MABUCHI AND AUBIN-YAU FUNCTIONALS OVER COMPLEX THREE-FOLDS

YI LI

ABSTRACT. In this paper we construct Mabuchi $\mathcal{L}^{\mathrm{M}}_{\omega}$ functional and Aubin-Yau functionals $\mathcal{I}^{\mathrm{AY}}_{\omega}, \mathcal{J}^{\mathrm{AY}}_{\omega}$ on any compact complex three-folds. The method presented here will be used in the forthcoming paper [5] on the construction of those functionals on any compact complex manifolds, which generalizes the previous work [4].

CONTENTS

1. INTRODUCTION

Mabuchi and Aubin-Yau functionals play a crucial role in studying Kähler-Einstein metrics and constant scalar curvatures. How to generalize these functionals from Kähler geometry to complex geometry is an interesting problem. In [4], the author solved this problem in dimension two and proved similar results in the Kähler setting. By carefully checking and using a trick, we can construct those functionals on any compact complex three-folds. Moreover, the idea in this paper will be used in the forthcoming paper in which we deal with higher dimension cases.

1.1. Mabuchi and Aubin-Yau functionals on Kähler manifolds.

Let (X, ω) be a compact Kähler manifold of the complex dimension n. then the volume

$$V_{\omega} := \int_X \omega^n \tag{1.1}$$

depends only on the Kähler class of ω. Let $\mathcal{P}_\omega$ denote the space of Kähler potentials and define the Mabuchi functional, for any smooth functions $\varphi', \varphi'' \in \mathcal{P}_\omega$, by

$$\mathcal{L}_\omega^{\mathrm{M,Kahler}}(\varphi', \varphi'') := \frac{1}{V_\omega}\int_0^1 \int_X \dot{\varphi}_t \omega_{\varphi_t}^n dt \tag{1.2}$$

where φ_t is any smooth path in $\mathcal{P}_\omega$ from φ' to φ''. Mabuchi [7] showed that (1.2) is well-defined.

Using (1.2) we can define Aubin-Yau functionals as follows:

$$\mathcal{I}_\omega^{\mathrm{AY,Kahler}}(\varphi) = \frac{1}{V_\omega}\int_Z \varphi(\omega^n - \omega_\varphi^n), \tag{1.3}$$

$$\mathcal{J}_\omega^{\mathrm{AY,Kahler}}(\varphi) = -\mathcal{L}_\omega^{\mathrm{M,Kahler}}(0, \varphi) + \frac{1}{V_\omega}\int_X \varphi\omega^n. \tag{1.4}$$

So Aubin-Yau functionals are also well-defined.

However, if ω is not closed, then the above definitions do not make sense. Hence we should add some extra terms on the definitions of those functionals. These extra terms should involve $\partial\omega$ and $\overline{\partial}\omega$, but, the essential question is to find the structure of the extra terms. In the next section, we will answer this question.

1.2. **Mabuchi and Aubin-Yau functionals on complex three-folds.** Throughout the rest part of this paper, we denote by (X, g) a compact complex manifold of the complex dimension 3, and ω the associated real $(1,1)$-form. Let

$$\mathcal{P}_\omega := \{\varphi \in C^\infty(X)_{\mathbb{R}} | \omega_\varphi := \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0\} \tag{1.5}$$

be the space of all real-valued smooth functions on X whose associated real $(1,1)$-forms are positive.

