CONSTANT GAUSS CURVATURE FOLIATIONS OF ADS SPACETIMES WITH PARTICLES

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Abstract. We prove that for any convex globally hyperbolic compact maximal (GHCM) anti-de Sitter (AdS) 3-dimensional space-time N with particles (cone singularities of angles less than π along time-like lines), the complement of the convex core in N admits a unique foliation by constant Gauss curvature surfaces. This extends, and provides a new proof of, a result of [6]. We also describe a parametrization of the space of convex GHCM AdS metrics on a given manifold, with particles of given angles, by the product of two copies of the Teichmüller space of hyperbolic metrics with cone singularities of fixed angles. Finally, we use the results on K-surfaces to extend to hyperbolic surfaces with cone singularities of angles less than π a number of results concerning landslides, which are smoother analogs of earthquakes sharing some of their key properties.

Keywords: convex GHCM AdS spacetime with particles; constant curvature surface; minimal Lagrangian map; landslide.

1. Introduction

Let $\theta = (\theta_1, ..., \theta_{n_0}) \in (0, \pi)^{n_0}$. In this paper we consider an oriented closed surface Σ of genus $\mathfrak{g}$ with n_0 marked points $p_1, ..., p_{n_0}$ and suppose that

$$2\pi(2 - 2\mathfrak{g}) + \sum_{i=1}^{n_0} (\theta_i - 2\pi) < 0.$$

This ensures that Σ can be equipped with a hyperbolic metric with cone singularities of angles θ_i at the marked points p_i for $i = 1, ..., n_0$ (see e.g. [39]). Denote by $\mathcal{T}_{\Sigma,\theta}$ the Teichmüller space of hyperbolic metrics on Σ with fixed cone angles, which is the space of hyperbolic metrics on Σ with cone singularities of angle θ_i at p_i, considered up to isotopies fixing each marked point (see more precisely Section 2.1).

1.1. Hyperbolic and anti-de Sitter manifolds associated to a surface.

1.1. Hyperbolic and anti-de Sitter manifolds associated to a surface. There is a deep, and to a large extend well-understood, connection between the space of conformal (or hyperbolic) structures on a closed surface S of genus at least two, and the space of *quasifuchsian* hyperbolic structures on $S \times \mathbb{R}$. A quasifuchsian hyperbolic structure on $S \times \mathbb{R}$ is a complete hyperbolic structure g on $S \times \mathbb{R}$ such that $S \times \mathbb{R}$ contains a non-empty, compact, geodesically convex subset. (We say that $K \subset S \times \mathbb{R}$ is *geodesically convex* if any geodesic segment in $(S \times \mathbb{R}, g)$ with endpoints in K is contained in K.) Given a quasifuchsian hyperbolic structure g on $S \times \mathbb{R}$, its ideal boundary is the disjoint union of two surfaces each homeomorphic to S, and each is equipped with a conformal structure well-defined up to isotopy, that is, an element of the Teichmüller space $\mathcal{T}_S$ of S.

The Bers Double Uniformization Theorem [8] asserts that any pair $(c_-, c_+) \in \mathcal{T}_S \times \mathcal{T}_S$ is obtained in this manner for a unique quasifuchsian hyperbolic structure on $S \times \mathbb{R}$ (considered up to isotopy). This homeomorphism between the moduli space of quasifuchsian hyperbolic structure on $S \times \mathbb{R}$ and $\mathcal{T}_S \times \mathcal{T}_S$ can be used for instance to recover the Weil-Petersson complex structure on $\mathcal{T}_S$, see e.g. [2].

Date: v2, September 1, 2019.

Q. Chen was partially supported by the International Program Funding for Ph.D. Candidates, Sun Yat-Sen University and the Shanghai Super Postdoctoral Incentive Program 2018.

JMS was partially supported by UL IRP grant NeoGeo and FNR grants INTER/ANR/15/11211745 and OPEN/16/11405402. The author also acknowledge support from U.S. National Science Foundation grants DMS-1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties" (the GEAR Network).

We are interested here in a Lorentzian cousin of hyperbolic geometry, the anti-de Sitter (AdS) geometry. The space AdS_3 is a 3-dimensional Lorentzian space of constant curvature -1, see Section 2.2. We will consider 3-dimensional manifolds endowed with an AdS structure, that is, a geometric structure locally modeled on AdS_3. Those AdS manifolds are also called AdS spacetimes, since they occur naturally in connection to gravitation. An AdS spacetime is *globally hyperbolic compact* (GHC) if it contains a closed Cauchy surface, and it is *globally hyperbolic compact maximal* (GHCM) if in addition any isometric embedding into a globally hyperbolic compact spacetime of the same dimension is an isometry. GHCM AdS spacetimes have been shown by G. Mess [3, 26] to present remarkable analogies with quasifuchsian hyperbolic manifolds, and they are now often called *quasifuchsian AdS spacetimes* in the mathematics literature.

1.2. **AdS spacetimes with particles.** Cone singularities play a significant role in hyperbolic geometry, see e.g. [9, 14, 15]. For quasifuchsian hyperbolic manifolds, it is interesting to consider cone singularities of angle less than π along infinite lines with endpoints on the two boundary components. Those infinite cone singularities are called "particles" for a reason that should be clear below. There is an extension of the Bers Double Uniformization Theorem to quasifuchsian hyperbolic manifolds with particles [24, 27] of fixed angles, with the conformal structure at infinity now marked by the position of the endpoints of the particles.

Here we are particularly interested in the AdS analogs of those quasifuchsian hyperbolic manifolds with particles: quasifuchsian AdS spacetimes with *particles*, that is, cone singularities of angles less than π along time-like lines. There is an extension of Mess' AdS version of double uniformization to this setting with particles, see [12].

Cone singularities of this type are used in the physics literature to model point particles in 3d gravity, see e.g. [34, 35]. (More details on quasifuchsian AdS spacetimes with particles, including a precise definition, can be found in Section 2.5.)

We say that a GHCM AdS spacetime with particles is *convex* if it contains a (locally) convex Cauchy surface. It turns out that convex GHCM AdS spacetimes with particles contain a smallest non-empty geodesically convex subset, called their *convex core*, see [12]. We denote by $\mathcal{GH}_{\Sigma,\theta}$ the space of convex GHCM AdS metrics on $\Sigma \times \mathbb{R}$ with cone singularities of angles θ_i along the lines $\{p_i\} \times \mathbb{R}$, considered up to isotopies fixing each singular line (see the definition in Section 2.5).

1.3. **Foliations of AdS spacetimes by K-surfaces.** Our main result (Theorem 1.1 below) asserts that in any convex GHCM AdS spacetime with particles, the complement of the convex core admits a unique foliation by constant Gauss curvature surfaces. This extends to spacetimes with particles a result of Béguin, Barbot and Zeghib [6] for non-singular GHCM AdS spacetimes.

Theorem 1.1. *Let (N, g) be a convex GHCM AdS spacetime with particles and let $C(N)$ be the convex core of N. Then $N \setminus C(N)$ admits a unique foliation by locally strictly convex, constant Gauss curvature surfaces which are orthogonal to the singular lines.*

By a *strictly convex* surface we mean, here and in the rest of the paper, a surface with a second fundamental form which is either positive definite or negative definite.

Note that an analog of this result in the hyperbolic and de Sitter context was obtained in [17], which can be considered as a continuation of the present paper (although it was published earlier).

We denote $\mathfrak{p} = (p_1, ..., p_{n_0})$ and let $\mathcal{ML}_{\Sigma,n_0}$ be the space of measured laminations on $\Sigma_{\mathfrak{p}} = \Sigma \setminus \{p_1, ..., p_{n_0}\}$. It is shown in [12, Lemma 2.2] that for each $g \in \mathcal{T}_{\Sigma,\theta}$, any $\lambda \in \mathcal{ML}_{\Sigma,n_0}$ can be uniquely realized as a geodesic lamination on $(\Sigma_{\mathfrak{p}}, g)$.

1.4. **Parameterization of the space of GHCM AdS spacetimes.** It is known that $\mathcal{GH}_{\Sigma,\theta}$ can be parameterized in several ways, such as the extension of Mess parameterization by $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ in terms of the left and right metrics [12, Theorem 1.4], and the parameterization by $\mathcal{T}_{\Sigma,\theta} \times \mathcal{ML}_{\Sigma,n_0}$ in terms of the embedding data (the induced metric and the bending lamination) of the past (or future) boundary of the convex core [12, Proposition 5.4, Proposition 5.8]. The first parameterization is equivalent to Thurston's Earthquake Theorem for hyperbolic metrics on Σ with cone singularities of fixed angles less than π (see [12, Theorem 1.2]).

Moreover, $\mathcal{GH}_{\Sigma,\theta}$ can also be parameterized by the cotangent bundle $T^*\mathcal{T}_{\Sigma,\theta}$ of $\mathcal{T}_{\Sigma,\theta}$, since $T^*\mathcal{T}_{\Sigma,\theta}$ is homeomorphic to the quotient of the space $\mathcal{H}_{\Sigma,\theta}$ of maximal surfaces in germs of AdS spacetimes with particles by diffeomorphisms isotopic to the identity fixing each marked point of Σ (see [23, Theorem 5.11]) and there is a bijection between this quotient space and $\mathcal{GH}_{\Sigma,\theta}$ (see [37, Theorem 1.4]).

We give a new parameterization of $\mathcal{GH}_{\Sigma,\theta}$ by $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ in terms of constant Gauss curvature surfaces. Specifically, we consider the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{GH}_{\Sigma,\theta}$, for each $K < -1$, which assigns to an element $(\tau, \tau') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ the isotopy class of the (unique) convex GHCM AdS spacetime (N, g) with particles, such that it contains a future-convex, spacelike, constant curvature K surface which is orthogonal to the singular lines, with induced metric $I \in \tau$ and third fundamental form $III \in \tau'$.

Theorem 1.2. *For any $K \in (-\infty, -1)$ and $\theta = (\theta_1, \cdots, \theta_{n_0}) \in (0, \pi)^{n_0}$, the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{GH}_{\Sigma,\theta}$ is a homeomorphism.*

Furthermore, we find that this result provides a convenient tool to prove the existence and uniqueness of the foliation of the complement of the convex core in a convex GHCM AdS spacetime with particles by locally strictly convex constant (Gauss) curvature surfaces which are orthogonal to the singular lines.

In the case of a non-singular 3-dimensional GHCM Lorentzian manifold of constant curvature, the corresponding result about the foliation by constant Gauss curvature surfaces has been proved by Barbot, Béguin and Zeghib (see Theorem 2.1 in [6]). For the existence part, the argument in [6] depends on the construction of barriers (see Definition 3.1 in [6]) and a barriers theorem of Gerhardt (see [20]) to find the surface of a given constant curvature from the barriers. Here by contrast, Theorem 1.1 is obtained as a consequence of Theorem 1.2, and we obtain a simpler approach to prove the existence of the foliation without using the barriers argument.

Remark 1.3. *For convenience, constant Gauss curvature surfaces are called constant curvature surfaces, or simply K-surfaces, henceforth.*

1.5. **Landslides on hyperbolic surfaces with cone singularities.** Finally, we use the results obtained on K-surfaces in GHCM AdS spacetimes with particles to extend some recent results on the landslide flow (see [10, 11]) to hyperbolic surfaces with cone singularities of fixed angles less than π.

Landslides are transformations of hyperbolic structures on a closed surface S of genus at least two, introduced in [10, 11] as "smoother" analogs of earthquakes. Earthquakes depend on the choice of a measured lamination $\lambda \in \mathcal{ML}_S$, so the earthquake flow can be defined as a map

$$\mathcal{E} : \mathcal{T}_S \times \mathcal{ML}_S \times \mathbb{R} \to \mathcal{T}_S$$

$$(h, \lambda, t) \mapsto E_{t\lambda}(h) ,$$

which for fixed $\lambda \in \mathcal{ML}_S$ defines an action of $\mathbb{R}$ on $\mathcal{T}_S$.

Landslide transformations, on the other hand, can be described as an action of S^1 on $\mathcal{T}_S \times \mathcal{T}_S$. For $e^{i\alpha} \in S^1$ and $(h, h') \in \mathcal{T}_S \times \mathcal{T}_S$, we denote by $L_{e^{i\alpha}}(h, h') \in \mathcal{T}_S \times \mathcal{T}_S$ the image of (h, h') by the landslide flow, and $L^1_{e^{i\alpha}}(h, h')$ its projection on the first factor. If $(t_n)_{n\in\mathbb{N}}$ and $(h'_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{R}_{>0}$ and $\mathcal{T}_S$, respectively, such that $t_n h'_n \to \lambda \in \mathcal{ML}_S$, then $L^1_{e^{it_n}}(h, h'_n) \to E_{\lambda/2}(h)$ as $n \to \infty$, see [10, Theorem 1.12].

In Section 5, we use Theorem 1.1 and other tools to extend the definition of landslide transformations to hyperbolic surfaces with cone singularities of fixed angles less than π. We show that the analog of Thurston's Earthquake Theorem extends to landslides on those hyperbolic cone surfaces: for all $h_1, h_2 \in \mathcal{T}_{\Sigma,\theta}$ and all $e^{i\alpha} \in S^1 \setminus \{1\}$, there exists a unique $h'_1 \in \mathcal{T}_{\Sigma,\theta}$ such that $L^1_{e^{i\alpha}}(h_1, h'_1) = h_2$, see Theorem 5.8.

We then go on to deduce from the properties of the landslide flow further results on the induced metrics and third fundamental forms of different K-surfaces in a given GHCM AdS spacetime with particles, see Theorem 5.15.

1.6. **Outline of the paper.** Section 2 is devoted to background material on different notions necessary for the rest of the paper: hyperbolic surfaces with cone singularities, AdS spacetimes with particles, etc. In Section 3 we prove Theorem 1.2, while Section 4 is devoted to the proof of Theorem 1.1. Section 5 describes applications to the landslide flow on the space of hyperbolic metrics with cone singuarities (of fixed angles) on a surface.

Acknowledgement. We are grateful to an anonymous referee for many smart and constructive comments that helped improve the content and exposition of the paper.

2. Background material

2.1. **Hyperbolic metrics with cone singularities.** First we recall the local model of a hyperbolic metric with a cone singularity of angle θ_0.

Let $\mathbb{H}^2$ be the Poincaré model of the hyperbolic plane. Denote by $\mathbb{H}^2_{\theta_0}$ the space obtained from $\mathbb{H}^2$ by taking a wedge of angle θ_0 bounded by two half-lines intersecting at the center 0 of $\mathbb{H}^2$ and gluing the two half-lines by a rotation fixing 0. We call $\mathbb{H}^2_{\theta_0}$ the *hyperbolic disk with cone singularity of angle* θ_0, which is a singular disk (with the singular point at the center) with the induced metric

$$g_{\theta_0} = dr^2 + \sinh^2(r)d\alpha^2,$$

where $(r,\alpha) \in \mathbb{R}_{\geq 0} \times \mathbb{R}/\theta_0\mathbb{Z}$ is a polar coordinate of $\mathbb{H}^2_{\theta_0}$.

Definition 2.1. *Let $\mathfrak{p} = (p_1, ..., p_{n_0})$ and $\theta = (\theta_1, ..., \theta_{n_0}) \in (0,\pi)^{n_0}$. A hyperbolic metric on Σ with cone singularities of angle θ at $\mathfrak{p}$ is a (singular) Riemannian metric which has constant curvature -1 on $\Sigma_\mathfrak{p}$ and such that each p_i has a neighborhood isometric to a neighborhood of the singular point in $\mathbb{H}^2_{\theta_0}$. Denote by $\mathfrak{M}^\theta_{-1}$ the space of hyperbolic metrics on Σ with cone singularities of angle θ at $\mathfrak{p}$.*

It is shown by Troyanov [39, Theorem A] and McOwen [25] that each conformal class of a metric on the surface Σ with marked points $p_1, ..., p_{n_0}$ admits a unique hyperbolic metric with cone singularities of angle θ_i at the p_i, as soon as

$$\chi(\Sigma,\theta) := \chi(\Sigma) + \sum_{i=1}^{n_0}(\frac{\theta_i}{2\pi} - 1) < 0 \ .$$

For each $g \in \mathfrak{M}^\theta_{-1}$, there exists a conformal coordinate z in a neighborhood U_i of p_i such that

$$g|_{U_i} = e^{2u_i(z)}|z|^{2(\beta_i-1)}|dz|^2,$$

where $u_i : U_i \to \mathbb{R}$ is $\mathcal{C}^2$ outside p_i and Hölder continuous on U_i (see the proof of the main theorem in [25]) and $\beta_i = \theta_i/(2\pi)$.

Let $\mathfrak{Diff}_0(\Sigma_\mathfrak{p})$ denote the space of diffeomorphisms on $\Sigma_\mathfrak{p}$ which are isotopic to the identity (fixing each marked point). They act by pull-back on $\mathfrak{M}^\theta_{-1}$. We say that two metrics $h_1, h_2 \in \mathfrak{M}^\theta_{-1}$ are *isotopic* if there exists a map $f \in \mathfrak{Diff}_0(\Sigma_\mathfrak{p})$ such that h_1 is the pull back by f of h_2.

