Remarks on contact and Jacobi geometry*

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Abstract

We present an approach to Jacobi and contact geometry that makes many facts, presented in the literature in an overcomplicated way, much more natural and clear. The key concepts are *Kirillov manifolds* and *Kirillov algebroids*, i.e. homogeneous Poisson manifolds and, respectively, homogeneous linear Poisson manifolds. The difference with the existing literature is that the homogeneity of the Poisson structure is related to a principal $\mathrm{GL}(1,\mathbb{R})$-bundle structure on the manifold and not just to a vector field. This allows for working with *Jacobi bundle structures* on nontrivial line bundles and drastically simplifies the picture of Jacobi and contact geometry. In this sense, the properly understood concept of a Jacobi structure is a specialisation rather than a generalisation of a Poission structure. Our results easily reduce to various basic theorems of Jacobi and contact geometry when the principal bundle structure is trivial, as well as give new insight in the theory. For instance, we describe the structure of Lie groupoids with a compatible principal G-bundle structure and the 'integrating objects' for Kirillov algebroids, define *canonical contact groupoids*, and show that any contact groupoid has a canonical realisation as a contact subgroupoid of the latter.

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1 Introduction

There is extensive literature in differential geometry devoted to Jacobi structures and derived concepts in which Jacobi structures are presented as generalising Poisson structures. The aim in this paper is to put some order in to the field, and to convince the reader that the properly understood concept of a Jacobi structure is a specialisation of a Poisson structure and not a generalisation. We present an approach to Jacobi and contact geometry which results in drastic simplification of many concepts, examples and proofs, gives a completely new insight into the theory, as well as novel discoveries and observations.

The main motivation for writing this paper was our irritation caused by the fact that many papers in the subject are unnecessarily complicated, because the authors generally ignore, to different extents, the fact that Jacobi geometry is nothing else but homogeneous Poisson geometry on principal $\mathrm{GL}(1,\mathbb{R})$-bundles ($\mathbb{R}^\times$-bundles, for short).

This is despite the fact that everyone knows there exists something like the *poissonisation* of a Jacobi structure. However, in most cases it is seen as the *poissonisation trick* used in only as a technical tool for proving particular results formulated in 'intrinsic' terms of Jacobi geometry. In consequence, the true landscape of Jacobi geometry, which is actually a homogeneous Poisson geometry, is hidden in the fog of the 'intrinsic Jacobi language'. In our understanding, the true poissonisation is not a trick but a genuine framework for Jacobi geometry, and it necessarily comes together with an additional structure of a $\mathbb{R}^\times$-bundle. Moreover, the insistence of working in terms of brackets, be they Poisson or Jacobi, often leads to complicated algebraic considerations in which the geometry is completely obscured. In this paper we will work with the corresponding tensor structures and the brackets themselves, which we view as secondary notions, will play no explicit rôle beyond initial motivation.

It is well known that the choice of one of equivalent definitions influences strongly our way of thinking and makes the formulations of some concepts and generalisations easier or harder, depending on the choice made. We will insist in this paper in understanding the 'poissonisation' as the genuine Jacobi structure with all consequences of this choice. However, in contrast to the standard definition, we understand the homogeneity not as associated with a vector field but a principal bundle structure, so our Poisson structure is 'homogeneous' in the sense that it homogeneous of degree -1 on a certain $\mathbb{R}^\times$-bundle. The appearance of this principal bundle structure is absolutely fundamental for the whole picture. In other words, the proper playground for Jacobi geometry will be the category of *Kirillov manifolds*, i.e. $\mathbb{R}^\times$-bundles equipped with a homogeneous Poisson structure. Moreover, all derived concepts like 'Jacobi algebroid', 'Jacobi bialgebroid', 'Jacobi/contact groupoid', etc., should be understood as the corresponding objects in Poisson geometry, equipped additionally with a principal $\mathbb{R}^\times$-action which is compatible with the other structures. The only thing to be decided is a reasonable notion of compatibility.

Compatibility with a vector bundle structure can be described, in the spirit of viewing linear structures as defined by a certain action of the monoid $(\mathbb{R},\cdot)$ of multiplicative reals [13, 14], as commutation of $\mathbb{R}$- and $\mathbb{R}^\times$-actions. As a side remark, actions of the monoid of complex numbers were explored in [28]. Following the observations made in [5] (cf. [1, 2, 3]), the compatibility with a Lie algebroid/groupoid structure is described as an $\mathbb{R}^\times$-action by the Lie algebroid/groupoid morphisms, etc. We will call such Lie groupoids $\mathbb{R}^\times$-*groupoids*, or more generally, for any Lie group G, we have the notion of a G-*groupoid*.

The compatibility of a symplectic or Poisson structure with the $\mathbb{R}^\times$-action is expressed in terms of homogeneity. We must stress, however, that this homogeneity is not defined in terms of a vector field like in [8, 26, 33] but in terms of the principal $\mathbb{R}^\times$-action. The fundamental vector fields define only the action of the connected component in the group. Another important ingredient of our framework, not really exploited in the literature, is the use of tangent and phase lifts of principal $\mathbb{R}^\times$-bundle structures.

We stress that our definition of a contact groupoid, i.e. a $\mathbb{R}^\times$-groupoid equipped with a homogeneous and multiplicative symplectic form, turns out to be equivalent to the definition of Dazord [9]. The corresponding objects are called by many authors *conformal contact groupoids*, for example [8]. The first and frequently used definition, presented in [22], is less general and involves an arbitrary multiplicative function, that is due to the fact that in this approach contact bundles are forced to be trivial. Also the definition proposed by Liebermann [25] requires a contact form. There are, however, no doubts that allowing for contact structures which do not come from a global contact form is fundamental for the completeness and elegance of the theory. On the other hand, although Dazord's definition is general and simple, it does not have any direct extension to Jacobi groupoids. Crainic and Salazar [7] also reaslise

that there are artificial complications that arise when insisting on working with trivial line bundles, but their way of solving the problem is completely different to ours and makes use of Spencer operators.

Our framework produces, for any Lie groupoid $\mathcal{G}$, canonical examples of contact groupoids $C(\mathcal{G}) \subset \mathsf{T}^*\mathcal{G}$, consisting of all covectors that vanish on vectors that are tangent to the source or target fibres. Interestingly, such examples are somewhat universal and generic; every contact groupoid can be represented or realised as a contact subgroupoid of a canonical one. We will of course be more precise in due course.

Among the main results of this paper:

- We give the description of any G-groupoid in terms of a splitting into a product of the G-bundle of units and the reduced groupoid (Theorem 4.9). In the simplest case, this description reduces to the well-known groupoid extension by the additive $\mathbb{R}$ with a help of a multiplicative function.
- We show that Kirillov algebroids and $\mathbb{R}^\times$-groupoids are related via Lie theory (Theorem 4.19).
- We present the Lie theory of Kirillov manifolds and contact groupoids, generalising and simplifying some known results of [8, 22] (Proposition 5.6).
- We prove that all contact groupoids have a realisation as a contact subgroupoid of a canonical contact groupoid (Theorem 5.7).

To sum up, our intention is to show how the setting of homogeneous Poisson geometry simplifies various concepts and problems of Jacobi and contact geometry. We would like to emphasize that, besides a number of new observations, the novelty of this paper lies mainly in the underlying approach that unifies and generalises various points of views, as well as establishes the proper language for the Jacobi and contact geometry. This results in a drastic simplification of proofs of various important facts spread over the literature and clarification of the strategies used. We hope that our work will put a new light on Jacobi and contact geometry as a whole, even if separate observations may seem to be known to the reader.

Remark 1.1. The picture of Jacobi geometry – or better Kirillov geometry – in terms of $\mathbb{R}^\times$-bundles and homogeneous Poisson structures allows for a natural generalisation thereof to the world of L_∞-algebras via replacing the Poisson structure with a higher Poisson structure (also known as a P_∞-structure). This leads to the notions of *higher Kirillov manifolds* and *homotopy Kirillov algebras*, see [4].

Arrangement of paper: In section 2 we recall the equivalence of line bundles and principal $\mathbb{R}^\times$-bundles. We also remind the reader of the tangent and phase lift of $\mathbb{R}^\times$ actions as these will feature heavily throughout this work. In section 3 we show how $\mathbb{R}^\times$-bundles appear in an essential way when dealing with Kirillov brackets and contact structures. We then turn our attention to Lie groupoids that have a compatible action of a Lie group upon them in section 4. In this section we examine the structure of such Lie groupoids and show how Kirillov algebroids and $\mathbb{R}^\times$-groupoids are related via the Lie functor. In the final section 5 we proceed to the main concept of this work: Kirillov and contact groupoids within the framework of $\mathbb{R}^\times$-groupoids.