For any $\varphi', \varphi'' \in \mathcal{P}_\omega$, we define

$$\begin{aligned} \mathcal{L}_\omega^{\mathrm{M}}(\varphi', \varphi'') &= \frac{1}{V_\omega}\int_0^1 \int_X \dot{\varphi}_t \omega_{\varphi_t}^3 dt \\ &- \frac{1}{V_\omega}\int_0^1 \int_X 3\sqrt{-1}\partial\omega \wedge (\overline{\partial}\dot{\varphi}_t \cdot \varphi_t) \wedge \omega_{\varphi_t} dt \\ &+ \frac{1}{V_\omega}\int_0^1 \int_X 3\sqrt{-1}\overline{\partial}\omega \wedge (\partial\dot{\varphi}_t \cdot \varphi_t) \wedge \omega_{\varphi_t} dt \\ &- \frac{1}{V_\omega}\int_0^1 \int_X \partial\varphi_t \wedge \overline{\partial}\varphi_t \wedge \partial\omega \wedge \overline{\partial}\dot{\varphi}_t - \frac{1}{V_\omega}\int_0^1 \int_X \overline{\partial}\varphi_t \wedge \partial\varphi_t \wedge \overline{\partial}\omega \wedge \partial\dot{\varphi}_t \end{aligned} \tag{1.6}$$

where $\{\varphi_t\}_{0\le t\le 1}$ is any smooth path in $\mathcal{P}_\omega$ from φ' to φ''. Our first result is

Theorem 1.1. *The functional (1.6) is independent of the choice of the smooth path $\{\varphi_t\}_{0\le t\le 1}$ in $\mathcal{P}_\omega$.*

For any $\varphi \in \mathcal{P}_\omega$ we set

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi) := \mathcal{L}_\omega^{\mathrm{M}}(0, \varphi). \tag{1.7}$$

Then we have an explicit formula of $\mathcal{L}_\omega^{\mathrm{M}}(\varphi)$:

Corollary 1.2. *One has*

$$\begin{aligned}\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi) &= \frac{1}{4V_{\omega}}\sum_{i=0}^{3}\int_X \varphi\omega^i_{\varphi}\wedge\omega^{3-i} \\ &- \sum_{i=0}^{1}\frac{i+1}{2V_{\omega}}\int_X \varphi\omega^i_{\varphi}\wedge\omega^{1-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\ &+ \sum_{i=0}^{1}\frac{i+1}{2V_{\omega}}\int_X \varphi\omega^i_{\varphi}\wedge\omega^{1-i}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.\end{aligned} \tag{1.8}$$

Now we define Aubin-Yau functionals $\mathcal{I}^{\mathrm{AY}}_{\omega}, \mathcal{J}^{\mathrm{AY}}_{\omega}$ for any compact complex three-fold (X,ω):

$$\begin{aligned}\mathcal{I}^{\mathrm{AY}}_{\omega}(\varphi) &= \frac{1}{V_{\omega}}\int_X \varphi(\omega^3-\omega^3_{\varphi}) \\ &- \frac{3}{2V_{\omega}}\int_X \varphi\omega_{\varphi}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi - \frac{3}{2V_{\omega}}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\ &+ \frac{3}{2V_{\omega}}\int_X \varphi\omega_{\varphi}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi + \frac{3}{2V_{\omega}}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi,\end{aligned} \tag{1.9}$$

$$\begin{aligned}\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) &= -\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi) + \frac{1}{V_{\omega}}\int_X \varphi\omega^3 \\ &- \frac{3}{2V_{\omega}}\int_X \varphi\omega_{\varphi}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi - \frac{3}{2V_{\omega}}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\ &+ \frac{3}{2V_{\omega}}\int_X \varphi\omega_{\varphi}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi + \frac{3}{2V_{\omega}}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.\end{aligned} \tag{1.10}$$

An important result is

Theorem 1.3. *For any $\varphi\in\mathcal{P}_{\omega}$, one has*

$$\frac{3}{4}\mathcal{I}^{\mathrm{AY}}_{\omega}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) \geq 0, \tag{1.11}$$

$$4\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) - \mathcal{I}^{\mathrm{AY}}_{\omega}(\varphi) \geq 0. \tag{1.12}$$