Denote by $\mathcal{T}_{\Sigma,\theta}$ the *Teichmüller space of hyperbolic metrics on Σ with fixed cone angle θ*, which is the space of isotopy classes of hyperbolic metrics on Σ with cone singularities of angle θ at $\mathfrak{p}$. Note that $\mathcal{T}_{\Sigma,\theta} = \mathfrak{M}^\theta_{-1}/\mathfrak{Diff}_0(\Sigma_\mathfrak{p})$ and $\mathfrak{M}^\theta_{-1}$ is a differentiable submanifold of the manifold consisting of all symmetric (0,2)-type $\mathcal{C}^2$ tensor fields on $\Sigma_\mathfrak{p}$. $\mathcal{T}_{\Sigma,\theta}$ is a finite-dimensional differentiable manifold which inherits a natural quotient topology. See e.g. [19, 29, 32, 33] for more details.

2.2. **The 3-dimensional anti-de Sitter space** AdS_3**.** Let $\mathbb{R}^{2,2}$ be $\mathbb{R}^4$ with the quadratic form $q(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2$. The *anti-de Sitter (AdS) 3-space* is defined as the quadric:

$$AdS_3 = \{x \in \mathbb{R}^{2,2} : q(x) = -1\}.$$

It is a 3-dimensional Lorentzian symmetric space of constant curvature -1 diffeomorphic to $\mathbb{D} \times S^1$, where $\mathbb{D}$ is a 2-dimensional disk.

Consider the projective map $\pi : \mathbb{R}^{2,2}\backslash\{0\} \to \mathbb{RP}^3$. The *Klein model* $\mathbb{ADS}_3$ of AdS 3-space is defined as the image of AdS_3 under the projection π. It is clear that $\mathbb{ADS}_3 = \pi(AdS_3) = AdS_3/\{\pm id\}$. The boundary $\partial\mathbb{ADS}_3$ is the image of the quadratic $Q = \{x \in \mathbb{R}^{2,2} : q(x) = 0\}$ under π, which is foliated by two families of projective lines, called the left and right leaves, respectively.

Geodesics in the Klein model $\mathbb{ADS}_3$ are given by projective lines: the spacelike geodesics correspond to the projective lines intersecting the boundary $\partial\mathbb{ADS}_3$ in two points, while lightlike geodesics are tangent to $\partial\mathbb{ADS}_3$, and timelike geodesics do not intersect $\partial\mathbb{ADS}_3$.

The group $\mathrm{Isom}_0(\mathbb{ADS}_3)$ of orienation and time-orientation preserving isometries of $\mathbb{ADS}_3$ can be identified as $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. One simple way to see this is because the boundary of the Klein model $\mathbb{ADS}_3$ is foliated by two families of lines, and those two foliations are invariant under any orientation and time-orientation isometry of AdS_3 since isometries act projectively in the Klein model. The set of leaves of each foliation is equipped with a real projective structure by the intersections with any leave of the other foliation, and the action of $\mathrm{Isom}_0(\mathbb{ADS}_3)$ defines in this way two elements of $PSL(2,\mathbb{R})$.

More details about the geometry of AdS_3 can be found in e.g. [3, 4, 26] or [18, Section 2.2].

2.3. **The singular AdS 3-space.** We now proceed to define the notion AdS spacetimes with cone singularities, see Section 2.4 below.

Let $\theta_0 > 0$. Define the *singular AdS 3-space* of angle θ_0 as

$$AdS^3_{\theta_0} := \{(t, r, \alpha) \in \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}/\theta_0\mathbb{Z}\} ,$$

with the metric

$$-\cosh^2(r)dt^2 + dr^2 + \sinh^2(r)d\alpha^2. \tag{1}$$

The set corresponding to $r = 0$ is called the *singular line* in $AdS^3_{\theta_0}$ and θ_0 is called the *total angle* around this singular line.

In the neighbourhood of the totally geodesic plane $P_0 = \{(t, r, \alpha) \in AdS^3_{\theta_0} : t = 0\}$ which consists of the points at time-like distance less than $\pi/2$ from P_0, this metric can also written (with different variables) as

$$-dt^2 + \cos^2(t)(dr^2 + \sinh^2(r)d\alpha^2) . \tag{2}$$

It is clear that $AdS^3_{\theta_0}$ is a Lorentzian spacetime of constant curvature -1 outside the singular line, that is, it is locally modelled on AdS_3. Indeed, it is obtained from the complete hyperbolic surface with a cone singularity of angle θ_0 by taking a warped product with $\mathbb{R}$ (see e.g. [16, 23, 37]).

There is a well-defined notion of totally geodesic plane orthogonal to the singular line in $AdS^3_{\theta_0}$. Those planes are precisely the subsets of equation $t = t_0$ in the expression (2). They are totally geodesic outside the singular locus, and can be considered to be orthogonal to this singular locus. There is one such plane going through each point of the singular locus.

2.4. **AdS spacetimes with particles.** An *AdS spacetime with particles* is a (singular) Lorentzian 3-manifold in which any point x has a neighbourhood isometric to a subset of $AdS^3_{\theta_0}$ for some $\theta_0 \in (0, \pi)$.

Definition 2.2. *Let $S \subset AdS^3_{\theta_0}$ be a spacelike surface which intersects the singular line at a point x. S is said to be orthogonal to the singular line at x if the time-like distance to the totally geodesic plane P orthogonal to the singular line at x satisfies:*

$$\lim_{y \to x, y \in S} \frac{d(y, P)}{d_S(x, y)} = 0 . \tag{3}$$

Here $d(y, P)$ is the length of the unique time-like geodesic segment through y orthogonal to P (this is the natural notion of Lorentzian distance) which is well-defined if y is close enough to P, and $d_S(x, y)$ is the distance between x and y along S. Note that $d_S(x, y)$ is finite since the expression (1) shows that the length of any curve on S is bounded from above by the length of its orthogonal projection on P.

Definition 2.3. *Let S be a spacelike surface in an AdS spacetime M with particles which intersects a singular line l at a point x'. S is said to be orthogonal to l at x' if there exists a neighborhood $U \subset M$ of x' which is isometric to a neighborhood of a singular point in $AdS^3_{\theta_0}$ such that the isometry sends $S \cap U$ to a surface orthogonal to the singular line in $AdS^3_{\theta_0}$.*

2.5. **Convex GHCM AdS spacetimes with particles.** We can now define the notion of convex GHCM spacetimes with particles appearing in Theorem 1.1.

Definition 2.4. *An AdS spacetime M with particles is convex GHCM if*

- *M is convex GHC: it contains a closed Cauchy surface (i.e. a spacelike surface intersecting each inextensible timelike curve exactly once) which is locally convex and orthogonal to the singular lines.*
- *M is maximal: if any isometric embedding of M into a convex GHC AdS spacetime is an isometry.*

It is well-known that a globally hyperbolic spacetime (with particles or not) is topologically a product of any of its Cauchy surfaces with an interval, see e.g. [7, Chapter 3] and [12, Proposition 2.4].

The convexity condition on Cauchy surfaces ensures the existence of convex core in a GHCM AdS spacetime with particles [12, Lemma 4.9]. Note that the convexity condition might be technical, since it is still unknown whether every GHCM AdS spacetime with particles always contains a convex Cauchy surface. On the other hand, the orthogonality condition above ensures that the induced metric on a Cauchy surface carries cone singularities of the same angle as the total angle around the singular lines at the intersection with the singular locus [23, Section 5.1].

Let $\mathcal{GH}'_{\Sigma,\theta}$ be the space of convex GHCM AdS metrics on $\Sigma\times\mathbb{R}$ with cone singularities of (total) angles θ_i along the lines $\{p_i\}\times\mathbb{R}$. Denote by $\mathfrak{Diff}_0(\Sigma\times\mathbb{R})$ the space of diffeomorphisms on $\Sigma\times\mathbb{R}$ isotopic the identity fixing each singular line. We say that two metrics $g_1, g_2\in\mathcal{GH}'_{\Sigma,\theta}$ are *isotopic* if there exists a map $f\in\mathfrak{Diff}_0(\Sigma\times\mathbb{R})$ such that g_1 is the pull back by f of g_2.

Denote by $\mathcal{GH}_{\Sigma,\theta}$ the *space of convex GHCM AdS metrics on* $\Sigma\times\mathbb{R}$ *with particles of fixed angle* θ, which is the space of isotopy classes of convex GHCM AdS metrics with cone singularities of angles θ_i along the lines $\{p_i\}\times\mathbb{R}$. Note that $\mathcal{GH}_{\Sigma,\theta}=\mathcal{GH}'_{\Sigma,\theta}/\mathfrak{Diff}_0(\Sigma\times\mathbb{R})$ and it is a finite-dimensional differentiable manifold with a natural quotient topology, see [12, 23].

2.6. **Convex spacelike surfaces in a convex GHCM AdS spacetime with particles.** Let (N, g) be a convex GHCM AdS spacetime with particles. Let $S\subset N$ be an (embedded) spacelike surface orthogonal to the singular lines with the induced metric I. The shape operator $B: TS\to TS$ of S is defined as

$$B(u)=\nabla_u n,$$

where n is the future-directed unit normal vector field on S and ∇ is the Levi-Civita connection of (N, g). The second and third fundamental forms of S are defined respectively as

$$II(u,v)=I(Bu,v),\qquad III(u,v)=I(Bu,Bv).$$

Definition 2.5. *Let S be a convex spacelike surface orthogonal to the singular lines in a convex GHCM AdS spacetime N with particles. We say that S is future-convex (resp. past-convex) if its future $I^+(S)$ (resp. its past $I^-(S)$) is geodesically convex. We say that S is strictly future-convex (resp. strictly past-convex) if $I^+(S)$ (resp. $I^-(S)$) is strictly geodesically convex.*

Note that if S is future-convex (resp. past-convex), then for each regular point x of S, both the principal curvatures at x are non-negative (resp. non-positive). If S is strictly future-convex (resp. strictly past-convex), then for each regular point x of S, both the principal curvatures at x are positive (resp. negative).

2.7. **The duality between strictly convex surfaces in convex GHCM AdS spacetimes with particles.** First we recall the duality between points and hyperplanes in AdS_3 (see e.g. [6, 11]).

Observe that AdS_3 is a quadric in $\mathbb{R}^{2,2}$. Every point x in AdS_3 is exactly the intersection in $\mathbb{R}^{2,2}$ of AdS_3 with a ray l starting from the origin 0 on which the quadratic form is negative definite. Denote by $l^\perp$ the hyperplane orthogonal to l in $\mathbb{R}^{2,2}$, with the induced metric of signature (2,1). The intersection between $l^\perp$ and AdS_3 is the disjoint union of two totally geodesic spacelike planes $P_x^\pm$, where P_x^+ (resp. P_x^-) is at a distance $\pi/2$ in the future (resp. in the past) of x.

Conversely, every totally geodesic spacelike plane P in AdS_3 is the intersection of AdS_3 with a hyperplane H of signature (2,1) in $\mathbb{R}^{2,2}$. The orthogonal $H^\perp$ of H intersects AdS_3 at two antipodal points $x_P^\pm$, where x_P^+ (resp. x_P^-) is at a distance $\pi/2$ in the future (resp. in the past) of P.

We define the dual P^* of P as the past (resp. future) intersection point x_P^- if the dual x^* of x is defined to be $AdS_3 \cap P_x^+$ (resp. $AdS_3 \cap P_x^-$). The *dual surface* S^* of a strictly convex surface $S \subset AdS_3$ is defined as the set of points on the convex side of S which are the dual points of the support planes of S. Equivalently, S^* can be obtained by pushing S along orthogonal geodesics on the convex side for a distance $\pi/2$ (see [6, Proposition 11.9]).

Note that $AdS^3_{\theta_0}$ can be obtained from the universal cover of AdS_3 by taking a wedge of angle θ_0 bounded by two time-like totally geodesic half-planes and gluing the two half-planes by a rotation fixing the common time-like geodesic. For a strictly convex spacelike surface $S \subset AdS^3_{\theta_0}$ orthogonal to the singular lines, there is a natural generalization for the dual surface S^*.

Since an AdS spacetime with particles is locally modelled on $AdS^3_{\theta_0}$ for some $\theta_0 \in (0, \pi)$, we can generalize to the singular case the duality between strictly convex spacelike surfaces.

Definition 2.6. *Let S be a strictly convex spacelike surface orthogonal to the singular lines in a convex GHCM AdS spacetime N with particles. The* dual surface *S^* of S is defined as the surface obtained by pushing S along orthogonal geodesics on the convex side for a distance $\pi/2$.*

Observe that the surface obtained by pushing a strictly convex surface $S \subset N$ (which is orthogonal to the singular lines) along orthogonal geodesics on the convex side for a distance $t \in [0, \pi/2]$ is still orthogonal to the singular lines. The strict convexity of S implies that S^* is a smooth (outside the singular locus), strictly convex surface. The relation between dual strictly convex surfaces in GHCM AdS spacetimes (see [6, Section 9.1] or [11, Section 2.6]) can be directly generalized to the following case with cone singularities.

Proposition 2.7. *Let (N, g) be a convex GHCM AdS spacetime with particles. Assume that $S \subset N$ is a strictly convex spacelike surface of constant curvature K orthogonal to singular lines. Then*

(1) S^* is a strictly convex spacelike surface of constant curvature K^* with the shape operator of opposite definiteness, which is orthogonal to the singular lines in N, where $K^* = -K/(1+K)$.
(2) The pull back of the induced metric on S^* through the duality map is the third fundamental form of S and vice versa.
(3) The dual surface $(S^*)^*$ of S^* is exactly S.

2.8. **Minimal Lagrangian maps between hyperbolic surfaces with cone singularities.** The construction of the parameterization of $\mathcal{GH}_{\Sigma,\theta}$ here depends strongly on minimal Lagrangian maps between hyperbolic surfaces with cone singularities.

Definition 2.8. *Given two hyperbolic metrics h, h' on Σ with cone singularities. A* minimal Lagrangian map *$m : (\Sigma, h) \to (\Sigma, h')$ is an area-preserving and orientation-preserving diffeomorphism such that its graph is a minimal surface in $(\Sigma \times \Sigma, h \oplus h')$.*

The following result is [37, Theorem 1.3].

Theorem 2.9 (Toulisse)**.** *Let $h, h' \in \mathfrak{M}^\theta_{-1}$. Then there exists a unique minimal Lagrangian diffeomorphism $m : (\Sigma, h) \to (\Sigma, h')$ isotopic to the identity.*

This is shown by proving the existence and uniqueness of maximal surfaces (see [37, Theorem 1.4]) in a convex GHCM AdS spacetime with particles.

Recall that a spacelike surface of a convex GHCM AdS spacetime (N, g) with particles is said to be a *maximal surface* if it is a locally area-maximizing Cauchy surface which is orthogonal to the singular lines. In particular, it has everywhere vanishing mean curvature and its principal curvatures are everywhere in $(-1, 1)$ (see [23, Lemma 5.15]) and tend to zero at the intersections with particles (see [37, Proposition 3.7]). It is shown in [23, Definition 5.10] that the space $\mathcal{H}_{\Sigma,\theta}$ of maximal surfaces in germs of AdS spacetimes with particles has the following convenient properties.

Lemma 2.10. *The space $\mathcal{H}_{\Sigma,\theta}$ is identified with the space of couples (g,h), where g is a smooth metric on $\Sigma_{\mathfrak{p}}$ with cone singularities of angle θ_i at the marked points p_i for $i=1,...,n_0$ and h is a symmetric bilinear form on $T\Sigma$ defined outside the marked points, such that*

- $tr_g(h)=0$.
- $d^\nabla h=0$, *where ∇ is the Levi-Civita connection of g.*
- $K_g=-1-\det_g(h)$.
- $\det_g(h)$ *is bounded.*

For the convenience of computation, we also introduce the following proposition, see [23, Proposition 3.12].

Proposition 2.11. *Let Σ be a surface with a Riemann metric g. Let $A: T\Sigma\to T\Sigma$ be a bundle morphism such that A is everywhere invertible and $d^\nabla A=0$, where ∇ is the Levi-Civita connection of g. Let h be the symmetric (0,2)-tensor defined by $h=g(A\bullet,A\bullet)$. Then the Levi-Civita connection of h is given by*

$$\nabla^h_u(v)=A^{-1}\nabla_u(Av),$$

and its curvature is given by

$$K_h=\frac{K_g}{\det(A)}.$$

Minimal Lagrangian maps between hyperbolic surfaces with metrics in $\mathfrak{M}^\theta_{-1}$ have an equivalent description in terms of morphisms between tangent bundles (see e.g. [37, Proposition 6.3]).

Proposition 2.12. *Let $h,h'\in\mathfrak{M}^\theta_{-1}$. Then a diffeomorphism $m:(\Sigma,h)\to(\Sigma,h')$ fixing the singular points is a minimal Lagrangian map if and only if there exists a bundle morphism $b: T\Sigma\to T\Sigma$ defined outside the singular locus which satisfies the following properties:*

- *b is self-adjoint for h with positive eigenvalues.*
- $\det(b)=1$.
- *b satisfies the Codazzi equation: $d^\nabla b=0$, where ∇ is the Levi-Civita connection of h.*
- *$h(b\bullet,b\bullet)$ is the pull back of h' by m.*
- *Both eigenvalues of b tend to 1 at the cone singularities.*

Proof. Note that Proposition 6.3 in [37] provides the equivalence between the existence of a minimal Lagrangian map $m:(\Sigma,h)\to(\Sigma,h')$ and the existence of a bundle morphism b which satisfies the first three properties. It suffices to check that given a minimal Lagrangian map $m:(\Sigma,h)\to(\Sigma,h')$, the bundle morphism b also satisfies the last property. Set

$$I'=\frac{1}{4}h((E+b)\bullet,(E+b)\bullet).$$

Denote by J' the complex structure of I' and set

$$B'=-J'(E+b)^{-1}(E-b).$$

Moreover, $J'=(E+b)^{-1}J(E+b)$, where J is the complex structure of h.