2 Principal $\mathbb{R}^\times$-bundles

In many cases, the *Jacobi bracket* is understood in the literature as a bracket on an algebra of functions on a manifold M, while a quick analysis shows its 'module nature', meaning that rank 1 modules (line bundles) form the natural and proper framework for such structures. The reason could be that, in the trivial case, sections of the bundle $\mathbb{R} \times M \to M$ are identified with the algebra $\mathcal{A} = C^\infty(M)$ of functions, and the regular $\mathcal{A}$-module structure on $\mathcal{A}$ looks exactly like the multiplication in $\mathcal{A}$, although morphisms in the category of modules are different from these in the category of rings.

Using Jacobi-type brackets on sections of a line bundle $L \to M$ is nothing else but working with 'local Lie algebras' in the sense of Kirillov [23], so we will call them *Kirillov brackets*. Moreover, the corresponding poissonisations live not on $M \times \mathbb{R}$ but on the dual bundle L^* with the removed 0-section, $(L^*)^\times = L^* \backslash \{0_M\}$, which can be recognized as a principal $\mathrm{GL}(1, \mathbb{R})$-bundle equipped with a homogeneous Poisson tensor. Such structures we will call *Kirillov manifolds* (*Kirillov structures*). We are working with a contact structure if this tensor is actually symplectic. We generally use the identification $\mathrm{GL}(1, \mathbb{R}) \simeq \mathbb{R}^\times$, where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ is the group of multiplicative reals. The appearance of the non-connected group $\mathbb{R}^\times$ is forced by the fact that real line bundles over M are classified by a $\mathbb{Z}_2 = \mathbb{R}^\times/\mathbb{R}_+$ cohomology of M.

2.1 Line bundles and principal $\mathbb{R}^\times$-bundles

For a vector bundle $E \to M$, with $E^\times$ we will denote the total space E with the zero-section removed, $E^\times = E \setminus \{0_M\}$. The latter is no longer a vector bundle, but a principal $\mathbb{R}^\times$-bundle, $E^\times \to \mathcal{P}(E) = E^\times/\mathbb{R}^\times$, where the principal $\mathbb{R}^\times$-action, $h : \mathbb{R}^\times \times E^\times \to E^\times$, depends on multiplication by non-zero reals. The bundle $\mathcal{P}(E) \to M$ is known as the *projectivisation* of E.

Since $\mathbb{R}^\times = \mathrm{GL}(1, \mathbb{R})$, in the case of vector bundles of rank 1, i.e. for line bundles, we have the following.

Proposition 2.1. *The association $L \mapsto L^\times$ establishes a one-to-one correspondence between line bundles over M and principal $\mathbb{R}^\times$-bundles over M.*

Denote the converse of the above association with $P \mapsto \bar{P}$. In other words, $\bar{P}^\times = P$. Let us introduce the notation $P^+ = \bar{P}^*$.

The fundamental vector field Δ_P of the $\mathbb{R}^\times$-action on $P = L^\times$ is nothing more than the Euler vector field Δ_L on L restricted to $L^\times$. We will refer to Δ_P, by some abuse of nomenclature, as the *Euler vector field* of P and the $\mathbb{R}^\times$-action as the *homogeneity structure* on P.

By employing t as the standard coordinate on $\mathbb{R}$, and so $\mathbb{R}^\times$, we can understand t as the fibre coordinate of P in some fixed local trivialisation. That is, (x^a, t) can serve as local coordinates on P, where (x^a) are understood as local coordinates on M; such coordinates we will call *homogeneous coordinates*. With respect to homogeneous coordinates the Euler vector field on P is simply $\Delta_P = t\partial_t$.

2.2 Tangent and phase lifts

The base manifold of a principal $\mathbb{R}^\times$-bundle P will be generally denoted with P_0, $P_0 = P/\mathbb{R}^\times$. A fundamental observation is that principal $\mathbb{R}^\times$-actions on P can be canonically lifted to principal $\mathbb{R}^\times$-actions on $\mathsf{T}P$ (*tangent lifts*) and T^*P (*phase lifts*), see e.g. [16].

Proposition 2.2. *Let $\pi : P \to P_0$ be a principal $\mathbb{R}^\times$-bundle with respect to an action* h. *Then,*

(a) $\mathsf{T}P$ is also canonically a principal $\mathbb{R}^\times$-bundle, with respect to the action

$$(\mathsf{T}\mathrm{h})_s := \mathsf{T}(\mathrm{h}_s)\,. \tag{2.1}$$

The base of the corresponding fibration is the Atiyah bundle $\mathsf{T}P/\mathbb{R}^\times$, whose sections are interpreted as $\mathbb{R}^\times$-invariant vector fields on $\mathsf{T}P$ or, equivalently, as the bundle $\mathrm{DO}^1(P^+, P^+)$ of first-order linear differential operators on the line bundle P^+ with values in P^+.

(b) T^*P is also canonically a principal $\mathbb{R}^\times$-bundle, with respect to the action

$$(\mathsf{T}^*\mathrm{h})_s := s \cdot (\mathsf{T}\mathrm{h}_{s^{-1}})^*\,. \tag{2.2}$$

The base of the corresponding fibration is the first jet bundle J^1P^+ of sections of the line bundle P^+.

In homogeneous local coordinates (t, x^a) on P, the naturally induced coordinates on the tangent bundle are $(t, x^a, \dot{t}, \dot{x}^b)$. Then,

$$(\mathsf{T}\mathrm{h})_s(t, x^a, \dot{t}, \dot{x}^b) = (s\,t, x^a, s\,\dot{t}, \dot{x}^b)\,,$$

Similarly, for the naturally induced coordinates (t, x^a, p, p_b) on the cotangent bundle,

$$(\mathsf{T}^*\mathrm{h})_s(t, x^a, p, p_b) = (s\,t, x^a, p, s \cdot p_b)\,.$$

Note that, since the lifted actions are linear, we have actually a whole series of lifted actions, since the multiplication of $(\mathsf{T}\mathrm{h})_s$ and $(\mathsf{T}^*\mathrm{h})_s$ by s^k, $k \in \mathbb{Z}$, gives a new principal action. The above one has the advantage that, for P coming from a vector bundle, $P = E^\times$, it can be extended to the lift of the corresponding action of the monoid $(\mathbb{R}, \cdot)$ of multiplicative reals, which in turn is the most efficient way to obtain the double vector bundle structures on $\mathsf{T}E$ and T^*E [13].

Let now Λ be a Poisson structure on a principal $\mathbb{R}^\times$-bundle (P, h), and let $\Lambda^\#$ be the corresponding vector bundle morphism

$$\Lambda^\# : \mathsf{T}^*P \to \mathsf{T}P\,. \tag{2.3}$$

The following is straightforward.

Theorem 2.3. *The map (2.3) intertwines the phase and the tangent lifts of the $\mathbb{R}^\times$ action if and only if Λ is homogeneous of degree -1, i.e. $(h_s)_*\Lambda = s^{-1}\Lambda$. In this case, we have the reduced map*

$$\Lambda_0^\# : \mathsf{T}^*P/\mathbb{R}^\times \simeq \mathrm{DO}^1(P^+, P^+) \longrightarrow \mathsf{T}P/\mathbb{R}^\times \simeq \mathsf{J}^1P^+ . \tag{2.4}$$

Remark 2.4. The above reductions of $\mathsf{T}P$ and T^*P can be extended to a reduction of the whole Courant algebroid structure on the Pontryagin bundle $\mathsf{T}P \oplus_P \mathsf{T}^*P$ to a structure of a *contact Courant algebroid* in the sense of [16] (or *Courant-Jacobi algebroid* in the sense of [12]) on $\mathrm{DO}^1(P^+, P^+) \oplus_{P_0} \mathsf{J}^1P^+$. As we have a canonical pairing

$$\mathrm{DO}^1(P^+, P^+) \oplus_{P_0} \mathsf{J}^1P^+ \longrightarrow P^+ ,$$

analogs of Dirac structures can also be naturally defined (see [39]).

3 Kirillov brackets and Kirillov manifolds

3.1 Kirillov manifolds

A line bundle equipped with a local Lie bracket on its sections, $(L, [\cdot,\cdot]_L)$, introduced by Kirillov [23], is known in the literature also as a *Jacobi bundle* following [33]. Locally this bracket is given by

$$[f,g]_L(x) = \Lambda^{ab}(x)\frac{\partial f}{\partial x^a}(x)\frac{\partial g}{\partial x^b}(x) + \Lambda^a(x)\left(f(x)\frac{\partial g}{\partial x^a}(x) - \frac{\partial f}{\partial x^a}(x)g(x)\right).$$

One can identify smooth sections of a line bundle L with smooth homogeneous functions of degree one on L^*, and further also with homogeneous functions of degree one on the principal $\mathbb{R}^\times$-bundle $L^{*\times} := (L^*)^\times$, i.e. functions $f : L^\times \to \mathbb{R}$ such that $f(h_t(v)) := f(t.v) = tf(v)$. We denote this identification via $u \rightsquigarrow \iota_u$, where $u \in \mathsf{Sec}(L)$.

