_{h,k,l}(\{b_i\},\{n_i\}) x^{h-2}y^k \prod_{i\geq 0} s_i^{b_i} t_i^{n_i},\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\widetilde{G}(x,y;\{s_i\};\{q_i\}) &= \sum_{b\geq 1} \widetilde{G}_b(x,y;\{s_i\};\{q_i\}),\\ \widetilde{G}_b(x,y;\{s_i\};\{q_i\}) &= \frac{1}{b!}\sum_{\sum_i b_i=b}\sum_{\{p_i\}} \widetilde{\mathcal{N}}_{h,k}(\{b_i\},\{p_i\}) x^{h-2}y^k \prod_{i\geq 0} s_i^{b_i} \prod_{i\geq 1} q_i^{p_i}.\end{aligned} \tag{4.2}$$

Generating functions of connected and disconnected partial chord diagrams are related by

$$\widetilde{Z}^{\mathrm{P}}(x,y;\{s_i\};\{t_i\}) = \exp\Big[\widetilde{F}(x,y;\{s_i\};\{t_i\})\Big], \tag{4.3}$$

$$\widetilde{Z}^{\mathrm{L}}(x,y;\{s_i\};\{q_i\}) = \exp\Big[\widetilde{G}(x,y;\{s_i\};\{q_i\})\Big]. \tag{4.4}$$

Furthermore, we introduce generating functions of non-oriented partial chord diagrams labeled by the boundary length and point spectrum.

Definition 4.2. Let $\widetilde{\mathcal{N}}_{h,k,l}(\{b_i\},\{n_{\boldsymbol{i}}\})$ denote the number of connected orientable and non-orientable partial chord diagrams of type $\{h,k,l;\{b_i\};\{n_{\boldsymbol{i}}\}\}$ with the boundary length and point spectrum $n_{\boldsymbol{i}}$. We define the generating functions

$$\begin{aligned}\widetilde{\mathcal{F}}(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}) &= \sum_{b\geq 1} \widetilde{\mathcal{F}}_b(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}),\\ \widetilde{\mathcal{F}}_b(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}) &= \frac{1}{b!}\sum_{\sum_i b_i=b}\sum_{\{n_{\boldsymbol{i}}\}} \widetilde{\mathcal{N}}_{h,k,l}(\{b_i\},\{n_{\boldsymbol{i}}\}) x^{h-2}y^k \prod_{i\geq 0} s_i^{b_i} \prod_{K\geq 1}\prod_{\{i_L\}_{L=1}^K} u_{\boldsymbol{i}}^{n_{\boldsymbol{i}}}.\end{aligned} \tag{4.5}$$

As usual, generating functions of connected and disconnected partial chord diagrams are related by

$$\widetilde{\mathcal{Z}}(x,y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \exp\left[\widetilde{\mathcal{F}}(x,y;\{s_i\};\{u_{\boldsymbol{i}}\})\right]. \tag{4.6}$$

Non-oriented analogue of partial chord diagrams and Wick contractions

A non-oriented partial chord diagram is a partial chord diagrams with each chord decorated by a binary variable, which indicates if it is twisted or not. Such non-oriented partial chord diagrams are enumerated by real symmetric[10] matrix integrals. The Gaussian average $\langle \mathcal{O}(M)\rangle_N^{\widetilde{\mathrm{G}}}$ over the space $\mathcal{H}_N(\mathbb{R})$ of real symmetric matrices is defined by

$$\langle \mathcal{O}(M)\rangle_N^{\widetilde{\mathrm{G}}} = \frac{1}{\mathrm{Vol}_N(\mathbb{R})}\int_{\mathcal{H}_N(\mathbb{R})} dM\ \mathcal{O}(M)\,\mathrm{e}^{-N\mathrm{Tr}\frac{M^2}{4}}, \tag{4.7}$$

where

$$\mathrm{Vol}_N(\mathbb{R}) = \int_{\mathcal{H}_N(\mathbb{R})} dM\ \mathrm{e}^{-N\mathrm{Tr}\frac{M^2}{4}} = N^{N(N+1)/2}\mathrm{Vol}(\mathcal{H}_N(\mathbb{R})), \tag{4.8}$$

For the choice of $\mathcal{O}(M) = M_{\alpha\beta}M_{\gamma\epsilon}$ $(\alpha,\beta,\gamma,\epsilon = 1,\dots,N)$, the Wick contraction is defined as

$$\overbracket{M_{\alpha\beta}M}_{\gamma\epsilon} := \langle M_{\alpha\beta}M_{\gamma\epsilon}\rangle_N^{\widetilde{\mathrm{G}}} = \frac{1}{N}(\delta_{\alpha\epsilon}\delta_{\beta\gamma} + \delta_{\alpha\gamma}\delta_{\beta\epsilon}). \tag{4.9}$$

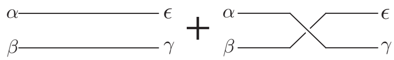
This Wick contraction consists of two terms, which encode the corresponding fatgraph as follows. The first term $\frac{1}{N}\delta_{\alpha\epsilon}\delta_{\beta\gamma}$ is the same as in the Hermitian matrix integral (2.5), and it can be identified with an untwisted band in the two dimensional surface Σ_c associated to the partial chord diagram c. The second term $\frac{1}{N}\delta_{\alpha\gamma}\delta_{\beta\epsilon}$ in (4.9) relates opposite matrix indices compared to the first term and can be identified with the twisted band in Σ_c, see Figure 14. Hence, for the real symmetric Gaussian average, the correspondence rules **C4, P5, L5** in Section 2 are replaced by the following rules [24, 54, 52, 26, 39, 23].

N5 The Wick contraction between $M_{\alpha_j\beta_j}$ and $M_{\alpha'_{j'}\beta'_{j'}}$ corresponds to a band or a twisted band connecting two chord ends. Each Wick contraction imposes either the constraint $\delta_{\alpha_j\beta'_{j'}}\delta_{\alpha'_{j'}\beta_j}$ or the constraint $\delta_{\alpha_j\alpha'_{j'}}\delta_{\beta_j\beta'_{j'}}$ for matrix indices assigned to edges of chord ends matched by Wick contractions.

In order to construct matrix models that enumerate non-oriented partial chord diagrams, we introduce two external real symmetric matrices matrices Ω_{P} and Ω_{L}

$$\Omega_{\mathrm{P}} = \Omega_{\mathrm{P}}^{\mathrm{T}}, \quad \Omega_{\mathrm{L}} = \Omega_{\mathrm{L}}^{\mathrm{T}},$$

[10] The Gaussian matrix integral over the space of real symmetric matrix is also referred to as the *Gaussian orthogonal ensemble* [21, 35].



$$\contraction{}{M}{_{\alpha\beta}}{M}M_{\alpha\beta}M_{\gamma\epsilon} = \frac{1}{N}(\delta_{\alpha\epsilon}\delta_{\beta\gamma} + \delta_{\alpha\gamma}\delta_{\beta\epsilon})$$

Figure 14: Wick contraction and the untwisted / twisted bands.

which take account of the fact that boundary cycles of non-oriented partial chord diagrams are not endowed with a specific orientation. To model the index structure $\boldsymbol{i}$ of the boundary length and point spectrum correctly, we assume these two matrices do not commute

$$[\Omega_{\mathrm{P}}, \Omega_{\mathrm{L}}] \neq 0.$$

Furthermore, we introduce corresponding generalized Miwa times

$$u_{(i_1,\dots,i_K)} = \frac{1}{N}\mathrm{Tr}\left(\Omega_{\mathrm{P}}^{i_1}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{i_2}\Omega_{\mathrm{L}}^{-1}\cdots\Omega_{\mathrm{P}}^{i_K}\Omega_{\mathrm{L}}^{-1}\right), \tag{4.10}$$

which are invariant under the symmetry

$$u_{(i_1,\dots,i_K)} = u_{(i_K,i_{K-1},\dots,i_1)}.$$

This assignment implies the bijective correspondence (analogous to the orientable case discussed earlier) between non-oriented partial chord diagrams and Wick contractions, which is summarized in Table 2.

Table 2: The correspondence between partial chord diagrams and operator products in the real symmetric matrix integral.

Partial chord diagram	Gaussian average
A chord end on a backbone	$\Omega_{\mathrm{L}}^{-1}M$
A marked point on a backbone	Ω_{P}
An underside of a backbone	$N\Omega_{\mathrm{L}}^{-1}$
A backbone	$N\mathrm{Tr}\left(\Omega_{\mathrm{P}}^{\alpha_1}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{\alpha_2}\Omega_{\mathrm{L}}^{-1}\cdots\Omega_{\mathrm{P}}^{\alpha_K}\Omega_{\mathrm{L}}^{-1}\right)$
A Chord	A Wick contraction $\contraction{}{M}{}{M}MM$

Using this correspondence, generating functions $\widetilde{Z}^{\mathrm{P}}(x, y; \{s_i\}; \{r_i\})$, $\widetilde{Z}^{\mathrm{L}}(x, y; \{s_i\}; \{q_i\})$, and $\widetilde{\mathcal{Z}}(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\})$ can be re-expressed in terms of matrix integrals. Repeating the same combinatorial arguments as for the orientable case in Section 2, we obtain the following three theorems.

Theorem 4.3. *Let $\widetilde{Z}_N^{\mathrm{P}}(y;\{s_i\};\{r_i\})$ be the real symmetric matrix integral with the external symmetric matrix Ω_{P} of rank N*

$$\begin{aligned}\widetilde{Z}_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}) = \\ = \frac{1}{\mathrm{Vol}_N(\mathbb{R})}\int_{\mathcal{H}_N(\mathbb{R})} dM \ \exp\left[-N\mathrm{Tr}\left(\frac{M^2}{4}-\sum_{i\geq 0} s_i y^{i/2}(M+y^{-1/2}\Omega_{\mathrm{P}})^i\right)\right],\end{aligned}\tag{4.11}$$

where r_i are reverse Miwa times

$$r_i = \frac{1}{N}\mathrm{Tr}\Omega_{\mathrm{P}}^i. \tag{4.12}$$

This matrix integral agrees with the generating function (4.3)

$$\widetilde{Z}_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}) = \widetilde{Z}^{\mathrm{P}}(N^{-1},y;\{s_i\},t_0=1,\{t_i=r_i\}_{i\geq 1}). \tag{4.13}$$

The t_0-dependence can be implemented by the following rescaling of parameters

$$\widetilde{Z}_{t_0N}^{\mathrm{P}}(t_0y;\{t_0^{-1}s_i\};\{t_0^{-1}t_i\}) = \widetilde{Z}^{\mathrm{P}}(N^{-1},y;\{s_i\};\{t_i=r_i\}). \tag{4.14}$$

Theorem 4.4. *Let $\widetilde{Z}_N^{\mathrm{L}}(y;\{s_i\};\{q_i\})$ be the real symmetric matrix integral with the external invertible symmetric matrix Ω_{L} of rank N*

$$\begin{aligned}\widetilde{Z}_N^{\mathrm{L}}(y;\{s_i\};\{q_i\}) = \\ = \frac{1}{\mathrm{Vol}_N(\mathbb{R})}\int_{\mathcal{H}_N(\mathbb{R})} dM \ \exp\left[-N\mathrm{Tr}\left(\frac{M^2}{4}-\sum_{i\geq 0} s_i y^{i/2}\left(\Omega_{\mathrm{L}}^{-1}M\right)^i\Omega_{\mathrm{L}}^{-1}\right)\right],\end{aligned}\tag{4.15}$$

where q_i are Miwa times

$$q_i = \frac{1}{N}\mathrm{Tr}\Omega_{\mathrm{L}}^{-i}. \tag{4.16}$$

This matrix integral agrees with the generating function (4.4)

$$\widetilde{Z}_N^{\mathrm{L}}(y;\{s_i\};\{q_i\}) = \widetilde{Z}^{\mathrm{L}}(N^{-1},y;\{s_i\};\{q_i\}). \tag{4.17}$$

As considered in (2.26) and Subsection 3.2, the specialization $s_i = s$ of the matrix integral (4.15) gives the following reduced model

$$\begin{aligned}\widetilde{Z}_N^{\mathrm{L}}(y;s;\{q_i\}) = \widetilde{Z}_N^{\mathrm{L}}(y;\{s_i=s\};\{q_i\}) = \\ = \frac{1}{\mathrm{Vol}_N(\mathbb{R})}\int_{\mathcal{H}_N(\mathbb{R})} dM \ \exp\left[-N\mathrm{Tr}\left(\frac{M^2}{4}+\frac{s}{y^{1/2}M-\Omega_{\mathrm{L}}}\right)\right].\end{aligned}\tag{4.18}$$

The cut-and-join equation that follows from this reduced model is derived in the next subsection.

Theorem 4.5. *Let $\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\})$ be the real symmetric matrix integral with the external invertible symmetric matrices Ω_{P} and Ω_{L} of rank N*

$$\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \tag{4.19}$$
$$= \frac{1}{\mathrm{Vol}_N(\mathbb{R})}\int_{\mathcal{H}_N(\mathbb{R})} dM\ \exp\left[-N\mathrm{Tr}\left(\frac{M^2}{4} - \sum_{i\geq 0} s_i(y^{1/2}\Omega_{\mathrm{L}}^{-1}M+\Omega_{\mathrm{P}})^i\Omega_{\mathrm{L}}^{-1}\right)\right],$$

and $u_{\boldsymbol{i}}$ be the generalized Miwa times defined in (4.10). This matrix integral agrees with the generating function (4.6)

$$\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \widetilde{\mathcal{Z}}(N^{-1},y;\{s_i\};\{u_{\boldsymbol{i}}\}). \tag{4.20}$$

4.2 Non-oriented analogues of cut-and-join equations

We derive now non-oriented analogues of cut-and-join equations discussed in Section 3. Analogously to the Hermitian matrix integral in (3.1), we introduce the following matrix integral.

Definition 4.6. Let $U = U^{\mathrm{T}}$ and $V = V^{\mathrm{T}}$ be rank N invertible symmetric matrices. We define a formal real symmetric matrix integral with parameters y, $\{g_i\}_{i=-\infty}^{+\infty}$ as follows

$$\widetilde{Z}_N(y;\{g_i\};U;V) = \tag{4.21}$$
$$= \frac{1}{\mathrm{Vol}_N(\mathbb{R})}\int_{\mathcal{H}_N(\mathbb{R})} dM\ \exp\left[-N\mathrm{Tr}\left(\frac{1}{4}M^2 - \sum_{i\in\mathbb{Z}} g_i(y^{1/2}V^{-1}M+U)^iV^{-1}\right)\right].$$

The matrix integrals discussed in the previous subsection follow from this matrix integral by specializations

$$\widetilde{Z}_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}): \quad g_{i<0}=0, \quad g_{i\geq 0}=s_i, \quad U=\Omega_{\mathrm{P}}, \quad V=I_N, \tag{4.22}$$
$$\widetilde{Z}_N^{\mathrm{L}}(y;\{s_i\};\{q_i\}): \quad g_{i<0}=0, \quad g_{i\geq 0}=s_i, \quad U=0, \quad V=\Omega_{\mathrm{L}}, \tag{4.23}$$
$$\widetilde{Z}_N^{\mathrm{L}}(y;s;\{q_i\}): \quad g_{i\neq -1}=0, \quad g_{-1}=-s, \quad U=\Omega_{\mathrm{L}}, \quad V=-I_N, \tag{4.24}$$
$$\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}): \quad g_{i<0}=0, \quad g_{i\geq 0}=s_i, \quad U=\Omega_{\mathrm{P}}, \quad V=\Omega_{\mathrm{L}}, \tag{4.25}$$

where I_N is the rank N identity matrix.

In Appendix A we prove the following proposition.

Proposition 4.7. *The matrix integral $\widetilde{Z}_N(y;\{g_i\};U;V)$ in (4.21) obeys the partial differential equation*

$$\left[\frac{\partial}{\partial y} - \frac{1}{4N}\mathrm{Tr}(V^{-1})^{\mathrm{T}}\frac{\partial}{\partial A}(V^{-1})^{\mathrm{T}}\frac{\partial}{\partial A}\right]\widetilde{Z}_N(y;\{g_i\};U;V) = 0, \tag{4.26}$$

where A is a matrix such that

$$U = A + A^{\mathrm{T}}.$$

From this proposition and by the specializations (4.22), (4.24), and (4.25) we find partial differential equations for the corresponding matrix integrals. For the specialization (4.23), because of $U = 0$ (and thus $A = 0$), the partial differential equation (4.26) cannot be reduced to a partial differential equation.

Corollary 4.8. *The matrix integral $\widetilde{Z}^{\mathrm{P}}_N(y;\{s_i\};\{r_i\})$ in (4.11), $\widetilde{Z}^{\mathrm{L}}_N(y;s;\{q_i\})$ in (4.18) and $\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\})$ in (4.19) obey partial differential equations*

$$\begin{aligned}
&\left[\frac{\partial}{\partial y} - \frac{1}{4N}\mathrm{Tr}\frac{\partial^2}{\partial \Lambda_{\mathrm{P}}^2}\right]\widetilde{Z}^{\mathrm{P}}_N(y;\{s_i\};\{r_i\}) = 0,\\
&\left[\frac{\partial}{\partial y} - \frac{1}{4N}\mathrm{Tr}\frac{\partial^2}{\partial \Lambda_{\mathrm{L}}^2}\right]\widetilde{Z}^{\mathrm{L}}_N(y;s;\{q_i\}) = 0,\\
&\left[\frac{\partial}{\partial y} - \frac{1}{4N}\mathrm{Tr}(\Omega_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}}(\Omega_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}}\right]\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = 0,
\end{aligned} \tag{4.27}$$

where Λ_{P} and Λ_{L} are matrices satisfying

$$\Omega_{\mathrm{P}} = \Lambda_{\mathrm{P}} + \Lambda_{\mathrm{P}}^{\mathrm{T}}, \qquad \Omega_{\mathrm{L}} = \Lambda_{\mathrm{L}} + \Lambda_{\mathrm{L}}^{\mathrm{T}}.$$

From this corollary we obtain non-oriented analogues of cut-and-join equations, by rewriting the derivatives with respect to the external matrices Λ_{P} and Λ_{L} in Corollary 4.8 in terms of Miwa times r_i in (4.12), q_i in (4.16), and $u_{\boldsymbol{i}}$ in (4.10) as follows.