Note that $J'B'=(E+b)^{-1}(E-b)$. It is not hard to check that B' satisfies the following conditions:

- B' is self-adjoint for I'. Indeed, choosing a suitable basis such that b is diagonal and using the fact that $\det(b)=1$, we have
$$tr(J'B')=0,$$
which implies that B' is self-adjoint for I'.
- $tr(B')=0$. This follows from the fact that $J'B'$ is self-adjoint for I', since $E\pm b$ is self-adjoint for h and $E+b$ commutes with $E-b$.
- $d^{\nabla'}B'=0$, where ∇' is the Levi-Civita connection of I'. Indeed, by Proposition 2.11 and a direct computation, we obtain that
$$d^{\nabla'}(J'B')=(E+b)^{-1}d^\nabla(E-b)=0.$$
Note that J' is parallel for ∇', it follows that $d^{\nabla'}B'=0$.

- $K_{I'} = -1 - \det(B')$. Indeed, by computation, we have
$$E + J'B' = 2(E + b)^{-1}.$$
By Proposition 2.11, it follows that
$$K_{I'} = \frac{K_h}{\det(\frac{1}{2}(E + b))} = -\det(2(E + b)^{-1}) = -\det(E + J'B') = -1 - \det(B').$$

Set $II' = I'(B'\bullet, \bullet)$. By Lemma 2.10, the couple (I', II') is exactly the first and second fundamental form of a maximal surface S' in a GHCM AdS spacetime (N', g') with particles, where (N', g') has the left metric
$$I'((E + J'B')\bullet, (E + J'B')\bullet) = h,$$
and the right metric
$$I'((E - J'B')\bullet, (E - J'B')\bullet) = h'.$$

Note that the eigenvalues of B' tend to zero at the intersections of S' with the particles (see [37, Proposition 3.7]) and $B' = -J'(E + b)^{-1}(E - b)$. Then both eigenvalues of b tend to 1 at the cone singularities.

We now prove that (N', g') is convex, using a remark appearing already in e.g. [13, Section 4.1]. Since the eigenvalues of B' tend to 0 at the intersections of S' with the particles, the absolute value of the eigenvalues are less than 1 over S', see [23, Lemma 3.11]. As a consequence, the GHCM spacetime (N', g') containing S' also contains the surfaces S_r equidistant from S' at oriented distance $r \in (-\pi/4, \pi/4)$. Those surfaces are smooth (outside the singular points) and, if the principal curvatures of S' at a point x are equal to $\pm\tan(t)$, then the principal curvatures of S_r at the corresponding point are $\tan(t + r)$ and $\tan(t - r)$, where $t \in [0, \pi/4)$. (This follows from the Riccati equation satisfied by the equdistant surfaces in AdS spacetimes.) As a consequence, both $S_{-\pi/4}$ and $S_{\pi/4}$ are convex, with convexities in opposite directions, and (N', g') is therefore convex GHCM. □

Corollary 2.13. *Let $h, h' \in \mathfrak{M}^{\theta}_{-1}$. Then there exists a unique bundle morphism $b : T\Sigma \to T\Sigma$ defined outside the singular locus, which is self-adjoint for h with positive eigenvalues, has determinant 1 and satisfies the Codazzi equation: $d^{\nabla} b = 0$, where ∇ is the Levi-Civita connection of h, such that $h(b\bullet, b\bullet)$ is isotopic to h'. Moreover, both eigenvalues of b tend to 1 at the cone singularities.*

Definition 2.14. *We say that a pair of hyperbolic metrics (h, h') is normalized if there exists a bundle morphism $b : T\Sigma \to T\Sigma$ defined outside the singular locus, which is self-adjoint for h, has determinant 1, and satisfies the Codazzi equation, such that $h' = h(b\bullet, b\bullet)$, or equivalently if the identity from (Σ, h) to (Σ, h') is a minimal Lagrangian diffeomorphism.*

Remark 2.15. *By Corollary 2.13, for any $(\tau, \tau') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$, we can realize (τ, τ') as a normalized representative (h, h'). Note that the normalized representative of (τ, τ') is unique up to isotopies acting diagonally on both h and h'.*

3. Parameterization of $\mathcal{GH}_{\Sigma,\theta}$ in terms of constant curvature surfaces.

3.1. **The definition of the map ϕ_K.** For the construction of the map ϕ_K, we introduce the following proposition which ensures the existence and the uniqueness (up to isometries) of the maximal extension of a convex GHC AdS spacetime with particles (see [16, Proposition 2.6]).

Proposition 3.1. *Let (N, g) be a convex GHC AdS spacetime with particles. There exists a unique (considered up to isometries) convex GHCM AdS spacetime (N', g') with particles, called the maximal extension of (N, g), in which (N, g) can be isometrically embedded.*

Lemma 3.2. *Let $K \in (-\infty, -1)$ and let $(h, h') \in \mathfrak{M}^{\theta}_{-1} \times \mathfrak{M}^{\theta}_{-1}$ be a pair of normalized metrics. Then there exists a unique GHCM AdS spacetime (N, g) that contains a future-convex, spacelike, constant curvature K surface which is orthogonal to the singular lines, with the induced metric $I = (1/|K|)h$ and the third fundamental form $III = (1/|K^*|)h'$, where $K^* = -K/(1 + K)$.*

Proof. Let $b : T\Sigma \to T\Sigma$ be the bundle morphism associated to h and h' by Definition 2.14, so that $h' = h(b\bullet, b\bullet)$.

Let $I = (1/|K|)h$. We equip Σ with the metric I and consider a bundle morphism $B : T\Sigma \to T\Sigma$, which is defined by $B = \sqrt{-1-K}b$. By the properties of h and b, it follows that

- (Σ, I) has constant curvature K.
- B is self-adjoint for I with positive eigenvalues.
- B satisfies the Codazzi equation: $d^{\nabla^I}B = 0$, where ∇^I is the Levi-Civita connection of I.
- B satisfies the Gauss equation: $K = -1 - \det(B)$.

Consider the manifold $\Sigma \times [0, \frac{\pi}{2})$ with the following metric:

$$g_0 = -dt^2 + I((\cos(t)E + \sin(t)B)\bullet, (\cos(t)E + \sin(t)B)\bullet) \ ,$$

where E is the identity isomorphism on $T\Sigma$ and $t \in [0, \frac{\pi}{2})$. Note that for each $t \in [0, \frac{\pi}{2})$, the surface $\Sigma \times \{t\}$ is the equidistant surface at distance t from the surface $\Sigma \times \{0\}$ on the convex side. The Lorentzian metric g_0 is a convex GHC AdS metric on $\Sigma \times [0, \frac{\pi}{2})$ with cone singularities of angle θ_i along the line $\{p_i\} \times [0, \frac{\pi}{2})$. (Note that g_0 is convex since the surface $\Sigma \times \{0\}$ is locally convex by construction.)

By Proposition 3.1, there exists a unique maximal extension (N, g) of the AdS spacetime $(\Sigma \times [0, \frac{\pi}{2}), g_0)$ with particles, which is a convex GHCM AdS spacetime with particles, such that the restriction of g to the subset $\Sigma \times [0, \frac{\pi}{2})$ of N is exactly g_0.

Since B has positive eigenvalues, the embedded surface $\Sigma \times \{0\}$ is future-convex. Hence, N contains a future-convex, spacelike, constant curvature K surface which is orthogonal to the singular lines, with the induced metric $I = (1/|K|)h$ and the third fundamental form

$$III = I(B\bullet, B\bullet) = \frac{1}{|K|}h(\sqrt{-1-K}b\bullet, \sqrt{-1-K}b\bullet) = \frac{1}{|K^*|}h',$$

where $K^* = -K/(1+K)$. This shows the existence of the required manifold (N, g).

Now we show the uniquess of (N, g). Suppose that (N_1, g_1) is another convex GHCM AdS spacetime with particles which contains a prescribed surface S_1. Then S_1 has the induced metric $I_1 = (1/|K|)h = I$ with shape operator B_1 and third fundamental form

$$III_1 = I(B_1\bullet, B_1\bullet) = \frac{1}{|K^*|}h' = I(B\bullet, B\bullet) = III.$$

Since S_1 is future-convex, then B_1 is positive definite. Therefore, the shape operator B_1 of S_1 in (N_1, g_1) is equal to B. Note that the embedding data (Σ, I, B) is exactly (Σ, I_1, B_1), then $(N_1, g_1) = (N, g)$. This completes the proof. □

Lemma 3.3. *Let $K \in (-\infty, -1)$. For any $(\tau, \tau') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$, let (h, h') and (h_1, h_1') be two normalized representatives of (τ, τ'). Let (N, g) and (N_1, g_1) be the convex GHCM AdS spacetimes with particles associated to (h, h') and (h_1, h_1'), as described in Lemma 3.2. Then (N, g) is isotopic to (N_1, g_1).*

Proof. Note that (h, h'), (h_1, h_1') are normalized representatives of (τ, τ'). By Remark 2.15, there exists a diffeomorphism φ from Σ to Σ which is isotopic to the identity (the isotopy fixes the marked points), such that $h_1 = \varphi^*h$ and $h_1' = \varphi^*h'$.

Let (Σ, I, B, III) and $(\Sigma, I_1, B_1, III_1)$ be the corresponding data of the surface contained in (N, g) and (N_1, g_1), as described in Lemma 3.2, respectively. Then $I = (1/|K|)h$, $III = (1/|K^*|)h'$ and $I_1 = (1/|K|)h_1$, $III_1 = (1/|K^*|)h_1'$. It follows that

$$I_1 = \varphi^*(I), \qquad III_1 = \varphi^*(III). \tag{4}$$

To see (N_1, g_1) is isotopic to (N, g), it suffices to prove that $II_1 = \varphi^*(II)$, where II is the second fundamental form of (Σ, I) in (N, g) and II_1 is the second fundamental form of (Σ, I_1) in (N_1, g_1).

By (4), we have

$$III_1 = \varphi^*(III) = \varphi^*(I(B\bullet, B\bullet)) = \varphi^*(I(B^2\bullet, \bullet)) = I(B^2 d\varphi\bullet, d\varphi\bullet),$$

where $d\varphi$ denotes the differential map (or the Jacobian matrix) of φ.

Note that

$$III_1 = I_1(B_1\bullet, B_1\bullet) = (\varphi^* I)(B_1\bullet, B_1\bullet) = (\varphi^* I)(B_1^2\bullet, \bullet) = I(d\varphi B_1^2\bullet, d\varphi\bullet)$$

Hence, $B^2 = (d\varphi)B_1^2(d\varphi)^{-1}$. Denote $A = (d\varphi)B_1(d\varphi)^{-1}$ and hence $B^2 = A^2$. Since both A and B are self-adjoint with positive eigenvalues, an elementary argument shows that $A = B$, that is, $(d\varphi)B_1 = B(d\varphi)$. Therefore,

$$\varphi^*(II) = \varphi^*(I(B\bullet, \bullet)) = I(Bd\varphi\bullet, d\varphi\bullet) = I(d\varphi B_1\bullet, d\varphi\bullet) = (\varphi^* I)(B_1\bullet, \bullet) = I_1(B_1\bullet, \bullet) = II_1.$$

This completes the proof of Lemma 3.3. □

Definition 3.4. *For any $K \in (-\infty, -1)$, define the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{GH}_{\Sigma,\theta}$ by assigning to an element $(\tau, \tau') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ the isotopy class of the convex GHCM AdS spacetime (N, g) with particles satisfying the prescribed property in Lemma 3.2. As a consequence of Lemma 3.2 and Lemma 3.3, this map is well-defined.*

Remark 3.5. *For convenience, for each pair $(\tau, \tau') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$, we always represent it by a pair of normalized hyperbolic metrics (h, h') and represent $\phi_K(\tau, \tau')$ by the convex GHCM AdS spacetime (N, g) with particles as constructed in the proof of Lemma 3.2.*

3.2. **The injectivity of the map ϕ_K.** We prove this property by applying the Maximum Principle outside the singular locus and a specialized analysis near cone singularities.

Proposition 3.6. *Let $(N, g) \in \mathcal{GH}'_{\Sigma,\theta}$ be a convex GHCM AdS spacetime with particles. Assume that S is a future-convex, spacelike, constant curvature K surface which is orthogonal to the singular lines. Then for each intersection point p_i of the surface S with the singular line l_i in N, both principal curvatures on S tend to $k = \sqrt{-1-K}$ at p_i for $i = 1, ..., n_0$.*

Proof. Let I and B be the induced metric and the shape operator of S in (N, g), respectively. Then we have

$$I = \frac{1}{|K|}h, \qquad III = \frac{1}{|K^*|}h',$$

where $h, h' \in \mathfrak{M}^{\theta}_{-1}$ and $K^* = -K/(1+K)$.

We claim that $id : (S, h) \to (S, h')$ is minimal Lagrangian. Indeed, set

$$b = \frac{1}{\sqrt{-1-K}}B.$$

Note that S is future-convex, then B is positive definite. One can easily check that

- b is self-adjoint for h with positive eigenvalues.
- $\det(b) = 1$.
- $d^{\nabla} b = 0$, where ∇ is the Levi-Civita connection of h.
- $h' = h(b\bullet, b\bullet)$.

Moreover, by Proposition 2.12, both eigenvalues of b tend to 1 at cone singularities. Hence, both eigenvalues of $B = \sqrt{-1-K}b$ tend to $k = \sqrt{-1-K}$ at the intersections of S with the singular lines. This implies the conclusion. □

Let (N, g) be a convex GHCM AdS spacetime with particles. Recall that a *geodesically convex* subset Ω of N is a subset of N such that any geodesic segment in N with endpoints in Ω is contained in N. It is proved in [16, Lemma 4.5, Lemma 4.9] that the following properties still hold for the case of convex GHCM AdS spacetimes with particles.

Lemma 3.7. *Each convex GHCM AdS spacetime (N, g) with particles contains a convex core $C(N)$, that is, the smallest non-empty geodesically convex subset of N. Moreover, for any point $x \in N \backslash C(N)$, the maximal timelike geodesic segment connecting x to $C(N)$ has length less than $\pi/2$.*

Remark 3.8. *The boundary of $C(N)$ is the union of two (possibly identified) surfaces, called the future boundary $\partial_+ C(N)$ and the past boundary $\partial_- C(N)$. In the Fuchsian case (i.e. the two metrics of the Mess parameterization are equal), $C(N) = \partial_+(N) = \partial_-(N)$ is a totally geodesic spacelike* *surface orthogonal to the singular lines. In the non-Fuchsian case, each boundary component of $C(N)$ is a spacelike surface orthogonal to the singular lines and is "pleated" along a measured geodesic lamination. In both cases, the induced metric on each boundary component of $C(N)$ is hyperbolic, with each cone singularity of angle equal to that of corresponding particle, as in [16, Lemma 1.5]. Moreover, the maximal geodesic segment starting from $x \in \partial_+ C(N)$ (resp. $\partial_- C(N)$) in the direction of a past-oriented (resp. future-oriented) normal vector at x has length $\pi/2$, see [16, Lemma 1.6].*

The following corollary is an immediate consequence of Lemma 3.7.

Corollary 3.9. *Let (N, g) be a convex GHCM AdS spacetime with particles.*

(1) If S is a strictly future-convex spacelike surface orthogonal to the singular lines in N, then S is in the past of the convex core $C(N)$ and stays at distance less than $\pi/2$ from $\partial_+ C(N)$.

(2) If S be a strictly past-convex spacelike surface orthogonal to the singular lines in N, then S is in the future of the convex core $C(N)$ and stays at distance less than $\pi/2$ from $\partial_- C(N)$.

The following theorem is an alternative version of the Maximum Principle Theorem (see [5, Lemma 2.3], [6, Proposition 4.6]) for the case of convex GHCM AdS spacetimes with particles.

Theorem 3.10. *(Maximum Principle) Let (N, g) be a convex GHCM AdS spacetime with particles. Let S and S' be two future-convex spacelike surfaces in N which are orthogonal to the singular lines. Assume that S and S' intersect at a point p which is not a singularity, and assume that S' is contained in the future of S. Then the principal curvatures of S' at p are greater than or equal to those of S.*

We will also use the following lemma on equidistant surfaces in AdS spacetimes, which follows by a direct computation, see e.g. [23, Lemma 3.22] and [6, Proposition 9.10]. Note that the signs of the principal curvatures here are opposite to those in [6], due to a sign difference of the definition of the second fundamental form.

Lemma 3.11. *Let (N, g) be a convex GHCM AdS spacetime with particles. Let S be a future-convex, spacelike surface in N orthogonal to the singular lines and let $\psi^t : S \to N$ be a map defined by $\psi^t(x) = \exp_x(t \cdot n(x))$, where $n(x)$ is the future-directed unit normal vector at x of S in N. Then for each $x \in S$ which is a regular point, we have*

(1) ψ^t is an embedding in a neighbourhood of x if t satisfies that $\lambda(x)\tan(t) \neq -1$ and $\mu(x)\tan(t) \neq -1$, where $\lambda(x)$ and $\mu(x)$ are the principal curvatures of S at x.