In particular

$$\begin{aligned}\frac{1}{4}\mathcal{I}^{\mathrm{AY}}_{\omega}(\varphi) &\leq \mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) \leq \frac{3}{4}\mathcal{I}^{\mathrm{AY}}_{\omega}(\varphi), \\ \frac{4}{3}\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) &\leq \mathcal{I}^{\mathrm{AY}}_{\omega}(\varphi) \leq 4\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi), \\ \frac{1}{3}\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) \leq \frac{1}{4}\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) &\leq \mathcal{I}^{\mathrm{AY}}_{\omega}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) \\ &\leq \frac{3}{4}\mathcal{I}^{\mathrm{AY}}_{\omega}(\varphi) \leq 3\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi).\end{aligned}$$

1.3. **Volume estimate.** For any compact Kähler manifold the volume (1.1) depends only on the Kähler class, but, this fact doesn't hold for the general compact Hermitian manifolds. To understand the change of volumes for compact Hermitian three-fold (X,ω), the author [4] introduced three quantities associated to ω:

$$\mathrm{Err}_\omega(\varphi) := \int_X \omega^n - \int_X \omega_\varphi^n, \quad \varphi \in \mathcal{P}_\omega, \tag{1.13}$$

$$\mathrm{SupErr}_\omega := \sup_{\varphi\in\mathcal{P}_\omega^0} (\mathrm{Err}_\omega(\varphi)), \tag{1.14}$$

$$\mathrm{InfErr}_\omega := \inf_{\varphi\in\mathcal{P}_\omega^0} (\mathrm{Err}_\omega(\varphi)). \tag{1.15}$$

where $\mathcal{P}_\omega^0 = \{\varphi \in \mathcal{P}_\omega | \sup_X \varphi = 0\}$ is the normalized subspace of $\mathcal{P}_\omega$. In any case, we have

$$\mathrm{InfErr}_\omega \le 0 \le \mathrm{SupErr}_\omega \le \int_X \omega^n. \tag{1.16}$$

It implies that the quantity SupErr_ω is bounded, but the quantity InfErr_ω may not have a lower bound. Notice that the lower boundedness of InfErr_ω is equivalent to the upper boundedness of $\sup_{\varphi\in\mathcal{P}_\omega^0}\int_X \omega_\varphi^n$. We can ask a natural question [6]:

Question 1.4. *Under what condition (weaker than the Kähler condition), the quantity*

$$\sup_{\varphi\in\mathcal{P}_\omega^0} \int_X \omega_\varphi^n \tag{1.17}$$

is bounded from above?

Remark 1.5. *(1) In* [4] *the author showed that if $\partial\overline{\partial}\omega = \partial\omega \wedge \overline{\partial}\omega = 0$, then the volume $\int_X \omega_\varphi^n$ does not depend on the choice of the smooth function $\varphi \in \mathcal{P}_\omega$. Consequently, $\int_X \omega_\varphi^n = \int_X \omega^n$ for any function $\varphi \in \mathcal{P}_\omega$, which gives an affirmative answer to Question 1.4. In the surface case, we can only assume $\partial\overline{\partial}\omega = 0$.*
(3) Let (X,ω) be a compact Hermitian manifold of the complex dimension 2*. By a theorem of Ganduchon* [2]*, there exists a smooth function u, unique up to a constant, such that*

$$\partial\overline{\partial}\omega_{\mathrm{G}} = 0, \quad \omega_{\mathrm{G}} := e^u\omega. \tag{1.18}$$

V. Tosatti and B. Weinkove [9] *showed that (1.17) is bounded by $\int_X (2e^{u-\inf_X u} - 1)\omega^2$ from above.*

Using the idea in [9] we can show that, under a suitable condition, InfErr_ω is bounded from below when $n = 3$ and, consequently, (1.17) has an upper bound.