Lemma 4.9. *Let L_1, K_1, M_1, and $M_2^{\vee}$ denote differential operators*

$$L_1 = \frac{1}{2}\sum_{i\geq 1} i(i+1)r_i\frac{\partial}{\partial r_{i+2}}, \tag{4.28}$$

$$K_1 = \frac{1}{2}\sum_{i\geq 3}(i-2)(i-1)q_i\frac{\partial}{\partial q_{i-2}}, \tag{4.29}$$

$$\begin{aligned}
M_1 = \frac{1}{2}\sum_{K\geq 1}\sum_{\{i_1,\dots,i_K\}}\sum_{1\leq I\neq M\leq K}\sum_{\ell=0}^{i_I-1}\sum_{m=0}^{i_M-1}\quad & u_{(m,i_{M-1},i_{M-2},\dots,i_{I+1},i_I-\ell-1,i_M-m-1,i_{M+1},\dots,i_{I-1},\ell)}\frac{\partial}{\partial u_{(i_1,\dots,i_K)}}\\
+\sum_{K\geq 1}\sum_{\{i_1,\dots,i_K\}}\sum_{L=1}^{K}\sum_{\ell+m\leq i_I-2}\quad & u_{(\ell,i_I-\ell-m-2,m,i_{I+1},\dots,i_{I-1})}\frac{\partial}{\partial u_{(i_1,\dots,i_K)}},
\end{aligned} \tag{4.30}$$

and

$$
\begin{aligned}
M_2^\vee = \frac{1}{2} \sum_{K,L\geq 1} \sum_{\{i_1,\dots,i_K\}} \sum_{\{j_1,\dots,j_L\}} \sum_{I=1}^{K} \sum_{J=1}^{L} \sum_{\ell=0}^{i_I-1} \sum_{m=0}^{j_J-1} \\
u_{(\ell,i_{I-1},\dots,i_{I+1},i_I-\ell-1,j_J-m-1,j_{J+1},\dots,j_{J-1},m)} \frac{\partial^2}{\partial u_{(i_1,\dots,i_K)} \partial u_{(j_1,\dots,j_L)}}.
\end{aligned} \tag{4.31}
$$

Then the derivatives with respect to Λ_{P} *and* Λ_{L} *in Corollary 4.8 are rewritten as*

$$
\begin{aligned}
&\frac{1}{4N}\mathrm{Tr}\frac{\partial^2}{\partial \Lambda_{\mathrm{P}}^2} f(\{r_i\}) = \left(L_0 + \frac{1}{N}L_1 + \frac{2}{N^2}L_2\right) f(\{r_i\}),\\
&\frac{1}{4N}\mathrm{Tr}\frac{\partial^2}{\partial \Lambda_{\mathrm{L}}^2} g(\{q_i\}) = \left(K_0 + \frac{1}{N}K_1 + \frac{2}{N^2}K_2\right) g(\{q_i\}),\\
&\frac{1}{4N}\mathrm{Tr}\left[(\Omega_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}}(\Omega_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}}\right] h(\{u_{\boldsymbol{i}}\})\\
&\qquad = \left(M_0 + \frac{1}{N}M_1 + \frac{1}{N^2}\big(M_2 + M_2^\vee\big)\right) h(\{u_{\boldsymbol{i}}\}),
\end{aligned} \tag{4.32}
$$

where $f(\{r_i\})$, $g(\{q_i\})$, *and* $h(\{u_{\boldsymbol{i}}\})$ *are functions of Miwa times* r_i, q_i, *and* $u_{\boldsymbol{i}}$, *respectively. Here* $L_{0,2}$, $K_{0,2}$, *and* $M_{0,2}$ *are defined in (3.7), (3.16), and (3.22), respectively.*

The proof of this lemma is given in Appendix B. By combining Corollary 4.8 with Lemma 4.9 one arrives at the following theorem.

Theorem 4.10. *The matrix integrals* $\widetilde{Z}_N^{\mathrm{P}}(y;\{s_i\};\{r_i\})$ *in (4.11),* $\widetilde{Z}_N^{\mathrm{L}}(y;s;\{q_i\})$ *in (4.18), and* $\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\})$ *in (4.19) obey the cut-and-join equations*

$$
\begin{aligned}
&\frac{\partial}{\partial y}\widetilde{Z}_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}) = \widetilde{\mathcal{L}}\widetilde{Z}_N^{\mathrm{P}}(y;\{s_i\};\{r_i\}),\\
&\frac{\partial}{\partial y}\widetilde{Z}_N^{\mathrm{L}}(y;s;\{q_i\}) = \widetilde{\mathcal{K}}\widetilde{Z}_N^{\mathrm{L}}(y;s;\{q_i\}),\\
&\frac{\partial}{\partial y}\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \widetilde{\mathcal{M}}\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}),
\end{aligned} \tag{4.33}
$$

where

$$
\begin{aligned}
\widetilde{\mathcal{L}} &= L_0 + \frac{1}{N}L_1 + \frac{2}{N^2}L_2,\\
\widetilde{\mathcal{K}} &= K_0 + \frac{1}{N}K_1 + \frac{2}{N^2}K_2,\\
\widetilde{\mathcal{M}} &= M_0 + \frac{1}{N}M_1 + \frac{1}{N^2}\big(M_2 + M_2^\vee\big).
\end{aligned}
$$

Assuming certain initial conditions at $y = 0$, one can iteratively determine the above matrix integrals by solving the cut-and-join equations

$$
\begin{aligned}
&\widetilde{Z}^{\mathrm{P}}_N(y;\{s_i\};\{r_i\}) = \mathrm{e}^{y\widetilde{\mathcal{L}}}\widetilde{Z}^{\mathrm{P}}_N(y=0;\{s_i\};\{r_i\}) = \mathrm{e}^{y\widetilde{\mathcal{L}}}\mathrm{e}^{N^2\sum_{i\geq 0} s_i r_i},\\
&\widetilde{Z}^{\mathrm{L}}_N(y;s;\{q_i\}) = \mathrm{e}^{y\widetilde{\mathcal{K}}}\widetilde{Z}^{\mathrm{L}}_N(y=0,s;\{q_i\}) = \mathrm{e}^{y\widetilde{\mathcal{K}}}\mathrm{e}^{N^2 s q_1},\\
&\widetilde{\mathcal{Z}}_N(y;\{s_i\};\{u_{\boldsymbol{i}}\}) = \mathrm{e}^{y\widetilde{\mathcal{M}}}\widetilde{\mathcal{Z}}_N(y=0;\{s_i\};\{u_{\boldsymbol{i}}\}) = \mathrm{e}^{y\widetilde{\mathcal{M}}}\mathrm{e}^{-N^2\sum_{i\geq 0} s_i u_{(i)}}.
\end{aligned}
\tag{4.34}
$$

The cut-and-join equations (4.33) agree with those of [2, 7]. Finally, from Theorem 4.10 we find non-oriented analogues of cut-and-join equations for 1-backbone generating functions.

Corollary 4.11. *The 1-backbone generating function $\widetilde{\mathcal{F}}_1(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\})$ obtained by picking up the $\mathcal{O}(s_i^1)$ term in $\widetilde{\mathcal{Z}}_N(y; \{s_i\}; \{u_{\boldsymbol{i}}\})$ is given by the following matrix integral*

$$
\begin{aligned}
&\widetilde{\mathcal{F}}_1(N^{-1}, y; \{s_i\}; \{u_{\boldsymbol{i}}\}) =\\
&= \frac{1}{\mathrm{Vol}_N(\mathbb{R})}\int_{\mathcal{H}_N(\mathbb{R})} dM\ \mathrm{e}^{-N\mathrm{Tr}\frac{M^2}{4}} N\sum_{i\geq 0} s_i \mathrm{Tr}(y^{1/2}\Omega_{\mathrm{L}}^{-1}M + \Omega_{\mathrm{P}})^i\Omega_{\mathrm{L}}^{-1},
\end{aligned}
\tag{4.35}
$$

and it obeys the cut-and-join equation

$$
\frac{\partial}{\partial y}\widetilde{\mathcal{F}}_1(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\}) = \widetilde{\mathcal{M}}\widetilde{\mathcal{F}}_1(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\}),
\tag{4.36}
$$

where $\widetilde{\mathcal{M}} = M_0 + xM_1 + x^2\big(M_2 + M_2^\vee\big)$. The solution is iteratively determined by

$$
\widetilde{\mathcal{F}}_1(x, y; \{s_i\}; \{u_{\boldsymbol{i}}\}) = \mathrm{e}^{y\widetilde{\mathcal{M}}}\widetilde{\mathcal{F}}_1(x, y=0; \{s_i\}; \{u_{\boldsymbol{i}}\}) = \mathrm{e}^{y\widetilde{\mathcal{M}}}\Bigg(x^{-2}\sum_{i\geq 0} s_i u_{(i)}\Bigg).
\tag{4.37}
$$

A Proof of Proposition 4.7

In this appendix we prove the Proposition 4.7, which states that the matrix integral

$$\widetilde{Z}_N(y;\{g_i\};U;V) = \\ = \frac{1}{\mathrm{Vol}_N(\mathbb{R})}\int_{\mathcal{H}_N(\mathbb{R})} dM\ \exp\left[-N\mathrm{Tr}\left(\frac{1}{4}M^2 - \sum_{i\in\mathbb{Z}} g_i(y^{1/2}V^{-1}M+U)^i V^{-1}\right)\right],$$

obeys the partial differential equation

$$\left[\frac{\partial}{\partial y} - \frac{1}{4N}\mathrm{Tr}(V^{-1})^{\mathrm{T}}\frac{\partial}{\partial A}(V^{-1})^{\mathrm{T}}\frac{\partial}{\partial A}\right]\widetilde{Z}_N(y;\{g_i\};U;V) = 0, \tag{A.1}$$

where A is a matrix that satisfies $U = A + A^{\mathrm{T}}$.

Proof. In order to differentiate the matrix integral $\widetilde{Z}_N(y;\{g_i\};U;V)$ with respect to A we use the identities

$$\frac{\partial U_{\alpha\beta}}{\partial A_{\gamma\epsilon}} = \delta_{\alpha\gamma}\delta_{\beta\epsilon} + \delta_{\alpha\epsilon}\delta_{\beta\gamma},$$

$$\frac{\partial (y^{1/2}V^{-1}X)^{-1}_{\alpha\beta}}{\partial A_{\gamma\epsilon}} = -(y^{1/2}V^{-1}X)^{-1}_{\alpha\gamma}(y^{1/2}V^{-1}X)^{-1}_{\beta\epsilon} - (y^{1/2}V^{-1}X)^{-1}_{\alpha\epsilon}(y^{1/2}V^{-1}X)^{-1}_{\beta\gamma},$$

where $X = M + y^{-1/2}VU$. Using this shifted variable X one obtains

$$\frac{1}{2N}\frac{\partial}{\partial A_{\alpha\beta}}\widetilde{Z}_N(y;\{g_i\};U;V) = \Bigg\langle \sum_{i=0}^{\infty} y^{(i-1)/2}g_i\sum_{j=0}^{i-1}\left((V^{-1}X)^jV^{-1}(V^{-1}X)^{i-j-1}\right)_{\alpha\beta} \\ - \sum_{i=1}^{\infty} y^{-(i+1)/2}g_{-i}\sum_{j=0}^{i-1}\left((V^{-1}X)^{-j-1}V^{-1}(V^{-1}X)^{-i+j}\right)_{\alpha\beta}\Bigg\rangle_{\mathbb{R}},$$

where $\langle\cdots\rangle_{\mathbb{R}}$ denotes the unnormalized average

$$\langle\mathcal{O}(X)\rangle_{\mathbb{R}} = \int_{\widetilde{\mathcal{H}}_N(\mathbb{R})} dX\ \mathcal{O}(X)\exp\left[-N\mathrm{Tr}\left(\frac{1}{4}(X-y^{-1/2}VU)^2 - \sum_{i\in\mathbb{Z}} y^{i/2}g_i(V^{-1}X)^iV^{-1}\right)\right].$$

Here $\widetilde{\mathcal{H}}_N(\mathbb{R})$ is the space of shifted matrices $X = M + y^{-1/2}VU$ with $M\in\mathcal{H}_N(\mathbb{R})$. It follows that

$$\frac{1}{2N^2}\mathrm{Tr}(V^{-1})^{\mathrm{T}}\frac{\partial}{\partial A}(V^{-1})^{\mathrm{T}}\frac{\partial}{\partial A}\widetilde{Z}_N(y;\{g_i\};U;V) = \\ = \Bigg\langle \sum_{i=0}^{\infty} y^{-1/2}g_i\sum_{j=0}^{i-1}\mathrm{Tr}(X - y^{-1/2}VU)V^{-1}(y^{1/2}V^{-1}X)^jV^{-1}(y^{1/2}V^{-1}X)^{i-j-1}$$

(A.2)

$$-\sum_{i=1}^{\infty} y^{-1/2} g_{-i} \sum_{j=0}^{i-1} \mathrm{Tr}(X - y^{-1/2}VU)V^{-1}(y^{1/2}V^{-1}X)^{-j-1}V^{-1}(y^{1/2}V^{-1}X)^{-i+j} \Big\rangle_{\mathbb{R}}.$$

On the other hand, by differentiating the matrix integral $\widetilde{Z}_N(y;\{g_i\};U;V)$ with respect to y one obtains the same expression as (A.2) times $N/2$, from which the partial differential equation (A.1) is obtained. □

B Proof of Lemma 4.9

In this appendix we prove the Lemma 4.9, which states that for functions $f(\{r_i\})$, $g(\{q_i\})$, and $h(\{u_{\boldsymbol{i}}\})$ of Miwa times r_i in (4.12), q_i in (4.16), and $u_{\boldsymbol{i}}$ in (4.10), we find

$$\frac{1}{4N}\mathrm{Tr}\frac{\partial^2}{\partial\Lambda_{\mathrm{P}}^2} f(\{r_i\}) = \left(L_0 + \frac{1}{N}L_1 + \frac{2}{N^2}L_2\right) f(\{r_i\}), \tag{B.1}$$

$$\frac{1}{4N}\mathrm{Tr}\frac{\partial^2}{\partial\Lambda_{\mathrm{L}}^2} g(\{q_i\}) = \left(K_0 + \frac{1}{N}K_1 + \frac{2}{N^2}K_2\right) g(\{q_i\}), \tag{B.2}$$

$$\begin{aligned} &\frac{1}{4N}\mathrm{Tr}\left[(\Omega_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}(\Omega_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}\right] h(\{u_{\boldsymbol{i}}\}) \\ &\qquad = \left(M_0 + \frac{1}{N}M_1 + \frac{1}{N^2}\big(M_2 + M_2^{\vee}\big)\right) h(\{u_{\boldsymbol{i}}\}), \end{aligned} \tag{B.3}$$

where $L_{0,1,2}$, $K_{0,1,2}$, $M_{0,1,2}$, and $M_2^{\vee}$ are defined in (3.7), (3.16), (3.22), and in Lemma 4.9. Here the matrices Λ_{P} and Λ_{L} satisfy $\Omega_{\mathrm{P}} = \Lambda_{\mathrm{P}} + \Lambda_{\mathrm{P}}^{\mathrm{T}}$ and $\Omega_{\mathrm{L}} = \Lambda_{\mathrm{L}} + \Lambda_{\mathrm{L}}^{\mathrm{T}}$.

Proof. First we prove (B.1). Consider the derivative $\partial/\partial\Lambda_{\mathrm{P}}$ of the reverse Miwa time r_i

$$\frac{\partial r_i}{\partial\Lambda_{\mathrm{P}\beta\alpha}} = \frac{2i}{N}\Omega_{\mathrm{P}\alpha\beta}^{i-1}, \qquad \mathrm{Tr}\frac{\partial^2 r_i}{\partial\Lambda_{\mathrm{P}}^2} = 2Ni\sum_{j=1}^{i-1} r_{j-1}r_{i-j-1} + 2i(i-1)r_{i-2}.$$

Using these relations the left hand side of (B.1) is rewritten as

$$\begin{aligned} \frac{1}{4N}\mathrm{Tr}\frac{\partial^2}{\partial\Lambda_{\mathrm{P}}^2} f(\{r_i\}) &= \frac{1}{4N}\sum_{i\geq 1}\mathrm{Tr}\frac{\partial^2 r_i}{\partial\Lambda_{\mathrm{P}}^2}\frac{\partial f(\{r_i\})}{\partial r_i} + \frac{1}{4N}\sum_{i,j\geq 1}\mathrm{Tr}\frac{\partial r_i}{\partial\Lambda_{\mathrm{P}}}\frac{\partial r_j}{\partial\Lambda_{\mathrm{P}}}\frac{\partial^2 f(\{r_i\})}{\partial r_i\partial r_j} \\ &= \frac{1}{2}\sum_{i\geq 2}\sum_{j=1}^{i-1} i r_{j-1}r_{i-j-1}\frac{\partial f(\{r_i\})}{\partial r_i} + \frac{1}{2N}\sum_{i\geq 2} i(i-1)r_{i-2}\frac{\partial f(\{r_i\})}{\partial r_i} \\ &\quad + \frac{1}{N^2}\sum_{i,j\geq 1} ij r_{i+j-2}\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j} f(\{r_i\}). \end{aligned}$$

This agrees with the right hand side of (B.1).