Having a Kirillov bracket $[\cdot,\cdot]_L$ on sections of L, we can try to define a Poisson bracket $\{\cdot,\cdot\}_\Lambda$, associated with a linear Poisson structure Λ on L^*, using the identity

$$\iota_{[u,v]_L} = \{\iota_u, \iota_v\}_\Lambda. \tag{3.1}$$

However, unlike the case of a Lie algebroid, this bracket is generally singular at points of the zero-section. Instead, one has to define a Poisson tensor on $L^{*\times}$. Indeed, in dual coordinates (x^a, t) on L^*,

$$\Lambda(x,t) = \frac{1}{2t}\Lambda^{ab}(x)\partial_{x^a}\wedge\partial_{x^b} + \Lambda^a(x)\partial_t\wedge\partial_{x^a} . \tag{3.2}$$

This identification allows for a very useful characterisation of Kirillov brackets (c.f. [16, 33]) in terms of *Kirillov manifolds* (*Kirillov structures*).

Definition 3.1 ([16])**.** A *principal Poisson $\mathbb{R}^\times$-bundle*, shortly *Kirillov manifold*, is a principal $\mathbb{R}^\times$-bundle (P,h) equipped with a Poisson structure Λ of degree -1. A *morphism of Kirillov manifolds* $\phi : P \to P'$ is a Poisson morphism that intertwines the respective $\mathbb{R}^\times$-actions.

Evidently, Kirillov manifolds form a category under the standard composition of smooth maps. We summarise all the above observations as:

Theorem 3.2. *There is a one-to-one correspondence between Kirillov brackets $[\cdot,\cdot]_L$ on a line bundle $L \to M$ and Kirillov manifold structures on the principal $\mathbb{R}^\times$-bundle $P = L^{*\times}$ given by (3.1).*

Remark 3.3. Local representation (3.2) identifies (locally) Kirillov manifolds with *Jacobi manifolds* in the sense of Lichnerowicz [27],

$$\Lambda \mapsto \left(\frac{1}{2}\Lambda^{ab}(x)\partial_{x^a}\wedge\partial_{x^b}\,, \Lambda^a(x)\partial_{x^a}\right) .$$

In other words, for a trivial principal $\mathbb{R}^\times$-bundle $P = M\times\mathbb{R}^\times$, the reduced tangent and cotangent bundles can be identified as

$$\mathsf{T}(M\times\mathbb{R}^\times)/\mathbb{R}^\times \simeq \mathsf{T}M\times\mathbb{R}\,, \quad \mathsf{T}^*(M\times\mathbb{R}^\times)/\mathbb{R}^\times \simeq \mathsf{T}^*M\times\mathbb{R}\,, \tag{3.3}$$

and the reduced map (2.4) is nothing but the vector bundle morphism

$$\Lambda_0^\# : \mathsf{T}^*M\times\mathbb{R} \to \mathsf{T}M\times\mathbb{R}\,,$$

induced by the Jacobi structure.

3.2 Coisotropic submanifolds of Kirillov manifolds

Recall that a submanifold S of a Poisson manifold (P, Λ) is called *coisotropic* if the ideal of functions vanishing on S is closed under the Poisson bracket. In the context of a Kirillov manifolds (P, h, Λ), of particular interest are coisotropic submanifolds which are simultaneously $\mathbb{R}^\times$-subbundles; we will call them *coisotropic Kirillov submanifolds* or simply *coisotropic subbundles*. The natural inclusion $S \hookrightarrow P$ implies $S_0 \hookrightarrow M$, where $S_0 = S/\mathbb{R}^\times$ is the reduced manifold. Of course, in the traditional language, S_0 is called a coisotropic submanifold of the corresponding Jacobi structure (cf. [24]).

Suppose now that S is a coisotropic subbundle of the Kirillov manifold P. It is clear that in local coordinates (t, x^α, y^i) on P adapted to S, so (t, x^α) form a coordinate system on S, that the Poission structure on P encoding the Kirillov manifold structure is of the form

$$\begin{aligned}\Lambda = &\frac{1}{2t}\Lambda^{\alpha\beta}(x,y)\frac{\partial}{\partial x^\beta}\wedge\frac{\partial}{\partial x^\alpha} + \frac{1}{t}\Lambda^{\alpha i}(x,y)\frac{\partial}{\partial y^i}\wedge\frac{\partial}{\partial x^\alpha}\\ &+\frac{1}{2t}\Lambda^{ij}(x,y)\frac{\partial}{\partial y^j}\wedge\frac{\partial}{\partial y^i} + \Lambda^\alpha(x,y)\frac{\partial}{\partial x^\alpha}\wedge\frac{\partial}{\partial t} + \Lambda^i(x,y)\frac{\partial}{\partial y^i}\wedge\frac{\partial}{\partial t},\end{aligned}$$

where we require $\Lambda^{ij} = 0$ and $\Lambda^i = 0$ on S.

Hamiltonian vector fields $X_h := \{h, \cdot\}_\Lambda$ where $h \in I_S$ are tangent to S and so form an integrable distribution. The corresponding foliation is known as the *characteristic foliation* of S. The space of leaves, if smooth, will inherit a Kirillov manifold structure. Studying reductions, deformations, etc., of coisotropic subbundles is an interesting but extensive task, which we postpone to a separate paper.

3.3 Contact structures

Proceeding to contact structures, first note that a nowhere-vanishing one-form α spans a trivial one dimensional vector subbundle $[\alpha]$ of T^*M. Associated with α is a canonical embedding $I_\alpha : \mathbb{R} \times M \to \mathsf{T}^*M$ which induces an isomorphisms of $\mathbb{R} \times M$ with $[\alpha]$. In natural local coordinates the canonical embedding is given by

$$I_\alpha^*(x^a, p_b) = (x^a, t\,\alpha_b(x)), \tag{3.4}$$

were t is the (global) coordinate on $\mathbb{R}$ and locally we have $\alpha = \mathrm{d}x^a\alpha_a(x)$.

Proposition 3.4. *The nowhere-vanishing one-form α is a contact form if and only if the trivial principal bundle $\mathbb{R}^\times \times M$ is, via I_α, a symplectic submanifold $[\alpha]^\times \subset \mathsf{T}^*M$.*

The above propositions is essentially a well-known rewording of the standard notion of the 'symplectisation' of a contact form. In particular, it is easy to see that in Darboux coordinates

$$I_\alpha^*(\mathrm{d}p_a \wedge \mathrm{d}x^a) := \omega = \mathrm{d}t \wedge \alpha + t \wedge \mathrm{d}\alpha, \tag{3.5}$$

which gives the symplectisation of α remembering that $t \neq 0$. Moreover, note the that contact form can be recovered from

$$i_\nabla \omega = t\alpha, \tag{3.6}$$

where ∇ is the Euler vector field on the principal $\mathbb{R}^\times$-bundle $[\alpha]^\times$, i.e. the fundamental vector field of the $\mathbb{R}^\times$-action. All this implies the following.

Proposition 3.5. [16] *A line subbundle C of T^*M is locally generated by contact one-forms if and only if $C^\times$ is a symplectic submanifold of T^*M.*

Definition 3.6. A principal $\mathbb{R}^\times$-bundle (P, h) equipped with a 1-homogeneous symplectic form ω, i.e. a symplectic form such that $(\mathrm{h}_t)^*\omega = t\,\omega$ $(t \neq 0)$, will be referred to as a *contact structure*. In other words, a contact structure is a Kirillov manifold whose Poisson structure is invertible (symplectic).

Let ∇ be the Euler vector field on P, $\nabla : P \to \mathsf{T}P$. It is easy to see that the composition $\eta = \omega^\flat \circ \nabla : P \to \mathsf{T}^*P$ is a one-form on P which takes values in basic covectors, $\eta(y) = \pi^*(\Psi(y)) \in \mathsf{T}^*_yP$, $\Psi(y) \in \mathsf{T}^*_{\pi(y)}P_0$, so can be viewed as a map $\Psi : P \to \mathsf{T}^*P_0$ which locally has the form (3.4). Consequently, the range $C^\times(P) = \{\Psi(y) \mid y \in P\}$ of Ψ is a a symplectic submanifold in T^*P_0. Thus we get the following.

Theorem 3.7. [16] *Any contact structure (P, ω, h), where P is an $\mathbb{R}^\times$-bundle over P_0, can be canonically symplectically embedded into T^*P_0 as a symplectic principal $\mathbb{R}^\times$-bundle of the form $C^\times$ for a line subbundle $C \subset \mathsf{T}^*P_0$.*

Remark 3.8. Commonly, a contact structure on a manifold M is understood as a maximally non-integrable hyperplane distribution $\mathcal{D} \subset \mathsf{T}M$, locally given as the polar (annihilator) of a line bundle $C \subset \mathsf{T}^*M$ generated by contact one-forms, $\mathcal{D} = C^0$. We will refer to such hyperplane distributions $\mathcal{D} \subset \mathsf{T}M$ as *contact distributions* to avoid confusion. In our language, contact structures are homogeneous symplectic structures on a principal $\mathbb{R}^\times$-bundle P, while in the classical language they are certain hyperplane distributions on the reduced manifold $M = P/\mathbb{R}^\times$.