Theorem 1.6. *Suppose that (X, g) is a compact Hermitian manifold of the complex dimension* 3 *and ω is its associated real $(1,1)$-form. If $\partial\overline{\partial}\omega = 0$, then* InfErr_ω *is bounded from below. More precisely, we have*

$$\mathrm{InfErr}_\omega \ge 3\left(1 - e^{2\cdot\mathrm{osc}(u)}\right) \cdot \int_X \omega^3. \tag{1.19}$$

Here u is a real-valued smooth function on X such that $\omega_G = e^u \cdot \omega$ is a Gauduchon metric, i.e., $\partial\overline{\partial}(\omega_G^2) = 0$. In particular

$$\int_X \omega^3 \le \sup_{\varphi\in\mathcal{P}_\omega^0} \int_X \omega_\varphi^3 \le \left(3e^{2\cdot\mathrm{osc}(u)} - 2\right) \int_X \omega^3. \tag{1.20}$$

Another interesting question [6] is

Conjecture 1.7. *For any compact Hermitian manifold (X,ω), one has*

$$\inf_{\varphi\in\mathcal{P}^0_\omega}\int_X \omega^n_\varphi > 0. \tag{1.21}$$

Remark 1.8. *(1) When ω is Kähler, for any smooth function $\varphi\in\mathcal{P}_\omega$, we have $\int_X\omega^n_\varphi=\int_X\omega^n$. Hence $\inf_{\varphi\in\mathcal{P}^0_\omega}\int_X\omega^n_\varphi=\int_X\omega^n>0$.*
(2) In [9]*, the authors confirmed Conjecture 1.7 for $n=2$ provided that u satisfies* $\sup_X(u)-\inf_X(u)<\log 2$*. More precisely*

$$\inf_{\varphi\in\mathcal{P}^0_\omega}\int_X\omega^n_\varphi \ge \int_X\left(2e^{u-\sup_X(u)}-1\right)\omega^2>0. \tag{1.22}$$

We can prove Conjecture 1.7 for $n=3$ provided that a similar condition holds for u.

Theorem 1.9. *Suppose that (X,g) is a compact Hermitian manifold of the complex dimension 3 and ω is its associated real $(1,1)$-form. We select a real-valued smooth function u on X so that $e^u\cdot\omega$ is a Gauduchon metric. If*

$$\mathrm{osc}(u)=\sup_X(u)-\inf_X(u)\le\frac{1}{2}\cdot\ln\frac{3}{2},\quad \partial\overline{\partial}\omega=0,$$

then

$$\inf_{\varphi\in\mathcal{P}^0_\omega}\int_X\omega^3_\varphi\ge\int_X\omega^3>0. \tag{1.23}$$

2. Mabuchi $\mathcal{L}^{\mathrm{M}}_\omega$ functional on compact complex three-folds

Let (X,g) be a compact complex manifold of the complex dimension 3, and ω be its associated real $(1,1)$-form. In this section we define Mabuchi $\mathcal{L}^{\mathrm{M}}_\omega$ functional and prove the independence of the choice of the smooth path. As a consequence, we give an explicit formula for $\mathcal{L}^{\mathrm{M}}_\omega$.

Let $\varphi',\varphi''\in\mathcal{P}_\omega$ and $\{\varphi_t\}_{0\le t\le 1}$ be a smooth path in $\mathcal{P}_\omega$ from φ' to φ''. We define

$$\mathcal{L}^0_\omega(\varphi',\varphi'')=\frac{1}{V_\omega}\int_0^1\int_X\dot{\varphi}_t\omega^3_{\varphi_t}dt. \tag{2.1}$$

Set

$$\psi(s,t):=s\cdot\varphi_t,\quad 0\le s,t\le 1. \tag{2.2}$$

Consider a 1-form on $[0,1]\times[0,1]$

$$\Psi^0:=\left(\int_X\frac{\partial\psi}{\partial s}\cdot\omega^3_\psi\right)ds+\left(\int_X\frac{\partial\psi}{\partial t}\cdot\omega^3_\psi\right)dt. \tag{2.3}$$

Taking the differential on Ψ^0, we have

$$d\Psi^0=I^0\cdot dt\wedge ds$$

where the quantity I^0 is given by

$$I^0=\int_X\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial s}\cdot\omega^3_\psi\right)-\int_X\frac{\partial}{\partial s}\left(\frac{\partial\psi}{\partial t}\cdot\omega^3_\psi\right). \tag{2.4}$$