Second, we prove (B.2). Using the identity

$$\frac{\partial \Omega^{-1}_{\mathrm{L}\gamma\epsilon}}{\partial \Lambda_{\mathrm{L}\alpha\beta}} = -\Omega^{-1}_{\mathrm{L}\beta\epsilon}\Omega^{-1}_{\mathrm{L}\gamma\alpha} - \Omega^{-1}_{\mathrm{L}\alpha\epsilon}\Omega^{-1}_{\mathrm{L}\gamma\beta},$$

one finds that

$$\frac{\partial q_i}{\partial \Lambda_{\mathrm{L}\alpha\beta}} = -2\frac{i}{N}\Omega^{-i-1}_{\mathrm{L}\beta\alpha}, \qquad \mathrm{Tr}\frac{\partial^2}{\partial \Lambda^2_{\mathrm{L}}} q_i = 2i(i+1)q_{i+2} + 2iN\sum_{j=0}^{i} q_{i+1}q_{j+1}.$$

Then the left hand side of (B.2) yields

$$\begin{aligned}\frac{1}{4N}\mathrm{Tr}\frac{\partial^2}{\partial \Lambda^2_{\mathrm{L}}} g(\{q_i\}) &= \frac{1}{2}\sum_{i\geq 1}\sum_{j=0}^{i} i q_{i+1}q_{j+1}\frac{\partial g(\{q_i\})}{\partial q_i} + \frac{1}{2N}\sum_{i\geq 1} i(i+1)q_{i+2}\frac{\partial g(\{q_i\})}{\partial q_i}\\ &\quad + \frac{1}{N^2}\sum_{i,j\geq 1} ij q_{i+j+2}\frac{\partial^2 g(\{q_i\})}{\partial q_i \partial q_j}.\end{aligned}$$

This agrees with the right hand side of (B.2).

Finally we prove (B.3). Using the chain rule, the derivative action on the left hand side of (B.3) is written as

$$\begin{aligned}&\mathrm{Tr}\left[(\Omega^{-1}_{\mathrm{L}})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}}(\Omega^{-1}_{\mathrm{L}})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}}\right] h(\{u_{\boldsymbol{i}}\}) =\\ &= \sum_{K\geq 1}\sum_{\{i_1,\dots,i_K\}} \mathrm{Tr}\left[(\Omega^{-1}_{\mathrm{L}})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}}(\Omega^{-1}_{\mathrm{L}})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}} u_{(i_1,\dots,i_K)}\right]\frac{\partial}{\partial u_{(i_1,\dots,i_K)}} h(\{u_{\boldsymbol{i}}\})\\ &\quad + \sum_{K,L\geq 1}\sum_{\{i_1,\dots,i_K\}}\sum_{\{j_1,\dots,j_L\}} \mathrm{Tr}\left[(\Omega^{-1}_{\mathrm{L}})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}} u_{(i_1,\dots,i_K)}(\Omega^{-1}_{\mathrm{L}})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}} u_{(j_1,\dots,j_L)}\right]\\ &\qquad\qquad \times \frac{\partial^2}{\partial u_{(i_1,\dots,i_K)}\partial u_{(j_1,\dots,j_L)}} h(\{u_{\boldsymbol{i}}\}).\end{aligned}$$

Each of the coefficients yields

$$\begin{aligned}&\mathrm{Tr}\left[(\Omega^{-1}_{\mathrm{L}})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}}(\Omega^{-1}_{\mathrm{L}})^{\mathrm{T}}\frac{\partial}{\partial \Lambda_{\mathrm{P}}} u_{(i_1,\dots,i_K)}\right] =\\ &= 2\sum_{1\leq I\neq M\leq K}^{K}\sum_{\ell=0}^{i_I-1}\sum_{m=0}^{i_M-1}\frac{1}{N}\Big(\mathrm{Tr}(\Omega_{\mathrm{P}}^{i_I-\ell-1}\Omega^{-1}_{\mathrm{L}}\Omega_{\mathrm{P}}^{i_{I+1}}\Omega^{-1}_{\mathrm{L}}\cdots\Omega_{\mathrm{P}}^{i_{M-1}}\Omega^{-1}_{\mathrm{L}}\Omega_{\mathrm{P}}^{m}\Omega^{-1}_{\mathrm{L}})\\ &\qquad\qquad \times \mathrm{Tr}(\Omega_{\mathrm{P}}^{i_M-m-1}\Omega^{-1}_{\mathrm{L}}\Omega_{\mathrm{P}}^{i_{M+1}}\Omega^{-1}_{\mathrm{L}}\cdots\Omega_{\mathrm{P}}^{i_{I-1}}\Omega^{-1}_{\mathrm{L}}\Omega_{\mathrm{P}}^{\ell}\Omega^{-1}_{\mathrm{L}})\\ &\qquad + \mathrm{Tr}(\Omega_{\mathrm{P}}^{m}\Omega^{-1}_{\mathrm{L}}\Omega_{\mathrm{P}}^{i_{M-1}}\Omega^{-1}_{\mathrm{L}}\Omega_{\mathrm{P}}^{i_{M-2}}\Omega^{-1}_{\mathrm{L}}\cdots\Omega_{\mathrm{P}}^{i_{I+1}}\Omega^{-1}_{\mathrm{L}}\Omega_{\mathrm{P}}^{i_I-\ell-1}\Omega^{-1}_{\mathrm{L}}\\ &\qquad\qquad \cdot \Omega_{\mathrm{P}}^{i_M-m-1}\Omega^{-1}_{\mathrm{L}}\Omega_{\mathrm{P}}^{i_{M+1}}\Omega^{-1}_{\mathrm{L}}\Omega_{\mathrm{P}}^{i_{M+2}}\Omega^{-1}_{\mathrm{L}}\cdots\Omega_{\mathrm{P}}^{i_{I-1}}\Omega^{-1}_{\mathrm{L}}\Omega_{\mathrm{P}}^{\ell}\Omega^{-1}_{\mathrm{L}})\Big)\end{aligned}$$

$$
\begin{aligned}
&+4\sum_{I=1}^{K}\sum_{\ell+m\le i_I-2}\Big(\frac{1}{N}\mathrm{Tr}(\Omega_{\mathrm{P}}^{\ell}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{m}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{i_{I+1}}\Omega_{\mathrm{L}}^{-1}\cdots\Omega_{\mathrm{P}}^{i_{I-1}}\Omega_{\mathrm{L}}^{-1})\mathrm{Tr}(\Omega_{\mathrm{P}}^{i_I-\ell-m-2}\Omega_{\mathrm{L}}^{-1})\\
&\qquad\qquad+\mathrm{Tr}(\Omega_{\mathrm{P}}^{\ell}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{m}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{i_I-\ell-m-2}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{i_{I+1}}\Omega_{\mathrm{L}}^{-1}\cdots\Omega_{\mathrm{P}}^{i_{I-1}}\Omega_{\mathrm{L}}^{-1})\Big)\\
&=2\sum_{1\le I\ne M\le K}\sum_{\ell=0}^{i_I-1}\sum_{m=0}^{i_M-1}\Big(Nu_{(i_I-\ell-1,i_{I+1},\dots,i_{M-1},m)}u_{(i_M-m-1,i_{M+1},\dots,i_{I-1},\ell)}\\
&\qquad\qquad+u_{(m,i_{M-1},i_{M-2}\dots,i_{I+1},i_I-\ell-1,i_M-m-1,i_{M+1},\dots,i_{I-1},\ell)}\Big)\\
&+4\sum_{I=0}^{K}\sum_{\ell+m\le i_I-2}\Big(Nu_{(\ell,m,i_{I+1},\dots,i_{I-1})}u_{(i_I-\ell-m-2)}+u_{(\ell,i_I-\ell-m-2,m,i_{I+1},\dots,i_{I-1})}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Tr}\left[(\Omega_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}u_{(i_1,\dots,i_K)}(\Omega_{\mathrm{L}}^{-1})^{\mathrm{T}}\frac{\partial}{\partial\Lambda_{\mathrm{P}}}u_{(j_1,\dots,j_L)}\right]=\\
&=\sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{i_I-1}\sum_{m=0}^{j_J-1}\frac{2}{N^2}\Big(\mathrm{Tr}(\Omega_{\mathrm{P}}^{i_I-\ell-1}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{i_{I+1}}\Omega_{\mathrm{L}}^{-1}\cdots\Omega^{i_{I-1}}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{\ell}\Omega_{\mathrm{L}}^{-1}\\
&\qquad\cdot\Omega_{\mathrm{P}}^{j_J-m-1}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{j_{J+1}}\Omega_{\mathrm{L}}^{-1}\cdots\Omega_{\mathrm{P}}^{j_{J-1}}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{m}\Omega_{\mathrm{L}}^{-1})\\
&\qquad+\mathrm{Tr}(\Omega_{\mathrm{P}}^{\ell}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{i_{I-1}}\Omega_{\mathrm{L}}^{-1}\cdots\Omega^{i_{I+1}}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{i_I-\ell-1}\Omega_{\mathrm{L}}^{-1}\\
&\qquad\cdot\Omega_{\mathrm{P}}^{j_J-m-1}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{j_{J+1}}\Omega_{\mathrm{L}}^{-1}\cdots\Omega_{\mathrm{P}}^{j_{J-1}}\Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{P}}^{m}\Omega_{\mathrm{L}}^{-1})\Big)\\
&=\frac{2}{N}\sum_{I=1}^{K}\sum_{J=1}^{L}\sum_{\ell=0}^{i_I-1}\sum_{m=0}^{j_J-1}\Big(u_{(i_I-\ell-1,i_{I+1},\dots,i_{I-1},\ell,j_J-m-1,j_{J+1},\dots,j_{J-1},m)}\\
&\qquad+u_{(\ell,i_{I-1},\dots,i_{I+1},i_I-\ell-1,j_J-m-1,j_{J+1},\dots,j_{J-1},m)}\Big).
\end{aligned}
$$

Then one obtains the right hand side of (B.3). □

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Jørgen Ellegaard Andersen
Centre for Quantum Geometry of Moduli Spaces,
Department of Mathematics, Aarhus University,
DK-8000 Aarhus C Denmark
jea.qgm@gmail.com

Hiroyuki Fuji
Faculty of Education, Kagawa University,
Takamatsu 760-8522 Japan
fuji@ed.kagawa-u.ac.jp

Masahide Manabe
Faculty of Physics, University of Warsaw,
ul. Pasteura 5, 02-093 Warsaw, Poland
masahidemanabe@gmail.com

Robert C. Penner
Institut des Hautes Études Scientifiques,
35 route de Chartres, 91440 Burs-sur-Yvette, France;
Division of Physics, Mathematics and Astronomy,
California Institute of Technology, Pasadena, CA 91125, USA
rpenner@caltech.edu, rpenner@ihes.fr

Piotr Sułkowski
Faculty of Physics, University of Warsaw,
ul. Pasteura 5, 02-093 Warsaw, Poland;
Walter Burke Institute for Theoretical Physics,
California Institute of Technology, Pasadena, CA 91125, USA
psulkows@fuw.edu.pl

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Enumeration of Chord Diagrams via Topological Recursion and Quantum Curve Techniques

by Jørgen Ellegaard Andersen, Hiroyuki Fuji, Masahide Manabe, Robert C. Penner, and Piotr Sułkowski[1]

Abstract

In this paper we consider the enumeration of orientable and non-orientable chord diagrams. We show that this enumeration is encoded in appropriate expectation values of the β-deformed Gaussian and RNA matrix models. We evaluate these expectation values by means of the β-deformed topological recursion, and – independently – using properties of quantum curves. We show that both these methods provide efficient and systematic algorithms for counting of chord diagrams with a given genus, number of backbones and number of chords.

1 Introduction

A *chord diagram* is a graph which can be realised in the plane as follows. It is comprised of a collection of b line segments (called *backbones*) on the real axis with k semi-circles (called *chords*) in the upper-half plane attached to the line segments. All chords are attached at different points on the backbones. A chord diagram comes from its realisation in the plane with a natural *fatgraph* structure, namely, half edges incident to each trivalent vertex are endowed with a cyclic order induced from the orientation of the plane. For a chord diagram c the fatgraph structure allows us, in the usual way, to define a surface Σ_c, which is simply just a

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small tubular neighbourhood of the realisation of the chord diagram in the plane, see left and middle diagrams in Figure 1. Let n be the number of boundary cycles, and g be the genus of the skinny surface Σ_c. Then the Euler characteristic χ of Σ_c is given by

$$\chi = 2 - 2g = b - k + n.$$

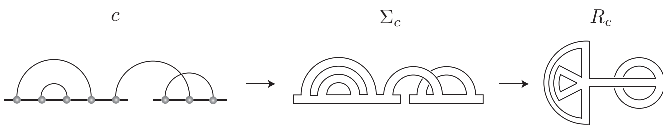

Figure 1: A chord diagram c, its skinny surface Σ_c, and the associated ribbon surface R_c.

To present a chord diagram c more simply and efficiently, we collapse each fattened backbone in Σ_c into a polyvalent fattened vertex, see the right side of Figure 1. We call the resulting surface R_c and we refer to it as the *Ribbon surface* associated to c. In order for c to be uniquely determined by the ribbon surface R_c, at each vertex we attach a tail at the place corresponding to the beginning of the backbone, see Figure 2.

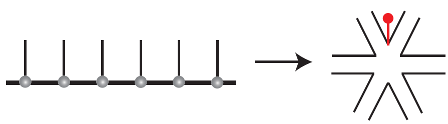

Figure 2: Polyvalent fatten vertex with a tail

In this paper we consider the enumeration of chord diagrams with the topological filtration induced by the genus and the number of backbones, employing matrix model techniques. Chord diagrams are widely used objects in pure and applied mathematics, see e.g. [19, 64, 14, 15, 8, 29]. Furthermore, they are now used also in the biological context for characterisation of secondary and pseudoknot structures of RNA molecules [77, 76, 75, 86, 85, 17, 80].[2] In particular, motivated by the study of RNA pseudoknot structures, a matrix model for the enumeration of chord diagrams – which we refer to as the RNA-matrix model – was constructed in [9, 10]. In this paper we study the β-deformed version of this model and present how it encodes orientable and non-orientable chord diagrams.

[2]The combinatorial aspects of interacting RNA molecules with the associated genus filtration are also discussed in [23, 78, 79, 87, 16, 17, 18, 4, 11, 12], and folding algorithms are studied in [22, 13, 81]. The matrix model approach to the enumeration of possible RNA structures is also studied in [20, 21, 47, 48, 49, 50, 51, 52, 53].

1.1 The RNA matrix model for orientable chord diagrams

Let $c_{g,b}(k)$ denote the number of connected chord diagrams with genus g, b backbones, and k chords.[3] We consider the following generating function

$$C_{g,b}(w) = \sum_{k=0}^{\infty} c_{g,b}(k) w^k. \tag{1.1}$$

In [17], the number $c_{g,1}(k)$ of chord diagrams with 1 backbone was studied. In particular for the class of planar graphs which have genus $g = 0$, the number $c_{0,1}(k)$ is shown to be equal to the Catalan number

$$C_k = \frac{(2k)!}{(k+1)!k!}.$$

We present explicitly the tailed ribbon surfaces with 1 backbone for $k = 1, 2, 3, 4$ in Figure 3.

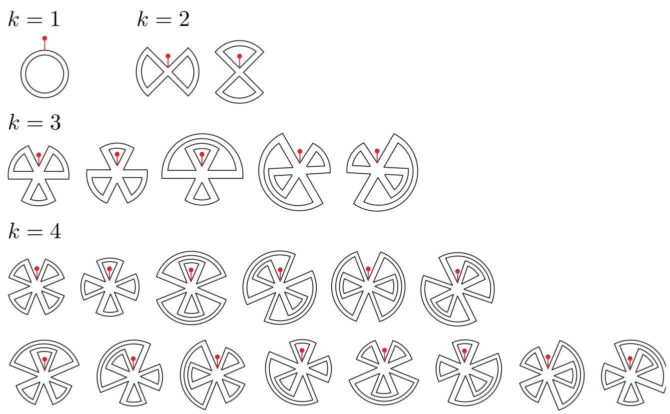

Figure 3: The planar ribbon surfaces with tails for $k = 1, 2, 3, 4$, whose counts agree with the Catalan numbers 1, 2, 5, 14.

In [9] the following theorem was established.

Theorem 1.1 (RNA matrix model for orientable chord diagrams [9]). *Let $\mathcal{H}_N$ be the space of rank N Hermitian matrices. We consider the matrix integral with*

[3]Harer and Zagier found a remarkable formula for $c_{g,1}(k)$, referred to as the *Harer-Zagier formula*, in their computation of the virtual Euler characteristic of Riemann moduli space for punctured surfaces [58].

the potential

$$V_{\mathrm{RNA}}(x) = \frac{x^2}{2} - \frac{stx}{1-tx}, \tag{1.2}$$

defined by

$$Z_N(s,t) = \frac{1}{\mathrm{Vol}_N}\int_{\mathcal{H}_N} dM\ \mathrm{e}^{-\frac{1}{\hbar}\mathrm{Tr}V_{\mathrm{RNA}}(M)}, \tag{1.3}$$

where

$$\mathrm{Vol}_N = \int_{\mathcal{H}_N} dM\ \mathrm{e}^{-N\mathrm{Tr}\frac{M^2}{2}} = N^{N(N+1)/2}\mathrm{Vol}(\mathcal{H}_N). \tag{1.4}$$

Under the 't Hooft limit

$$\hbar \to 0, \qquad N \to \infty, \qquad \mu = \hbar N, \tag{1.5}$$

with the 't Hooft parameter μ kept finite, the logarithm of the above matrix integral (called the free energy) $F_N(s,t) = \log Z_N(s,t)$ has an asymptotic expansion

$$F_N(s,t) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(s,t). \tag{1.6}$$

Moreover, this free energy encodes generating functions (1.1) for the numbers $c_{g,b}(k)$ of chord diagrams

$$\mu^{2g-2}F_g(s,t) = \sum_{b=1}^{\infty} \frac{s^b}{\mu^b b!} C_{g,b}(\mu t^2) - \frac{s}{\mu}\delta_{g,0}. \tag{1.7}$$

By (1.7) we see that $F_g(s,t)$ enumerates orientable chord diagrams with genus g, and s and t^2 are generating parameters respectively for the number of backbones and chords.

1.2 RNA matrix model for non-oriented chord diagrams

As a natural generalization of the above enumerative problem, we consider the non-oriented[4] analogue of the enumeration of chord diagrams. A non-oriented chord diagram c is a chord diagram, where a binary quantity, twisted or untwisted, is assigned to each chord. There is a natural non-oriented fatgraph structure associated with a non-oriented chord diagram and the corresponding non-oriented surface Σ_c, where the binary quantity assigned to each chord determines if the band along the chord for the associated ribbon surface is twisted or not, as depicted in Figure 4.

Figure 4: A band and a twisted band.