Example 3.9. The canonical symplectic structure on the cotangent bundle T^*M is linear, thus homogeneous on $(\mathsf{T}^*M)^\times$. The symplectic homogeneous manifold $P = (\mathsf{T}^*M)^\times$ represents therefore a contact structure. In the traditional language it is a canonical contact structure on the reduced manifold $(\mathsf{T}^*M)^\times/\mathbb{R}^\times$, i.e. on the projectivisation $\mathcal{P}(\mathsf{T}^*M)$ of the cotangent bundle.

Example 3.10. Consider a principal $\mathbb{R}^\times$-bundle (P, h). It is easy to see that the canonical symplectic form on the cotangent bundle T^*P is homogeneous with respect to the lifted action T^*h, so T^*P represents canonically a contact structure. If we write $P = L^{*\times}$, then in the traditional language this is exactly the canonical contact structure C on the reduced manifold $\mathsf{T}^*P/\mathbb{R}^\times$ which is the first jet bundle $J^1P^+ = J^1L$. For the trivial bundle $L = \mathbb{R} \times M$, the canonical contact structure C is the trivial line subbundle of $\mathsf{T}^*(\mathbb{R} \times \mathsf{T}^*M)$ generated by the contact form $\alpha = \mathrm{d}z - p_a\mathrm{d}x^a$. Thus we have $P = C^\times = \mathbb{R}^\times \times \mathbb{R} \times \mathsf{T}^*M$, which we equip with local coordinates (t, z, x^a, p_b) and thus the symplectic structure on P is

$$\omega = \mathrm{d}t \wedge \mathrm{d}z - p_a\mathrm{d}t \wedge \mathrm{d}x^a - t\mathrm{d}p_a \wedge \mathrm{d}x^a\,.$$

Remark 3.11. Contact structures on nonnegatively graded manifolds further equipped with homological contact vector fields were studied by Mehta [34] using a more traditional language than put forward here. In particular, for the degree 1 case he showed that there is a one-to-one correspondence between such structures (with a global contact form) and Jacobi manifolds. The line bundle approach to the concept of a *generalized contact bundle* can be found in the work of Vitagliano Wade [40]. Furthermore, the $\mathbb{R}^\times$-principal bundle approach can also be applied to the notion of a *contact structure on a Lie algebroid* following Ida & Popescu [17, Remakr 4.2]

4 Principal bundle Lie groupoids and algebroids

4.1 Morphisms of Lie groupoids and Lie algebroids

Our general reference to the theory of Lie groupoids and Lie algebroids will be Mackenzie's book [32].

Let $\mathcal{G} \rightrightarrows M$ be an arbitrary Lie groupoid with *source map* $s : \mathcal{G} \to M$ and *target map* $t : \mathcal{G} \to M$. There is also the inclusion map $\iota_M : M \to \mathcal{G}$, $\iota_M(x) = \mathbb{1}_x$, and a *partial multiplication* $(g, h) \mapsto gh$ which is defined on $\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G} : s(g) = t(h)\}$. Moreover, the manifold $\mathcal{G}$ is foliated by s-fibres $\mathcal{G}_x = \{\, g \in \mathcal{G} |\, s(g) = x\}$, where $x \in M$. As by definition the source and target maps are submersions, the s-fibres are themselves smooth manifolds. Geometric objects associated with this foliation will be given the superscript s. In particular, the distribution tangent to the leaves of the foliation will be denoted by $\mathsf{T}^s\mathcal{G}$. To ensure no misunderstanding with the notion of a Lie groupoid morphism we recall the definition we will be using.

Definition 4.1. Let $\mathcal{G}_i \rightrightarrows M_i$ $(i = 1, 2)$ be a pair of Lie groupoids. Then a *Lie groupoid morphisms* is a pair of maps (Φ, ϕ) such that the following diagram is commutative

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "\Phi"] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "t_1"] & \mathcal{G}_2 \arrow[d, shift right, "s_2"'] \arrow[d, shift left, "t_2"] \\ M_1 \arrow[r, "\phi"'] & M_2 \end{tikzcd}$$

in the sense that

$$s_2 \circ \Phi = \phi \circ s_1, \qquad \text{and} \qquad t_2 \circ \Phi = \phi \circ t_1$$

subject to the further condition that Φ respects the (partial) multiplication; if $g, h \in \mathcal{G}_1$ are composable, then $\Phi(gh) = \Phi(g)\Phi(h)$. It then follows that for $x \in M_1$ we have $\Phi(\mathbb{1}_x) = \mathbb{1}_{\phi(x)}$ and $\Phi(g^{-1}) = \Phi(g)^{-1}$.

Consider a Lie groupoid $\mathcal{G} \rightrightarrows M$. A subset $\mathcal{H} \subset \mathcal{G}$ is a *Lie subgroupoid* if it is closed under multiplication (when defined) and inversion. In particular, we have that

$$h \in \mathcal{H} \Rightarrow h^{-1} \in \mathcal{H} \qquad \Longrightarrow \qquad \mathbb{1}_{s(h)} \in \mathcal{H} \quad \text{and} \quad \mathbb{1}_{t(h)} \in \mathcal{H}.$$

Thus we have a Lie groupoid over $s(\mathcal{H}) = t(\mathcal{H})$. However, in general a Lie subgroupoid need not be a Lie groupoid over the entire manifold M. If $s(\mathcal{H}) = t(\mathcal{H}) = M$ then $\mathcal{H}$ is said to be a *wide subgroupoid*. The Cartesian product $\mathcal{G}_1 \times \mathcal{G}_2$ of two Lie groupoids is canonically a Lie groupoid, and it follows immediately from the above definition that Φ is a Lie groupoid morphism if and only if its graph is a Lie subgroupoid in $\mathcal{G} \times \mathcal{H}$.

A similar fact holds true for Lie algebroids, but as we have many alternative definitions of a Lie algebroid, there are many alternative definitions of a Lie algebroid morphism (see e.g. [15, Theorem 14] in a little more general setting). Dealing with homogeneous Poisson structures in this paper, we will mainly understand a Lie algebroid on a vector bundle E as a linear Poisson structure on E^*. Then, as is commonly known (see e.g. Mackenzie [32] page 400), Lie subalgebroids of E correspond to coisotropic subbundles in the Poisson manifold E^* by passing to the polar.

It is also well known that via a differentiation procedure one can construct the *Lie functor*

$$\mathtt{Grpd} \xrightarrow{\text{Lie}} \mathtt{Algd},$$

that sends a Lie groupoid to its Lie algebroid, and sends morphisms of Lie groupoids to morphisms of the corresponding Lie algebroids. However, as is also well known, we do not have an equivalence of categories as not all Lie algebroids arise as the infinitesimal versions of Lie groupoids. There is no direct generalisation of Lie III, apart from the local case. The obstruction to the integrability of Lie algebroids, the so called *monodromy groups*, were first uncovered by Crainic & Ferandes [6]. To set some notation and nomenclature, given a Lie groupoid $\mathcal{G}$, we say that $\mathcal{G}$ *integrates* $\mathrm{Lie}(\mathcal{G}) = \mathrm{A}(\mathcal{G})$. Moreover, if $\Phi : \mathcal{G} \to \mathcal{H}$ is a morphism of Lie groupoids, then we will write $\Phi' = \mathrm{Lie}(\Phi) : \mathrm{A}(\mathcal{G}) \to \mathrm{A}(\mathcal{H})$ for the corresponding Lie algebroid morphism, which actually comes from the differential $\mathsf{T}\Phi : \mathsf{T}\mathcal{G} \to \mathsf{T}\mathcal{H}$ restricted to s-fibres at submanifold of M.

Let us just recall Lie II theorem as we will need it later on;

Theorem 4.2. (Lie II)
Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be Lie groupoids. Suppose that $\mathcal{G}$ is source simply-connected and that $\phi : \mathrm{A}(\mathcal{G}) \to \mathrm{A}(\mathcal{H})$ is a Lie algebroid morphism between the associated Lie algebroids. Then, ϕ integrates to a unique Lie groupoid morphisms $\Phi : \mathcal{G} \to \mathcal{H}$ such that $\Phi' = \phi$.

This generalisation of Lie II to the groupoid case was first proved by Mackenzie & Xu [31]. A simplified proof was obtained shortly after by Moerdijk & J. Mrčun [35]. Note that the assumption that the Lie groupoid $\mathcal{G}$ is source simply-connected is essential.