(2) The principal curvatures of $\psi^t(S)$ at the point $\psi^t(x)$ are given by

$$\lambda^t(\psi^t(x)) = \frac{\lambda(x) - \tan(t)}{1 + \lambda(x)\tan(t)}, \qquad \mu^t(\psi^t(x)) = \frac{\mu(x) - \tan(t)}{1 + \mu(x)\tan(t)}.$$

(3) Fix $x \in S$, $\lambda^t(\psi^t(x))$ and $\mu^t(\psi^t(x))$ are both strictly decreasing in $t \in (t_0(x) - \pi/2, \pi/2)$, where $t_0(x) = \min\{\arctan\lambda(x), \arctan\mu(x)\}$.

The following lemma gives a comparison between the principal curvatures at a common singular point p of two spacelike surfaces (orthogonal to the singular lines) which behave umbilically (i.e. principal curvatures extend and coincide) at p and locate at particular positions near p, which can be viewed as a "singular" version of the Maximum Principle Theorem.

Lemma 3.12. *Let (N, g) be a convex GHCM AdS spacetime with particles. Let S, S' be two spacelike surfaces in N which are orthogonal to the singular lines. Assume that S and S' intersect at a singular point p such that the limits of both principal curvatures of S at p are equal to $k > 0$, and the limits of both principal curvatures of S' at p are equal to $k' > 0$. If there exists a neighbourhood U of p in S and a neighbourhood U' of p in S' such that U' is in the future of U, then $k' \geq k$.*

Proof. Let l_i be the singular line through p in N with total angle θ_i. Note that S, S' are orthogonal to l_i at p. By Definition 2.3, there is a sufficiently small neighbourhood V of p in N isometric to a neighbourhood W of a singular point $x \in AdS^3_{\theta_i}$ such that the isometry sends $S \cap V \subset U$ and $S' \cap V \subset U'$ to surfaces X and X' orthogonal to the singular line say l in $AdS^3_{\theta_i}$ at x.

Recall that $AdS^3_{\theta_i}$ is obtained from the universal cover $\widetilde{AdS}_3$ of AdS_3 by taking a wedge $\tilde{W}$ of angle θ_i bounded by two totally geodesic time-like half-planes P_1, P_2 and gluing P_1, P_2 by a rotation fixing the common time-like geodesic $\tilde{l} := P_1 \cap P_2$. Restricted to the picture before gluing, the surfaces X, X' correspond to the surfaces say $\tilde{X}$, $\tilde{X}'$ in $\widetilde{AdS}_3 \cap \tilde{W}$ respectively, which are tangent to the totally geodesic plane P orthogonal to $\tilde{l}$ at $\tilde{x}$, where $\tilde{x}$ corresponds to $x \in AdS^3_{\theta_i}$. This picture is simple thanks to the assumptions on S and S': $\tilde{X}$ and $\tilde{X}'$ behave umbilically at the common point $\tilde{x}$, and $\tilde{X}'$ lies in the future of $\tilde{X}$. Similarly to the regular case (see [5, Lemma 2.3]), we can now write the surfaces $\tilde{X}$, $\tilde{X}'$ as the graphs of two functions f, f' over their common (sectorial) tangent plane $P \cap \tilde{W}$ at $\tilde{x}$. The limit of the second fundamental form of S (resp. S') at p is equal to the Hessian of f (resp. f') at $\tilde{x}$ and, since $f' \geq f$ in the neighborhood of $\tilde{x}$ by construction, $k' \geq k$. □

Lemma 3.13. *Let S_i be a future-convex spacelike surface of constant curvature K_i which is orthogonal to the singular lines in a convex GHCM AdS spacetime with particles for $i = 1, 2$. Then we have the following statements:*

(1) $K_1 = K_2$ if and only if S_1 coincides with S_2.
(2) $K_1 > K_2$ if and only if S_1 is strictly in the future of S_2.
(3) $K_1 < K_2$ if and only if S_1 is strictly in the past of S_2.

Proof. First we claim that if some part of S_1 lies strictly in the future of S_2, then $K_1 > K_2$. To see this, set $t_0 = \sup\{d(x, S_1) : x \in S_2 \text{ is in the past of } S_1\}$, where $d(x, S_1)$ is the maximum of the Lorentzian lengths of causal segments connecting x to S_1. It is clear that $t_0 > 0$ by assumption.

Note that $d(x, S_1)$ is continuous (see Lemma 4.3 in [16]) and S_1, S_2 are compact, thus t_0 is attained at some point $x_0 \in S_2$. In particular, if x_0 is a regular point, the distance t_0 is realized by a geodesic segment with the endpoints orthogonal to S_1 and S_2 which avoids the singularities. If x_0 is a singular point, the distance t_0 is realized by the segment contained in the singular line through x_0 which connects x_0 to S_1.

Denote $S_2^t = \psi^t(S_2)$, where ψ^t is the map defined in Lemma 3.11. Consider $S_2^{t_0}$, it intersects S_1 at the point $y_0 = \psi^{t_0}(x_0)$ and it is in the future of S_1. We discuss it in the following two cases.

Case 1: x_0 is a regular point. By Corollary 3.9, $t_0 \in (0, \pi/2)$. By Statement (3) of Lemma 3.11, $\lambda_2^t(\psi^t(x_0))$ and $\mu_2^t(\psi^t(x_0))$ are both strictly decreasing in $t \in (0, \pi/2)$. Then we have

$$\lambda_2^{t_0}(y_0)\mu_2^{t_0}(y_0) < \lambda_2(x_0)\mu_2(x_0) = -1 - K_2. \tag{5}$$

On the other hand, Theorem 3.10 implies that

$$\lambda_2^{t_0}(y_0)\mu_2^{t_0}(y_0) \geq \lambda_1(y_0)\mu_1(y_0) = -1 - K_1, \tag{6}$$

where $\lambda_2^{t_0}(y_0) \geq \lambda_1(y_0) > 0$ and $\mu_2^{t_0}(y_0) \geq \mu_1(y_0) > 0$. Combining (5) and (6), we get $K_1 > K_2$.

Case 2: x_0 is a singularity. We first claim that S_2^t is orthogonal to the singular lines in N for any $t \in [0, \pi/2)$. By assumption, S_2 is orthogonal to each singular line (say l_i) at a singular point (say x_i). Denote $x_i^t = \psi^t(x_i)$. We will show that S_2^t is orthogonal to l_i at x_i^t. Let P_i be the totally geodesic plane in N orthogonal to l_i at x_i, and P_i^t be obtained by pushing along geodesics orthogonal to P_i in the future direction for a distance t. Let Q_i^t denote the set of points in the future of x_i lying at a constant timelike distance t from x_i. By definition, it is not hard to check that P_i^t and Q_i^t are both spacelike surfaces orthogonal to l_i at x_i^t. Since S_2 intersects P_i at x_i and lies in the future of P_i near x_i, S_2^t intersects P_i^t at x_i^t and lies in the future of P_i^t near x_i^t. Moreover, the Lorentzian length of any timelike geodesic segment starting from a point (near x_i^t) in Q_i^t and ending in S_2 orthogonally is at least t (equal to t exactly when it starts from x_i^t), which implies that S_2^t intersects Q_i^t at x_i^t and lies in the past of Q_i^t near x_i^t. Therefore, near each singular point x_i^t, S_2^t lies between two surfaces P_i^t and Q_i^t (which are orthogonal to l_i at x_i^t), and is thus orthogonal to l_i at x_i^t. In particular, $S_2^{t_0}$ is orthogonal to the singular lines.

By Proposition 3.6 and Lemma 3.11, the limits of both principal curvatures of S_2 at the singularity x_0 are equal to

$$\lambda_2(x_0) = \mu_2(x_0) := k_2,$$

and the limits of both principal curvatures of $S_2^{t_0}$ at $y_0 = \psi^{t_0}(x_0)$ are equal to

$$\lambda_2^{t_0}(y_0) = \mu_2^{t_0}(y_0) := k_2^{t_0} < k_2 \tag{7}$$

where $k_2 = \sqrt{-1-K_2}$ and $k_2^{t_0} = (k_2 - \tan(t_0))/(1 + k_2\tan(t_0))$. Moreover, S_1 and $S_2^{t_0}$ intersects at a singularity y_0 and $S_2^{t_0}$ is in the future of S_1. By Lemma 3.12, we have

$$k_2^{t_0} \geq k_1 = \sqrt{-1-K_1}. \tag{8}$$

Combining (7) and (8), we have $K_1 > K_2$. The claim follows. Similarly, we can show that if some part of S_2 lies strictly in the future of S_1 (equivalently, some part of S_1 lies strictly in the past of S_2), then $K_2 > K_1$.

Now we are ready to show Statement (1). The sufficiency is obvious. It suffices to show the necessity. Suppose by contradiction that S_1 does not coincide with S_2, then either some part of S_1 lies strictly in the future of S_2, or some part of S_1 lies strictly in the past of S_2. In the first case $K_1 > K_2$, while in the second case $K_2 < K_1$. Statement (1) follows.

By the symmetry between Statement (2) and Statement (3), it suffices to prove Statement (2). The sufficiency is clear by the above claim. We are left to prove that if $K_1 > K_2$, then S_1 lies strictly in the future of S_2. Indeed, no part of S_1 lies strictly in the past of S_2, otherwise, by the above result, $K_1 < K_2$. Therefore, if S_1 is not strictly in the future of S_2, there must be some common point say p of S_1 and S_2 near which S_1 is in the future of S_2. By Theorem 3.10 and Lemma 3.12, the (limit of) principal curvatures of S_1 at p are greater than or equal to those of S_2 at p. This implies that $K_1 \leq K_2$, which contradicts our assumption. As a consequence, S_1 is strictly in the future of S_2. The lemma follows. □

Lemma 3.14. *For any $K \in (-\infty, -1)$, the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{GH}_{\Sigma,\theta}$ is injective.*

Proof. Assume that (h, h'), $(h_1, h_1') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ satisfy that $\phi_K(h, h') = \phi_K(h_1, h_1') := (N, g)$. Then (N, g) contains a future-convex, spacelike surface S of constant curvature K orthogonal to the singular lines, with the induced metric $I = (1/|K|)h$ and the third fundamental form $III = (1/|K^*|)h'$ and a future-convex, spacelike surface S_1 of constant curvature K orthogonal to the singular lines, with the induced metric $I_1 = (1/|K|)h_1$ and the third fundamental form $III = (1/|K^*|)h_1'$. By Lemma 3.13, we have $S = S_1$. Then $h = h_1$ and $h' = h_1'$, which implies that $(h, h') = (h_1, h_1')$. □

3.3. The continuity of the map ϕ_K.

To see this, we relate minimal Lagrangian maps to harmonic maps and use some basic facts on the properties of harmonic maps and energy.

Let $f : (M, g) \to (N, h)$ be a C^1 map between two closed Riemannian surfaces (possibly with punctures). The differential df of f is a section of $T^*M \otimes f^*(TN)$ with the metric $g^* \otimes f^*h$, where g^* is the metric on T^*M dual to g. The *energy* of f is defined as

$$E(f, g, h) = \int_M e(f)\, d\sigma_g,$$

where $d\sigma_g$ is the area element of (M, g), and $e(f) = \frac{1}{2}||df||^2_{g^*\otimes f^*h}$ is called the *energy density* of f. We call f a *harmonic* map if it is a critical point of the energy E.

It is known that the value of the energy functional E at such a triple (f, g, h) depends only on the conformal class of g. In particular, set $M = N = \Sigma$ and $g, h \in \mathfrak{M}^\theta_{-1}$, the energy functional E depends only on the conformal class c of g (see [21, equality (3.4)]). This implies that the harmonicity is conformally invariant on the domain.

The *Hopf differential* of f is defined as the (2,0) part of the pull-back by f of h in the conformal coordinate of c, which is denoted by $\Phi(f)$. It measures the difference between the conformal class of $f^*(h)$ and c. It is shown (cf. [21, Lemma 5.1]) that for f harmonic, $\Phi(f)$ is a meromorphic quadratic differential on (Σ, c) with at most simple poles at cone singularities.

Theorem 3.15. (J. Gell-Redman [21, Theorem 2]) *Given $g \in \mathfrak{M}^\theta_{-1}$ and $c \in \mathcal{T}_{\Sigma,\theta}$, there exists a unique harmonic map $u_{c,g} : (\Sigma, c) \to (\Sigma, g)$ isotopic to the identity fixing each marked point, and $u_{c,g}$ is a* *diffeomorphism on $\Sigma_{\mathfrak{p}}$. Moreover, the harmonic maps $u_{c,g}$ vary smoothly with respect to the target metric g.*

Minimal Lagrangian maps between hyperbolic surfaces (with cone singularities of angles less than π) are related to harmonic maps (see e.g. [10, 31, 37]).

Theorem 3.16. (Toulisse [37, Theorem 6.4]) *Let $h_1, h_2 \in \mathfrak{M}^{\theta}_{-1}$. Then there exists a unique conformal structure c on Σ such that*

$$\Phi(u_1) + \Phi(u_2) = 0,$$

where $\Phi(u_i)$ is the Hopf differential of the unique harmonic map $u_i : (\Sigma, c) \to (\Sigma, h_i)$ isotopic to the identity for $i = 1, 2$. Moreover, the map $u_2 \circ u_1^{-1} : (\Sigma, h_1) \to (\Sigma, h_2)$ is minimal Lagrangian and isotopic to the identity.

It is known that (see [36, Proposition 2.14]) for each $c \in \mathcal{T}_{\Sigma,\theta}$, the tangent space $T_c\mathcal{T}_{\Sigma,\theta}$ of $\mathcal{T}_{\Sigma,\theta}$ at c consists of those trace free, divergence free symmetric (0,2)-tensors on $\Sigma_{\mathfrak{p}}$ of class $\mathcal{C}^2$. It is identified with the space $\mathcal{QD}_c(\Sigma)$ of meromorphic quadratic differentials (with respect to the complex structure c) on Σ with at most simple poles at singularities, by assigning $q \in \mathcal{QD}_c(\Sigma)$ to the real part $\Re(q) \in T_c\mathcal{T}_{\Sigma,\theta}$.

Recall that the L^2-metric defined on $T_c\mathcal{T}_{\Sigma,\theta}$ is given by the inner product:

$$\langle\langle h, k\rangle\rangle_c = \frac{1}{2}\int_M \mathrm{tr}(HK)d\mu_g,$$

where H, K are the (1,1)-tensors on $\Sigma_{\mathfrak{p}}$ obtained from h and k via the representative (hyperbolic) metric g of c (by raising an index), μ_g is the volume element induced on $\Sigma_{\mathfrak{p}}$ by g.

Let $\xi dz^2, \eta dz^2 \in \mathcal{QD}_c(\Sigma)$ and write $g = \lambda|dz|^2$ under the conformal coordinate $z = x + iy$ of c. As for the Teichmüller space of non-singular closed surfaces (see [38, Section 2.6] and [33, Definition 3.1]), the Weil-Petersson metric on $\mathcal{T}_{\Sigma,\theta}$ is defined as

$$\langle \xi, \eta\rangle_{WP} = \Re\int_{\Sigma} \frac{\xi\bar{\eta}}{\lambda}dxdy.$$

One can check that the Weil-Petersson metric on $T_c\mathcal{T}_{\Sigma,\theta}$ is equal to the L^2-metric :

$$\langle \xi, \eta\rangle_{WP} = \langle\langle \Re(\xi), \Re(\eta)\rangle\rangle_c.$$

Fix $g_0 \in \mathfrak{M}^{\theta}_{-1}$. Let $E(\bullet, g_0) : \mathcal{T}_{\Sigma,\theta} \to \mathbb{R}$ be a map which assigns to $c \in \mathcal{T}_{\Sigma,\theta}$ the energy of the (unique) harmonic map u_{c,g_0} as indicated in Theorem 3.15. Similarly, we can define $E(c_0, \bullet)$ by fixing a point c_0 in the source space of harmonic maps. Note that $E(\bullet, g_0)$ and $E(c_0, \bullet)$ are both smooth functions, see e.g. [38, Chapter 3.1] and [40, Remarks (iv) and Theorem 5.7]. The following lemma provides the properties of $E(\bullet, g_0)$ we need, see [36, Theorem 3.2], [38, Chapter 3.1 and Theorem 3.1.3].

Lemma 3.17. *$E(\bullet, g_0)$ has the following properties:*

(1) $E(\bullet, g_0)$ is proper.
(2) The Weil-Petersson gradient $\nabla E(\bullet, g_0)$ of $E(\bullet, g_0)$ at $g \in \mathcal{T}_{\Sigma,\theta}$ is (up to a factor) $\Re(\Phi(u_{g,g_0}))$.
(3) The second derivative of $E(\bullet, g_0)$ at a critical point is (up to a positive factor) Weil-Petersson metric (hence, positive definite).
(4) The isotopy class associated to g_0 is the only critical point of $E(\bullet, g_0)$.

Let $h_1, h_2 \in \mathfrak{M}^{\theta}_{-1}$. Define the functional $E_{h_1,h_2}(\bullet) = E(\bullet, h_1) + E(\bullet, h_2)$ over $\mathcal{T}_{\Sigma,\theta}$. By Lemma 3.17, $E_{h_1,h_2}(\bullet)$ is proper and has a unique critical point $c \in \mathcal{T}_{\Sigma,\theta}$ such that $\Phi(u_{c,h_1}) + \Phi(u_{c,h_2}) = 0$. As a consequence, we have the following proposition.