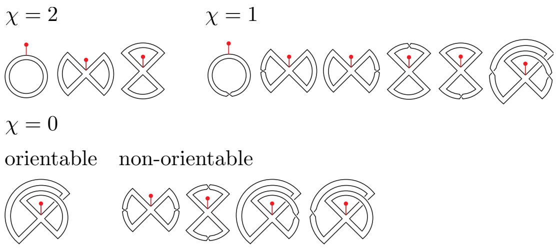

Figure 5: Non-oriented ribbon surfaces with 1 backbone for $k = 1, 2$.

The non-oriented analogue of the ribbon surface is again constructed by collapsing the fattened backbones into polyvalent fattened vertices and, as before, adding a tail for each fattened vertex. Some examples of non-oriented ribbon surfaces are depicted in Figure 5 and in Appendix A.

Instead of the genus, for a surface Σ_c the *cross-cap number* (or the *non-oriented genus*) h is well-defined, and the Euler characteristic is given by

$$\chi = 2 - h = b - k + n.$$

Let $c^{\mathsf{r}}_{h,b}(k)$ denote the number of non-oriented chord diagrams with the cross-cap number h, b backbones, and k chords. In analogy to the oriented case, we introduce the generating function $C^{\mathsf{r}}_{h,b}(w)$

$$C^{\mathsf{r}}_{h,b}(w) = \sum_{k=0}^{\infty} c^{\mathsf{r}}_{h,b}(k) w^k. \tag{1.8}$$

As proven in [4], the non-oriented analogue of Theorem 1.1 is straightforwardly obtained by replacing the integration over Hermitian matrices in (1.3) with the integration over real symmetric matrices.

Theorem 1.2 (RNA matrix model for non-oriented chord diagrams)**.** *Let $\mathcal{H}_N(\mathbb{R})$ be the space of rank N real symmetric matrices. We consider the real symmetric matrix integral with the potential (1.2) defined by*

$$Z^{\mathsf{r}}_N(s,t) = \frac{1}{\mathrm{Vol}_N(\mathbb{R})} \int_{\mathcal{H}_N(\mathbb{R})} dM \; \mathrm{e}^{-\frac{1}{2\hbar}\mathrm{Tr} V_{\mathrm{RNA}}(M)}, \tag{1.9}$$

[4]Non-oriented is a shorthand for the union of orientable and non-orientable.

where

$$\mathrm{Vol}_N(\mathbb{R}) = \int_{\mathcal{H}_N(\mathbb{R})} dM\ \mathrm{e}^{-N\mathrm{Tr}\frac{M^2}{4}} = N^{N(N+1)/2}\mathrm{Vol}(\mathcal{H}_N(\mathbb{R})). \tag{1.10}$$

Under the 't Hooft limit

$$\hbar \to 0, \qquad N \to \infty, \qquad \mu = \hbar N, \tag{1.11}$$

with the 't Hooft parameter μ kept finite and fixed, the free energy $F^{\mathsf{r}}_N(s,t) = \log Z^{\mathsf{r}}_N(s,t)$ has an asymptotic expansion

$$F^{\mathsf{r}}_N(s,t) = \frac{1}{2}\sum_{h=0}^{\infty} \hbar^{h-2} F^{\mathsf{r}}_h(s,t). \tag{1.12}$$

This free energy encodes generating functions (1.8) for the numbers $c^{\mathsf{r}}_{h,b}(k)$ of non-oriented chord diagrams

$$\mu^{h-2}F^{\mathsf{r}}_h(s,t) = \sum_{b=1}^{\infty} \frac{s^b}{\mu^b b!} C^{\mathsf{r}}_{h,b}(\mu t^2) - \frac{s}{\mu}\delta_{g,0}. \tag{1.13}$$

Note that matrix model techniques for the enumeration of non-oriented chord diagrams are also considered in [55, 84, 83, 60, 73, 54].

1.3 β-deformed RNA matrix model as a unified model

In the context of matrix models, it is known that their β-deformation implements the enumeration of non-oriented chord diagrams [70], as we shall now recall[5].

Definition 1.3 (β-deformed RNA matrix model)**.** The β-deformed eigenvalue integral with the potential (1.2) is defined by

$$Z^{\beta}_N(s,t) = \frac{1}{\mathrm{Vol}^{\beta}_N} \int_{\mathbb{R}^N} \prod_{a=1}^{N} dz_a \Delta(z)^{2\beta} \mathrm{e}^{-\frac{\sqrt{\beta}}{\hbar}\sum_{a=1}^{N} V_{\mathrm{RNA}}(z_a)}, \tag{1.14}$$

where $\Delta(z)$ denotes the *Vandermonde determinant*

$$\Delta(z) = \prod_{a<b}(z_a - z_b),$$

and

$$\mathrm{Vol}^{\beta}_N = \int_{\mathbb{R}^N} \prod_{a=1}^{N} dz_a \Delta(z)^{2\beta} \mathrm{e}^{-\frac{\sqrt{\beta}}{2\hbar}\sum_{a=1}^{N} z_a^2}. \tag{1.15}$$

[5]The β-deformed Dyson's model is solved in various ways. See e.g. [61, 62, 63].

In the cases of $\beta = 1$ and $\beta = 1/2$, the β-deformed eigenvalue integral (1.14) reduces to the eigenvalue representation of the Hermitian matrix integral (1.3) and the real symmetric matrix integral (1.9) upon the redefinition $\hbar \to \sqrt{2}\hbar$, respectively. Here $z_a \in \mathbb{R}$ $(a = 1, \cdots, N)$ correspond to the eigenvalues of the matrix M in each matrix integral. The other special case is $\beta = 2$, for which the eigenvalue integral (1.14) represents the quaternionic matrix integral. Under the 't Hooft limit

$$\hbar \to 0, \qquad N \to \infty, \qquad \mu = \beta^{1/2}\hbar N, \tag{1.16}$$

with the fixed 't Hooft parameter μ, the free energy $F_N^\beta(s,t) = \log Z_N^\beta(s,t)$ has the asymptotic expansion

$$F_N^\beta(s,t) = \sum_{g,\ell=0}^{\infty} \hbar^{2g-2+\ell}\gamma^\ell F_{g,\ell}(s,t), \tag{1.17}$$

where

$$\gamma = \beta^{1/2} - \beta^{-1/2}.$$

The free energies (1.6) for $\beta = 1$ and (1.12) for $\beta = 1/2$ satisfy

$$\begin{aligned} F_g(s,t) &= F_{g,0}(s,t), \\ F_h^{\mathsf{r}}(s,t) &= \sum_{\substack{g,\ell=0 \\ 2g+\ell=h}}^{\infty} 2^g(-1)^\ell F_{g,\ell}(s,t). \end{aligned} \tag{1.18}$$

Combining (1.7) in Theorem 1.1 for $\beta = 1$ and (1.13) in Theorem 1.2 for $\beta = 1/2$, we obtain the following proposition.

Proposition 1.4. *Let $\widetilde{C}_{g,\ell,b}(w)$ be defined by*

$$(-\mu)^{2g-2+\ell}F_{g,\ell}(s,t) = \sum_{b=1}^{\infty} \frac{s^b}{\mu^b b!}\widetilde{C}_{g,\ell,b}(\mu t^2) - \frac{s}{\mu}\delta_{g,0}\delta_{\ell,0}. \tag{1.19}$$

Then we have the following relations

$$\begin{aligned} C_{g,b}(w) &= \widetilde{C}_{g,0,b}(w), \\ C_{h,b}^{\mathsf{r}}(w) &= \sum_{\substack{g,\ell=0 \\ 2g+\ell=h}}^{\infty} 2^g\widetilde{C}_{g,\ell,b}(w). \end{aligned} \tag{1.20}$$

In the following sections, using β-deformed topological recursion and the theory of quantum curves, we will develop the computational techniques of determining the functions $F_{g,\ell}$.

1.4 β-deformed topological recursion and Gaussian resolvents

In our first approach, we will employ an analytical method, referred to as the *β-deformed topological recursion* [33, 28, 24, 68, 67]. β-deformed topological recursion is a powerful machinery for analyzing the asymptotic expansion of a matrix model free energy in the $N \to \infty$ limit. Let $Z_N^\beta(\{r_n\})$ and $F_N^\beta(\{r_n\}) = \log Z_N^\beta(\{r_n\})$ denote the partition function and the free energy of the β-deformed matrix model with a general potential

$$V(x) = \sum_{n=0}^{K} r_n x^n. \tag{1.21}$$

By the asymptotic expansion of the free energy $F_N^\beta(\{r_n\})$ we mean the following expansion in the 't Hooft limit (1.16)

$$F_N^\beta(\{r_n\}) = \sum_{g,\ell=0}^{\infty} \hbar^{2g-2+\ell}\gamma^\ell F_{g,\ell}(\{r_n\}). \tag{1.22}$$

The β-deformed topological recursion in fact can be formulated more generally, as a tool that assigns a series of multi-linear differentials to a given algebraic curve. In the context of matrix models these multi-linear differentials are identified with various matrix model correlators (and the topological recursion represents Ward identities between these correlators), while the underlying algebraic curve is identified with a *spectral curve* $\mathcal{C}$ (of the matrix model with $\beta = 1$)

$$\begin{aligned} &\mathcal{C} = \left\{ \ (x,\omega) \in \mathbb{C}^2 \ \middle| \ H(x,\omega) = \omega^2 - 2V'(x)\omega - f(x) = 0 \ \right\}, \\ &f(x) = \lim_{\substack{N\to\infty,\hbar\to 0\\ \beta\to 1}} -4\beta^{1/2}\hbar \left\langle \sum_{a=1}^{N} \frac{V'(x)-V'(z_a)}{x-z_a} \right\rangle_N^\beta, \end{aligned} \tag{1.23}$$

where $\langle \mathcal{O}(\{z_a\})\rangle_N^\beta$ denotes the β-deformed eigenvalue integral

$$\langle \mathcal{O}(\{z_a\})\rangle_N^\beta = \frac{1}{Z_N^\beta(\{r_n\})\mathrm{Vol}_N^\beta} \int_{\mathbb{R}^N} \prod_{a=1}^{N} dz_a \Delta(z)^{2\beta} \mathcal{O}(\{z_a\}) \mathrm{e}^{-\frac{\sqrt{\beta}}{\hbar}\sum_{a=1}^{N} V(z_a)}. \tag{1.24}$$

In particular for the RNA matrix model presented in Definition 1.3, the spectral curve takes the following form [9].

Example 1.5. For the potential $V_{\mathrm{RNA}}(x)$ given in (1.2), the defining equation for $\mathcal{C}$ in (1.23) takes form

$$y_{\mathrm{RNA}}^2 = M_{\mathrm{RNA}}(x)^2(x-a)(x-b), \quad y_{\mathrm{RNA}} = V'_{\mathrm{RNA}}(x) - \omega, \tag{1.25}$$

where the branch points $a < b$ are solutions of the equations

$$S = \frac{D^2 - 4\mu}{t(3D^2 - 4\mu)},$$
$$4s^2t^4(3D^2 - 4\mu)^6 = D^2(D^2 - 4\mu)^2\big(4D^2 - t^2(3D^2 - 4\mu)^2\big)^3,$$

and σ and δ are defined as

$$\sigma = St = \frac{a+b}{2}t, \qquad \delta = Dt = \frac{a-b}{2}t.$$

The function $M_{\mathrm{RNA}}(x)$ is given by

$$M_{\mathrm{RNA}}(x) = \frac{(2tx - 2 + \sigma)^2 + \eta}{8(tx-1)^2}, \qquad \eta := \frac{\sigma(4 - 4\delta^2 - 7\sigma + 3\sigma^2)}{\sigma - 1}.$$

The multi-linear differentials computed by the β-deformed topological recursion are defined as follows [33].

Definition 1.6. The connected h-point symmetric multi-linear differential $W_h \in \mathcal{M}^1(\mathcal{C}^{\times h})^s$ is defined as

$$W_h(x_1, \ldots, x_h) = \beta^{h/2} \left\langle \prod_{i=1}^{h} \sum_{a=1}^{N} \frac{dx_i}{x_i - z_a} \right\rangle_{N,\beta}^{(\mathrm{c})}, \tag{1.26}$$

where $\langle \mathcal{O} \rangle_{N,\beta}^{(\mathrm{c})}$ denotes the connected part [46] of $\langle \mathcal{O} \rangle_N^{\beta}$ introduced in (1.24). In the 't Hooft limit (1.16), h-point multi-linear differentials admit an asymptotic expansion

$$W_h(x_1, \ldots, x_h) = \sum_{g,\ell=0}^{\infty} \hbar^{2g-2+h+\ell} \gamma^{\ell} W_{\ell}^{(g,h)}(x_1, \ldots, x_h). \tag{1.27}$$

For the class of genus 0 spectral curves of the form

$$\begin{aligned} &y(x)^2 = M(x)^2 \sigma(x), \\ &\sigma(x) = (x-a)(x-b), \qquad M(x) = c \prod_{i=1}^{f} (x - \alpha_i)^{m_i}, \end{aligned} \tag{1.28}$$

the 2-point multi-linear differential $W_0^{(0,2)}(x_1, x_2)$ takes form

$$W_0^{(0,2)}(x_1, x_2) = B(x_1, x_2) - \frac{dx_1 dx_2}{(x_1 - x_2)^2}, \tag{1.29}$$

where the *Bergman kernel* $B(x_1, x_2)$ is a bilinear differential

$$B(x_1, x_2) = \frac{dx_1 dx_2}{2(x_1 - x_2)^2} \left[1 + \frac{x_1 x_2 - \frac{1}{2}(a+b)(x_1 + x_2) + ab}{\sqrt{\sigma(x_1)\sigma(x_2)}} \right]. \tag{1.30}$$

An exact formula for $W_1^{(0,1)}(x)$ is also known [32, 33, 28],

$$W_1^{(0,1)}(x) = -\frac{dy(x)}{2y(x)} + \frac{dx}{2\sqrt{\sigma(x)}}\left[1 + \sum_{i=1}^{f} m_i\left(1 + \frac{\sqrt{\sigma(\alpha_i)}}{x - \alpha_i}\right)\right]. \tag{1.31}$$

All other differentials $W_\ell^{(g,h)}(x_1, \ldots, x_h)$ in the asymptotic expansion (1.27) can be determined recursively by means of the topological recursion, as stated in the theorem below.

Theorem 1.7 ([33, 31, 28]). *The differentials $W_\ell^{(g,h)}(x_H)$ for $(g, h, \ell) \neq (0, 1, 0)$, $(0, 2, 0)$, $(0, 1, 1)$ in the asymptotic expansion (1.27) obey the β-deformed topological recursion*

$$\begin{aligned}
&\mathcal{W}_0^{(0,1)}(x) = 0, \quad \mathcal{W}_0^{(0,2)}(x_1, x_2) = W_0^{(0,2)}(x_1, x_2) + \frac{dx_1 dx_2}{2(x_1 - x_2)^2}, \\
&\mathcal{W}_\ell^{(g,h)}(x_H) = W_\ell^{(g,h)}(x_H) \text{ for } (g, h, \ell) \neq (0, 1, 0),\ (0, 2, 0), \\
&W_\ell^{(g,h+1)}(x, x_H) = \oint_{\mathcal{A}} \frac{1}{2\pi i} \frac{dS(x, z)}{y(z) dz}\Bigg[W_\ell^{(g-1,h+2)}(z, z, x_H) \\
&\qquad + \sum_{k=0}^{g} \sum_{n=0}^{\ell} \sum_{\emptyset = J \subseteq H} \mathcal{W}_{\ell-n}^{(g-k,|J|+1)}(z, x_J) \mathcal{W}_n^{(k,|H|-|J|+1)}(z, x_{H \setminus J}) \\
&\qquad + dz^2 \frac{\partial}{\partial z} \frac{W_{\ell-1}^{(g,h+1)}(z, x_H)}{dz}\Bigg],
\end{aligned} \tag{1.32}$$

where $H = \{1, 2, \ldots, h\} \supset J = \{i_1, i_2, \ldots, i_j\}$, $H \backslash J = \{i_{j+1}, i_{j+2}, \ldots, i_h\}$, and $\mathcal{A}$ is the counterclockwise contour around the branch cut $[a, b]$. Here $dS(x_1, x_2)$ is the third type differential, which for the genus 0 spectral curve (1.28) takes form

$$dS(x_1, x_2) = \frac{\sqrt{\sigma(x_2)}}{\sqrt{\sigma(x_1)}} \frac{dx_1}{x_1 - x_2}. \tag{1.33}$$

Our task in what follows is to determine generating functions of chord diagrams encoded in the free energy of the RNA matrix model (1.19). To this end we take advantage of the following trick [9]. The potential of the RNA matrix model $V_{\mathrm{RNA}}(x)$ can be separated into the Gaussian part $V_{\mathrm{G}}(x)$ and the rational part $V_{\mathrm{rat}}(x)$

$$V_{\mathrm{RNA}}(x) = V_{\mathrm{G}}(x) + V_{\mathrm{rat}}(x) + s, \qquad V_{\mathrm{G}}(x) = \frac{1}{2}x^2, \qquad V_{\mathrm{rat}}(x) = -\frac{t^{-1}s}{t^{-1} - x}. \tag{1.34}$$

Adopting this separation into the definition of $Z_N^\beta(s, t)$ in (1.14), we find that the partition function of the RNA matrix model can be re-expressed as an expectation value in the β-deformed Gaussian model

$$Z_N^\beta(s,t) = \mathrm{e}^{-\frac{\sqrt{\beta}}{\hbar}sN}\left\langle \exp\left[\frac{s\sqrt{\beta}}{t\hbar}\sum_{a=1}^{N}\frac{1}{t^{-1}-z_a}\right]\right\rangle_{N,\beta}^{\mathrm{G}},$$

where $\langle \mathcal{O}(\{z_a\})\rangle_{N,\beta}^{\mathrm{G}}$ denotes $\langle \mathcal{O}(\{z_a\})\rangle_N^\beta$ defined in (1.24) with the Gaussian potential $V(x) = V_{\mathrm{G}}(x)$. It follows that the right hand side of this equation is given by a sum of h-point multi-linear differentials with $x_i = t^{-1}$ $(i = 1, \ldots, h)$ in the Gaussian model.

