From modular curves to Shimura curves: a p-adic approach

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1. Modular curves and elliptic curves

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2. **Quaternion algebras and Shimura curves**

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3. ***p*-adic uniformisation of Shimura curves**
4. **Bad reduction of Shimura curves**

1. MODULAR CURVES AND ELLIPTIC CURVES

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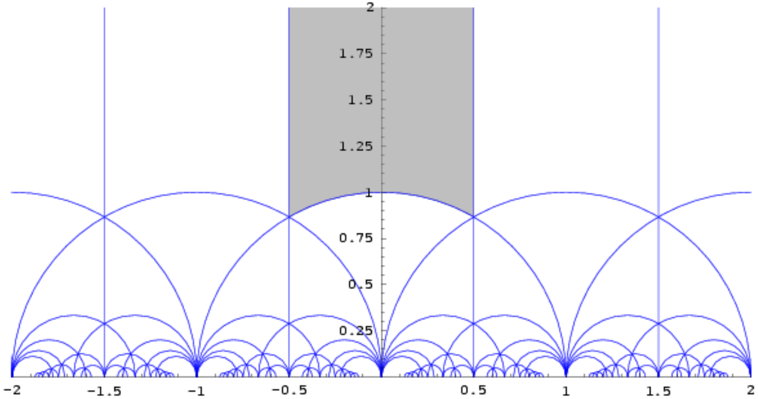

Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}_\infty$

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This is a meromorphic function on $\mathbb{C}$, invariant under Λ, and the map

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \to & \mathbb{P}^2(\mathbb{C}) \\ [z] & \mapsto & (\mathscr{P}(z) : \mathscr{P}'(z) : 1) \end{array}$$

defines an isomorphism of Riemann surfaces from $\mathbb{C}/\Lambda$ to $E(\mathbb{C})$, where E is the **elliptic curve**

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3,$$

with g_2 and g_3 are determined from the lattice Λ.

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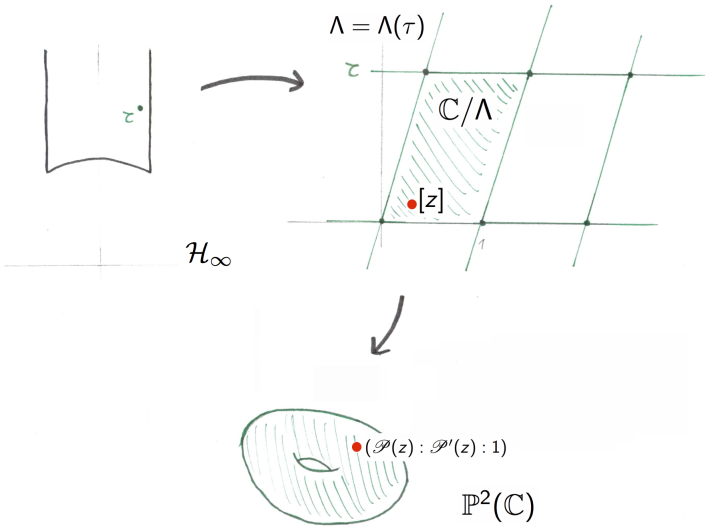

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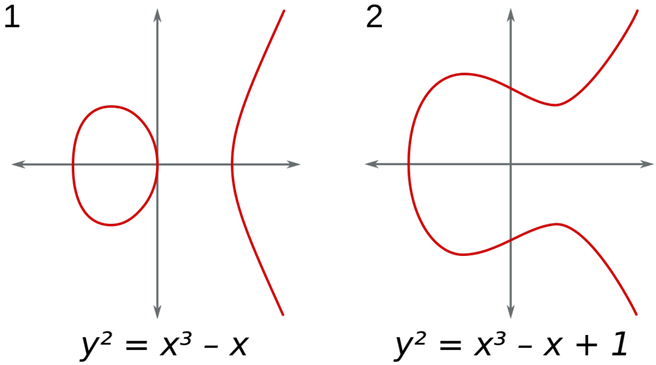

Real points of two elliptic curves

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Then there exist homolorphic functions $F, G : U \subseteq \mathbb{C} \to \mathbb{C}$ invariant under the group $\langle \tau \rangle$ such that there is a bijection

$$\begin{array}{ccc} \langle \tau \rangle \setminus \mathbb{C} & \to & S^1(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C}) \\ t & \mapsto & (F(t) : G(t) : 1) \end{array}$$

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where

$$j(z) := q^{-1} + 744 + 196884q + 21493760q^2 + \dots, \quad q = e^{2\pi i z}$$

is the Klein j-function.