4.2 Compatible group actions on Lie groupoids and algebroids

In our study of Jacobi and contact groupoids we will encounter Lie groupoids that have a compatible action of $\mathbb{R}^\times$ upon them; compatibility to be defined shortly. However, as the basic theory of compatible group actions on Lie groupoids is independent of the actual Lie group, we discuss the general setting here focusing on what we will need later in this paper.

Definition 4.3. An action $h : G \times \mathcal{G} \to \mathcal{G}$ of a Lie group G on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is called *compatible with the groupoid structure* if $h_g : \mathcal{G} \to \mathcal{G}$ are groupoid isomorphisms for all $g \in G$. A principal G-bundle $\pi : \mathcal{G} \to \mathcal{G}_0$ is a *principal bundle G-groupoid* (*G-groupoid in short*) if the principal action of G on $\mathcal{G}$ is compatible with the groupoid structure.

Similarly, a G-action on a Lie algebroid A is *compatible* if the group acts by Lie algebroid isomorphisms, and we get a *G-algebroid* if a principal G-action is compatible with the Lie algebroid structure.

Remark 4.4. The reader should immediately be reminded of Mackenzie's notion of a *PBG-groupoid* [29, 30], which is close to ours, although Mackenzie, being interested in extensions of principal bundles, starts with a principal G-structure on the manifold M of units extended accordingly to a Lie groupoid $\mathcal{G}$. The other difference is that what we call a Lie groupoid is a *differentiable groupoid* in the sense of Mackenzie, and his *Lie groupoids* in [29, 30] (or *locally trivial Lie groupoids* in [32]) form much smaller class and are understood as particular *transitive Lie groupoids*. However, the following observations are independent of these details and so are probably already known to Mackenzie.

1. The action of G on $\mathcal{G}$ commutes with the source and target maps, thus projects onto a G-action on the manifold M. Moreover, M as an immersed submanifold of $\mathcal{G}$ is invariant with respect to the G-action, and the projected and restricted actions coincide.

2. As the action of G on $\mathcal{G}$ is principal, it is also principal on the immersed submanifold M, so M inherits a structure of a principal G-bundle. In particular, the quotient manifold $M_0 = M/G$ exists.

3. The reduced manifold $\mathcal{G}/G = \mathcal{G}_0$ is a Lie groupoid $\mathcal{G}/G = \mathcal{G}_0 \rightrightarrows M/G = M_0$, with the set of units M_0, defined by the following structure:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\pi"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & \mathcal{G}_0 \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] \\ M \arrow[r, "\mathrm{p}"'] & M_0 \end{tikzcd}$$

$$\begin{aligned} &\sigma \circ \pi = \mathrm{p} \circ s\,, \\ &\tau \circ \pi = \mathrm{p} \circ t\,, \\ &\mathbb{1}_{\mathrm{p}(x)} = \pi(\mathbb{1}_x) \quad \text{for all} \quad x \in M\,, \\ &\pi(y)^{-1} = \pi(y^{-1}) \quad \text{for all} \quad y \in \mathcal{G}\,, \\ &\pi(yy') = \pi(y)\pi(y') \quad \text{for all} \quad (y, y') \in \mathcal{G}^{(2)}\,, \end{aligned}$$

where $\pi : \mathcal{G} \to \mathcal{G}_0$ is the canonical projection. In fact, the above constructions imply, tautologically, that $(\pi, \mathrm{p}) : \mathcal{G} \rightrightarrows M \to \mathcal{G}_0 \rightrightarrows M_0$ is a morphism of Lie groupoids with the above structures. This concept is essentially of double nature: a G-groupoid is a principal G-bundle object in the category of Lie groupoids. From the point of view of Jacobi and contact geometry, the most important will be of course $\mathbb{R}^\times$-groupoids.

Remark 4.5. It is well known that a G-action on a set X is equivalent to a groupoid morphism of G into the pair groupoid $X \times X$. It can be shown that if $X = \mathcal{G}$ is a (Lie) groupoid and the action is by automorphism, then the morphism of G into $\mathcal{G} \times \mathcal{G}$ is simultaneously a morphism with respect to the other, namely Cartesian product groupoid structure on $\mathcal{G} \times \mathcal{G}$ (with $M \times M$ as the set of units, which is simultaneously the pair groupoid over M). A G-groupoid can be therefore also defined as a (double) groupoid morphism of G (viewed as a double groupoid) into the double groupoid (in the sense of Ehresmann) $\mathcal{G} \times \mathcal{G}$, with the diagram

$$\begin{tikzcd} \mathcal{G} \times \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right, "s \times s"'] \arrow[d, shift left, "t \times t"] & \mathcal{G} \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ M \times M \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \tag{4.1}$$

Of course, we are unable to develop the corresponding theory here: replacing the group G with a groupoid leads to groupoid morphisms in the sense of Zakrzewski [36, 44], which are nowadays also called *groupoid comorphisms*.

It is easy to see that a compatible principal G-structure on a Lie groupoid $\mathcal{G}$ induces canonically a compatible principal G-structure on the Lie algebroid $\mathrm{Lie}(\mathcal{G})$. Indeed, if $h : G \times \mathcal{G} \to \mathcal{G}$ is such a structure, then *via* the first Lie theorem,

$$h'_g = \mathrm{Lie}(h_g) \tag{4.2}$$

define a free G-action on $\mathrm{Lie}(\mathcal{G})$ by automorphisms. This action is also proper, as the Lie functor is a restriction of the tangent functor; the tangent lift of a proper group action is a proper group action. Actually we have the following theorem on integrability of G-algebroids.

Theorem 4.6. *A source simply-connected Lie groupoid $\mathcal{G}$ is a G-groupoid if and only if* $\mathrm{Lie}(\mathcal{G})$ *is a G-algebroid, with (4.2) describing the relation of the corresponding principal G-structures.*

Proof. It remains to prove that if $g \mapsto h'_g$ gives rise to a compatible principal G-action on $\mathrm{Lie}(G)$, then $g \mapsto h_g$ is also principal and compatible. *Via* the Lie second theorem we know that the latter is a free G-group action as Lie groupoid automorphisms. This action is smooth, as it comes from the induced G-action on the Weinstein groupoid of admissible paths in $\mathrm{Lie}(\mathcal{G})$. It only remains to show that this group action is proper.

Of course, M is an invariant submanifold of this action and the 'integrated' action coincides with the original action on $M \subset \mathrm{Lie}(\mathcal{G})$, thus is proper. Moreover, the integrated action on $\mathcal{G}$ projects *via* the source map s to the action on M which implies that the integrated action is proper. Indeed, having two compact sets K_i, $i = 1, 2$, in $\mathcal{G}$, we have that $\{g \in G \mid h_g(K_1) \cap K_2 \neq \emptyset\}$ is a closed subset of the compact set $\{g \in G \mid h_g(s(K_1)) \cap s(K_2) \neq \emptyset\}$, thus compact. $\square$

Remark 4.7. The above theorem can be derived from the main results of Stefanini [37, 38] describing, roughly speaking, integrability conditions for G-algebroids with G being a Lie groupoid. However, the Lie group case is substantially simpler, so we decided to present the direct proof.

4.3 Structure of G-groupoids

Let now $\mathcal{G}$ be a G-groupoid with the structure diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\pi"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & \mathcal{G}_0 \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] \\ M \arrow[r, "\mathrm{p}"'] & M_0 \end{tikzcd} \quad . \tag{4.3}$$

Proposition 4.8. *The map*

$$\mathcal{S} : \mathcal{G} \to \mathcal{G}_0 \times_{M_0} M := \{(y_0, x) \in \mathcal{G}_0 \times M \mid \mathrm{p}(x) = \sigma(y_0)\} \ , \quad \mathcal{S}(y) = (\pi(y), s(y)) \ , \tag{4.4}$$

is a diffeomorphism.

Proof. The map $\mathcal{S}$ is clearly a diffeomorphism if the principal G-bundle $\mathcal{G}$ is trivial. Indeed, if $\mathcal{G} = \mathcal{G}_0 \times G$, then $M \simeq M_0 \times G$, the manifold $\mathcal{G}_0 \times_{M_0} M$ can be identified with $\mathcal{G}_0 \times G$ by $(y_0, g) \mapsto (y_0, (\sigma(y_0), g))$, and with these identifications the map $\mathcal{S} : \mathcal{G}_0 \times G \to \mathcal{G}_0 \times G$ is the identity. As the bundle $\mathcal{G} \to \mathcal{G}_0$ is locally trivial, the map $\mathcal{S}$ is generally a surjective local diffeomorphism. It is also globally injective, thus a global diffeomorphism. Indeed, $\mathcal{S}(y) = \mathcal{S}(y')$implies that $\pi(y) = \pi(y')$, so $y' = yg$ for some $g \in G$, and therefore $s(y') = s(y)g$. But $\mathcal{S}(y) = \mathcal{S}(y')$ implies also $s(y') = s(y)$, so that $g = e$ (the action is free) and $y = y'$. □

Using the above identification, we can transmit the G-groupoid structure from $\mathcal{G}$ onto $\mathcal{G}_0 \times_{M_0} M$. The G-action is clearly $(y_0, x)g = (y_0, xg)$, the embedding of units is $\iota_M(x) = (\mathbb{1}_x, x)$, and the source map reads $s(y_0, x) = x$. Knowing the inverse we could define the target map and the composition. It is easy to see that the inverse of $y = (y_0, x)$ is $y^{-1} = (y_0^{-1}, t(y_0, x),)$, where t is the target map.