As in [4] we simplify I^0 in two slightly different ways. Directly computing shows

$$\begin{aligned} I^0 &= \int_X \left[\frac{\partial^2\psi}{\partial t\partial s}\cdot\omega_\psi^3 + \frac{\partial\psi}{\partial s}\cdot 3\omega_\psi^2\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\right] \\ &- \int_X \left[\frac{\partial^2\psi}{\partial s\partial t}\cdot\omega_\psi^3 + \frac{\partial\psi}{\partial t}\wedge 3\omega_\psi^2\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right] \\ &= \int_X 3\frac{\partial\psi}{\partial s}\omega_\psi^2\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) + \int_X 3\frac{\partial\psi}{\partial t}\omega_\psi^2\wedge\sqrt{-1}\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right) \\ &= -\int_X 3\frac{\partial\psi}{\partial s}\omega_\psi^2\wedge\sqrt{-1}\overline{\partial}\partial\left(\frac{\partial\psi}{\partial t}\right) - \int_X 3\frac{\partial\psi}{\partial t}\omega_\psi^2\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right). \end{aligned}$$

Notice that the last two steps are the differential expressions for I^0. Using the first expression, we have

$$\begin{aligned} I^0 &= \int_X 3\frac{\partial\psi}{\partial s}\omega_\psi^2\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) + \int_X 3\frac{\partial\psi}{\partial t}\omega_\psi^2\wedge\sqrt{-1}\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right) \\ &= \int_X -3\sqrt{-1}\partial\left(\frac{\partial\psi}{\partial s}\omega_\psi^2\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) + \int_X -3\sqrt{-1}\overline{\partial}\left(\frac{\partial\psi}{\partial t}\omega_\psi^2\right)\wedge\partial\left(\frac{\partial\psi}{\partial s}\right) \\ &= \int_X -3\sqrt{-1}\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^2 + \frac{\partial\psi}{\partial s}2\omega_\psi\wedge\partial\omega\right]\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\ &= \int_X -3\sqrt{-1}\left[\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^2 + \frac{\partial\psi}{\partial t}2\omega_\psi\wedge\overline{\partial}\omega\right]\wedge\partial\left(\frac{\partial\psi}{\partial s}\right) \end{aligned}$$

$$= \int_X -6\sqrt{-1}\frac{\partial\psi}{\partial s}\omega_\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) + \int_X -6\sqrt{-1}\frac{\partial\psi}{\partial t}\omega_\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right). \tag{2.5}$$

Similarly, we have

$$I^0 = -\int_X 3\frac{\partial\psi}{\partial s}\omega_\psi^2\wedge\sqrt{-1}\overline{\partial}\partial\left(\frac{\partial\psi}{\partial t}\right) - \int_X 3\frac{\partial\psi}{\partial t}\omega_\psi^2\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)$$

$$= \int_X 6\sqrt{-1}\frac{\partial\psi}{\partial s}\omega_\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial t}\right) + \int_X 6\sqrt{-1}\frac{\partial\psi}{\partial t}\omega_\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right). \tag{2.6}$$

Therefore, from (2.5) and (2.6) it follows that

$$\begin{aligned} \frac{2I^0}{6\sqrt{-1}} &= \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\frac{\partial\psi}{\partial s}\wedge\omega_\psi\wedge\partial\omega + \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\frac{\partial\psi}{\partial t}\wedge\omega_\psi\wedge\overline{\partial}\omega \\ &- \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\frac{\partial\psi}{\partial t}\wedge\omega_\psi\wedge\partial\omega - \int_X \partial\left(\frac{\partial\psi}{\partial t}\right)\frac{\partial\psi}{\partial s}\wedge\omega_\psi\wedge\overline{\partial}\omega. \end{aligned} \tag{2.7}$$