2. QUATERNION ALGEBRAS AND SHIMURA CURVES

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B is **indefinite** when $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ and **definite** when $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{H}$.

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We want to study the bad reduction of Shimura curves, i.e. its reduction modulo some prime $p \mid D_B$.

3. p-ADIC UNIFORMISATION OF SHIMURA CURVES

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The Theorem of Čerednik and Drinfel'd

Theorem. There is an isomorphism

$$\Gamma_p \backslash (\mathcal{H}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2}) \simeq (X(Dp, N) \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\mathrm{rig}}$$

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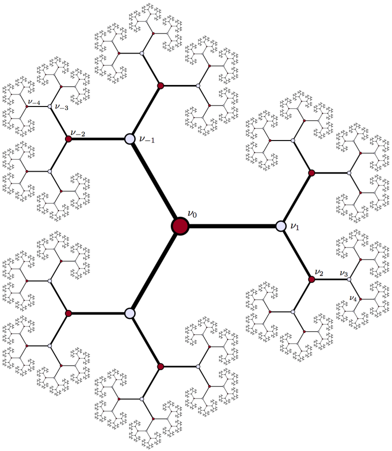
The group $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts transitively on $\mathrm{Ver}(\mathcal{T}_p)$: if $M = \langle u, v \rangle \subseteq \mathbb{Q}_p^2$ and $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$ then $\gamma \cdot M := \langle \gamma u, \gamma v \rangle$.

The Bruhat-Tits tree



Bruhat-Tits tree $\mathcal{T}_p$ for $p = 2$

Picture taken from: *The Bruhat-Tits tree of SL(2), Bill Casselman*

4. BAD REDUCTION OF SHIMURA CURVES

Mumford theory

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Theorem. Let $\Gamma \subseteq \mathrm{PGL}_2(\mathbb{Q}_p)$ be a discontinuous and finitely generated group. Then there exists a normal subgroup Γ^{Sch} of finite index which is torsion-free. In particular Γ^{Sch} is a p-adic Schottky group.

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If $t_\xi(p) = 0$, we will say that p satisfies the **null-trace condition** with respect to $\xi\mathcal{O}$.

Technical conditions

D	H	N	$\mathcal{O}$	$\#(\mathcal{O}^\times/\mathbb{Z}^\times)$	ξ	$\mathrm{Nm}(\xi)$
2	$\left(\frac{-1,-1}{\mathbb{Q}}\right)$	1	$\mathbb{Z}\left[1, i, j, \frac{1}{2}(1+i+j+k)\right]$	12	2	4
		3	$\mathbb{Z}\left[1, 3i, -2i+j, \frac{1}{2}(1-i+j+k)\right]$	3	$(-i+k)$	2
		9	$\mathbb{Z}\left[1, 9i, -4i+j, \frac{1}{2}(1-3i+j+k)\right]$	1	1	1
		11	$\mathbb{Z}\left[1, 11i, -10i+j, \frac{1}{2}(1-3i+j+k)\right]$	1	1	1
3	$\left(\frac{-1,-3}{\mathbb{Q}}\right)$	1	$\mathbb{Z}\left[1, i, \frac{1}{2}(i+j), \frac{1}{2}(1+k)\right]$	6	2	4
		2	$\mathbb{Z}\left[1, 2i, \frac{1}{2}(-i+j), \frac{1}{2}-i+\frac{1}{2}k\right]$	2	$\frac{1}{2}(-1-i-j+k)$	2
		4	$\mathbb{Z}\left[1, 4i, \frac{1}{2}(-5i+j), \frac{1}{2}-3i+\frac{1}{2}k\right]$	1	1	1
5	$\left(\frac{-2,-5}{\mathbb{Q}}\right)$	1	$\mathbb{Z}\left[1, \frac{1}{2}(1+i+j), j, \frac{1}{4}(2+i+k)\right]$	3	$\frac{1}{2}(-1+i-j)$	2
		2	$\mathbb{Z}\left[1, 1+i+j, \frac{1}{2}(-1-i+j), \frac{1}{4}(-i-2j+k)\right]$	1	1	1
13	$\left(\frac{-2,-13}{\mathbb{Q}}\right)$	1	$\mathbb{Z}\left[1, \frac{1}{2}(1+i+j), j, \frac{1}{4}(2+i+k)\right]$	1	1	1

Table : Definite orders $\mathcal{O}$ with $\xi \in \mathcal{O}$ satisfying the right-unit property

About the groups Γ_p and $\Gamma_{p,+}$

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Let $\mathcal{O}$ with $h(D, N) = 1$, $\xi \in \mathcal{O}$. Then every $\alpha \in \mathcal{O}[1/p]^{\times}$ can be decomposed as a product

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Moreover, if $2 \notin \xi\mathcal{O}$ this decomposition is unique, and if $2 \in \xi\mathcal{O}$, the decomposition is unique up to sign.