Proposition 3.18. *The conformal structure c in Theorem 3.16 is the unique critical (minimum) point of the functional $E_{h_1,h_2}(\bullet) : \mathcal{T}_{\Sigma,\theta} \to \mathbb{R}$.*

We are now ready to prove the following lemma.

Lemma 3.19. *For any $K \in (-\infty, -1)$, the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{GH}_{\Sigma,\theta}$ is continuous.*

Proof. It suffices to prove that if the sequence $(h_k, h'_k)_{k\in\mathbb{N}}$ converges to $(h, h') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$, then the sequence $(\phi_K(h_k, h'_k))_{k\in\mathbb{N}}$ converges to $\phi_K(h, h') \in \mathcal{GH}_{\Sigma,\theta}$. Denote by m_k the unique minimal Lagrangian map between (Σ, h_k) and (Σ, h'_k) isotopic to the identity and by m the unique minimal Lagrangian map between (Σ, h) and (Σ, h') isotopic to the identity.

We claim that the sequence $(m_k)_{k\in\mathbb{N}}$ converges to m. Indeed, by Proposition 3.18, denote by c_k the unique critical point of $E_{h_k,h'_k}(\bullet)$ and by c the unique critical point of $E_{h,h'}(\bullet)$.

Now we prove that c_k converges to c. Note that both $E(\bullet, g_0)$ and $E(c_0, \bullet)$ are smooth functions on $\mathcal{T}_{\Sigma,\theta}$. By the assumption that $(h_k, h'_k)_{k\in\mathbb{N}} \to (h, h')$, we have

$$E_{h_k,h'_k}(\bullet) \to E_{h,h'}(\bullet),$$

and

$$\nabla E_{h_k,h'_k}(\bullet) = C\Re(\Phi(u_{\bullet,h_k}) + \Phi(u_{\bullet,h'_k})) \to C\Re(\Phi(u_{\bullet,h}) + \Phi(u_{\bullet,h'})) = \nabla E_{h,h'}(\bullet),$$

in the sense of compact-open topology as $k \to \infty$, where C is a non-zero constant. Note that $\nabla E_{h_k,h'_k}(c_k) = 0$, $\nabla E_{h,h'}(c) = 0$ and $E_{h,h'}(\bullet)$ has non-degenerate second derivative at c. By the Implicit Function Theorem on Banach spaces (see [1, Theorem 2.5.7]), c is the limit of the critical points c_k of $E_{h_k,h'_k}(\bullet)$. By a closeness result for harmonic maps (see Theorem 7.1 in [21]), u_{c_k,h_k} (resp. u_{c_k,h'_k}) converges to the harmonic map $u_{c,h}$ (resp. $u_{c,h'}$). Combined with Theorem 3.16, $m_k = u_{c_k,h_k} \circ (u_{c_k,h'_k})^{-1}$ converges to $u_{c,h} \circ (u_{c,h'})^{-1} = m$.

Let $b_k : T\Sigma \to T\Sigma$ be the bundle morphism defined outside the singular locus which is described in Proposition 2.12 with the property that $m_k^*(h'_k) = h_k(b_k\bullet, b_k\bullet)$. Then b_k converges to a bundle morphism from $T\Sigma$ to $T\Sigma$, which is denoted by b.

Let $I_k = (1/|K|)h_k$ and $B_k = \sqrt{-1-K}b_k$. Then $(\Sigma, I_k, B_k)_{k\in\mathbb{N}}$ converges to (Σ, I, B), in the sense that I_k, B_k converges to $I = (1/|K|)h$, $B = \sqrt{-1-K}b$, respectively. This implies that $(\phi_K(h_k, h'_k))_{k\in\mathbb{N}}$ converges to $\phi_K(h, h')$ in $\mathcal{GH}_{\Sigma,\theta}$. The proof is complete. □

Proposition 3.20. *For any $K \in (-\infty, -1)$, the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{GH}_{\Sigma,\theta}$ is a local homeomorphism.*

Proof. By the extension of Mess parameterization (see [16, Theorem 1.4]), $\mathcal{GH}_{\Sigma,\theta}$ is homeomorphic to $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$. Thus, $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ and $\mathcal{GH}_{\Sigma,\theta}$ have the same dimension and have no boundary. Moreover, it follows from Lemma 3.14 and Lemma 3.19 that ϕ_K is injective and continuous. By the invariance of domain theorem for manifolds, ϕ_K is a local homeomorphism. □

3.4. **The properness of the map ϕ_K.** To prove this property of ϕ_K, we recall some elementary facts about hyperbolic surfaces with cone singularities of angles less than π.

First we introduce the following Collar Lemma for hyperbolic cone surfaces (see [19, Theorem 3]).

Lemma 3.21. (Collar Lemma) *Let S be a hyperbolic cone-surface of genus $\mathfrak{g}$ with n_0 cone points $p_1, ..., p_{n_0}$ with cone angles $\theta_1, ..., \theta_{n_0} \in (0, \pi)$ and $(\mathfrak{g}, n_0) \geq (0, 4)$. Let α be the largest cone angle. Let $\{\gamma_1, ..., \gamma_m\}$ be a maximal collection of mutually disjoint simple closed geodesics on S, where $m = 3\mathfrak{g} - 3 + n_0$. Then the collars*

$$C(\gamma_k) = \{x \in S : d(x, \gamma_k) \leq \operatorname{arcsinh}\left(\cos\alpha / \sinh\frac{\ell(\gamma_k)}{2}\right)\}$$

and

$$C(p_l) = \{x \in S : d(x, p_l) \leq \operatorname{arccosh}(1/\sin\theta_l)\}$$

are pairwise disjoint for $k = 1, ..., m$ and $l = 1, ..., n_0$, where $\ell(\gamma_k)$ is the length of the geodesic γ_k.

Lemma 3.22. *Let $(\tau_i)_{i\in\mathbb{N}} \subset \mathcal{T}_{\Sigma,\theta}$ be a sequence which escape from any compact subset of $\mathcal{T}_{\Sigma,\theta}$. Then there exists a simple closed curve γ on Σ such that, up to extracting a subsequence, the length of γ under τ_i tends to infinity.*

Proof. Note that the underlying surface Σ we consider satisfies the condition $\chi(\Sigma, \theta) < 0$. Each marked hyperbolic cone-surface in $\mathcal{T}_{\Sigma,\theta}$ admits a pants decomposition $\mathcal{P} = \{C_i\}_{i=1}^{3g-3+n_0}$ such that each pair of pants obtained from $\mathcal{P}$ is either a hyperbolic pair of pants with three boundary components or a generalized hyperbolic pair of pants with exactly one or two boundary components degenerating into cone points of the given angles. In the latter case, the pair of pants is uniquely determined by the lengths of the non-degenerate boundary components and the cone angles at the cone points.

With the angles of the cone points fixed, $\mathcal{T}_{\Sigma,\theta}$ has Fenchel-Nielsen coordinates analogous to those of the usual Teichmüller space of non-singular hyperbolic surfaces, see e.g. [29, Section 3]. Moreover, the twist parameter along each pant curve C_i is determined by the length of the shortest simple closed geodesic α_i which intersects C_i and the length of the geodesics $T^k_{C_i}(\alpha_i)$ obtained by taking a positive k times Dehn-twist along C_i along α_i for $k = 1, 2$. This line of arguments can be used to show that there exist finitely many simple closed curves on Σ whose lengths completely determine an element in $\mathcal{T}_{\Sigma,\theta}$, see [29, Theorem A].

By assumption, $(\tau_i)_{i\in\mathbb{N}}$ escapes from any compact subset of $\mathcal{T}_{\Sigma,\theta}$. Therefore there must be some simple closed curve γ whose length under τ_i tends to either infinite or zero (in the latter case, it follows from Lemma 3.21 that any simple closed curve intersecting γ is becoming infinitely long). Therefore, there always exists a simple closed curve (still denoted by γ) on Σ such that, up to extracting a subsequence, the length of γ under τ_i tends to infinity. This completes the proof. □

The following lemma gives a comparison between the lengths of simple closed geodesics in the same isotopy class on the past boundary $\partial_-C(N)$ of the convex core and on a spacelike surface in its past in a convex GHCM AdS spacetime (N, g).

Lemma 3.23. *Let (N, g) be a convex GHCM AdS spacetime with particles. Let S be a spacelike surface in the past of $\partial_-C(N)$ which is orthogonal to the singular lines in N. Then for any closed geodesic γ on $\partial_-C(N)$, the length of γ is larger than the length of any closed minimizing geodesic γ' on S homotopic to γ.*

Proof. Let λ_- be the bending lamination of $\partial_-C(N)$. The set of isotopy classes of weighted non-trivial simple closed curves is dense in the space $\mathcal{ML}_{\Sigma,n_0}$ of measured laminations on $\Sigma_\mathfrak{p}$ (see [16, Proposition 3.1]). It suffices to consider the case where λ_- is a disjoint finite union of weighted simple closed geodesics on $\partial_-C(N)$. Assume that $\mathrm{supp}(\lambda_-) = \cup_{i=1}^m \alpha_i$, where α_i is a simple closed geodesic on $\partial_-C(N)$ disjoint from α_j for $j \neq i$. Then $\partial_-C(N) \setminus \mathrm{supp}\lambda_-$ is a disjoint finite union of spacelike subsurfaces of $\partial_-C(N)$ which are totally geodesic in N.

Let $\Sigma_0 = \partial_-C(N)$, and let h_0 be the induced metric on Σ_0. First we construct a family $(\Sigma_t)_{t\in[0,\pi/2]}$ of future-convex equidistant surfaces from Σ_0 in $I^-(\Sigma_0)$. For each $t \in (0, \pi/2]$, let

$$\Omega_t = \{x \in I^-(\Sigma_0) \mid d(x, \Sigma_0) \leq t\} ,$$

and let

$$\Sigma_t = \partial\Omega_t \cap I^-(\Sigma_0) .$$

Note that Σ_t is a future-convex (non-smooth) spacelike surface orthogonal to the singular lines (see e.g. [16, Lemma 4.2]) and Σ_t can be disconnected when it is close to the past singularity of N and even empty when t tends to $\pi/2$. It is clear that $\cup_{t\in(0,\pi/2)}\Sigma_t = I^-(\Sigma_0)$.

Let $x \in \Sigma_t$, for some $t \in (0, \pi/2)$, and let n be a unit future-oriented vector orthogonal to a support plane of Σ_t at x. Let $\gamma_{x,n}$ be the intersection with $I^-(\Sigma_0) \cup \Sigma_0$ of the geodesic starting from x with velocity n. Since Σ_t is future-convex, the $\gamma_{x,n}$ are disjoint. We define an "orthonormal projection" p_t to Σ_t, sending a point $y \in I^+(\Sigma_t) \cap (I^-(\Sigma_0) \cup \Sigma_0)$ to $x \in \Sigma_t$ if $y \in \gamma_{x,n}$ for a certain time-like unit vector n orthogonal to a support plane of Σ_t at x. Since Σ_t is future-convex, x is then the unique point on Σ_t realizing the distance to x. Denote by $Dom(p_t)$ the domain of p_t, which is a subset of $I^+(\Sigma_t) \cap (I^-(\Sigma_0) \cup \Sigma_0)$.

Let $r, s > 0$ and let $y \in Dom(p_{r+s})$. Then $p_{r+s}(y) = p_s(p_r(y))$, because the time-like geodesic segment between y and $p_{r+s}(y)$ must intersect Σ_r at a point which maximizes both the distance between y and Σ_r and the distance between $p_r(y)$ and $p_{r+s}(y)$.

It follows that there exists a flow $(\phi_t)_{t\in[0,\pi/2]}$, defined for each t on a subset of $I^-(\Sigma_0)$, such that if $y \in \Sigma_r \cap Dom(p_{r+s})$, then $p_{r+s}(y) = \phi_s(y)$. By definition, $(\phi_t)_{t\in[0,\pi/2]}$ is the flow of a past-oriented unit time-like vector field X, which is however not continuous. At each point $x \in \Sigma_r$, for $r \in [0, \pi/2)$,

X is normal to a support plane of Σ_r. Although X is discontinuous, it follows from its definition that the flow of X exists (but the flow of $-X$ is not well-defined).

A direct examination shows that the restriction of p_r to Σ_0 is distance-decreasing. In fact, regions near a pleating line of Σ_0 are typically sent to a line, and the length along pleating geodesics is contracted by a factor $\cos(r)$. Similarly, on flat regions of Σ_0 which are sent to smooth regions on Σ_r, lengths are contracted by a factor $\cos(r)$. So, if we denote by h_r the pull-back on Σ_0 by p_r of the induced metric on Σ_r, then $(h_r)_{r\in(0,\pi/2)}$ is a decreasing family of pseudo-metrics (each defined on a subset of Σ_0, this subset being also decreasing with r).

We now consider the map $\phi:\Sigma_0\to S$, with $\phi(x)$ defined by following the flow of X from x to the first intersection point with S. For all $x\in\Sigma_0$, we also denote by $t(x)$ the time needed to reach $\phi(x)$, so that $\phi(x)\in\Sigma_{t(x)}$. Finally we denote by h the pseudo-metric obtained on Σ_0 as the pull-back by ϕ of the induced metric on S. (Note that h is defined on the whole of Σ_0 because ϕ is defined on the whole of Σ_0 since any integral curve of X starting from Σ_0 must intersect S. For the same reason, $h_{t(x)}$ is well-defined at x.)

Let $x\in\Sigma_0$. At $\phi(x)$, the tangent plane $T_{\phi(x)}S$ can be identified to the tangent P to any support plane of $\Sigma_{t(x)}$ by projection along the normal to P. Under this identification, the induced metric on $T_{\phi(x)}S$ is smaller than the induced metric on P (the difference being dt^2, where t denotes now the distance to Σ_0).

It follows that, at all $x\in\Sigma_0$, $h\le h_{t(x)}$, and therefore $h\le h_0$.

Let γ be a closed geodesic on Σ_0, and let $\gamma'=\phi(\gamma)\subset S$. The length of γ for h is smaller than the length of γ for h_0, so that the length of γ' for the induced metric on S is less than the length of γ for the induced metric on Σ_0. It follows that the length on S of any minimizing geodesic homotopic to γ' (and therefore to γ) is smaller than the length of γ. □

Note that a much simpler proof of the Lemma 3.23 can be given if S is a future-convex spacelike surface (orthogonal to the singular lines) of constant curvature and if there is a foliation of the region between $\partial_-C(N)$ and S by smooth (outside the singular locus) future-convex surfaces (orthogonal to the singular lines). The existence of such a foliation clearly follows from Theorem 1.1. However, at this point of the proof, we couldn't find a simple way to prove the existence of such a foliation by smooth future-convex surfaces. Therefore, we give an alternative method, which also generalizes the case of a future-convex surface S (orthogonal to the singular lines) to the case of a spacelike surface (orthogonal to the singular lines) in the past of $\partial_-C(N)$.

The following corollary is an analogue of Lemma 3.23.

Corollary 3.24. *Let (N,g) be a convex GHCM AdS spacetime with particles. Let S be a spacelike Cauchy surface in the future of $\partial_+C(N)$ which is orthogonal to the singular lines in N. Then for any closed geodesic γ on $\partial_+C(N)$, the length of γ is larger than the length of the closed geodesic γ' on S homotopic to γ.*

Proposition 3.25. *For any $K\in(-\infty,-1)$, the map $\phi_K:\mathcal{T}_{\Sigma,\theta}\times\mathcal{T}_{\Sigma,\theta}\to\mathcal{GH}_{\Sigma,\theta}$ is proper.*

Proof. Let $(N_k,g_k):=\phi_K(h_k,h'_k)$. It suffices to verify that if a sequence $(h_k,h'_k)_{k\in\mathbb{N}}$ escape from any compact subset of $\mathcal{T}_{\Sigma,\theta}\times\mathcal{T}_{\Sigma,\theta}$, then $(N_k,g_k)_{k\in\mathbb{N}}$ escape from any compact subset of $\mathcal{GH}_{\Sigma,\theta}$. Indeed, if $(h_k,h'_k)_{k\in\mathbb{N}}$ escape from any compact subset of $\mathcal{T}_{\Sigma,\theta}\times\mathcal{T}_{\Sigma,\theta}$, then $(h_k)_{k\in\mathbb{N}}$ or $(h'_k)_{k\in\mathbb{N}}$ escape from any compact subset of $\mathcal{T}_{\Sigma,\theta}$. We discuss in the following two cases.

Case 1: If $(h_k)_{k\in\mathbb{N}}$ escape from any compact subset of $\mathcal{T}_{\Sigma,\theta}$. By Lemma 3.22, there is a simple closed curve γ on Σ, such that up to extracting a subsequence, $\ell_{h_k}(\gamma)\to\infty$.

Denote by S_k the future-convex, constant curvature K surface which is orthogonal to the singular lines with the induced metric $I_k=(1/|K|)h_k$ in (N_k,g_k). Denote the induced metric on $\partial_-C(N_k)$ by I_k^-. It is shown in [16, Lemma 5.4] that I_k^- is a hyperbolic metric with cone singularities of angles equal to the given angles at the intersections with the corresponding singular lines. By Lemma 3.23, $\ell_{I_k^-}(\gamma)\ge\ell_{I_k}(\gamma)\to\infty$. Note that $\mathcal{GH}_{\Sigma,\theta}$ can be parameterized by the embedding data (including the induced metric and the bending lamination) of the past (or future) boundary of the convex core (see e.g. [16]). This implies that $(N_k,g_k)_{k\in\mathbb{N}}$ are not contained in any compact subset of $\mathcal{GH}_{\Sigma,\theta}$.