Proposition 1.8. *Let $\omega_b(x_1, \ldots, x_b)$ denote the connected b-resolvent in the β-deformed Gaussian model, which we also refer to as the Gaussian b-resolvent,*

$$\omega_b(x_1,\ldots,x_b) = \frac{W_h(x_1,\ldots,x_b)}{dx_1\cdots dx_b} = \beta^{b/2}\left\langle \prod_{i=1}^{b}\sum_{a=1}^{N}\frac{1}{x_i - z_a}\right\rangle_{N,\beta}^{(\mathrm{c})}. \tag{1.35}$$

Then the free energy $F_N^\beta(s,t)$ of the β-deformed RNA matrix model is given by

$$F_N^\beta(s,t) = -\frac{\sqrt{\beta}}{\hbar}sN + \sum_{b=1}^{\infty}\frac{s^b}{b!t^b\hbar^b}\omega_b\big(t^{-1},\ldots,t^{-1}\big). \tag{1.36}$$

Combining the above relation with the form of the free energy in (1.19), we obtain the key theorem for our enumeration of chord diagrams.

Theorem 1.9. *The coefficients $\widetilde{C}_{g,\ell,b}(w)$ in (1.19) are obtained directly from the principal specialization of the Gaussian b-resolvents*

$$\omega_b(x,\ldots,x) = \frac{1}{x^b}\sum_{g=0,\ell=0}^{\infty}(\mu^{-1}\hbar)^{2g-2+b+\ell}(-\gamma)^\ell\widetilde{C}_{g,\ell,b}(\mu x^{-2}). \tag{1.37}$$

On the basis of this theorem, one can compute the generating function $\widetilde{C}_{g,\ell,b}(w)$ of the numbers of non-oriented chord diagrams using the β-deformed topological recursion for the Gaussian model. We will present some details of such a computation in Section 2, and compare its results with other methods of enumeration.

Finally, in order to find the complete form of the asymptotic expansion, we also need to consider the *unstable* part of the free energy, which is not determined by the topological recursion and must be computed independently. The unstable part consists of four terms: $F_{0,0}(s,t)$, $F_{0,1}(s,t)$, $F_{0,2}(s,t)$, and $F_{1,0}(s,t)$. The two terms $F_{0,0}(s,t)$ and $F_{1,0}(s,t)$ are the same as the terms $F_0(s,t)$ and $F_1(s,t)$ in the asymptotic expansion (1.6) of the Hermitian matrix model. On the other hand, $F_{0,1}(\{r_n\})$ and $F_{0,2}(\{r_n\})$ can be computed using famous formulae, referred to respectively as the Dyson's formula and Wiegmann-Zabrodin formula, which for the genus 0 spectral curve (1.28) take the following form.

Theorem 1.10. *For the class of genus* 0 *spectral curves (1.28) determined by the general potential (1.21), the unstable parts* $F_{0,1}(\{r_n\})$ *and* $F_{0,2}(\{r_n\})$ *of the free energy are given by*

(1.38)
$$\frac{\partial}{\partial\mu}F_{0,1}(\{r_n\}) = 1 + \log|c| + \frac{1}{2}\log\left(\frac{a-b}{4}\right)^2 + \sum_{i=1}^{f} m_i \log\left[\frac{1}{2}\left(\alpha_i - \frac{a+b}{2} + \sqrt{\sigma(\alpha_i)}\right)\right],$$

$$F_{0,2}(\{r_n\}) = -\frac{1}{2}\sum_{i=1}^{f} m_i \log\left(1-\mathfrak{s}_i^2\right) - \frac{1}{2}\sum_{i,j=1}^{f} m_i m_j \log\left(1-\mathfrak{s}_i\mathfrak{s}_j\right)$$
$$+\frac{1}{24}\log\left|M(a)M(b)(a-b)^4\right|, \qquad (1.39)$$

where $\mathfrak{s}_i$, $i = 1, \ldots, f$ *are defined as*

$$\alpha_i(\mathfrak{s}_i) = \frac{a+b}{2} - \frac{a-b}{4}(\mathfrak{s}_i + \mathfrak{s}_i^{-1}), \qquad |\mathfrak{s}_i| < 1.$$

The formula (1.38) for $F_{0,1}(\{r_n\})$ was found in [33, 31, 28]. On the other hand, the formula (1.39) for $F_{0,2}(\{r_n\})$ is proven in appendix B.

1.5 A recursion relation from the quantum curve

The main object that we consider in our second approach is the *wave-function* for the matrix model. The wave-function is the 1-point function defined as follows.

Definition 1.11. Let $\epsilon_{1,2}$ denote the parameters

$$\epsilon_1 = -\beta^{1/2}g_s, \quad \epsilon_2 = \beta^{-1/2}g_s, \quad g_s = 2\hbar. \qquad (1.40)$$

For the general type potential (1.21), the 1-point function $Z_\alpha(x;\{r_n\})$ ($\alpha = 1, 2$)

$$Z_\alpha(x;\{r_n\}) = \frac{1}{\mathrm{Vol}_N^\beta}\int_{\mathbb{R}^N}\prod_{a=1}^{N} dz_a \Delta(z)^{2\beta}\psi_\alpha(x)\mathrm{e}^{-\frac{\sqrt{\beta}}{\hbar}\sum_{a=1}^{N}V(z_a)}, \qquad (1.41)$$

with the eigenvalue operator

$$\psi_\alpha(x) = \prod_{a=1}^{N}(x-z_a)^{\frac{\epsilon_1}{\epsilon_\alpha}},$$

is referred to as the wave-function. For the RNA matrix model we denote the wave-function by $Z_\alpha(x;s,t)$.

A prominent property of the wave-function [2, 1, 67, 34] is that it satisfies a partial differential equation, which is called the *quantum curve*[6] or the *time-dependent Schrödinger equation.*

Proposition 1.12 ([1, 67]). *The wave-function $Z_\alpha(x;\{r_n\})$ satisfies the partial differential equation*

$$\left[-\epsilon_\alpha^2\frac{\partial^2}{\partial x^2}-2\epsilon_\alpha V'(x)\frac{\partial}{\partial x}+\hat{f}(x)\right]Z_\alpha(x;\{r_n\})=0, \tag{1.42}$$

where $\hat{f}(x)$ is the differential operator

$$\hat{f}(x)=g_s^2\sum_{n=0}^{K}x^n\partial_{(n)},\qquad \partial_{(n)}=\sum_{k=n+2}^{K+2}kr_k\frac{\partial}{\partial r_{k-n-2}},$$

and we denoted $\partial/\partial r_0=-N/(2\epsilon_2)$.

As its name suggests, the partial differential equation (1.42) is interpreted as the quantization of the spectral curve $\mathcal{C}$ of the matrix model. Promoting the parameters (x,ω) in (1.23) to the non-commutative operators $(\hat{x},\hat{p})$, such that

$$\hat{p}=\epsilon_\alpha\frac{\partial}{\partial x},\qquad \hat{x}=x,\qquad [\hat{p},\hat{x}]=\epsilon_\alpha,$$

the partial differential equation (1.42) can be written in the form

$$\widehat{A}Z_\alpha(x;\{r_n\})=0,\qquad \widehat{A}\equiv A(\hat{x},\hat{p}),$$

for an appropriate choice of $A(\hat{x},\hat{p})$. The operator $\widehat{A}$ is interpreted as a quantization of the spectral curve $\mathcal{C}$, and the equation (1.42) reduces to the defining equation of the spectral curve in the classical limit [1, 67].

Note that, more generally, to a given spectral curve one may associate an infinite family of wave-functions and corresponding quantum curves, which take form of Virasoro singular vectors [67, 34]. In particular the quantum curves in (1.42), for both values of $\alpha=1,2$, correspond to singular vectors at level 2. In this work we do not consider quantum curves at levels higher than 2.

In what follows we use the partial differential equation (1.42) as a tool to determine the partition function $Z_N^\beta(s,t)$ of the β-deformed RNA matrix model. The point is that the partition function $Z_N^\beta(\{r_n\})$ of the matrix model with the

[6] The name "quantum curve" is also used for the ordinary differential equation for the wave function [36, 35, 59, 57, 88, 72, 71, 66, 41, 44, 39, 82, 37, 42, 74, 56, 43, 38, 26, 40, 25]. In this article, we also use this name for a partial differential equation which arises from the conformal field theoretical description of the matrix model [65].

general type potential (1.21) appears as the leading order term in the expansion of the wave-function $Z_\alpha(x;\{r_n\})$ around $x=\infty$

$$Z_\alpha(x;\{r_n\}) = x^{\epsilon_1 N/\epsilon_\alpha}\left(Z_N^\beta(\{r_n\}) + \mathcal{O}(x^{-1})\right).$$

Therefore, from the expansion of the wave-function with $\alpha = 2$ for the RNA matrix model

$$Z_{\alpha=2}(x;s,t) = \exp\left[S(x,s,t)\right],$$
$$S(x,s,t) = -\frac{2\mu}{\epsilon_2}\log x + \sum_{b=0}^{\infty}\sum_{p=0}^{\infty} S_{b,p}(t)s^b x^{-p},$$

the free energy $F_N^\beta(s,t)$ can be written as

$$F_N^\beta(s,t) = \sum_{b=0}^{\infty} S_{b,p=0}(t)s^b.$$

We use the partial differential equation (1.42) to determine each $S_{b,p}(t)$. The main result of this approach is summarized in the following theorem.

Theorem 1.13. *The partial differential equation for the phase function $S(x,s,t)$ for the RNA matrix model takes form*

$$\epsilon_2^2\left[\frac{\partial^2}{\partial x^2}S(x,s,t) + \left(\frac{\partial}{\partial x}S(x,s,t)\right)^2\right] + 2\epsilon_2\left(x - \frac{st}{(1-tx)^2}\right)\frac{\partial}{\partial x}S(x,s,t)$$
$$+ 4\mu\left(1 - \frac{st^2(2-tx)}{(1-tx)^2}\right) + \frac{\epsilon_1\epsilon_2 st^2(2-tx)}{(1-tx)^2}\frac{\partial}{\partial s}S(x,s,t) + \frac{\epsilon_1\epsilon_2 t^3}{1-tx}\frac{\partial}{\partial t}S(x,s,t) = 0. \tag{1.43}$$

This equation generates a hierarchy of differential equations for the coefficients $S_{b,p}(t)$ of the phase function, and the phase function is determined recursively with respect to the backbone number b.

As an independent verification of this algorithm, we checked iterative computations for $b = 1,2,3,4,5$ up to $\mathcal{O}(t^{12})$, and confirmed that they agree with the results of the β-deformed topological recursion from the first approach.

2 Enumeration of chord diagrams via the topological recursion

In the introduction, we extended the RNA matrix model proposed in [9] to the β-deformed RNA matrix model given by (1.14), that enumerates both orientable and non-orientable chord diagrams via Proposition 1.4. In this section we enumerate orientable and non-orientable chord diagrams by the formalism of the β-deformed topological recursion (1.32).

2.1 Enumeration of chord diagrams via Gaussian resolvents

In principle, by applying the β-deformed topological recursion formalism to the spectral curve (1.25) of the RNA matrix model, one can recursively compute the asymptotic expansion (1.17) of the free energy $F_N^\beta(s,t)$. In [9], using the topological recursion [5, 45, 46] for the $\beta=1$ (which does not involve $W_{\ell\geq 1}^{(g,h)}(x_H)$ terms) RNA matrix model (1.3), $F_2(s,t)=F_{2,0}(s,t)$ and $F_3(s,t)=F_{3,0}(s,t)$ were explicitly computed. However, because of the complicated form of the curve (1.25), an explicit computation of $W_{\ell\geq 1}^{(g,h)}(x_H)$ is not easy. In this section we consider instead the β-deformed Gaussian matrix model with the Gaussian potential $V_{\mathrm{G}}(x)$ in (1.34).

Using Theorem 1.9, one can compute the coefficients $\widetilde{C}_{g,\ell,b}(w)$ in (1.19) of the free energy $F_N^\beta(s,t)$ from the Gaussian b-resolvents (1.35). For the Gaussian matrix model, the spectral curve takes form

$$y_{\mathrm{G}}(x)^2 = x^2 - 4\mu. \tag{2.1}$$

In order to apply the β-deformed topological recursion (1.32) to this Gaussian curve, it is convenient to introduce the Zhukovsky variable z as

$$x(\mathrm{z}) = \sqrt{\mu}(\mathrm{z}+\mathrm{z}^{-1}).$$

In this variable the branch points $x=\pm 2\sqrt{\mu}$ are mapped to $\mathrm{z}=\pm 1$. To express the β-deformed topological recursion (1.32) in this Zhukovsky variable, we define

$$\begin{aligned}
&\widehat{y}_{\mathrm{G}}(\mathrm{z})d\mathrm{z} = y_{\mathrm{G}}(x(\mathrm{z}))dx = \sqrt{\mu}(\mathrm{z}-\mathrm{z}^{-1})d\mathrm{z},\\
&d\widehat{S}(\mathrm{z}_1,\mathrm{z}_2) = dS(x_1(\mathrm{z}_1),x_2(\mathrm{z}_2)) = \frac{d\mathrm{z}_1}{\mathrm{z}_1-\mathrm{z}_2} - \frac{d\mathrm{z}_1}{\mathrm{z}_1-\mathrm{z}_2^{-1}},\\
&\widehat{W}_\ell^{(g,h)}(\mathrm{z}_1,\ldots,\mathrm{z}_h) = W_\ell^{(g,h)}(x_1(\mathrm{z}_1),\ldots,x_h(\mathrm{z}_h)).
\end{aligned} \tag{2.2}$$

We then obtain the β-deformed topological recursion (1.32) for the Gaussian spectral curve in the Zhukovsky variable. More generally, the β-deformed topological recursion for genus 0 spectral curves (1.28) in the Zhukovsky is discussed in detail in [68, 67].

Corollary 2.1. *The differentials $\widehat{W}_\ell^{(g,h)}(\mathrm{z}_H)$ for $(g,h,\ell)\neq(0,1,0)$, $(0,2,0)$, $(0,1,1)$ obey the β-deformed topological recursion in the Zhukovsky variable*

$$\widehat{W}_\ell^{(g,h+1)}(\mathrm{z},\mathrm{z}_H) = \oint_{\widetilde{\mathcal{A}}} \frac{1}{2\pi i}\frac{d\widehat{S}(\mathrm{z},\zeta)}{\widehat{y}_{\mathrm{G}}(\zeta)d\zeta}\mathrm{Rec}_\ell^{(g,h+1)}(\zeta,\mathrm{z}_H). \tag{2.3}$$

Here

$$\mathrm{Rec}_\ell^{(g,h+1)}(\zeta,\mathrm{z}_H) = \widehat{W}_\ell^{(g-1,h+2)}(\zeta,\zeta,\mathrm{z}_H)$$

$$+\sum_{k=0}^{g}\sum_{n=0}^{\ell}\sum_{\emptyset=J\subseteq H}\widehat{\mathcal{W}}_{\ell-n}^{(g-k,|J|+1)}(\zeta,\mathrm{z}_J)\widehat{\mathcal{W}}_{n}^{(k,|H|-|J|+1)}(\zeta,\mathrm{z}_{H\setminus J})$$
$$+\,d\zeta^2\left[\frac{\partial}{\partial\zeta}+\frac{\partial^2\zeta}{\partial w^2}\left(\frac{\partial w}{\partial\zeta}\right)^2\right]\frac{\widehat{W}_{\ell-1}^{(g,h+1)}(\zeta,\mathrm{z}_H)}{d\zeta},\tag{2.4}$$

with $w=\sqrt{\mu}(\zeta+\zeta^{-1})$, $|\zeta|>1$, *and*

$$\widehat{\mathcal{W}}_0^{(0,1)}(\mathrm{z})=0,\quad \widehat{\mathcal{W}}_0^{(0,2)}(\mathrm{z}_1,\mathrm{z}_2)=\widehat{W}_0^{(0,2)}(\mathrm{z}_1,\mathrm{z}_2)+\frac{(\mathrm{z}_1^2-1)(\mathrm{z}_2^2-1)d\mathrm{z}_1d\mathrm{z}_2}{2(\mathrm{z}_1-\mathrm{z}_2)^2(\mathrm{z}_1\mathrm{z}_2-1)^2},$$
$$\widehat{\mathcal{W}}_\ell^{(g,h)}(\mathrm{z}_H)=\widehat{W}_\ell^{(g,h)}(\mathrm{z}_H)\ \textit{for}\ (g,h,\ell)\neq(0,1,0),\ (0,2,0),\tag{2.5}$$

where $\widetilde{\mathcal{A}}$ *is the contour surrounding the unit disk* $|\zeta|=1$*. For the Gaussian model the initial data of the recursion (1.30) and (1.31) takes form*

$$\widehat{W}_0^{(0,2)}(\mathrm{z}_1,\mathrm{z}_2)=\frac{d\mathrm{z}_1d\mathrm{z}_2}{(\mathrm{z}_1\mathrm{z}_2-1)^2},\tag{2.6}$$
$$\widehat{W}_1^{(0,1)}(\mathrm{z})=\left(\frac{1}{\mathrm{z}}-\frac{1}{2(\mathrm{z}-1)}-\frac{1}{2(\mathrm{z}+1)}\right)d\mathrm{z}.\tag{2.7}$$