How to obtain a Schottky group in Γ_p

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Then

$$S = \{[\Phi_p(\alpha_1)], \ldots, [\Phi_p(\alpha_s)], [\Phi_p(\beta_1)], \ldots, [\Phi_p(\beta_t)]\} \subseteq \mathrm{PGL}_2(\mathbb{Q}_p)$$

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is a system of generators of $\Gamma_p(\xi)$.

In particular, if p satisfies the null-trace condition with respect to $\xi\mathcal{O}$, then $\Gamma_p(\xi) \subseteq \Gamma_p$ is a Schottky group of rank s.

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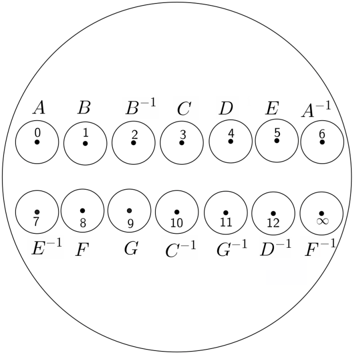
where $\widetilde{\alpha}_\gamma$ and $\widetilde{\alpha}_{\gamma^{-1}}$ are defined as the reduction in $\mathbb{P}^1(\mathbb{F}_p)$ of the fixed points of the transformations $\{\gamma, \gamma^{-1}\}$.

Our example



Fundamental domain for the action of $\Gamma_p(\xi)$ on $\mathcal{H}_p$

Third consequence

(d) The stable reduction-graph of $X_p(\xi)$ is the open subtree $\mathcal{T}_p^{(1)} \smallsetminus \{v_0^{(1)}, \ldots, v_{p-1}^{(1)}, v_\infty^{(1)}\}$ of $\mathcal{T}_p$

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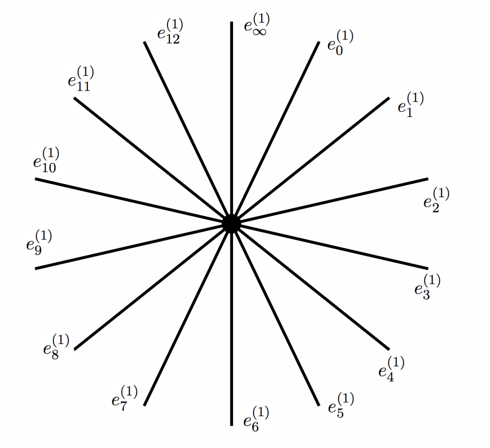
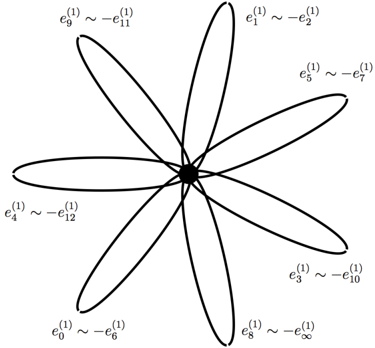


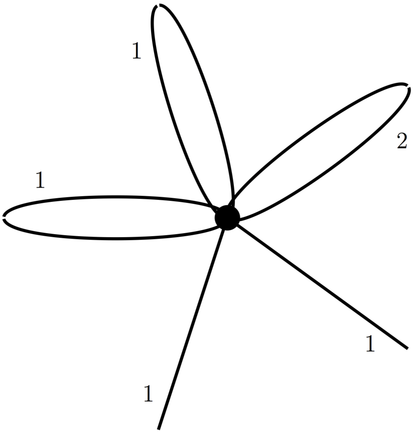
Reduction of the fundamental domain $\mathcal{F}_{13}(\xi)$

Our example



Stable reduction-graph of the Mumford curve associated to $\Gamma_{13}(\xi)$

Our example



Reduction-graphs with lengths $\Gamma_{13} \backslash \mathcal{T}_{13}$

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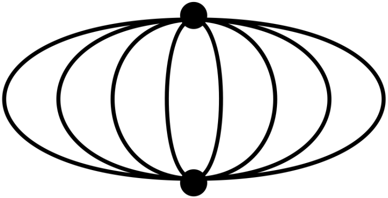

Reduction-graphs with lengths $\Gamma_{13,+}\backslash\mathcal{T}_{13}$ for the Shimura curve $X(3 \cdot 13, 2)$

Thank you!