Case 2: If $(h'_k)_{k\in\mathbb{N}}$ escape from any compact subset of $\mathcal{T}_{\Sigma,\theta}$. By Lemma 3.2, the future-convex constant curvature K surface S_k in (N_k, g_k) has third fundamental form $III_k = (1/|K^*|)h'_k$, where $K^* = -K/(1+K)$. By Proposition 2.7, the dual surface S_k^* of S_k is a past-convex constant curvature K^* surface which is orthogonal to the singular lines with the induced metric I_k^* such that the pull back of I_k^* on S_k^* through the duality map is III_k.

Using a similar argument as in the first case and applying Corollary 3.24, there exists a simple closed curve γ' on Σ, such that up to extracting a subsequence, $\ell_{I_k^+}(\gamma') \geq \ell_{I_k^*}(\gamma') \to \infty$, where I_k^+ denotes the induced metric on $\partial_+ C(N_k)$. This implies that $(N_k, g_k)_{k\in\mathbb{N}}$ are not contained in any compact subset of $\mathcal{GH}_{\Sigma,\theta}$.

Combining these two cases, the proof is complete. □

Proof of Theorem 1.2. Note that $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ and $\mathcal{GH}_{\Sigma,\theta}$ are simply connected. By Proposition 3.20 and Proposition 3.25, for each $K < -1$, ϕ_K is both a local homeomorphism and a proper map, which implies that ϕ_K is a homeomorphism.

4. The existence and uniqueness of foliations.

In this section, we prove Theorem 1.1, as an application of Theorem 1.2. Let (N, g) be a convex GHCM AdS spacetime with particles. Denote by B^+ and B^- the future and the past component of $N \setminus C(N)$.

To prove Theorem 1.1, we first show that B^- admits a unique foliation by future-convex constant curvature surfaces orthogonal to the singular lines. Note that there is a duality between future-convex and past-convex surfaces orthogonal to the singular lines in GHCM AdS spacetimes with particles (see Proposition 2.7). It is a direct consequence that B^+ admits a unique foliation by past-convex constant curvature surfaces orthogonal to the singular lines.

Indeed, Theorem 1.2 says that for each $K \in (-\infty, -1)$, the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{GH}_{\Sigma,\theta}$ is a homeomorphism. In particular, ϕ_K is a surjection. This implies that there exists an embedded future-convex spacelike surface S_K of constant curvature K which is orthogonal to the singular lines in N. Moreover, it follows from the injectivity of ϕ_K and Corollary 3.9 that this surface S_K is unique and contained in B^-. This implies that the union of S_K over all $K \in (-\infty, -1)$ is contained in B^-. It remains to show that the union of S_K over all $K \in (-\infty, -1)$ is exactly B^-.

To prove this, we first generalize the notion of the uniformly spacelike (see Definition 3.7 in [6]) property of a sequence of spacelike surfaces to the case with cone singularities as follows.

Definition 4.1. *A sequence $(S_k)_{k\in\mathbb{N}}$ of spacelike surfaces orthogonal to the singular lines in N is said to be* uniformly spacelike *if, for every sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in S_k$, it falls into exactly one of the following two classes:*

(1) $x_k \in S_k$ escapes from any compact subset of N.

(2) Up to extracting a subsequence, the sequence $(x_k, P_k)_{k\in\mathbb{N}}$ converges to a limit (x, P), with $x \in N$ and P a totally geodesic spacelike plane through x (and orthogonal to the singular line through x if x is a singular point).

Here P_k is the tangent plane of S_k at x_k if x_k is a regular point, and P_k is the totally geodesic plane orthogonal to the singular line through x_k if x_k is a singular point. For convenience, we call P_k the support plane *of S_k at x_k whether x_k is regular or not.*

Let $(S_k)_{k\in\mathbb{N}}$ be a sequence of future-convex spacelike surfaces orthogonal to the singular lines in N, such that S_{k+1} is strictly in the past of S_k for all $k \in \mathbb{N}$. Denote by Ω the union of the future $I^+(S_k)$ of S_k over all $k \in \mathbb{N}$ and denote by $S_\infty = \partial\Omega$ the boundary of Ω (that is contained in the past of the convex core $C(N)$).

Note that after pushing along geodesics orthogonal to a future-convex spacelike surface S (orthogonal to the singular lines) in the future direction for the distance $t \in [0, \pi/2]$, the obtained surface is still orthogonal to the singular lines. In the case $\Omega \neq N$, the property of $\partial\Omega$ and the uniformly spacelike property of $(S_k)_{k\in\mathbb{N}}$ (see Theorem 3.6 and Corollary 3.8 in [6]) can be directly generalized to the case with cone singularities as follows.

Lemma 4.2. *Let Ω and S_∞ be the domain and the surface in N as described above. Assume that $\Omega \neq N$, then*
(1) S_∞ is the set of limits in N of sequences $(x_k)_{k\in\mathbb{N}}$ with $x_k \in S_k$.
(2) S_∞ is a future-convex spacelike surface which is orthogonal to the singular lines in N.
(3) $(S_k)_{k\in\mathbb{N}}$ is uniformly spacelike.

To prove Theorem 1.1, we need the following compactness result, which is an elementary fact about $\mathcal{T}_{\Sigma,\theta}$. In the case of hyperbolic metrics on closed surfaces, we refer to Lemma 9.4 in [11]. In our case with cone singularities, it is a direct consequence of Lemma 3.22.

Lemma 4.3. *Let $C > 1$ and $h \in \mathcal{T}_{\Sigma,\theta}$. Let $\mathcal{B}(h)$ be the set consisting of $h' \in \mathcal{T}_{\Sigma,\theta}$ such that for all simple closed curves γ on Σ, $\ell_\gamma(h') \leq C\ell_\gamma(h)$. Then $\mathcal{B}(h)$ is compact.*

Proposition 4.4. *Let (N, g) be a convex GHCM AdS spacetime with particles. Let $(S_i)_{i\in\mathbb{N}^+}$ be a sequence of future-convex spacelike surfaces of constant curvatures K_i which are orthogonal to the singular lines in N such that $K_{i+1} < K_i$ for all $i \in \mathbb{N}^+$. Then the following statements hold.*
(1) If $K_i \to -\infty$, then the union of $I^+(S_i)$ over $i \in \mathbb{N}^+$ is exactly the whole manifold N.
(2) If $K_i \to K$ with $-\infty < K < -1$, then the sequence $(S_i)_{i\in\mathbb{N}^+}$ converges to a future-convex spacelike surface S_∞ of constant curvature K (which is orthogonal to the singular lines) in the $\mathcal{C}^2$-topology outside the singular locus.

Proof. Proof of Statement (1): assume that the union Ω of $I^+(S_i)$ over all $i \in \mathbb{N}^+$ is not N. By Lemma 4.2, the boundary $S_\infty = \partial\Omega$ of Ω is a future-convex spacelike surface. Moreover, $(S_i)_{i\in\mathbb{N}^+}$ is uniformly spacelike. Therefore, the area of S_i does not tend to zero as $i \to \infty$. However, by the Gauss-Bonnet formula for surfaces with cone singularities (see e.g. [39, Proposition 1]), the area of S_i is equal to $2\pi\chi(\Sigma, \theta)/K_i$, where Σ is the surface such that N is homeomorphic to $\Sigma \times \mathbb{R}$. Since $K_i \to -\infty$, this implies that the area of S_i tends to zero, which leads to a contradiction.

Proof of Statement (2): Denote by Ω the union of $I^+(S_i)$ over all $i \in \mathbb{N}^+$. First we claim that Ω is not the whole manifold N, and more specifically that Ω is contained in the future of a Cauchy surface of N. To see this, we take a number $K' < K$, it follows from Theorem 1.2 and Lemma 3.13 that there exists an embedded future-convex spacelike surface $S_{K'} \subset N$ of constant curvature K' (which is orthogonal to the singular lines), such that $S_{K'}$ is strictly in the past of the surfaces S_i for all $i \in \mathbb{N}^+$. Hence, the closure of Ω is contained in the closure of $I^+(S_{K'})$.

Denote by $S_\infty = \partial\Omega$ the boundary of Ω. By Lemma 4.2, S_∞ is a future-convex spacelike surface which is orthogonal to the singular lines in N. Moreover, it is the $\mathcal{C}^0$ limit of $(S_i)_{i\in\mathbb{N}^+}$. We claim that S_∞ is a Cauchy surface. Indeed, as the boundary of a future domain Ω (i.e. any future-directed timelike curve starting from a point of Ω is contained in Ω), S_∞ is a closed achronal subset of N (see e.g. [28, Corollary 27] and [6, Section 7]). Hence, there is no timelike curve in N meeting S_∞ more than once. Combined with the fact that S_∞ is spacelike, it suffices to show that S_∞ intersects each inextensible timelike curve in N. But if l is an inextensible timelike curve in N, each S_i intersects l in a unique point $x_i \in l$. Since $S_i \to S_\infty$, the sequences $(x_i)_{i\in\mathbb{N}_{>0}}$ converges to a limit $x \in S_\infty$, so that $S_\infty \cap l \neq \emptyset$.

Let $S_0 = \partial_- C(N)$. Then S_0 is a spacelike surface of constant curvature $K_0 = -1$ which is orthogonal to the singular lines. Denote by g_i, g_∞ the metrics induced on S_i, S_∞ by the Lorentzian metric g on N for all $i \in \mathbb{N}$.

Note that all the surfaces S_i are orthogonal to the singular lines l_k (corresponding to $\{p_k\} \times \mathbb{R}$) and the cone angle of the singularity on S_i at the intersection with l_k is $\theta_k \in (0, \pi)$ for $k = 1, ..., n_0$. Therefore, the metrics g_i can be written as follows:

$$g_i = (1/|K_i|)\widehat{g_i},$$

where $\widehat{g_i} \in \mathfrak{M}^\theta_{-1}$ for all $i \in \mathbb{N}$.

By Lemma 3.23, for any simple closed curve γ on Σ, if γ_i denotes the closed geodesic on S_i which is homotopic to γ then

$$\ell_{\gamma_i}(g_i) \leq \ell_{\gamma_0}(g_0),$$

for all $i \in \mathbb{N}$. As a consequence,

$$\ell_{\gamma_i}(\widehat{g_i}) = \ell_{\gamma_i}(|K_i|g_i) = \sqrt{|K_i|}\ell_{\gamma_i}(g_i) \leq \sqrt{|K|}\,\ell_{\gamma_0}(g_0) = \sqrt{K/K_0}\,\ell_{\gamma_0}(\widehat{g_0}),$$

for all $i \in \mathbb{N}$. Here $K/K_0 > 1$. Lemma 4.3 therefore shows that the isotopy classes of the hyperbolic metrics $\widehat{g_i}$ remain in a compact subset of $\mathcal{T}_{\Sigma,\theta}$ and, after extracting a subsequence, they converge to a limit $\tau \in \mathcal{T}_{\Sigma,\theta}$.

The same result can be applied to the dual surfaces S_i^*, which are past-convex surfaces in N of constant curvature $K_i^* = -K_i/(K_i+1)$. The isotopy classes of their induced metrics suitably to have curvature -1, say $\widehat{g_i}^*$, converge, after extracting a subsequence, to a limit $\tau^* \in \mathcal{T}_{\Sigma,\theta}$. Since the $\widehat{g_i}^*$ are the third fundamental forms of the S_i (up to a scaling factor), the duality map δ_i between $(S_i, \widehat{g_i})$ and $(S_i^*, \widehat{g_i}^*)$ is the unique minimal Lagrangian diffeomorphism in its isotopy class. It therefore converges to the minimal Lagrangian diffeomorphism between τ and τ^* (see the proof of Lemma 3.19). As a consequence, the norms of the differentials of the δ_i (with respect to $\widehat{g_i}$ and $\widehat{g_i}^*$) are uniformly bounded. This means that $\widehat{g_i}^* = \widehat{g_i}(b_i\cdot, b_i\cdot)$, where the $b_i : TS_i \to TS_i$ are Codazzi tensors of determinant 1 which are uniformly bounded. It follows that the shape operators of the S_i, and therefore their principal curvatures, are uniformly bounded.

Since the surfaces S_i are Cauchy surfaces with uniformly bounded principal curvatures, and they are in the future of a Cauchy surface, they converge in the $C^{1,1}$ topology to S_∞. Since the induced metric on S_∞ is the limit of that of the S_i, and the S_i have constant curvature $K_i \to K$, this surface S_∞ has constant curvature K.

Since the S_i and S_∞ have constant curvature, the elliptic regularity shows that $S_i \to S_\infty$ in the C^2 topology (outside the singular locus). $\square$

Recall that the *cosmological time* of a spacetime (M, g) is the function $\tau : M \to [0, +\infty]$ associating to $x \in M$ the supremum of the Lorentzian lengths of all past-oriented inextensible causal curves starting from x. It is said to be *regular* if $\tau(x) < +\infty$ for all $x \in M$ and for each past-oriented inextensible causal curve $\gamma : [0, +\infty) \to M$, the limit $\tau(\gamma(t)) \to 0$ as $t \to +\infty$.

Replace "past-oriented" by "future-oriented" in the definition of the cosmological time τ, we define the *reverse of the cosmological time* $\breve{\tau}$.

In general, the cosmological time of a spacetime is not regular (e.g. Minkowski space and de Sitter space). In our case, a convex GHCM AdS spacetime (N, g) with particles has a regular cosmological time τ. By Remark 3.8, we have $B^+ = \{x \in N : \tau(x) > \pi/2\}$ and $B^- = \{x \in N : \breve{\tau}(x) > \pi/2\}$.

Proposition 4.5. *Let (N, g) be a convex GHCM AdS spacetime with particles. Then B^- is exactly the union of the surface S_K over $K \in (-\infty, -1)$, where S_K is the future-convex spacelike surface of constant curvature K which is orthogonal to the singular lines in N.*

Proof. Denote by V the union of the surface S_K over $K \in (-\infty, -1)$. Moreover, Lemma 3.13 implies that S_K is disjoint from $S_{K'}$ for all $K \neq K' \in (-\infty, -1)$. Note that V is contained in B^-. We only need to prove that B^- is contained in V.

Fix a number $K_1 < -1$. Consider the union V_1 of the surfaces S_K over $K \in (-\infty, K_1)$. By Proposition 4.4, we have $V_1 \cap B^- = I^-(S_{K_1}) \cap B^-$. Let $V_2 = V \setminus V_1$, that is, the union of the surface S_K over $K \in [K_1, -1)$. It is enough to show that $B^- \setminus V_1 \subset V_2$. The argument is similar to that of Claim 11.14 in [6]. For completeness, we include the proof as follows.

Denote by V_2^* the union of the surfaces S_K^* dual to S_K over all $K \in [K_1, -1)$. By Proposition 2.7, the surface S_K^* is a past-convex spacelike surface in B^+ of constant curvature $K^* = -K/(1+K)$, which is orthogonal to the singular lines in N. Observe that $K^* \to -\infty$ iff $K \to -1$.

Note that Proposition 4.4 is applicable to the family $\{S_{K^*} := S_K^*\}_{K^* \in (-\infty, K_1^*]} \subset B^+$ (it follows directly from reversing the time orientation of N), where $K_1^* = -K_1/(1+K_1)$. This implies that

$$\lim_{K^* \to -\infty} \sup_{x \in S_{K^*}} \breve{\tau}(x) = \lim_{K \to -1} \sup_{x \in S_K^*} \breve{\tau}(x) = 0.$$

where $\breve{\tau}$ is the reverse cosmological time of (N, g).

By Proposition 2.7, the dual surface S_K^* of S_K is obtained by pushing S_K along orthogonal geodesics in the future direction for a distance $\pi/2$. Then the length of a timelike curve joining S_K to S_K^* is at most $\pi/2$. Hence,

$$\lim_{K\to -1} \sup_{x\in S_K} \breve{\tau}(x) \le \pi/2.$$

Note that $B^- = \{x \in N : \breve{\tau}(x) > \pi/2\}$ and S_K is contained in B^- for all $K < -1$. Therefore,

$$\lim_{K\to -1} \sup_{x\in S_K} \breve{\tau}(x) = \pi/2. \tag{9}$$

For any point $x \in B^- \setminus V_1$, we have $\breve{\tau}(x) > \pi/2$. By (9), there exists a future-convex surface S_{K_x} of constant curvature $K_x \in [K_1, -1)$ such that x is in the past of S_{K_x}. Therefore, $x \in V_2$. This completes the proof. □

Proposition 4.6. *Let (N, g) be a convex GHCM AdS spacetime with particles. Then B^- admits a unique foliation by future-convex spacelike surfaces of constant curvature. Moreover, the curvature varies from -1 near the past boundary component of $C(N)$ to $-\infty$ near the past singularity of N.*

Proof. By Lemma 3.13, the future-convex spacelike surface S_K of constant curvature K (which is orthogonal to the singular lines) is unique. Moreover, S_K and $S_{K'}$ are disjoint for all $K \neq K' \in (-\infty, -1)$. Combining this and Proposition 4.5, we obtain the existence and uniqueness. □

By Proposition 2.7, the corresponding result of Proposition 4.6 also holds for B^+ as follows.