Next we define

$$\mathcal{L}^1_\omega(\varphi',\varphi'') = \frac{1}{V_\omega}\int_0^1\int_X a_1\cdot\partial\omega\wedge\omega_{\varphi_t}\wedge(\overline{\partial}\dot{\varphi}_t\cdot\varphi_t)dt, \tag{2.8}$$

$$\mathcal{L}^2_\omega(\varphi',\varphi'') = \frac{1}{V_\omega}\int_0^1\int_X a_2\cdot\overline{\partial}\omega\wedge\omega_{\varphi_t}\wedge(\partial\dot{\varphi}_t\cdot\varphi_t)dt. \tag{2.9}$$

Here a_1, a_2 are non-zero constants and we require $\overline{a_1} = a_2$. Similarly, we consider

$$\begin{aligned} \Psi^1 &= \left[\int_X a_1\partial\omega\wedge\omega_\psi\wedge\left(\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\psi\right)\right]ds + \left[\int_X a_1\partial\omega\wedge\omega_\psi\wedge\left(\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\psi\right)\right]dt, \\ \Psi^2 &= \left[\int_X a_2\overline{\partial}\omega\wedge\omega_\psi\wedge\left(\partial\left(\frac{\partial\psi}{\partial s}\right)\psi\right)\right]ds + \left[\int_X a_2\overline{\partial}\omega\wedge\omega_\psi\wedge\left(\partial\left(\frac{\partial\psi}{\partial t}\right)\psi\right)\right]dt. \end{aligned}$$

Therefore

$$d\Psi^1 = I^1 \cdot dt \wedge ds$$

where

$$I^1 = \int_X a_1 \frac{\partial}{\partial t}\left[\partial\omega \wedge \omega_\psi \wedge \left(\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\cdot\psi\right)\right] - \int_X a_1 \frac{\partial}{\partial s}\left[\partial\omega \wedge \omega_\psi \wedge \left(\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\cdot\psi\right)\right].$$

Consequently,

$$\begin{aligned}
\frac{I^1}{a_1} &= \int_X -\frac{\partial}{\partial t}\left[\left(\psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right)\wedge\omega_\psi\wedge\partial\omega\right] + \int_X \frac{\partial}{\partial s}\left[\left(\psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\right)\wedge\omega_\psi\wedge\partial\omega\right] \\
&= \int_X -\left[\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) + \psi\cdot\overline{\partial}\left(\frac{\partial^2\psi}{\partial t\partial s}\right)\right]\wedge\omega_\psi\wedge\partial\omega \\
&+ \int_X \left[\frac{\partial\psi}{\partial s}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) + \psi\cdot\overline{\partial}\left(\frac{\partial^2\psi}{\partial s\partial t}\right)\right]\wedge\omega_\psi\wedge\partial\omega \\
&+ \int_X \psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge -\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega \\
&+ \int_X \psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega \\
&= \int_X -\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi\wedge\partial\omega + \int_X \frac{\partial\psi}{\partial s}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi\wedge\partial\omega \\
&+ \int_X \psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge -\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega \\
&+ \int_X \psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega.
\end{aligned}$$

In the same fashion way, we have

$$d\Psi^2 = I^2 \cdot dt \wedge ds,$$

where

$$\begin{aligned}
\frac{I^2}{a_2} &= \int_X -\frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi\wedge\overline{\partial}\omega + \int_X \frac{\partial\psi}{\partial s}\cdot\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi\wedge\overline{\partial}\omega \\
&+ \int_X \psi\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge \sqrt{-1}\overline{\partial}\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\omega \\
&+ \int_X \psi\cdot\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge -\sqrt{-1}\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\omega.
\end{aligned}$$

To simplify notation we set

$$\mathcal{A} := \int_X \psi\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge -\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right), \tag{2.10}$$

$$\mathcal{B} := \int_X \psi\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right). \tag{2.11}$$