From Theorem 1.9 the coefficients $\widetilde{C}_{g,\ell,b}(w)$ of the free energy $F_N^\beta(s,t)$ can now be computed. For example we find, for $b=1$,

$$\widetilde{C}_{0,0,1}(w)=\frac{1-\sqrt{1-4w}}{2w},\quad \widetilde{C}_{0,1,1}(w)=\frac{1-\sqrt{1-4w}}{2(1-4w)},\quad \widetilde{C}_{1,0,1}(w)=\frac{w^2}{(1-4w)^{5/2}},$$
$$\widetilde{C}_{0,2,1}(w)=\frac{w\big(1+w-\sqrt{1-4w}\big)}{(1-4w)^{5/2}},\quad \widetilde{C}_{1,1,1}(w)=\frac{w^2\big(1+30w-(1+6w)\sqrt{1-4w}\big)}{2(1-4w)^4},$$
$$\widetilde{C}_{0,3,1}(w)=\frac{5w^2\big(1+2w-(1+w)\sqrt{1-4w}\big)}{(1-4w)^4},$$

for $b=2$,

$$\widetilde{C}_{0,0,2}(w)=\frac{w}{(1-4w)^2},\quad \widetilde{C}_{0,1,2}(w)=\frac{w\big(1+18w-(1+4w)\sqrt{1-4w}\big)}{2(1-4w)^{7/2}},$$
$$\widetilde{C}_{1,0,2}(w)=\frac{w^3(21+20w)}{(1-4w)^5},\quad \widetilde{C}_{0,2,2}(w)=\frac{w^2\big(8+98w+38w^2-(8+45w)\sqrt{1-4w}\big)}{(1-4w)^5},$$
$$\widetilde{C}_{1,1,2}(w)=\frac{w^3\big(21+1462w+2700w^2-(21+376w+240w^2)\sqrt{1-4w}\big)}{2(1-4w)^{13/2}},$$
$$\widetilde{C}_{0,3,2}(w)=\frac{w^3\big(117+1316w+1182w^2-(117+854w+292w^2)\sqrt{1-4w}\big)}{(1-4w)^{13/2}},$$

for $b = 3$,

$$\widetilde{C}_{0,0,3}(w) = \frac{2w^2(3+4w)}{(1-4w)^{9/2}},$$
$$\widetilde{C}_{0,1,3}(w) = \frac{w^2\left(3+160w+354w^2-(3+50w+40w^2)\sqrt{1-4w}\right)}{(1-4w)^6},$$
$$\widetilde{C}_{1,0,3}(w) = \frac{12w^4(45+207w+68w^2)}{(1-4w)^{15/2}},$$
$$\widetilde{C}_{0,2,3}(w) = \frac{2w^3\left(58+1797w+5232w^2+1004w^3-(58+977w+1416w^2)\sqrt{1-4w}\right)}{(1-4w)^{15/2}}.$$

Then, by (1.20) in Proposition 1.4, we obtain the generating functions $C_{g,b}(w)$ for orientable chord diagrams and $C^{\mathfrak{r}}_{h,b}(w)$ for non-oriented chord diagrams. For instance, we obtain

$$\begin{aligned}
C_{0,1}(w) &= 1 + w + 2w^2 + 5w^3 + 14w^4 + 42w^5 + 132w^6 + 429w^7 + \mathcal{O}(w^8),\\
C_{1,1}(w) &= w^2 + 10w^3 + 70w^4 + 420w^5 + 2310w^6 + 12012w^7 + \mathcal{O}(w^8),\\
C_{0,2}(w) &= w + 8w^2 + 48w^3 + 256w^4 + 1280w^5 + 6144w^6 + \mathcal{O}(w^7),\\
C_{1,2}(w) &= 21w^3 + 440w^4 + 5440w^5 + 51840w^6 + 421120w^7 + \mathcal{O}(w^8),\\
C_{0,3}(w) &= 6w^2 + 116w^3 + 1332w^4 + 11880w^5 + 90948w^6 + \mathcal{O}(w^7),\\
C_{1,3}(w) &= 540w^4 + 18684w^5 + 350736w^6 + 4779720w^7 + \mathcal{O}(w^8),
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
C^{\mathfrak{r}}_{1,1}(w) &= w + 5w^2 + 22w^3 + 93w^4 + 386w^5 + 1586w^6 + 6476w^7 + \mathcal{O}(w^8),\\
C^{\mathfrak{r}}_{2,1}(w) &= 5w^2 + 52w^3 + 374w^4 + 2290w^5 + 12798w^6 + 67424w^7 + \mathcal{O}(w^8),\\
C^{\mathfrak{r}}_{1,2}(w) &= 8w^2 + 117w^3 + 1084w^4 + 8119w^5 + 53640w^6 + \mathcal{O}(w^7),\\
C^{\mathfrak{r}}_{2,2}(w) &= 111w^3 + 2404w^4 + 30442w^5 + 295500w^6 + \mathcal{O}(w^7),\\
C^{\mathfrak{r}}_{1,3}(w) &= 116w^3 + 3204w^4 + 49248w^5 + 561782w^6 + \mathcal{O}(w^7),\\
C^{\mathfrak{r}}_{2,3}(w) &= 2952w^4 + 105300w^5 + 2021396w^6 + \mathcal{O}(w^7).
\end{aligned} \tag{2.9}$$

We note that the generating function $C^{\mathfrak{r}}_{h,b}(w)$ with an even cross-cap number $h = 2g$ enumerates both orientable and non-orientable chord diagrams, and therefore non-orientable chord diagrams are enumerated by

$$C^{\mathfrak{r}}_{2g,b}(w) - C_{g,b}(w). \tag{2.10}$$

2.2 The unstable part of the free energy

For a matrix model with the general potential (1.21), the unstable coefficients $F_{0,0}(\{r_n\})$, $F_{0,1}(\{r_n\})$, $F_{1,0}(\{r_n\})$ and $F_{0,2}(\{r_n\})$ in the asymptotic expansion (1.22) of the free energy $F_N^\beta(\{r_n\})$ must be computed separately. For the general potential (1.21) and the genus 0 spectral curve (1.28), the coefficients $F_{0,0}(\{r_n\})$ [27, 69] and $F_{1,0}(\{r_n\})$ [7, 6] (see [3, 30] for multi-cut solutions) are given by

$$F_{0,0}(\{r_n\}) = -\mu \int_{[a,b]} dz \rho(z) V(z) + \mu^2 \int_{[a,b]^2} dz dz' \rho(z)\rho(z') \log|z-z'|, \tag{2.11}$$

$$F_{1,0}(\{r_n\}) = -\frac{1}{24} \log \left| M(a) M(b) (a-b)^4 \right|, \tag{2.12}$$

where $\rho(z) = \lim_{N\to\infty} \frac{1}{N} \sum_{a=1}^N \delta(z-z_a)$ is the eigenvalue density given by

$$\rho(z) = \frac{1}{2\pi i \mu}\left(W_0^{(0,1)}(z-i\epsilon) - W_0^{(0,1)}(z+i\epsilon)\right) = \frac{1}{2\pi i\mu} y(z), \qquad z \in [a,b].$$

In [9], $F_{0,0}(s,t)$ and $F_{1,0}(s,t)$ for the RNA matrix model (1.3) were computed using the above formulae.

Furthermore, the coefficients $F_{0,1}(s,t)$ and $F_{0,2}(s,t)$ for the genus 0 spectral curve (1.28) can be computed using Theorem 1.10. In particular, by Proposition 1.4 the generating function $C^{\mathfrak{r}}_{2,b}(w) - C_{1,b}(w)$ for the numbers of chord diagrams with the topology of the Klein bottle takes form

$$F_{1,0}(\{r_n\}) + F_{0,2}(\{r_n\}) = -\frac{1}{2}\sum_{i=1}^{f} m_i \log\left(1-\mathfrak{s}_i^2\right) - \frac{1}{2}\sum_{i,j=1}^{f} m_i m_j \log\left(1-\mathfrak{s}_i \mathfrak{s}_j\right). \tag{2.13}$$

Using the above formulae for the spectral curve (1.25) of the RNA matrix model, we determined the generating functions $C_{0,b}(w)$, $C_{1,b}(w)$, $C^{\mathfrak{r}}_{1,b}(w)$, and $C^{\mathfrak{r}}_{2,b}(w)$. We checked that the results coincide with the results obtained from the Gaussian b-resolvents discussed in the previous subsection. Compared with the method discussed in the previous subsection, the advantage of this method is that we find all order generating functions for the backbone number b.

3 Enumeration of chord diagrams via quantum curve techniques

In this section we consider a recursive computation of the numbers of chord diagrams, based on the quantum curve equation (1.42) for the wave-function of the RNA matrix model defined in Definition 1.11. For the β-deformed RNA matrix model the quantum curve equation reduces to a partial differential equation in three parameters x, s, and t. We solve this partial differential equation recursively and obtain the generating function for the numbers of chord diagrams as the leading term in the expansion of the wave-function near $x \to \infty$.

3.1 Differential equation for the wave-function from the quantum curve

The quantum curve for the β-deformed RNA matrix model is the key equation we take advantage of in this section.

Proposition 3.1. *Let $Z_\alpha(x;s,t)$ $(\alpha = 1,2)$ denote the wave-function for the β-deformed RNA matrix model*

$$Z_\alpha(x;s,t) = \frac{1}{\mathrm{Vol}_N^\beta} \int_{\mathbb{R}^N} \prod_{a=1}^{N} dz_a \Delta(z)^{2\beta} \psi_\alpha(x) \mathrm{e}^{-\frac{\sqrt{\beta}}{\hbar}\sum_{a=1}^N V_{\mathrm{RNA}}(z_a)}, \tag{3.1}$$

where $\psi_\alpha(x) = \prod_{a=1}^N (x - z_a)^{\frac{\epsilon_1}{\epsilon_\alpha}}$ and the potential $V(x) = V_{\mathrm{RNA}}(x)$ is chosen as in equation (1.2). Then the partial differential equation (1.42) reduces to

$$\begin{aligned}\Bigg[- \left(\epsilon_\alpha \frac{\partial}{\partial x}\right)^2 - 2\epsilon_\alpha \left(x - \frac{st}{(1-tx)^2}\right)\frac{\partial}{\partial x} - 4\mu \left(1 - \frac{st^2(2-tx)}{(1-tx)^2}\right) \\ - \frac{\epsilon_1\epsilon_2 st^2(2-tx)}{(1-tx)^2}\frac{\partial}{\partial s} - \frac{\epsilon_1\epsilon_2 t^3}{1-tx}\frac{\partial}{\partial t}\Bigg] Z_\alpha(x;s,t) = 0,\end{aligned} \tag{3.2}$$

where μ denotes the 't Hooft parameter $\mu = \beta^{1/2}\hbar N = -\epsilon_1 N/2$.

Proof. The RNA matrix model has the potential (1.2), with the coefficients r_n in (1.21) that take form

$$r_0 = 0, \qquad r_2 = \frac{1}{2} - st^2, \qquad r_n = -st^n \ \ (n \neq 0,2).$$

Adopting this choice of coefficients, the action of $\hat{f}(x)$ in (1.42) on $Z_\alpha(x;s,t)$ can be rewritten solely in terms of derivatives with respect to s and t

$$\begin{aligned}\hat{f}(x)Z_\alpha(x;s,t) &= -\frac{2}{\epsilon_2} g_s^2 N Z_\alpha(x;s,t) + g_s^2 \sum_{n=0}^{\infty} x^n t^{n+3} \frac{\partial}{\partial t} Z_\alpha(x;s,t) \\ &\quad + g_s^2 s \sum_{n=0}^{\infty} (n+2) x^n t^{n+2} \left(\frac{2}{\epsilon_2}N + \frac{\partial}{\partial s}\right) Z_\alpha(x;s,t) \\ &= g_s^2 \left[-\frac{2}{\epsilon_2}N + \frac{t^3}{1-tx}\frac{\partial}{\partial t} + s\frac{t^2(2-tx)}{(1-tx)^2}\left(\frac{2N}{\epsilon_2} + \frac{\partial}{\partial s}\right)\right] Z_\alpha(x;s,t).\end{aligned}$$

Using the relation between N and μ stated in Proposition 3.1, one obtains the partial differential equation (3.2). □

For simplicity, in the following we choose $\alpha = 2$ and consider the wave-function $Z_2(x;s,t)$. On the basis of Proposition 3.1 we describe an algorithm to compute the free energy of the β-deformed RNA matrix model. To this end we consider the wave-function in the two following limits.

The first limit we consider is such that $x \to \infty$. By the definition of the wave-function, the partition function $Z_N^\beta(s,t)$ is encoded in this limit as follows

$$Z_2(x;s,t) = x^{\epsilon_1 N/\epsilon_2}\left(Z_N^\beta(s,t) + \mathcal{O}(x^{-1})\right). \tag{3.3}$$

Equivalently, in the $x \to \infty$ limit the free energy $F_N^\beta(x;s,t) = \log Z_N^\beta(x;s,t)$ is found from the *phase function*

$$S(x,s,t) = \log Z_2(x;s,t). \tag{3.4}$$

Taking advantage of Proposition 3.1, we find the nonlinear partial differential equation (1.43) for the phase function $S(x,s,t)$ in Theorem 1.13.

The second limit we consider is $s \to 0$. In this limit the RNA matrix model reduces to the Gaussian matrix model, and we denote by $Z_\alpha^{\mathrm{G}}(x)$ the corresponding Gaussian wave-function

$$Z_\alpha^{\mathrm{G}}(x) = \lim_{s\to 0} Z_\alpha(x;s,t) = \frac{1}{\mathrm{Vol}_N^\beta}\int_{\mathbb{R}^N}\prod_{a=1}^N dz_a \Delta(z)^{2\beta}\psi_\alpha(x) \mathrm{e}^{-\frac{\sqrt{\beta}}{2\hbar}\sum_{a=1}^N z_a^2}. \tag{3.5}$$

This Gaussian wave-function obeys an ordinary differential equation

$$\left[-\left(\epsilon_\alpha\frac{\partial}{\partial x}\right)^2 - 2\epsilon_\alpha x\frac{\partial}{\partial x} - 4\mu\right] Z_\alpha^{\mathrm{G}}(x) = 0, \tag{3.6}$$

which is obtained in the $s \to 0$ limit of equation (3.2). Here we denote the phase function of the Gaussian wave-function for $\alpha = 2$ by

$$S_0(x) = \log Z_2^{\mathrm{G}}(x). \tag{3.7}$$

In order to determine the free energy $F_N^\beta(s,t) = \log Z_N^\beta(s,t)$, we consider now the expansion of the phase function $S(x,s,t)$. First, we consider the expansion of the phase function with respect to the backbone parameter s

$$S(x,s,t) = \sum_{b=0}^\infty S_b(x,t)s^b, \tag{3.8}$$

which has the following properties:

- The phase function $S_0(x)$ agrees with that of the Gaussian model

$$S_0(x) = S_0(x,t). \tag{3.9}$$

- The expansion of the phase function $S_b(x,t)$ around $x=\infty$ takes form

$$S_b(x,t) = -\frac{2\mu}{\epsilon_2}\delta_{b,0}\log x + \sum_{p=0}^{\infty} S_{b,p}(t)x^{-p}, \tag{3.10}$$

where an additional $\log x$ term for $b=0$ appears from the limit of $\psi_\alpha(x)$. In particular $S_{0,p} \equiv S_{0,p}(t)$ do not depend on t.

- The free energy $F_N^\beta(s,t)$ is obtained as the generating function of $S_{b,0}(t)$

$$F_N^\beta(s,t) = \sum_{b=1}^{\infty} S_{b,0}(t)s^b. \tag{3.11}$$

Applying the expansion (3.8) to the nonlinear partial differential equation (1.43), one finds a hierarchy of differential equations that determine the functions $S_{b,p}(t)$ recursively. Although our main task is to determine the functions $S_{b,0}(t)$ in (3.11), we need the extra data of the higher order terms $S_{b,p\geq 1}(t)$ to determine $S_{b,0}(t)$.[7] In the following we solve the equation (1.43) systematically in four steps.

3.2 Solving the recursion relations in four steps

Now we solve the recursion relation for $S_{b,p}(t)$ in the following steps.

Step 1 Determine the hierarchy of differential equations by expanding the equation (1.43) in the parameter s.

Step 2 Solve the ordinary differential equation (3.6) for the Gaussian wave-function.

Step 3 Determine $S_1(x,t)$ iteratively by solving the differential equation to order $\mathcal{O}(s^1)$ in Step 1, with the initial data $S_0(x)$ obtained in Step 2.

Step 4 Repeat the same analysis for $S_b(x,t)$, by adopting the $S_{b'(\leq b-1)}(x,t)$ as an input data.

Step 1: Hierarchy of differential equations for the phase function. We determine the form of the hierarchy of differential equations by substituting the expansion (3.8) in the differential equation (1.43). Picking up coefficients of s^0, s^1, and s^b $(b\geq 2)$ respectively, we obtain nonlinear partial differential equations for $S_b(x,t)$

(3.12)

$$\epsilon_2^2\frac{\partial^2}{\partial x^2}S_0(x) + \epsilon_2^2\left(\frac{\partial}{\partial x}S_0(x)\right)^2 + 2\epsilon_2 x\frac{\partial}{\partial x}S_0(x) + 4\mu = 0,$$

[7]It is not easy to reduce this recursion relation between $S_{b,p}(t)$ to a simple recursion relation involving $S_{b,0}(t)$ only.

$$\begin{aligned}&\epsilon_2^2\frac{\partial^2}{\partial x^2}S_1(x,t)+2\epsilon_2^2\frac{\partial}{\partial x}S_0(x)\frac{\partial}{\partial x}S_1(x,t)+2\epsilon_2 x\frac{\partial}{\partial x}S_1(x,t)-\frac{2\epsilon_2 t}{(1-tx)^2}\frac{\partial}{\partial x}S_0(x)\\&\quad-\frac{4\mu t^2(2-tx)}{(1-tx)^2}+\frac{\epsilon_1\epsilon_2 t^2(2-tx)}{(1-tx)^2}S_1(x,t)+\frac{\epsilon_1\epsilon_2 t^3}{1-tx}\frac{\partial}{\partial t}S_1(x,t)=0,\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}&\epsilon_2^2\frac{\partial^2}{\partial x^2}S_b(x,t)+\epsilon_2^2\sum_{a=0}^{b}\frac{\partial}{\partial x}S_a(x,t)\frac{\partial}{\partial x}S_{b-a}(x,t)+2\epsilon_2 x\frac{\partial}{\partial x}S_b(x,t)\\&\quad-\frac{2\epsilon_2 t}{(1-tx)^2}\frac{\partial}{\partial x}S_{b-1}(x,t)+\frac{b\epsilon_1\epsilon_2 t^2(2-tx)}{(1-tx)^2}S_b(x,t)+\frac{\epsilon_1\epsilon_2 t^3}{1-tx}\frac{\partial}{\partial t}S_b(x,t)=0.\end{aligned}\tag{3.14}$$

The first differential equation (3.12) for $S_0(x)$ is equivalent to the quantum curve equation (3.6) for the Gaussian model, and we find that (3.12)–(3.14) can be solved successively for $S_b(x,t)$ $(b=1,2,3,\ldots)$.