Corollary 4.7. *Let (N, g) be a convex GHCM AdS spacetime with particles. Then B^+ admits a unique foliation of past-convex spacelike surfaces of constant curvature. Moreover, the curvature varies from -1 near the upper boundary component of $C(N)$ to $-\infty$ near the future singularity of N.*

Proof of Theorem 1.1. It follows directly from Proposition 4.6 and Corollary 4.7.

Remark 4.8. *It follows from Statement (2) of Proposition 4.4 and Proposition 2.7 that the future-convex (resp. past-convex) spacelike surface S_K of constant curvature K (which is orthogonal to the singular lines) depends continuously on $K \in (-\infty, -1)$. This implies that the (unique) foliation of B^- (resp. B^+) by K-surfaces is a continuous foliation. In particular, it provides a $\mathcal{C}^2$ foliation of the regular part of B^- (resp. B^+).*

5. Applications.

In this section, we use the results obtained above on K-surfaces in convex GHCM AdS spacetimes with particles to extend to hyperbolic surfaces with cone singularities (of fixed angles less than π) a number of results concerning the landslide flow (see e.g. [10]). Hence we give a partial answer to the last question posed in Section 9 of [10].

Using Theorem 2.12 and Proposition 2.12, we extend the definition of a landslide action of S^1 on $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$. Moreover, as an application of Theorem 1.1, we extend to hyperbolic surfaces with cone singularities an analog of Thurston's Earthquake Theorem for the landslide flow on $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$. Finally, we show that the relation between the AdS geometry and landslides provides more details about the parametrization map ϕ_K.

5.1. The landslide action of S^1 on $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$.

First we define the landslide transformation on $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$.

Let $(h, h') \in \mathfrak{M}^\theta_{-1} \times \mathfrak{M}^\theta_{-1}$, let $b : T\Sigma \to T\Sigma$ be the bundle morphism associated to h and h' by Corollary 2.13, and let $\alpha \in \mathbb{R}$. Set

$$\beta_\alpha = \cos\left(\frac{\alpha}{2}\right)E + \sin\left(\frac{\alpha}{2}\right) Jb,$$

where $E : T\Sigma \to T\Sigma$ is the identity morphism and J is the complex structure induced by h.

Let $h_\alpha = h(\beta_\alpha \bullet, \beta_\alpha \bullet)$ and define

$$L_{e^{i\alpha}}(h, h') := (h_\alpha, h_{\alpha+\pi}).$$

In particular, we have $L_1(h, h') = (h, h')$, and $L_{-1}(h, h') = (h', h)$. Denote by $L^1_{e^{i\alpha}}$ (resp. $L^2_{e^{i\alpha}}$) the composition of $L_{e^{i\alpha}}$ with the projection on the first (resp. second) factor.

Proposition 5.1. *For all $\alpha \in \mathbb{R}$, h_α is a hyperbolic metric with cone singularities of the same angles as h.*

Proof. It can be checked (as in the proof of Lemma 3.2 in [10]) that $d^\nabla \beta_\alpha = 0$ and $\det(\beta_\alpha) = 1$, where ∇ is the Levi-Civita connection of h. By Theorem 2.11, the Levi-Civita connection ∇^α of h_α is given by $\nabla^\alpha_u v = \beta_{-\alpha}\nabla_u(\beta_\alpha v)$. The curvature of h_α outside the singular locus is

$$K_\alpha = \frac{K_h}{\det(\beta_\alpha)} = -1. \tag{10}$$

Note that $\beta_\alpha = \cos(\frac{\alpha}{2})E + \sin(\frac{\alpha}{2})Jb$, where $b : T\Sigma \to T\Sigma$ is a bundle morphism associated to (h, h') by Corollary 2.13. In particular, the bundle morphism $b \to E$ at the marked points of Σ. Therefore, near the cone singularities of h, $\beta_\alpha \to \cos(\alpha/2)E + \sin(\alpha/2)J$, and as a consequence

$$\begin{array}{rcl} h_\alpha & = & h(\beta_\alpha\cdot, \beta_\alpha\cdot) \\ & \to & h((\cos(\alpha/2)E + \sin(\alpha/2)J)\cdot, (\cos(\alpha/2)E + \sin(\alpha/2)J)\cdot) \\ & = & h\ . \end{array}$$

Since h_α is asymptotic to h at the cone singularities, the cone angle of h_α is equal to the cone angle of h at those points. ☐

The following lemma shows that the landslide flow is well-defined on $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$.

Lemma 5.2. *Let $(\tau, \tau') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$, and let (h, h'), $(\bar{h}, \bar{h}')$ be two normalized representatives of (τ, τ'). Then $h_\alpha = L^1_{e^{i\alpha}}(h, h')$ and $\bar{h}_\alpha = L^1_{e^{i\alpha}}(\bar{h}, \bar{h}')$ are isotopic in $\mathcal{T}_{\Sigma,\theta}$ for all $\alpha \in \mathbb{R}$.*

Proof. By definition, for all $\alpha \in \mathbb{R}$,

$$h_\alpha = h(\beta_\alpha \bullet, \beta_\alpha \bullet), \qquad \bar{h}_\alpha = \bar{h}(\bar{\beta}_\alpha \bullet, \bar{\beta}_\alpha \bullet),$$

where $\beta_\alpha = \cos(\frac{\alpha}{2}) + \sin(\frac{\alpha}{2})Jb$, and $\bar{\beta}_\alpha = \cos(\frac{\alpha}{2}) + \sin(\frac{\alpha}{2})\bar{J}\,\bar{b}$. Here b (resp. $\bar{b}$) is the bundle morphism associated to h, h' (resp. $\bar{h}, \bar{h}'$) by Corollary 2.13. J (resp. $\bar{J}$) is the complex structure induced by h (resp. $\bar{h}$).

By Remark 2.15, there exists a diffeomorphism f from Σ to Σ, which is isotopic to the identity (the isotopy fixes each marked point) such that $\bar{h} = f^*h$, $\bar{h}' = f^*h'$. Using a similar argument as in the proof of Lemma 3.3, we can prove that

$$\bar{b} = (df)^{-1}b(df)\ , \qquad \bar{J} = (df)^{-1}J(df)\ , \tag{11}$$

where df is the differential of f. Applying (11) to the expression of $\bar{\beta}_\alpha$, we obtain that

$$\bar{\beta}_\alpha = (df)^{-1}(\cos(\frac{\alpha}{2}) + \sin(\frac{\alpha}{2})Jb)(df) = (df)^{-1}\beta_\alpha(df). \tag{12}$$

Substituting (12) and $\bar{h} = f^*h$ into $\bar{h}_\alpha = \bar{h}(\bar{\beta}_\alpha\bullet, \bar{\beta}_\alpha\bullet)$, we see that $\bar{h}_\alpha = f^*(h_\alpha)$. This implies that $\bar{h}_\alpha$ is isotopic to h_α for all $\alpha \in \mathbb{R}$. ☐

Remark 5.3. *For simplicity, henceforth we denote by (h, h') both a pair of normalized metrics in $\mathfrak{M}^\theta_{-1} \times \mathfrak{M}^\theta_{-1}$ and its equivalence class in $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$.*

Definition 5.4. *For all $\alpha \in \mathbb{R}$, the map $L_{e^{i\alpha}} : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ sending an element $(h, h') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ to the pair of isotopy classes of h_α, $h_{\alpha+\pi}$ is well-defined. We call $L_{e^{i\alpha}}$ the landslide (transformation) of parameter α.*

Note that the argument for the flow property of landslides on the product of two copies of the Teichmüller space of a closed surface (see Theorem 1.8 in [10]) can be directly applied to the case with cone singularities. It leads to the following proposition.

Proposition 5.5. *The landslide $L_{e^{i\alpha}}$ given by Definition 5.4 is a flow: for any $\alpha, \alpha' \in \mathbb{R}$,*

$$L_{e^{i\alpha'}} \circ L_{e^{i\alpha}} = L_{e^{i(\alpha+\alpha')}}.$$

In other words, the map $L : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \times S^1 \to \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ associating to $(h, h', e^{i\alpha})$ the image $L_{e^{i\alpha}}(h, h')$ defines an action of S^1 on $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$. We call L the landslide flow, or the landslide action on $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$.

5.2. **The extension of Thurston's Earthquake Theorem.** In this section we extend to hyperbolic surfaces with cone singularities (of fixed angles less than π) an analog of the Earthquake Theorem, already proved for the landslide flow on non-singular hyperbolic surfaces in [10]. To prove this theorem, we give the following lemma, as a generalization of Lemma 1.9 in [10] to the case with cone singularities.

Lemma 5.6. *Let $(h, h') \in \mathfrak{M}^\theta_{-1} \times \mathfrak{M}^\theta_{-1}$ be a pair of normalized metrics and let $\alpha \in (0, \pi)$. Then there exists a unique GHCM AdS spacetime (N, g) with particles which contains a future-convex spacelike surface orthogonal to the singular lines with the induced metric $I_\alpha = \cos^2(\frac{\alpha}{2})h$ and the third fundamental form $III_\alpha = \sin^2(\frac{\alpha}{2})h'$. Moreover, $L^1_{e^{i\alpha}}(h, h')$ and $L^1_{e^{-i\alpha}}(h, h')$ are the left and right metrics of (N, g), respectively.*

Proof. Note that $\cos^2(\frac{\alpha}{2}), \sin^2(\frac{\alpha}{2}) \in (0, 1)$. The first statement is a direct consequence of Lemma 3.2 applied with $K = -1/\cos^2(\frac{\alpha}{2})$ and $K^* = -1/\sin^2(\frac{\alpha}{2})$.

Denote by B_α the shape operator of the future-convex spacelike surface of constant curvature K in (N, g) and denote by J_α the complex structure of I_α. A simple computation shows that $B_\alpha = \tan(\frac{\alpha}{2})b$, where b is the bundle morphism associated to h, h' by Corollary 2.13, and $J_\alpha = J$, where J is the complex structure of h. By the extension of Mess' parametrization (see Theorem 1.4 in [16]), the left and right metrics of (N, g) can be expressed as

$$\mu_l = I_\alpha((E + J_\alpha B_\alpha)\bullet, (E + J_\alpha B_\alpha)\bullet), \qquad \mu_r = I_\alpha((E - J_\alpha B_\alpha)\bullet, (E - J_\alpha B_\alpha)\bullet). \tag{13}$$

Substituting $B_\alpha = \tan(\frac{\alpha}{2})b$ and $J_\alpha = J$ into (13), we obtain that

$$\mu_l = h(\beta_\alpha\bullet, \beta_\alpha\bullet) = L^1_{e^{i\alpha}}(h, h').$$

Similarly, we can prove that the right metric of (N, g) is

$$\mu_r = h(\beta_{-\alpha}\bullet, \beta_{-\alpha}\bullet) = L^1_{e^{-i\alpha}}(h, h').$$

This completes the proof. □

Corollary 5.7. *Let $(\mu_l, \mu_r) \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ and $\alpha \in (0, \pi)$. There exists a unique $(h, h') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ such that*

$$L^1_{e^{i\alpha}}(h, h') = \mu_l, \qquad L^1_{e^{-i\alpha}}(h, h') = \mu_r.$$

Proof. Given μ_l and μ_r, by the extension of Mess' parametrization (see Theorem 1.4 in [16]), there exists a unique convex GHCM AdS spacetime (N, g) with particles which has the left and right metrics μ_l and μ_r. By Theorem 1.1, (N, g) contains a unique future-convex surface S_K of constant curvature $K = -1/\cos^2(\frac{\alpha}{2})$. Denote by I and III the first and third fundamental form on S_K. Then III has constant curvature $K^* = -1/\sin^2(\frac{\alpha}{2})$. Set $h = |K|I$ and $h' = |K^*|III$. It can be checked that (h, h') is a pair of normalized metrics. It follows from Lemma 5.6 that $L^1_{e^{i\alpha}}(h, h') = \mu_l$, $L^1_{e^{-i\alpha}}(h, h') = \mu_r$. This shows the existence.

Now we show the uniqueness. Suppose $(\bar{h}, \bar{h}') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ is another pair such that

$$L^1_{e^{i\alpha}}(\bar{h}, \bar{h}') = \mu_l, \qquad L^1_{e^{-i\alpha}}(\bar{h}, \bar{h}') = \mu_r. \tag{14}$$

By Lemma 5.6, there exists a unique GHCM AdS spacetime $(\bar{N}, \bar{g})$ with particles which contains a future-convex spacelike surface orthogonal to the singular lines, with the induced metric $\cos^2(\frac{\alpha}{2})\bar{h}$ and the third fundamental form $\sin^2(\frac{\alpha}{2})\bar{h}'$. Moreover, by (14), the left and right metrics of $(\bar{N}, \bar{g})$ are μ_l and μ_r, respectively. The extension of Mess' parametrization implies that $(\bar{N}, \bar{g})$ is (N, g) up to isotopy. The uniqueness in Theorem 1.1 shows that $(\bar{h}, \bar{h}') = (h, h')$ in $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$. □

Now we are ready to prove the extension of Thurston's Earthquake Theorem to the case with cone singularities, which generalizes Theorem 1.14 in [10].

Theorem 5.8. *Let $h_1, h_2 \in \mathcal{T}_{\Sigma,\theta}$ and let $e^{i\alpha} \in S^1 \setminus \{1\}$. Then there exists a unique $h'_1 \in \mathcal{T}_{\Sigma,\theta}$ such that $L^1_{e^{i\alpha}}(h_1, h'_1) = h_2$.*

Proof. First we show the existence. Corollary 5.7 applied with $\mu_l = h_2$, $\mu_r = h_1$ and $\varphi = \alpha/2$ shows that there exists a unique $(h_0, h_0') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ such that $L^1_{e^{i\varphi}}(h_0, h_0') = h_2$ and $L^1_{e^{-i\varphi}}(h_0, h_0') = h_1$. Set $h_1' = L^2_{e^{-i\varphi}}(h_0, h_0')$. Then we get

$$L^1_{e^{i\alpha}}(h_1, h_1') = L^1_{e^{i2\varphi}}(L_{e^{-i\varphi}}(h_0, h_0')) = h_2.$$

Assume that $\bar{h}_1'$ is another element in $\mathcal{T}_{\Sigma,\theta}$ such that $L^1_{e^{i\alpha}}(h_1, \bar{h}_1') = h_2$. Set $(h, h') = L_{e^{i\alpha/2}}(h_1, \bar{h}_1')$. By computation, we have $L^1_{e^{i\alpha/2}}(h, h') = h_2$, and $L^1_{e^{-i\alpha/2}}(h, h') = h_1$. The uniqueness in Corollary 5.7 implies that $(h_0, h_0') = (h, h')$. Hence $\bar{h}_1' = L^2_{e^{-i\varphi}}(h, h') = L^2_{e^{-i\varphi}}(h_0, h_0') = h_1'$. This completes the proof. □

5.3. **The landslide flow in terms of harmonic maps.** Recall that in the non-singular case, landslides can also be defined in terms of multiplication of the Hopf differential of harmonic maps by complex numbers of modulus 1 (see Definition 1.2 in [10]).

Consider a map $\Phi : \mathcal{T}_{\Sigma,\theta} \to \mathcal{QD}_c(\Sigma)$, which associates to $g \in \mathcal{T}_{\Sigma,\theta}$ the Hopf differential (with respect to the conformal structure c) of the harmonic map $u_{c,\bar{g}}$ from (Σ, c) to $(\Sigma, \bar{g})$ isotopic to the identity, where $\bar{g}$ is a representative of g. By the uniqueness in Theorem 3.15 and the fact that harmonic maps remain harmonic when composed from the left with isometries, Φ is well-defined (i.e. independent of the choice of the representatives of g).

For simplicity, we use the same notation for both $g \in \mathcal{T}_{\Sigma,\theta}$ and its representative henceforth. We want to show that Φ is a homeomorphism. The argument is similar to that for Theorem 3.1 (the well-known Wolf's parameterization of Teichmüller space of hyperbolic closed surfaces without cone singularities) in [40], but we need to analyze the behaviour near the cone singularities particularly by analysing the regularity of the map at the singular points.

Define the map $E : \mathcal{T}_{\Sigma,\theta} \to \mathbb{R}$ as $E(g) = E(u_{c,g})$. Before showing Φ is a homeomorphism, we first prove the following lemma.

Lemma 5.9. *The map E is proper.*

Proof. It is enough to show that $B = \{g \in \mathcal{T}_{\Sigma,\theta} : E(g) < C_0\}$ is compact. By Lemma 4.3, it suffices to show that

$$\ell_\gamma(g) \leq C \ell_\gamma(g_0), \tag{15}$$

for all $g \in B$ and all simple closed curves γ on Σ.