One can easily check what properties of t ensure that the axioms of a groupoid hold true.

Theorem 4.9. *Let* $\mathrm{p} : M \to M_0$ *be a principal G-bundle with the right G-action $M \times G \ni (x, g) \mapsto xg \in M$, and $\mathcal{G}_0 \rightrightarrows M_0$ be a Lie groupoid with the source and the target map σ and τ, respectively.*

Then, any G-groupoid structure on the manifold $\mathcal{G}_0 \times_{M_0} M$ equipped with the principal G-action $(y_0, x)g = (y_0, xg)$, the source map $s(y_0, x) = x$, and such that the projection $(y_0, x) \mapsto y_0$ is a groupoid morphism, is uniquely determined by its target map t. On the other hand, a map $t : \mathcal{G}_0 \times_{M_0} M \to M$, $t(y_0, x) =: y_0.x$, can serve as such a target map if and only if it has the following properties (holding for all $x \in M$):

(i) $\mathrm{p}(y_0.x) = \tau(y_0)$ *for all* $y_0 \in \mathcal{G}_0$,

(ii) $y_0.(y_0'.x) = (y_0 y_0').x$ *for all* $(y_0, y_0') \in \mathcal{G}_0^2$,

(iii) $\mathbb{1}_{\mathrm{p}(x)}.x = x$,

(iv) $y_0.(xg) = (y_0.x)g$ *for all* $y_0 \in \mathcal{G}_0$ *and all* $g \in G$.

Note that $(i) - (iii)$ mean that t is an action of $\mathcal{G}_0$ on $\mathrm{p} : M \to M_0$ (c.f. [32, Definition 1.6.1]), and (iv) means that the action is G-equivariant. The G-groupoid determined by t as above we will denote $\mathcal{G}_0 \times^t_{M_0} M$ and called *t-split G-groupoid*. Thus, any G-groupoid (4.3) is t-split for some $t(y_0, x) = y_0.x$ satisfying $(i) - (iv)$.

There are two particular cases of the above construction which are of great importance. The first is the case of a trivial principal bundle, $M = M_0 \times G$ which is always a local form of any G-groupoid. In this case we can use the identification $\mathcal{G}_0 \times_{M_0} M \simeq \mathcal{G}_0 \times G$ and replace the map t satisfying (i) with a map $\mathrm{b} : \mathcal{G}_0 \to G$. Indeed, any map on a Lie group commuting with all the right-translations is a left-translation, so can we write $t(y_0, \sigma(y_0), g) = (\tau(y_0), \mathrm{b}(y_0)g)$. Now, the properties $(i) - (iv)$ can be reduced to

$$\mathrm{b}(y_0)\mathrm{b}(y_0') = \mathrm{b}(y_0 y_0') \tag{4.5}$$

for all $(y_0, y_0') \in \mathcal{G}_0^2$, i.e. to the fact that $\mathrm{b} : \mathcal{G}_0 \to G$ is a groupoid morphism. This is of course always the local form of any G-groupoid. The corresponding G-groupoid structure, denoted with $\mathcal{G}_0 \times^{\mathrm{b}} G$, is an obvious generalisation of the groupoid extension by the additive $\mathbb{R}$ with a help of a multiplicative function considered in the literature (cf. [8, Definition 2.3]), and we have shown that this construction is in a sense universal. Thus we get the following.

Theorem 4.10. *For any G-groupoid structure on the trivial G-bundle $\mathcal{G} = \mathcal{G}_0 \times G$ there is a Lie groupoid structure on $\mathcal{G}_0$ with the source and target maps $\sigma, \tau : \mathcal{G}_0 \to M_0$ and a groupoid morphism $\mathrm{b} : \mathcal{G}_0 \to G$ such that the source map s, the target map t and the partial multiplication in $\mathcal{G}$ read*

$$s(y_0, g) = (\sigma(y_0), g)\,, \quad t(y_0, g) = (\tau(y_0), \mathrm{b}(y_0)g)\,, \quad (y_0, g_1)(y_0', g_2) = (y_0 y_0', g_2)\,.$$

Another particular case is that of a bundle of groups, i.e. a groupoid in which the source and the target map coincide (the anchor map $\rho = (s, t)$ is *diagonal*). This means that $\mathcal{G}_0$ is a bundle of groups and the map t is trivial, $t(y_0, x) = x$. Any G-groupoid with diagonal anchor splits therefore as the product $\mathcal{G} = \mathcal{G}_0 \times_{M_0} M$ in which all groupoid operations come from $\mathcal{G}_0$ and the principal G-action from M.

This is in particular the case of a G-vector bundle, i.e. a vector bundle $\tau : P \to M$ on which G acts principally by vector bundle automorphisms, which means in this case that the G-action commutes with the natural homogeneity structure $l : \mathbb{R} \times P \to P$ that is associated with homotheties of the said vector bundle structure. In other words, $(tv)g = t(vg)$ and we have the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \pi\ } & P_0 \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau_0} \\ M & \xrightarrow{\ \pi_0\ } & M_0 \end{array} \tag{4.6}$$

where, as we already know, τ, τ_0 are vector bundles, and π, π_0 are principal $\mathbb{R}^\times$-bundles. We stress that this double structure is not a double vector bundle. In particular, P_0 is *not* canonically embedded in P, but we have a variant of isomorphism (4.4),

$$(\tau, \pi) : P \to M \times_{M_0} P_0 = \{(x, y_0) \in M \times P_0 : \pi_0(x) = \tau_0(y_0)\}\,. \tag{4.7}$$

In other words, we get the following generalisation of [16, Theorem 3.2].

Theorem 4.11. *If $\tau : P \to M$ is a G-vector bundle, then there is an induced principal G-action on M and the splitting $E = M \times_{M_0} P_0$, where $P_0 = P/G$ is a vector bundle over $M_0 = M/G$. The G-vector bundle structure on P comes directly from this splitting in the obvious manner.*

4.4 Linear $\mathbb{R}^\times$-bundles

When dealing with Jacobi and contact geometry, principal $\mathbb{R}^\times$-bundles $\pi : P \to P_0$ that also carry a compatible vector bundle structure $\tau : P \to M$ are an essential part of the theory. We will refer to such structures as *linear $\mathbb{R}^\times$-bundles*.

We are free to employ local homogeneous coordinates of the form (t, x^α, y^i) on P, where (t, x^α) represent coordinates on M and (x^α, y^i) on P_0, so that the $\mathbb{R}^\times$-action h reads

$$h_s(t, x^\alpha, y^i) = (s\,t, x^\alpha, y^i) \tag{4.8}$$

and $(t, x^\alpha, y^i) \mapsto (t, x^\alpha)$ is a vector fibration.

We will use the following fundamental fact.

Theorem 4.12. *For any principal $\mathbb{R}^\times$ bundle $M \to M_0$, the tangent $\mathsf{T}M$ and the cotangent T^*M bundle are canonically linear $\mathbb{R}^\times$-bundles with the $\mathbb{R}^\times$ action described in Proposition 2.2.*

It is easy to see that, starting with coordinates (t, x^a) in M, where $t \in \mathbb{R}^\times$, identification (4.7) takes in the above cases the form

$$\mathsf{T}M = M \times_{M_0} (\mathsf{T}M/\mathbb{R}^\times)\,, \quad (t, x^a, \dot{t}, \dot{x}^b) \mapsto (t, x^a, \dot{\mathbf{t}}, \dot{x}^b)\,,$$

where $\dot{\mathbf{t}} = t^{-1}\,\dot{t}$, and

$$\mathsf{T}^*M = M \times_{M_0} \mathsf{J}^1M^+\,, \quad (t, x^a, z, p_b) \mapsto (t, x^a, z, \mathbf{p}_b)\,,$$

where $\mathbf{p}_b = t^{-1}\,p_b$.

4.5 Kirillov algebroids vs $\mathbb{R}^\times$-groupoids

Assuming that a Kirillov manifold is equipped simultaneously with a compatible vector bundle, we get the following.

Definition 4.13. [16] A *Kirillov algebroid* is a linear Poisson $\mathbb{R}^\times$-bundle, i.e. a linear $\mathbb{R}^\times$-bundle equipped with a Poisson structure which is linear and homogeneous of degree -1 with respect to the $\mathbb{R}^\times$-action. If the principal $\mathbb{R}^\times$-bundle is trivial, then we speak about a *Jacobi algebroid*. A *morphism of Kirillov algebroids* is Poisson morphism that intertwines the respective pairs of $\mathbb{R}$- and $\mathbb{R}^\times$-actions.