Step 2: The Gaussian phase function. The Gaussian part of the wave-function is necessary as an input data in order to solve the equation (3.13).[8] Substituting the expansion (3.10) for $S_0(x)$ in the differential equation (3.12), we obtain the recursion relation for the t-independent (as follows from the t-independence of $S_0(x)$) coefficients $S_{0,p}\equiv S_{0,p}(t)$.

Proposition 3.2. *The coefficients $S_{0,p}$ in the $1/x$ expansion of the Gaussian phase function $S_0(x)$ obey the recursion relation*

$$\begin{aligned}&S_{0,2p-3}=0,\quad S_{0,2}=\frac{\mu}{2\epsilon_2}(2\mu+\epsilon_2),\\&S_{0,2p}=\frac{1}{2p}\Big\{\epsilon_2(p-1)(2p-1)S_{0,2p-2}-4\mu(p-1)S_{0,2p-2}\\&\qquad+2\epsilon_2\sum_{q=1}^{p-2}q(p-q-1)S_{0,2q}S_{0,2p-2q-2}\Big\},\end{aligned}\tag{3.15}$$

where p is a positive integer with $p\geq 2$.

Solving this recursion relation iteratively, one finds the expansion

$$\begin{aligned}S_0(x)=&-\frac{2\mu}{\epsilon_2}\log x+\frac{\mu}{2\epsilon_2 x^2}(2\mu+\epsilon_2)+\frac{\mu}{8\epsilon_2 x^4}(4\mu+3\epsilon_2)(2\mu+\epsilon_2)\\&+\frac{\mu}{24\epsilon_2 x^6}(2\mu+\epsilon_2)(15\epsilon_2^2+34\mu\epsilon_2+20\mu^2)\end{aligned}$$

[8] Any solution of the ordinary differential equation (3.6) can be expressed in terms of Hermite polynomials.

$$+\frac{\mu}{64\epsilon_2 x^8}(2\mu+\epsilon_2)(105\epsilon_2^3+310\mu\epsilon_2^2+316\mu^2\epsilon_2+112\mu^3)$$
$$+\frac{\mu}{160\epsilon_2 x^{10}}(2\mu+\epsilon_2)(945\epsilon_2^4+3288\mu\epsilon_2^3+4424\mu^2\epsilon_2^2+2752\mu^3\epsilon_2+672\mu^4)$$
$$+\mathcal{O}(x^{-12}). \tag{3.16}$$

Step 3: The 1-backbone phase function $S_1(x,t)$.

In this step we continue our analysis of the 1-backbone phase function $S_1(x,t)$. For this purpose, we expand equation (3.13) with respect to x^{-1} and consider differential equations obtained for the coefficients of x^1, x^0, x^{-1}, and x^{-p} $(p \geq 2)$.

Corollary 3.3. *The coefficients $S_{1,p}(t)$ $(p=0,1,\ldots)$ obey the following differential equations in the parameter t:*

$$2S_{1,1}(t)+\epsilon_1 t(\Theta_t+1)S_{1,0}(t)-\frac{4\mu t}{\epsilon_2}=0, \tag{3.17}$$

$$4tS_{1,2}(t)-4S_{1,1}(t)+\epsilon_1 t^2(\Theta_t+1)S_{1,1}(t)-\epsilon_1 t(\Theta_t+2)S_{1,0}(t)+\frac{8\mu t}{\epsilon_2}=0, \tag{3.18}$$

$$\begin{aligned}&6t^2S_{1,3}(t)-8tS_{1,2}(t)+\epsilon_1 t^3(\Theta_t+1)S_{1,2}(t)-\epsilon_1 t^2(\Theta_t+2)S_{1,1}(t)\\ &\qquad -2\epsilon_2 t^2 S_{1,1}(t)-4\mu t^2 S_{1,1}(t)+2S_{1,1}(t)-\frac{4\mu t}{\epsilon_2}=0,\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}&2(p+2)t^2S_{1,p+2}(t)-4(p+1)tS_{1,p+1}(t)+\epsilon_1 t^3(\Theta_t+1)S_{1,p+1}(t)\\ &-\epsilon_1 t^2(\Theta_t+2)S_{1,p}(t)-p(p+1)\epsilon_2 t^2 S_{1,p}(t)-4p\mu t^2 S_{1,p}(t)+2pS_{1,p}(t)\\ &+2(p-1)p\epsilon_2 tS_{1,p-1}(t)+8(p-1)\mu tS_{1,p-1}(t)-(p-2)(p-1)\epsilon_2 S_{1,p-2}(t)\\ &-4(p-2)\mu S_{1,p-2}(t)-2(p-1)tS_{0,p-1}-2\epsilon_2 t^2\sum_{q=2}^{p-1}q(p-q)S_{0,q}S_{1,p-q}(t)\\ &+4\epsilon_2 t\sum_{q=3}^{p-1}(q-1)(p-q)S_{0,q-1}S_{1,p-q}(t)-2\epsilon_2\sum_{q=4}^{p-1}(q-2)(p-q)S_{0,q-2}S_{1,p-q}(t)=0,\end{aligned} \tag{3.20}$$

where $\Theta_t = t\partial/\partial t$, and $S_{0,p}$ are solutions of the recursion relation (3.15).

In Step 2, $S_{0,p}$ have been already determined iteratively. Therefore, substituting the solution (3.16) in equations (3.17)–(3.20), we obtain differential equations for $S_{1,p}(t)$. Furthermore, the following condition follows from the recursive structure

$$S_{1,p}(t)=\sum_{k=1}^{\infty}S_{1,p,k}t^k, \qquad S_{1,p,k}=0 \ \text{ for } \ p+k \ \text{ odd},$$

whose implementation accelerates the iteration. We have implemented this iteration on a computer and found iteratively solutions for $S_{1,p}(t)$ $(p = 0, 1, 2, \ldots)$, which are summarized in appendix C.

Step 4: The multi-backbone phase functions $S_b(x, t)$ $(b \geq 2)$.

To extend our analysis to the multi-backbone phase functions $S_b(x, t)$ $(b \geq 2)$ we expand the differential equation (3.14) with respect to the parameter x, in the same way as in the previous steps.

Corollary 3.4. *The coefficients $S_{b,p}(t)$ of the multi-backbone phase function $S_b(x, t)$ obey*

$$\begin{aligned}
&2(p+2)t^2 S_{b,p+2}(t) - 4(p+1)tS_{b,p+1}(t) + \epsilon_1 t^3(\Theta_t + b)S_{b,p+1}(t) - \epsilon_1 t^2(\Theta_t + 2b)S_{b,p}(t) \\
&\quad - p(p+1)\epsilon_2 t^2 S_{b,p}(t) - 4p\mu t^2 S_{b,p}(t) + 2pS_{b,p}(t) + 2(p-1)p\epsilon_2 tS_{b,p-1}(t) \\
&\quad + 8(p-1)\mu t S_{b,p-1}(t) - (p-2)(p-1)\epsilon_2 S_{b,p-2}(t) - 4(p-2)\mu S_{b,p-2}(t) \\
&\quad - 2(p-1)tS_{b-1,p-1}(t) - \epsilon_2 t^2 \sum_{a=0}^{b}\sum_{q=1}^{p-1} q(p-q)S_{a,q}(t)S_{b-a,p-q}(t) \\
&\quad + 2\epsilon_2 t \sum_{a=0}^{b}\sum_{q=2}^{p-1}(q-1)(p-q)S_{a,q-1}(t)S_{b-a,p-q}(t) \\
&\quad - \epsilon_2 \sum_{a=0}^{b}\sum_{q=3}^{p-1}(q-2)(p-q)S_{a,q-2}(t)S_{b-a,p-q}(t) = 0,
\end{aligned} \tag{3.21}$$

where $S_{b,p}(t) = 0$ for $p \leq -1$.

The following conditions that follow from (3.21) again accelerate the iteration

$$S_{b,p}(t) = \sum_{k=b}^{\infty} S_{b,p,k} t^k, \qquad \begin{aligned} S_{b,0,k} &= 0 \ \text{ for } \ b \geq 2, \quad k \leq 2b-3, \\ S_{b,p,k} &= 0 \ \text{ for } \ p+k \ \text{ odd.} \end{aligned}$$

Programming this recursion on a computer, we determined the multi-backbone phase function $S_b(x, t)$ iteratively. Computational results for $S_{b,p}(t)$ $b = 2, 3$ are summarized in appendix C.

3.3 The free energy

Finally, we collect all $S_{b,0}(t)$ obtained in the above four steps together, and substitute them into the equation (3.11). Rewriting ϵ_α $(\alpha = 1, 2)$ in (1.40) in terms of the parameters $\hbar$ and $\gamma = \beta^{1/2} - \beta^{-1/2}$, we obtain the free energy of the β-deformed RNA matrix model

$$F_N^\beta(s, t) = \sum_{\ell=0}^{\infty} s^b F_b(\mu, \hbar, \gamma; t),$$

$$
\begin{aligned}
F_1(\mu,\hbar,\gamma;t) = &\left(\frac{\mu^2}{\hbar^2}-\frac{\mu}{\hbar}\gamma\right)t^2+\left(\frac{2\mu^3}{\hbar^2}-\frac{5\mu^2}{\hbar}\gamma+\left(3\gamma^2+1\right)\mu\right)t^4\\
&+\left(\frac{5\mu^4}{\hbar^2}-\frac{22\mu^3}{\hbar}\gamma+\left(32\gamma^2+10\right)\mu^2-\left(15\gamma^3+13\gamma\right)\mu\hbar\right)t^6\\
&+\left(\frac{14\mu^5}{\hbar^2}-\frac{93\mu^4}{\hbar}\gamma+\left(234\gamma^2+70\right)\mu^3+\left(52\gamma^3+43\gamma\right)\mu^2\hbar\right.\\
&\qquad\left.+\left(105\gamma^4+160\gamma^2+1\right)\mu\hbar^2\right)t^8+\mathcal{O}(t^{10}),
\end{aligned}
$$

$$
\begin{aligned}
F_2(\mu,\hbar,\gamma;t) = &\frac{\mu}{2\hbar^2}t^2+\left(\frac{4\mu^2}{\hbar^2}-\frac{4\mu}{\hbar}\gamma\right)t^4\\
&+\left(\frac{24\mu^3}{\hbar^2}-\frac{117\mu^2}{2\hbar}\gamma+\frac{\mu}{2}\left(69\gamma^2+21\right)\right)t^6\\
&+\left(\frac{128\mu^4}{\hbar^2}-\frac{542\mu^3}{\hbar}\gamma+\mu^2\left(762\gamma^2+220\right)-\mu\hbar\left(348\gamma^3+282\gamma\right)\right)t^8\\
&+\mathcal{O}(t^{10}),\\
F_3(\mu,\hbar,\gamma;t) = &\frac{\mu}{\hbar^2}t^4+\left(\frac{58\mu^2}{3\hbar^2}-\frac{58\mu}{3}\gamma\right)t^6\\
&+\left(\frac{222\mu^3}{\hbar^2}-\frac{534\mu^2}{\hbar}\gamma+\mu\left(312\gamma^2+90\right)\right)t^8+\mathcal{O}(t^{10}).
\end{aligned}
$$

From (1.17) and (1.19) we obtain the coefficients $\widetilde{C}_{g,\ell,b}(w)$ in $F_b(\mu,\hbar,\gamma;t)$ via the formula

$$
F_b(\mu,\hbar,\gamma;t)=\sum_{g,\ell=0}^{\infty}\frac{(-\gamma)^{\ell}\hbar^{2g-2+\ell}}{\mu^{b-2+2g+\ell}b!}\widetilde{C}_{g,\ell,b}(\mu t^2)-\mu\hbar^{-2}\delta_{b,1}. \tag{3.22}
$$

Taking into account the relations (1.20) for $C_{g,b}(w)$ and $C^{\mathfrak{r}}_{h,b}(w)$

$$
C_{g,b}(w)=\widetilde{C}_{g,0,b}(w),\qquad C^{\mathfrak{r}}_{h,b}(w)=\sum_{\substack{g,\ell=0\\2g+\ell=h}}^{\infty}2^g\widetilde{C}_{g,\ell,b}(w),
$$

we find complete agreement with the computational results (2.8) and (2.9) obtained from the β-deformed topological recursion via Gaussian resolvents.

A Enumeration of non-orientable fatgraphs with $k=3$

For the case of one backbone (i.e. one vertex) and $k=3$ chords, we list the non-oriented fatgraphs with $\chi=1$ in Figure 6 and $\chi=0$ in Figure 7.

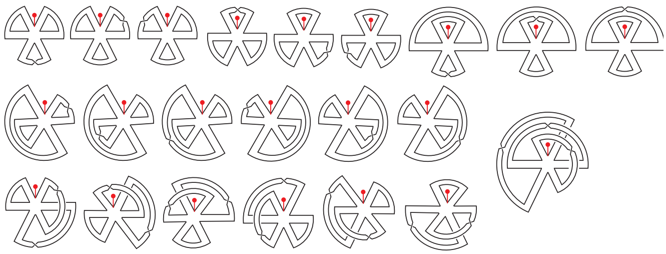

Figure 6: Non-oriented tailed fatgraphs with $b = 1$, $k = 3$ and $\chi = 1$. The total number of graphs is 22.

Both of these numbers of graphs agree with the number computed from the cut-and-join method and the time-dependent Schrödinger equation.

B The free energy $F_{0,2}$ in 1-cut β-deformed models

In this appendix, we determine the unstable term $F_{0,2}(\{r_n\})$ in the free energy (1.22) for the β-deformed eigenvalue integral (1.14) with the general potential (1.21)

$$F_{0,2}(\{r_n\}) = F^I_{0,2}(\{r_n\}) + F^A_{0,2}(\{r_n\}). \tag{B.1}$$

We consider genus $\mathsf{s}-1$ spectral curves of the form

$$\begin{aligned} &y(x)^2 = M(x)^2\sigma(x), \\ &\sigma(x) = \prod_{i=1}^{2\mathsf{s}}(x-\mathsf{q}_i), \qquad M(x) = c\prod_{i=1}^{f}(x-\alpha_i)^{m_i}, \end{aligned} \tag{B.2}$$

whose form is determined by the choice of the potential. The terms $F^I_{0,2}(\{r_n\})$ and $F^A_{0,2}(\{r_n\})$ are given by [33, 31]

$$F^I_{0,2}(\{r_n\}) = -\frac{1}{8\pi^2}\oint_{\mathcal{A}}\frac{dy(z')}{y(z')}\int_D dS(z,z')\log|y(z)|, \tag{B.3}$$

$$F^A_{0,2}(\{r_n\}) = -\frac{1}{12}\log\left|\prod_{i=1}^{2\mathsf{s}}M(\mathsf{q}_i)\cdot\prod_{1\le i<j\le 2\mathsf{s}}(\mathsf{q}_i-\mathsf{q}_j)\right|, \tag{B.4}$$

where $\mathcal{A} = \bigcup_{i=1}^{\mathsf{s}}\mathcal{A}_i$ is the contour around the branch cut $D = \bigcup_{i=1}^{\mathsf{s}} D_i$, $D_i = [\mathsf{q}_{2i-1}, \mathsf{q}_{2i}]$. Then $dS(x_1, x_2)$ is the third type differential, which is a 1-form in x_1

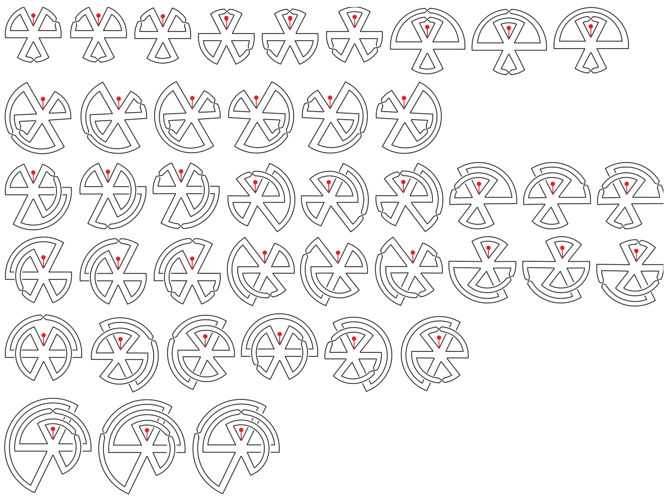

Figure 7: Non-oriented tailed fatgraphs with $b = 1$, $k = 3$ and $\chi = 0$. The total number of graphs is 42.

and a multivalued function of x_2, determined by the conditions

$$\bullet\ dS(x_1, x_2) \underset{x_1 \to x_2}{\sim} \frac{dx_1}{x_1 - x_2} + \text{reg.}, \qquad \bullet\ dS(x_1, x_2) \underset{x_1 \to \overline{x}_2}{\sim} -\frac{dx_1}{x_1 - x_2} + \text{reg.},$$

$$\bullet\ \oint_{x_2 \in \mathcal{A}_i} dS(x_1, x_2) = 0, \quad i = 1, \ldots, \mathsf{s} - 1. \tag{B.5}$$

Here $\overline{x}$ is the conjugate point of a point x on the spectral curve (B.2), such that

$$\sqrt{\sigma(x)} = -\sqrt{\sigma(\overline{x})}, \quad M(x) = M(\overline{x}). \tag{B.6}$$

In the following we consider the $\mathsf{s} = 1$ case in (B.2)

$$\sigma(x) = (x - a)(x - b), \quad a < b, \tag{B.7}$$

and by applying the method used to derive the unstable term $F_{0,2}(\{r_n\})$ in a 1-cut matrix model with a hard edge [24], we prove the formula (1.39).