By Lemma 3.21, there exists a uniform lower bound for the injectivity radius of the singularities over $\mathcal{T}_{\Sigma,\theta}$. Denote by $\overline{\Sigma}^{g_0}$ (resp. $\overline{\Sigma}^{g}$) the metric completion of $(\Sigma_{\mathfrak{p}}, g_0)$ (resp. $(\Sigma_{\mathfrak{p}}, g)$) by adding the set $\mathfrak{p}$ and denote by $\mathrm{inj}(g_0)$ the injectivity radius of $\overline{\Sigma}^{g_0}$. Then $\mathrm{inj}(g_0) > 0$. Let $c_1(g_0) = \min\{1, (\mathrm{inj}(g_0))^2\}$. The Courant-Lebesgue Lemma (see [40, Proposition 3] and [22, Lemma 3.1]) can be applied to the harmonic map $u : \overline{\Sigma}^{g_0} \to \overline{\Sigma}^{g}$, that is, for any $x_1, x_2 \in \overline{\Sigma}^{g_0}$ with $d_{g_0}(x_1, x_2) < \delta < c_1(g_0)$, we have

$$d_g(u(x_1), u(x_2)) < 4\sqrt{2}\pi {C_0}^{1/2} (\log(1/\delta))^{-1/2}.$$

This implies (15), where C depends on g_0 and C_0. The proof is complete. □

Proposition 5.10. *The map Φ is a homeomorphism.*

Proof. Observe that $\mathcal{T}_{\Sigma,\theta}$ and $\mathcal{QD}_c(\Sigma)$ are both $6g-6+2n$-dimensional cells. By Brouwer's Invariance of Domain Theorem, it suffices to show that Φ is continuous, injective and proper.

The continuity is obvious, since the harmonic maps $u_{c,g}$ vary smoothly with respect to the target metric g (see Theorem 3.15).

For the injectivity of Φ, we use the maximum principle as applied in [40, Theorem 3.1]. Suppose that $g_1, g_2 \in \mathcal{T}_{\Sigma,\theta}$ satisfy that $\Phi(g_1) = \Phi(g_2)$. Denote $\Phi(g_i) = \Phi_i$ (i always takes values in $\{1, 2\}$ in this proof), so that $\Phi_1 = \Phi_2$. Let z be a conformal coordinate on (Σ, c). Set $c = g_0 = \sigma(z)|dz|^2$, $g_i = \rho_i(u_i(z))|du_i|^2$, where $u_i = u_{c,g_i}$. By computation, we obtain that $\Phi_i = \rho_i(u_i(z))(u_i)_z\overline{(u_i)_{\bar{z}}}$. Set $H_i = \sigma^{-1}(z)\rho_i(u_i(z))|(u_i)_z|^2$, and $L_i = \sigma^{-1}(z)\rho_i(u_i(z))|(u_i)_{\bar{z}}|^2$. We have the following quantities (see [40, Section 2]):

(a) The energy density $e_i = H_i + L_i$.

(b) The Jacobian $J_i = H_i - L_i > 0$.
(c) The norm of the quadratic differential $|\Phi_i|^2/\sigma^2 = H_i L_i$.
(d) The Beltrami differential $\nu_i = (u_i)_{\bar{z}}/(u_i)_z = \overline{\Phi_i}/\sigma H_i$.
(e) The pull-bak metric of g_i by u_i is $u_i{}^*(g_i) = 2\Re(\Phi_i dz^2) + \sigma e_i dz d\bar{z}$.

Set $h_i = \log H_i$ and $\Delta = 4\sigma^{-1}\partial^2_{z\bar{z}}$. We claim that $h_1 = h_2$. Indeed, near the cone singularity p_k of angle θ_k, it is known (see [21, Form 2.3] and [36, Section 4.2])) that the harmonic map u_i is expressed as $u_i(z) = \xi_i z + r^{1+\varepsilon} f_i(z)$, where z is the conformal coordinate centered at p_k, $\xi_i \in \mathbb{C}\setminus\{0\}$, $r = |z|$, $\varepsilon > 0$ and $f_i \in \mathcal{C}^{2,\gamma}(D(R))$, where $D(R) = \{z \in \mathbb{C}, |z| \in [0, R)\}$ for some small $R > 0$ and $\gamma \in (0, 1)$. Moreover, $\sigma(z) = e^{\lambda(z)}|z|^{2(\beta_k-1)}$, $\rho_i(u_i(z)) = e^{\zeta_i(u_i(z))}|u_i(z)|^{2(\beta_k-1)}$ in $D(R)$, where $\beta_k = \theta_k/(2\pi)$, and $\lambda(z)$, $\zeta_i(u_i(z))$ are continuous functions on $D(R)$. It is computed in [36, Section 4.2] that

$$H_i = \sigma^{-1}(z) e^{\zeta_i(u_i(z))}|\xi_i|^{2\beta_k} r^{2(\beta_k-1)}(1 + O(r^\varepsilon)). \tag{16}$$

Substituting $\sigma(z) = e^{\lambda(z)}|z|^{2(\beta_k-1)}$ into (16), we obtain that

$$H_i = e^{\zeta_i(u_i(z))-\lambda(z)}|\xi_i|^{2\beta_k}(1 + O(r^\varepsilon)). \tag{17}$$

Note that we can make a metric completion of the punctured surfaces $(\Sigma_{\mathfrak{p}}, g_0)$ and $(\Sigma_{\mathfrak{p}}, g_i)$ by directly adding the set $\mathfrak{p}$. Then $h_1 - h_2 = \log(H_1/H_2)$ is continuous on a compact surface Σ and achieves its maximum at a point of Σ, called x_0. Indeed, $h_1 - h_2 \in \mathcal{C}^2(\Sigma_{\mathfrak{p}}) \cap \mathcal{C}^{0,\delta}(\Sigma)$ for a number $\delta \in (0, 1)$. This follows from (17), the regularity of u_i at cone singularities, and the fact that the functions λ, ζ_i are in the class of $\mathcal{C}^2(\Sigma_{\mathfrak{p}}) \cap \mathcal{C}^{0,\delta}(\Sigma)$ (see the proof of the main theorem in [25]).

It is well-known that the following identity (see [30, Section 1, equation (16)]) holds on $\Sigma_{\mathfrak{p}}$:

$$\Delta h_i = 2(H_i - L_i - 1) = 2(e^{h_i} - \sigma^{-2}|\Phi_i|^2 e^{-h_i} - 1). \tag{18}$$

We claim that $h_1 - h_2 \le 0$. Otherwise, $h_1 - h_2 > 0$ at the maximum point x_0. By (18) and the continuity of h_i, $\Delta(h_1 - h_2) = 2\{(e^{h_1} - e^{h_2}) - \sigma^{-2}|\Phi|^2(e^{-h_1} - e^{-h_2})\} > 0$ in a small neighbourhood U of x_0, here $\Phi = \Phi_1 = \Phi_2$. If x_0 is a regular point, $\Delta(h_1 - h_2) \le 0$ at x_0, which is a contradiction. Hence x_0 is a singular point. Recall that a function $f \in \mathcal{C}^2(U\setminus\{x_0\}) \cap \mathcal{C}^{0,\delta}(U)$ with $\Delta f \ge 0$ in $U\setminus\{x_0\}$ has the property that $\Delta f \ge 0$ in U in the sense of distribution and the mean value inequality holds in U (i.e. the average of the integral of f over any ball $B_r(x) \subset U$ centered at x is not less than $f(x)$). Hence, $h_1 - h_2$ is subharmonic in U. Note that $h_1 - h_2$ achieves its maximum at an interior point x_0 of U, then $h_1 - h_2 \equiv (h_1 - h_2)(x_0) > 0$ in U and thus $h_1 - h_2$ also achieves a maximum at a regular point, which contradicts the above result.

Therefore, $h_1 \le h_2$. Symmetrically, we have $h_2 \le h_1$. Hence, $h_1 = h_2$, which implies that $H_1 = H_2$. By equality (c) and (a), $L_1 = L_2$ and $e_1 = e_2$. Combined with equality (e), we get $u_1{}^*(g_1) = u_2{}^*(g_2)$. Note that u_1, u_2 are isotopic to identity, then $g_1 = g_2 \in \mathcal{T}_{\Sigma,\theta}$.

To show the properness of Φ, we first state the fact that $||\Phi(g)|| \to \infty$ iff $E(g) \to \infty$. Indeed, applying equalities (b),(c),(d) and the Gauss-Bonnet formula for surfaces with cone singularities (see e.g. [39, Proposition 1]):

$$\int J\sigma dz d\bar{z} = \mathrm{Area}_g(\Sigma) = -2\pi\chi(\Sigma, \theta),$$

as in [40, Theorem 3.1], we have

$$\int H\sigma dz d\bar{z} + 2\pi\chi(\Sigma,\theta) = \int L\sigma dz d\bar{z} \le \int |\Phi(g)| dz d\bar{z} \le \int H\sigma dz d\bar{z} = \int L\sigma dz d\bar{z} - 2\pi\chi(\Sigma,\theta).$$

Adding the first two and last two integrals and applying equality (a), we obtain

$$E(g) + 2\pi\chi(\Sigma,\theta) \le 2\int |\Phi(g)| dz d\bar{z} \le E(g) - 2\pi\chi(\Sigma,\theta).$$

Combined with Lemma 5.9, Φ is proper. The proof is complete. □

Proposition 5.10 shows that given a meromorphic quadratic differential $q \in \mathcal{QD}_c(\Sigma)$ with at most simple poles at singularities, there exists a unique $h \in \mathcal{T}_{\Sigma,\theta}$ such that the identity map $id : (\Sigma, c) \to (S, h)$ is harmonic with Hopf differential q.

This statement, combined with Lemma 3.15, makes it possible to generalize the definition of the landslide flow in terms of harmonic maps to hyperbolic surfaces with cone singularities as follows.

Definition 5.11. *Let $c, h \in \mathcal{T}_{\Sigma,\theta}$ and let $e^{i\alpha} \in S^1$. Define $R_{c,\alpha}(h)$ as the (unique) metric $h^\alpha \in \mathcal{T}_{\Sigma,\theta}$ such that if $f : (\Sigma, c) \to (\Sigma, h)$ and $f^\alpha : (\Sigma, c) \to (\Sigma, h^\alpha)$ are the harmonic maps isotopic to the identity (fixing each marked point), then $\Phi(f^\alpha) = e^{i\alpha}\Phi(f)$.*

Let $h, h' \in \mathcal{T}_{\Sigma,\theta}$. Recall that if h_α is used to denote $L^1_{e^{i\alpha}}(h, h')$, then $L_{e^{i\alpha}}(h, h') = (h_\alpha, h_{\alpha+\pi})$. Denote by c_α the conformal structure of the metric $h_\alpha + m^*_\alpha(h_{\alpha+\pi})$, where $m_\alpha : (\Sigma, h_\alpha) \to (\Sigma, h_{\alpha+\pi})$ is the unique minimal Lagrangian map isotopic to the identity, which is called the *center* of $(h_\alpha, h_{\alpha+\pi})$. Applying the analogous argument as Theorem 1.10 in [10] to the case with cone singularities, we have the following proposition.

Proposition 5.12. *Let $h, h' \in \mathcal{T}_{\Sigma,\theta}$ and let c_α be the center of $(h_\alpha, h_{\alpha+\pi})$. Then*

(1) The identity $id : (\Sigma, h_\alpha) \to (\Sigma, h_{\alpha+\pi})$ is minimal Lagrangian.
(2) c_α is independent of α — we denote it by c.
(3) For any $\alpha \in \mathbb{R}$, $\Phi(f_\alpha) = e^{i\alpha}\Phi(f)$, where $f_\alpha : (\Sigma, c) \to (\Sigma, h_\alpha)$ is the unique harmonic map isotopic to the identity.

The following corollary is a direct consequence of Definition 5.11, Proposition 5.10 and Proposition 5.12.

Corollary 5.13. *Let $(h, h') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ be a normalized representative, and let c be the conformal class of $h + h'$. Then for any $e^{i\alpha} \in S^1$, we have*

$$L_{e^{i\alpha}}(h, h') = (R_{c,\alpha}(h), R_{c,\alpha+\pi}(h)).$$

5.4. **An application of the landslide flow.** We now go in the reverse direction, and use the properties of the landslide flow to obtain new results on the geometry of K-surfaces in convex GHCM AdS spacetimes with particles. We first state a lemma on landslides on hyperbolic surfaces with cone singularities, and then use it to obtain Theorem 5.15 below on K-surfaces.

Lemma 5.14. *Let $(h, h') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ be a normalized representative. Define the map $L_\bullet(h, h') : S^1 \to \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ by associating $L_{e^{i\alpha}}(h, h')$ to $e^{i\alpha} \in S^1$. Then the following two statements hold:*

(1) If $h \neq h'$, then the map $e^{i\alpha} \mapsto L_{e^{i\alpha}}(h, h')$ is injective.
(2) If $h = h'$, then this map $e^{i\alpha} \mapsto L_{e^{i\alpha}}(h, h')$ is constant, that is, $L_{e^{i\alpha}}(h, h') = (h, h)$ for all $e^{i\alpha} \in S^1$.

Proof. First we show the first statement. Assume that $h \neq h'$ and $L_{e^{i\alpha_1}}(h, h') = L_{e^{i\alpha_2}}(h, h')$. By Corollary 5.13, we have

$$R_{c,\alpha_1}(h) = R_{c,\alpha_2}(h), \qquad \Phi(f^{\alpha_i}) = e^{i\alpha_i}\Phi(f), \tag{19}$$

for $i = 1, 2$, where $f : (\Sigma, c) \to (\Sigma, h)$ and $f^{\alpha_i} : (\Sigma, c) \to (\Sigma, R_{c,\alpha_i}(h))$ are the (unique) harmonic maps isotopic to the identity, c is the conformal structure of $h+h'$. Moreover, (19) implies $\Phi(f^{\alpha_1}) = \Phi(f^{\alpha_2})$, that is,

$$e^{i(\alpha_1-\alpha_2)}\Phi(f) = 0.$$

Note that $\Phi(f) \neq 0$ since $h \neq h'$. This implies that $\alpha_1 = \alpha_2$.

Assume that $h = h'$, then c is the conformal structure of h. It follows that the harmonic map $f : (\Sigma, c) \to (\Sigma, h)$ isotopic to the identity is exactly the identity by choosing the representative metric h of c. Hence, $\Phi(f) = 0$ and $\Phi(f^\alpha) = e^{i\alpha}\Phi(f) = 0$ for all $\alpha \in S^1$. By Proposition 5.10, Definition 5.11 and Corollary 5.13, we obtain

$$L_{e^{i\alpha}}(h, h') = (R_{c,\alpha}(h), R_{c,\alpha+\pi}(h)) = (h, h),$$

for all $e^{i\alpha} \in S^1$. □

Theorem 5.15. *Let $(N, g) \in \mathcal{GH}_{\Sigma,\theta}$ and $K_1, K_2 \in (\infty, -1)$. Then the following two statements are equivalent:*

(1) The preimages under ϕ_{K_1} and ϕ_{K_2} of (N, g) are the same point in $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$.

(2) $K_1 = K_2$ *or* (N, g) *is Fuchsian.*

Proof. First we show Statement (1) implies Statement (2). Denote by (h, h') the same preimage under ϕ_{K_1} and ϕ_{K_2} of (N, g). Let $\alpha_i \in (0, \pi)$ such that $K_i = -1/\cos^2\alpha_i$ for $i = 1, 2$. From the definition of ϕ_{K_i}, (N, g) contains a future-convex spacelike surface S_{K_i} orthogonal to the singular lines, with the induced metric $(1/|K_i|)h$ and the third fundamental form $(1/|K^*)|h'$, where $K_i^* = -K_i/(1 + K_i) = -1/\sin^2\alpha_i$ for $i = 1, 2$. Apply Lemma 5.6 with $(h, h') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ and $\alpha_i \in (0, \pi)$, the left and right metrics of (N, g) are respectively

$$\mu_l = L^1_{e^{i\alpha_1}}(h, h') = L^1_{e^{i\alpha_2}}(h, h'), \qquad \mu_r = L^1_{e^{-i\alpha_1}}(h, h') = L^1_{e^{-i\alpha_2}}(h, h'). \tag{20}$$

We claim that if (N, g) is not Fuchsian, then $h \neq h'$. Otherwise, by (20) and Statement (2) of Lemma 5.14, $h = h'$ implies that $\mu_l = \mu_r$ and hence (N, g) is Fuchsian. This leads to contradiction. By Statement (1) of Lemma 5.14, we have $\alpha_1 = \alpha_2$. This implies that $K_1 = K_2$.

Now it suffices to prove that Statement (2) implies Statement (1). (2) is clear if $K_1 = K_2$ since then $\phi_{K_1} = \phi_{K_2}$. If (N, g) is Fuchsian, denote by (h_1, h'_1) and (h_2, h'_2) the preimages under the maps ϕ_{K_1} and ϕ_{K_2} of (N, g). Note that $\alpha_i \in (0, \pi)$. By Lemma 5.6, we have

$$\mu_l = L^1_{e^{i\alpha_1}}(h_1, h'_1) = L^1_{e^{-i\alpha_1}}(h_1, h'_1) = \mu_r, \qquad \mu_l = L^1_{e^{i\alpha_2}}(h_2, h'_2) = L^1_{e^{-i\alpha_2}}(h_2, h'_2) = \mu_r.$$

By Statement (1) of Lemma 5.14, we obtain $h_1 = h'_1$ and $h_2 = h'_2$. By Statement (2) of Lemma 5.14, we get that

$$h_1 = L^1_{e^{i\alpha_1}}(h_1, h'_1) = \mu_l = L^1_{e^{i\alpha_2}}(h_2, h'_2) = h_2\ .$$

This implies that $(h_1, h'_1) = (h_2, h'_2)$. The proof is complete. □

Remark 5.16. *Note that Theorem 5.15 also holds for the non-singular case. This implies that for a non-Fuchsian convex GHCM AdS spacetime N (with particles or not), any two spacelike surfaces of distinct constant curvatures are not isotopic.*

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