Remark 4.14. If the principal $\mathbb{R}^\times$-bundle is trivial, our concept of a Jacobi algebroids coincides with the one introduced and studied in [11, 12] and that of a *generalized Lie algebroid* in [18]. In full generality, the notion of a Kirillov algebroid is equivalent to that of an *abstract Jacobi algebroid*, defined in [24] as a Lie algebroid and together with a representation thereof on a line bundle, see [16] for a closer description and proof of this equivalence.

We will denote a Kirillov algebroid as the quadruple $(P, \mathrm{h}, \mathrm{l}, \Lambda)$, where h and l are $\mathbb{R}^\times$ and $\mathbb{R}$-actions, respectively, or simply (P, Λ) where no risk of confusion can occur. In local homogeneous coordinates, the Poisson structure must be of the form

$$\Lambda = \frac{1}{t}\Lambda^{i\alpha}(x)\frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial y^i} + \frac{1}{2t}y^k\Lambda_k^{ij}(x)\frac{\partial}{\partial y^j} \wedge \frac{\partial}{\partial y^i} + \Lambda^i(x)\frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial t}\,. \tag{4.9}$$

where (t, x, y) are coordinates of (h, l)-bidegrees $(1, 0)$, $(0, 0)$, and $(0, 1)$, respectively.

Example 4.15. There is a canonical Kirillov algebroid associated with a given Kirillov manifold (P, h, Λ). It is simply the linear $\mathbb{R}^\times$-bundle $\mathsf{T}P$ (with the tangent lift of $\mathbb{R}^\times$-action) equipped with the tangent lift $\mathrm{d}_{\mathsf{T}}\,\Lambda$ of the Poisson structure Λ.

Remark 4.16. The above construction is in principle equivalent to the one described in [11, Remark 2] and, for trivial $\mathbb{R}^\times$-bundles, it leads to the construction of a Lie algebroid associated with a given Jacobi structure, as presented in [22]. The above description, however, is strikingly simple.

We must draw attention to the similarities with Lie algebroids. In particular Lie algebroid structures on a vector bundle are equivalent to linear Poisson structures on the dual vector bundle; there is an equivalence of categories here. In [16] the following was proved (expressed now in our language).

Proposition 4.17. *A Kirillov algebroid can be equivalently defined as a $\mathbb{R}^\times$-algebroid.*

One has to take a little care here, as the above picture is dual to the description in terms of homogeneous Poisson structures. In particular, in the above proposition we make the identification $E = P^*$ as vector bundles over $P_0 = P/\mathbb{R}^\times$.

Definition 4.18. A Kirillov algebroid (P, Λ) is said to be an *integrable* if the associated Lie algebroid ΠP^* is globally integrable.

In view of Proposition 4.17, Theorem 4.6 immediately implies the following.

Theorem 4.19. *There is a one-to-one correspondence between integrable Kirillov algebroids and source simply-connected $\mathbb{R}^\times$-groupoids.*

5 Kirillov and contact groupoids

5.1 Kirillov and Jacobi groupoids

Definition 5.1. A *Kirillov groupoid* is a $\mathbb{R}^\times$-groupoid equipped with a homogeneous multiplicative Poisson structure. Kirillov groupoids with trivial $\mathbb{R}^\times$-bundle will be called *Jacobi groupoids*. If the Poisson structure is non-degenerate, i.e. a symplectic structure, then we will speak of a *contact groupoid*.

Remark 5.2. In other words, a contact groupoid is a *homogeneous symplectic groupoid*, i.e. a symplectic groupoid $(\mathcal{G}, \omega)$ equipped additionally with a compatible principal $\mathbb{R}^\times$-bundle structure h such that $\mathbb{R}^\times$ acts by groupoid isomorphisms and ω is homogeneous of degree 1 with respect to this action, $\mathrm{h}_t^*\omega = t\,\omega$. Symplectic groupoids have been defined by Weinstein [41] and, under different names, independently by Karasev [21] and Zakrzewski [43, 44]. They can be understood as groupoids $\mathcal{G} \rightrightarrows M$ equipped with a *multiplicative* symplectic form ω. The notion of a homogeneous symplectic groupoid can be traced back to Libermann [26], however her notion of homogeneity is in terms of a vector field and not an action of $\mathbb{R}^\times$, so does not cover the case of an arbitrary line bundle.

Example 5.3. Let $\mathcal{G}$ be a Lie groupoid. Then, the cotangent bundle $\mathsf{T}^*\mathcal{G}$ is canonically a symplectic groupoid [41] with respect to the canonical symplectic form $\omega_{\mathcal{G}}$ on $\mathsf{T}^*\mathcal{G}$. The manifold of units is the dual $\mathrm{A}^*(\mathcal{G})$ of the Lie algebroid $\mathrm{A}(\mathcal{G})$ of $\mathcal{G}$, embedded into $\mathsf{T}^*\mathcal{G}$ as the conormal bundle ν^*M. We will refer to it as to the *canonical symplectic groupoid of* $\mathcal{G}$. It has a vector bundle structure compatible with the groupoid structure in the sense that homotheties $\mathrm{l}_t(\theta_y) = t.\theta_y$ in the vector bundle $\mathsf{T}^*\mathcal{G} \to \mathcal{G}$ act as groupoid morphisms (it is a $\mathcal{VB}$*-groupoid*). The source and the target maps $s, t : \mathsf{T}^*\mathcal{G} \to \mathrm{A}^*(\mathcal{G})$ intertwine the homotheties in $\mathsf{T}^*\mathcal{G} \to \mathcal{G}$ with that in $\mathrm{A}^*(\mathcal{G}) \to M$. It is now clear that removing the level sets $Z_s = s^{-1}(\{0\})$ and $Z_t = t^{-1}(\{0\})$ gives us an open-dense subgroupoid

$$\mathcal{C}(\mathcal{G}) = \mathsf{T}^*\mathcal{G} \setminus \{Z_s \cup Z_t\} \rightrightarrows \mathrm{A}^*(\mathcal{G}) \setminus \{0\} \tag{5.1}$$

of $\mathsf{T}^*\mathcal{G}$. In other words, $\mathcal{C}(\mathcal{G})$ consists of covectors from $\mathsf{T}^*\mathcal{G}$ which vanish on vectors tangent to source or target fibers. Of course, being an open subgroupoid of $\mathsf{T}^*\mathcal{G}$ it is still a symplectic groupoid, but as the zero section of $\mathsf{T}^*\mathcal{G}$ has been removed and as $\mathcal{C}(\mathcal{G})$ remains $\mathbb{R}^\times$-invariant, the group $\mathbb{R}^\times$ acts on $\mathcal{C}(\mathcal{G})$ by non-zero homotheties in a free and proper way. The symplectic form remains homogeneous of degree 1 with respect to this action, so we are dealing with a contact groupoid. The contact groupoid $\mathcal{C}(\mathcal{G})$ is canonically associated with the groupoid $\mathcal{G}$ and will be called the *canonical contact groupoid of* $\mathcal{G}$. In the traditional picture, it should be viewed as the reduced groupoid $\mathcal{C}(\mathcal{G})/\mathbb{R}^\times$ which is an open-dense submanifold of the projectivisation bundle $\mathcal{P}(\mathsf{T}^*\mathcal{G})$.

Remark 5.4. We will show that Jacobi groupoids in our sense coincide with the Jacobi groupoids defined by Iglesias-Ponte & Marrero in [19] (also see [20]), while our contact groupoids are contact groupoids in the sense of Dazord [9]. The latter are more general and the same time conceptually simpler than that of Kerbrat & Souici-Benhammadi [22], which require a globally defined contact form (whose multiplicativity is twisted by a multiplicative function).