Proposition B.1. *For the above genus* 0 *spectral curve, the unstable term* $F_{0,2}(\{r_n\})$ *takes form*

$$\begin{aligned} F_{0,2}(\{r_n\}) = &-\frac{1}{2}\sum_{i=1}^{f} m_i \log\left(1 - \mathfrak{s}_i^2\right) - \frac{1}{2}\sum_{i,j=1}^{f} m_i m_j \log\left(1 - \mathfrak{s}_i \mathfrak{s}_j\right) \\ &+ \frac{1}{24}\log\left|M(a)M(b)(a-b)^4\right|, \end{aligned} \tag{B.8}$$

where $\mathfrak{s}_i$, $i = 1, \ldots, f$ *are defined by*

$$\alpha_i(\mathfrak{s}_i) = \frac{a+b}{2} - \frac{a-b}{4}(\mathfrak{s}_i + \mathfrak{s}_i^{-1}), \qquad |\mathfrak{s}_i| < 1. \tag{B.9}$$

Proof. In the above genus 0 case the third type differential $dS(x_1, x_2)$ is given by

$$dS(x_1, x_2) = \frac{\sqrt{\sigma(x_2)}}{\sqrt{\sigma(x_1)}} \frac{dx_1}{x_1 - x_2}. \tag{B.10}$$

We introduce the Zhukovsky variable z by

$$x(\mathrm{z}) = \frac{a+b}{2} - \frac{a-b}{4}(\mathrm{z} + \mathrm{z}^{-1}). \tag{B.11}$$

Then the branch points $x = a, b$ are mapped to $\mathrm{z} = -1, +1$, and the first and second sheet of the spectral curve are mapped to the regions $|\mathrm{z}| \geq 1$ and $|\mathrm{z}| \leq 1$, respectively. Under this map we obtain

$$\sqrt{\sigma(x)} = \frac{b-a}{4}(\mathrm{z} - \mathrm{z}^{-1}), \qquad \frac{dx}{\sqrt{\sigma(x)}} = \frac{d\mathrm{z}}{\mathrm{z}}, \qquad \frac{\sqrt{\sigma(x_2)}}{x_1 - x_2} = \frac{\mathrm{z}_1}{\mathrm{z}_1 - \mathrm{z}_2} - \frac{\mathrm{z}_1}{\mathrm{z}_1 - \mathrm{z}_2^{-1}},$$

and the third type differential (B.10) is rewritten as

$$dS(x_1, x_2) = \frac{d\mathrm{z}_1}{\mathrm{z}_1 - \mathrm{z}_2} - \frac{d\mathrm{z}_1}{\mathrm{z}_1 - \mathrm{z}_2^{-1}}. \tag{B.12}$$

Under the map (B.11), the zeros or poles α_i of the moment function $M(x)$ on the spectral curve (B.2) are mapped to $2f$ points $\mathfrak{s}_i^{\pm 1}$, $i = 1, \ldots, f$,

$$\alpha_i(\mathfrak{s}_i) = \frac{a+b}{2} - \frac{a-b}{4}(\mathfrak{s}_i + \mathfrak{s}_i^{-1}),$$

and we obtain

$$x - \alpha_i = \frac{b-a}{4} \frac{(\mathrm{z} - \mathfrak{s}_i)(\mathrm{z} - \mathfrak{s}_i^{-1})}{\mathrm{z}}.$$

Without loss of generality, in this proof we can assume

$$|\mathfrak{s}_i| > 1.$$

First, let us consider (B.3). Since the variable z of the integrand is on the branch cut $D = [a, b]$, we can put $\mathrm{z} = e^{i\theta}$ in the Zhukovsky variable, and by (B.12)

$$dS(z, z') = -id\theta \left[\frac{\mathrm{z}}{\mathrm{z}'} \frac{1}{1 - \frac{\mathrm{z}}{\mathrm{z}'}} + \frac{1}{1 - \frac{1}{\mathrm{z}\mathrm{z}'}} \right] = -id\theta \left[1 + \sum_{k=1}^{\infty} \frac{2}{\mathrm{z}'^k} \cos k\theta \right]$$

is obtained, where we have used $|\mathrm{z}'| > |\mathrm{z}| = 1$. Then we obtain

$$
\begin{aligned}
\widehat{F}^I_{0,2}(z') &:= \int_D dS(z,z')\log|y(z)| \\
&= \int_0^\pi id\theta\Big[1+\sum_{k=1}^\infty \frac{2}{\mathrm{z}'^k}\cos k\theta\Big]\Big[\log\Big|c\frac{b-a}{4}\prod_{i=1}^f\Big(\frac{b-a}{4}\Big)^{m_i}\Big| + \log(2\sin\theta) \\
&\qquad\qquad + \log\prod_{i=1}^f|e^{i\theta}-\mathfrak{s}_i|^{m_i}|e^{i\theta}-\mathfrak{s}_i^{-1}|^{m_i}\Big] \\
&= i\pi\log\Big|c\frac{b-a}{4}\prod_{i=1}^f\Big(\frac{b-a}{4}\Big)^{m_i}\Big| - i\pi\sum_{k=1}^\infty\frac{1}{k\mathrm{z}_2^{2k}} \\
&\quad + i\sum_{i=1}^f m_i\int_0^\pi d\theta\Big[1+\sum_{k=1}^\infty\frac{2}{\mathrm{z}_2^k}\cos k\theta\Big]\log|e^{i\theta}-\mathfrak{s}_i||e^{i\theta}-\mathfrak{s}_i^{-1}|,
\end{aligned}
\tag{B.13}
$$

where we have used that

$$
\begin{aligned}
&\int_0^\pi d\theta\log(2\sin\theta) = 0,\\
&\int_0^\pi d\theta\cos k\theta\log\sin\theta = \begin{cases} -\frac{\pi}{k} & \text{if } k \text{ is a nonzero even integer,}\\ 0 & \text{if } k \text{ is an odd integer.}\end{cases}
\end{aligned}
$$

By

$$
\begin{aligned}
\log|e^{i\theta}-\mathfrak{s}_i| &= \log|\mathfrak{s}_i| - \frac12\sum_{k=1}^\infty\frac1k\Big(\frac{1}{\mathfrak{s}_i^k}+\frac{1}{\bar{\mathfrak{s}}_i^k}\Big)\cos k\theta - \frac{i}{2}\sum_{k=1}^\infty\frac1k\Big(\frac{1}{\mathfrak{s}_i^k}-\frac{1}{\bar{\mathfrak{s}}_i^k}\Big)\sin k\theta,\\
\log|e^{i\theta}-\mathfrak{s}_i^{-1}| &= -\frac12\sum_{k=1}^\infty\frac1k\Big(\frac{1}{\mathfrak{s}_i^k}+\frac{1}{\bar{\mathfrak{s}}_i^k}\Big)\cos k\theta + \frac{i}{2}\sum_{k=1}^\infty\frac1k\Big(\frac{1}{\mathfrak{s}_i^k}-\frac{1}{\bar{\mathfrak{s}}_i^k}\Big)\sin k\theta,
\end{aligned}
$$

(B.13) is written as

$$
\widehat{F}^I_{0,2}(z) = i\pi\Big[\log\Big|c\frac{b-a}{4}\prod_{i=1}^f\Big(\frac{b-a}{4}\mathfrak{s}_i\Big)^{m_i}\Big| - \sum_{k=1}^\infty\frac{1}{k\mathrm{z}^{2k}} - \sum_{i=1}^f m_i\sum_{k=1}^\infty\frac{2}{k\mathrm{z}^k\mathfrak{s}_i^k}\Big]. \tag{B.14}
$$

Here we have used

$$
\int_0^\pi d\theta\cos k\theta\cos\ell\theta = \frac{\pi}{2}\delta_{k,\ell},
$$

and the fact that for an arbitrary $\bar{\mathfrak{s}}_i$, there exists an $\mathfrak{s}_j$ such that $\bar{\mathfrak{s}}_i = \mathfrak{s}_j$. By

$$
\frac{dy(z)}{y(z)} = \Big[\frac{1}{\mathrm{z}-1}+\frac{1}{\mathrm{z}+1}-\frac{1}{\mathrm{z}}+\sum_{i=1}^f m_i\Big(\frac{1}{\mathrm{z}-\mathfrak{s}_i}+\frac{1}{\mathrm{z}-\mathfrak{s}_i^{-1}}-\frac{1}{\mathrm{z}}\Big)\Big]d\mathrm{z},
$$

and by (B.14), (B.3) is rewritten as

$$F^I_{0,2}(\{r_n\}) = \frac{1}{4}\oint_{\widetilde{\mathcal{A}}} \frac{dz}{2\pi i}\left[\frac{1}{\mathrm{z}-1} + \frac{1}{\mathrm{z}+1} - \frac{1}{\mathrm{z}} + \sum_{i=1}^{f} m_i\Big(\frac{1}{\mathrm{z}-\mathfrak{s}_i} + \frac{1}{\mathrm{z}-\mathfrak{s}_i^{-1}} - \frac{1}{\mathrm{z}}\Big)\right]$$
$$\times\left[\log\left|c\frac{b-a}{4}\prod_{i=1}^{f}\Big(\frac{b-a}{4}\mathfrak{s}_i\Big)^{m_i}\right| - \sum_{k=1}^{\infty}\frac{1}{k\mathrm{z}^{2k}} - \sum_{i=1}^{f} m_i \sum_{k=1}^{\infty}\frac{2}{k\mathrm{z}^k\mathfrak{s}_i^k}\right],$$

where $\widetilde{\mathcal{A}}$ is the contour around the unit disk $|\mathrm{z}| = 1$. This contour can be changed as

$$\oint_{\widetilde{\mathcal{A}}} \longrightarrow -\sum_{i=1}^{f}\oint_{\widetilde{\mathcal{A}}_i} - \oint_{\widetilde{\mathcal{A}}_\infty},$$

where $\widetilde{\mathcal{A}}_i$ and $\widetilde{\mathcal{A}}_\infty$ are the contours around $\mathfrak{s}_i$ and infinity, respectively. Then by

$$\sum_{k=1}^{\infty}\frac{u^k}{k} = -\log(1-u), \qquad |u| < 1,$$

we obtain

$$F^I_{0,2}(\{r_n\}) = -\frac{1}{2}\sum_{i=1}^{f} m_i \log\Big(1 - \frac{1}{\mathfrak{s}_i^2}\Big) - \frac{1}{2}\sum_{i,j=1}^{f} m_i m_j \log\Big(1 - \frac{1}{\mathfrak{s}_i\mathfrak{s}_j}\Big)$$
$$+ \frac{1}{8}\log\left|M(a)M(b)\Big(\frac{a-b}{4}\Big)^2\right|, \tag{B.15}$$

where we used

$$\log\left|c\prod_{i=1}^{f}\Big(\frac{b-a}{4}\mathfrak{s}_i\Big)^{m_i}\right| = -\sum_{i=1}^{f} m_i \log\Big(1-\frac{1}{\mathfrak{s}_i^2}\Big) + \frac{1}{2}\log\big|M(a)M(b)\big|.$$

With (B.4) for the genus 0 case, after changing $\mathfrak{s}_i$ to $\mathfrak{s}_i^{-1}$, we finally obtain the formula (B.8), where we ignored the constant term $-\log\sqrt{2}$. □

C An iterative solution using the quantum curve

The 1-backbone phase function $S_1(x,t)$

The solutions $S_{1,p}(t)$ $(p = 0, 1, 2)$ of the equations (3.17) – (3.20) are found successively as follows

$$S_{1,0}(t) = -t^3\frac{2\mu(2\mu+\epsilon_1+\epsilon_2)}{\epsilon_2\epsilon_2} - t^4\frac{\mu}{\epsilon_1\epsilon_2}\big(5\epsilon_1\epsilon_2 + 8\mu^2 + 10\mu(\epsilon_1+\epsilon_2) + 3(\epsilon_1^2+\epsilon_2^2)\big)$$
$$- t^6\frac{\mu}{2\epsilon_1\epsilon_2}\big(40\mu^3 + 88\mu^2(\epsilon_1+\epsilon_2) + 64\mu(\epsilon_1^2+\epsilon_2^2) + 108\mu\epsilon_1\epsilon_2 + 15(\epsilon_1^3+\epsilon_2^3)$$

$$
\begin{aligned}
&\quad + 32\epsilon_1\epsilon_2(\epsilon_1+\epsilon_1)\big)\\
&- t^8\frac{\mu}{4\epsilon_1\epsilon_2}\big(224\mu^4 + 744\mu^3(\epsilon_1+\epsilon_2) + 936\mu^2(\epsilon_1^2+\epsilon_2^2) + 1592\mu^2\epsilon_1\epsilon_2\\
&\quad + 520\mu(\epsilon_1^3+\epsilon_2^3) + 1130\mu\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2) + 260\epsilon_1\epsilon_1(\epsilon_1^2+\epsilon_2^2)\\
&\quad + 331\epsilon_1^2\epsilon_2^2 + 105(\epsilon_1^4+\epsilon_2^4)\big) + \mathcal{O}(t^{10}),
\end{aligned}
$$

$$
\begin{aligned}
S_{1,1}(t) &= t\frac{2\mu}{\epsilon_2} + t^3\frac{3\mu(2\mu+\epsilon_1+\epsilon_2)}{\epsilon_2} + t^5\frac{5\mu}{2\epsilon_2}\big(8\mu^2 + 10\mu(\epsilon_1+\epsilon_2) + 3(\epsilon_1^2+\epsilon_2^2) + 5\epsilon_1\epsilon_2\big)\\
&+ t^7\frac{7\mu}{4\epsilon_2}\big(40\mu^3 + 88\mu^2(\epsilon_1+\epsilon_2) + 64\mu(\epsilon_1^2+\epsilon_2^2) + 108\mu\epsilon_1\epsilon_2 + 15(\epsilon_1^3+\epsilon_2^3)\\
&\quad + 32\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)\big) + \mathcal{O}(t^9),
\end{aligned}
$$

$$
\begin{aligned}
S_{1,2}(t) &= t^2\frac{\mu(2\mu+\epsilon_2)}{\epsilon_2} + t^4\frac{\mu(2\mu+\epsilon_2)(4\mu+2\epsilon_1+3\epsilon_2)}{\epsilon_2}\\
&+ t^6\frac{3\mu(2\mu+\epsilon_2)\big(20\mu^2 + 2\mu(12\epsilon_1+17\epsilon_2) + 7\epsilon_1^2 + 15\epsilon_2^2 + 17\epsilon_1\epsilon_2\big)}{4\epsilon_2} + \mathcal{O}(t^8).
\end{aligned}
$$

The 2-backbone phase function $S_2(x,t)$

The solutions $S_{2,p}(t)$ $(p=0,1,2)$ of equation (3.21) take form

$$
\begin{aligned}
S_{2,0}(t) &= -t^2\frac{2\mu}{\epsilon_1\epsilon_2} - t^4\frac{8\mu(2\mu+\epsilon_1+\epsilon_2)}{\epsilon_1\epsilon_2}\\
&- t^6\frac{3\mu}{2\epsilon_1\epsilon_2}\big(64\mu^2 + 78\mu(\epsilon_1+\epsilon_2) + 23(\epsilon_1^2+\epsilon_2)^2 + 39\epsilon_1\epsilon_2\big)\\
&- t^8\frac{\mu}{\epsilon_1\epsilon_2}\big(512\mu^3 + 1084\mu^2(\epsilon_1+\epsilon_2) + 762\mu(\epsilon_1^2+\epsilon_2^2) + 1304\mu\epsilon_1\epsilon_2\\
&\quad + 174(\epsilon_1^3+\epsilon_2^3) + 381\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)\big) + \mathcal{O}(t^{10}),
\end{aligned}
$$

$$
\begin{aligned}
S_{2,1}(t) &= t^3\frac{4\mu}{\epsilon_2} + t^5\frac{24\mu(2\mu+\epsilon_1+\epsilon_2)}{\epsilon_2}\\
&+ t^7\frac{6\mu}{\epsilon_2}\big(64\mu^2 + 78\mu(\epsilon_1+\epsilon_2) + 23(\epsilon_1^2+\epsilon_2^2) + 39\epsilon_1\epsilon_2\big) + \mathcal{O}(t^9),
\end{aligned}
$$

$$
\begin{aligned}
S_{2,2}(t) &= t^2\frac{\mu}{\epsilon_2} + t^4\frac{\mu(16\mu+3\epsilon_1+8\epsilon_2)}{\epsilon_2}\\
&+ t^6\frac{3\mu}{4\epsilon_2}\big(192\mu^2 + 2\mu(61\epsilon_1+117\epsilon_2) + 13\epsilon_1^2 + 69\epsilon_2^2 + 61\epsilon_1\epsilon_2\big) + \mathcal{O}(t^8).
\end{aligned}
$$

The 3-backbone phase function $S_3(x,t)$

The solutions $S_{3,p}(t)$ $(p=0,1,2)$ of equation (3.21) take form

$$
S_{3,0}(t) = -t^4\frac{4\mu}{\epsilon_1\epsilon_2} - t^6\frac{116\mu(2\mu+\epsilon_1+\epsilon_2)}{3\epsilon_1\epsilon_2}
$$

$$- t^8 \frac{6\mu}{\epsilon_1\epsilon_2}\big(148\mu^2 + 178\mu(\epsilon_1+\epsilon_2) + 52(\epsilon_1^2+\epsilon_2^2) + 89\epsilon_1\epsilon_2\big) + \mathcal{O}(t^{10}),$$

$$S_{3,1}(t) = t^5\frac{14\mu}{\epsilon_2} + t^7\frac{174\mu(2\mu+\epsilon_1+\epsilon_2)}{\epsilon_2} + \mathcal{O}(t^9),$$

$$S_{3,2}(t) = t^4\frac{4\mu}{\epsilon_2} + t^6\frac{2\mu(58\mu+15\epsilon_1+29\epsilon_2)}{\epsilon_2} + \mathcal{O}(t^8).$$

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Jørgen Ellegaard Andersen
Centre for Quantum Geometry of Moduli Spaces,
Department of Mathematics, Aarhus University,
DK-8000 Aarhus C Denmark
jea.qgm@gmail.com

Hiroyuki Fuji
Faculty of Education, Kagawa University,
Takamatsu 760-8522 Japan
fuji@ed.kagawa-u.ac.jp

Masahide Manabe
Faculty of Physics, University of Warsaw,
ul. Pasteura 5, 02-093 Warsaw, Poland
masahidemanabe@gmail.com

Robert C. Penner
Institut des Hautes Études Scientifiques,
35 route de Chartres, 91440 Burs-sur-Yvette, France;
Division of Physics, Mathematics and Astronomy,
California Institute of Technology, Pasadena, CA 91125, USA
rpenner@caltech.edu, rpenner@ihes.fr

Piotr Sułkowski
Faculty of Physics, University of Warsaw,
ul. Pasteura 5, 02-093 Warsaw, Poland;
Walter Burke Institute for Theoretical Physics,
California Institute of Technology, Pasadena, CA 91125, USA
psulkows@fuw.edu.pl

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