Since a Kirillov groupoid is both: an $\mathbb{R}^\times$-groupoid and a Poisson groupoid as defined by Weinstein [42], let us decipher this definition for the Kirillov groupoid $(\mathcal{G} \rightrightarrows M, \mathrm{h}, \Lambda)$;

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\pi"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & \mathcal{G}_0 \arrow[d, shift right, "\sigma"'] \arrow[d, shift left, "\tau"] \\
M \arrow[r, "\mathrm{p}"'] & M_0
\end{tikzcd}$$

1. π and p are principal $\mathbb{R}^\times$-bundles.
2. M is a coisotropic submanifold of $(\mathcal{G}, \Lambda)$.
3. There is a unique homogeneous Poisson structure on M such that the source map is a Poisson morphism and the target map an anti-Poisson morphism. Thus $(M, \mathrm{h}, s_*\Lambda)$, where h is the homogeneity structure restricted to M, is a Kirillov manifold.
4. The Poisson structure induces a morphism of Lie groupoids:

$$\begin{tikzcd}
\mathsf{T}^*\mathcal{G} \arrow[r, "\Lambda^\#"] \arrow[d, shift right] \arrow[d, shift left] & \mathsf{T}\mathcal{G} \arrow[d, shift right] \arrow[d, shift left] \\
\mathrm{A}^*(\mathcal{G}) \arrow[r, "\rho_\Lambda"'] & \mathsf{T}M
\end{tikzcd}$$

   According to Theorem 2.3, $\Lambda^\#$ intertwines the lifted actions of $\mathbb{R}^\times$, so induces a morphism of the reduced groupoids

$$\Lambda_0^\# : \mathsf{T}^*\mathcal{G}/\mathbb{R}^\times \to \mathsf{T}\mathcal{G}/\mathbb{R}^\times \,. \tag{5.2}$$

5. According to Remark 3.3, in the case of the trivial $\mathbb{R}^\times$-bundle, $\mathcal{G} = \mathcal{G}_0 \times \mathbb{R}^\times$, we have the identifications

$$\mathsf{T}^*(\mathcal{G}_0 \times \mathbb{R}^\times)/\mathbb{R}^\times \simeq \mathsf{T}^*\mathcal{G}_0 \times \mathbb{R}\,, \quad \mathsf{T}(\mathcal{G}_0 \times \mathbb{R}^\times)/\mathbb{R}^\times \simeq \mathsf{T}\mathcal{G}_0 \times \mathbb{R}\,,$$

and the reduced morphism can be viewed as a map

$$\Lambda_0^{\#} : \mathsf{T}^*\mathcal{G}_0 \times \mathbb{R} \to \mathsf{T}\mathcal{G}_0 \times \mathbb{R}\,, \tag{5.3}$$

associated with a Jacobi structure on $\mathcal{G}_0$. This map is a groupoid morphism for the groupoid structures determined by that on $\mathcal{G}_0 \times \mathbb{R}^\times$, so according to Theorem 4.10, by the groupoid structure of $\mathcal{G}_0$ and a multiplicative function $\mathrm{b} : \mathcal{G}_0 \to \mathbb{R}^\times$, which reduces to a multiplicative function $\log|\mathrm{b}|$ into the additive group of reals. This leads to the definition of a Jacobi groupoid as presented in [19], although the explicit form of the groupoid structures on $\mathsf{T}^*\mathcal{G}_0 \times \mathbb{R}$ and $\mathsf{T}\mathcal{G}_0 \times \mathbb{R}$, expressed in terms of $\mathcal{G}_0$ and b, is quite complicated. Our 'Kirillov version' of the Jacobi groupoid is not only more general, but conceptually simpler. The technical complications of the definition in [19], together with the presence of the multiplicative function $\log|\mathrm{b}|$, come from insisting on working with trivialisations of principal bundles.

The above observations do not change if we consider a contact groupoid; in particular we still have a Kirillov manifold M which is generally not a contact manifold.

Definition 5.5. A Kirillov manifold is said to be an *integrable Kirillov manifold* if it arises from a contact groupoid as described above.

By minor modification of the classical results on the integrability of Poisson manifolds and in view of Theorem 4.19, we are led to the following;

Proposition 5.6. *The following statements are equivalent:*

1. *The Kirillov manifold* (M, h, Λ) *is integrable.*
2. *The Kirillov algebroid* $(\mathsf{T}M, \mathsf{T}\,\mathrm{h}, \mathrm{d}_{\mathsf{T}}\Lambda)$ *is integrable.*
3. *The Lie algebroid structure on* T^*M *corresponding to the linear Poisson structure* $\mathrm{d}_{\mathsf{T}}\,\Lambda$ *is integrable.*
4. *The Poisson structure* Λ *is integrable.*

The above proposition includes the main result of [8]. The connection to the language of the splitting $\mathcal{G} \times_r \mathbb{R}$ used there is given by our theorem 4.10.

5.2 Contact groupoids

Let us restrict attention to a contact groupoid $(\mathcal{G}, \mathrm{h}, \omega)$. We know, *via* Remark 3.8, that the homogeneous symplectic structure ω on the $\mathbb{R}^\times$-bundle $(\mathcal{G}, \mathrm{h})$ is equivalent to a contact structure $C = C(\mathcal{G}, \mathrm{h}, \omega) \subset \mathsf{T}^*\mathcal{G}_0$, and further to a contact distribution $\mathcal{D} = \mathcal{D}(\mathcal{G}, \mathrm{h}, \omega) = C^0 \subset \mathsf{T}\mathcal{G}_0$. Such a distribution is called a *contact groupoid* by Dazord [9, 10] (and *conformal contact groupoid* by the authors understanding contact groupoids as groupoids equipped with a globally defined contact form) if the contact distribution is closed with respect to the operation in the tangent groupoid $\mathsf{T}\mathcal{G}_0$: it is invariant with respect to inversion and $\mathcal{D} \bullet \mathcal{D} \subset \mathcal{D}$, where $\bullet$ is the (partial) multiplication in $\mathsf{T}\mathcal{G}_0$. In other words, $\mathcal{D}$ is a Lie subgroupoid of $\mathsf{T}\mathcal{G}_0 \rightrightarrows \mathsf{T}M$. We will show that Dazord's definition is equivalent to ours.

Theorem 5.7. *Any contact groupoid* $(\mathcal{G}, \mathrm{h}, \omega)$*, with* $\mathcal{G}/\mathbb{R}^\times = \mathcal{G}_0$*, has a canonical and equivalent realization as each of the following:*

- *a contact subgroupoid* $C^\times(\mathcal{G}, \mathrm{h}, \omega)$ *of the canonical contact groupoid* $\mathcal{C}(\mathcal{G}_0)$*;*
- *a Dazord groupoid* $\mathcal{D}(\mathcal{G}, \mathrm{h}, \omega)$*, being simultaneously a contact distribution and a subgroupoid of* $\mathsf{T}\mathcal{G}_0$*.*

Proof. Let ∇ be the Euler vector field on $\mathcal{G}$. Since ∇ generates groupoid isomorphisms, it is a multiplicative vector field on $\mathcal{G}$, thus $\nabla : \mathcal{G} \to \mathsf{T}\mathcal{G}$ is a groupoid morphism (over $\nabla_{|M} : M \to \mathsf{T}M$). Since the symplectic form ω is multiplicative, it defines an isomorphism of groupoids $\omega^\flat : \mathsf{T}\mathcal{G} \to \mathsf{T}^*\mathcal{G}$. The one-form $\eta = \omega^\flat \circ \nabla : \mathcal{G} \to \mathsf{T}^*\mathcal{G}$ is a groupoid morphism, thus multiplicative, $\eta(yy') = \eta(y) \star \eta(y')$, where $\star$ is the groupoid multiplication in the cotangent groupoid $\mathsf{T}^*\mathcal{G} \rightrightarrows \mathrm{A}^*(\mathcal{G})$. We know that it takes values in basic covectors, $\eta(y) = \pi^*(\Psi(y)) \in \mathsf{T}^*_y\mathcal{G}$, $\Psi(y) \in \mathsf{T}^*_{\pi(y)}\mathcal{G}_0$, so can be viewed as a map $\Psi : \mathcal{G} \to \mathsf{T}^*\mathcal{G}_0$. Consequently, the range $C^\times(\mathcal{G}, \mathrm{h}, \omega) = \{\Psi(y) : y \in \mathcal{G}\}$ of Ψ is a subgroupoid in the cotangent groupoid $\mathsf{T}^*\mathcal{G}_0$. According to theorem 3.7, Ψ is also an embedding of contact structures (homogeneous symplectic $\mathbb{R}^\times$-bundles), so Ψ is just a realization of $(\mathcal{G}, \mathrm{h}, \omega)$ as a contact subgroupoid of $\mathcal{C}(\mathcal{G}_0)$.

Note finally that the contact distribution $\mathcal{D}(\mathcal{G}, \mathrm{h}, \omega)$ and the contact structure $C(\mathcal{G}, \mathrm{h}, \omega)$ are related by the polar condition: one annihilates the other in the canonical pairing between the tangent and the cotangent bundle. Since the partial multiplication in the cotangent and tangent groupoid are related by the condition

$$\theta_g \star \theta'_h(X_g \bullet X'_h) = \theta_g(X_g) + \theta'_h(X'_h)\,,$$

it can be easily seen that $C(\mathcal{G}, \mathrm{h}, \omega)$ is a subgroupoid if and only if $\mathcal{D}(\mathcal{G}, \mathrm{h}, \omega)$ is a subgroupoid. □

Remark 5.8. To be very clear, $\Psi : \mathcal{G} \to \mathsf{T}^*\mathcal{G}_0$ is in general *not* a Lie groupoid morphism (a contact form on $\mathcal{G}_0$ need not to be multiplicative), however the range of Ψ is a contact subgroupoid of $\mathsf{T}^*\mathcal{G}_0$. Thus we have a canonical realisation of $\mathcal{G}$ rather than a genuine morphism between contact groupoids.

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