Using the above symbols gives

$$\begin{aligned}
\frac{I^1}{a_1} &= \int_X -\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi\wedge\partial\omega + \int_X \frac{\partial\psi}{\partial s}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi\wedge\partial\omega + \mathcal{A} + \mathcal{B}, \\
\frac{I^2}{a_2} &= \int_X -\frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi\wedge\overline{\partial}\omega + \int_X \frac{\partial\psi}{\partial s}\cdot\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi\wedge\overline{\partial}\omega + \overline{\mathcal{A}} + \overline{\mathcal{B}},
\end{aligned}$$

and the last two terms can be determined completely as follows:

$$\begin{aligned}
\mathcal{A} &= \int_X \sqrt{-1}\partial\left[\psi\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\right]\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X \sqrt{-1}\left[\partial\left(\psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right)\wedge\partial\omega\right]\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X \sqrt{-1}\left[\partial\psi\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)+\psi\cdot\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right]\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X -\psi\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\\
&+ \int_X -\sqrt{-1}\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= -\mathcal{B}+\int_X -\sqrt{-1}\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right).
\end{aligned}$$

Adding the term $\mathcal{B}$ on both sides we obtain

$$\mathcal{A}+\mathcal{B} = \int_X -\sqrt{-1}\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right), \tag{2.12}$$

$$\overline{\mathcal{A}}+\overline{\mathcal{B}} = \int_X \sqrt{-1}\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\frac{\partial\psi}{\partial t}\right). \tag{2.13}$$

The final step is to introduce

$$\mathcal{L}^3_\omega(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_3\partial\varphi_t\wedge\partial\omega\wedge\overline{\partial}\dot{\varphi}_t\wedge\overline{\partial}\varphi_t, \tag{2.14}$$

$$\mathcal{L}^4_\omega(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_4\overline{\partial}\varphi_t\wedge\overline{\partial}\omega\wedge\partial\dot{\varphi}_t\wedge\partial\varphi_t, \tag{2.15}$$

where a_3, a_4 are non-zero constants determined later and we require $\overline{a_3} = a_4$. Consider

$$\begin{aligned}
\Psi^3 &= \left[\int_X a_3\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\right]ds\\
&+ \left[\int_X a_3\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\right]dt,\\
\Psi^4 &= \left[\int_X a_4\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\right]ds\\
&+ \left[\int_X a_4\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\psi\right]dt.
\end{aligned}$$

We take the differential on Ψ^3 and Ψ^4, and these differentials can be written as

$$d\Psi^3 = I^3\cdot dt\wedge ds,\quad d\Psi^4 = I^4\cdot dt\wedge ds$$

where

$$\begin{aligned} I^3 &= \int_X a_3 \frac{\partial}{\partial t}\left[\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\right] \\ &- \int_X a_3 \frac{\partial}{\partial s}\left[\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\right], \\ I^4 &= \int_X a_4 \frac{\partial}{\partial t}\left[\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\right] \\ &- \int_X a_4 \frac{\partial}{\partial s}\left[\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\psi\right]. \end{aligned}$$

Calculate

$$\begin{aligned} \frac{I^3}{a_3} &= \int_X \left[\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi + \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial^2\psi}{\partial t\partial s}\right)\wedge\overline{\partial}\psi\right. \\ &+ \left.\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\right] \\ &- \int_X \left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi + \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial^2\psi}{\partial s\partial t}\right)\wedge\overline{\partial}\psi\right. \\ &+ \left.\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right] \\ &= \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi + \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi \\ &+ 2\int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\ &= \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi + \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi \\ &+ \frac{2}{-\sqrt{-1}}(\mathcal{A}+\mathcal{B}). \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{I^4}{a_4} &= \int_X -\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\omega\wedge\partial\psi + \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\omega\wedge\partial\psi \\ &+ \frac{2}{\sqrt{-1}}(\overline{\mathcal{A}}+\overline{\mathcal{B}}). \end{aligned}$$

If we set

$$\mathcal{H} := \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi + \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi,$$

then those two terms have the shorted expressions:

$$\frac{I^3}{a_3} = \mathcal{H} + \frac{2}{-\sqrt{-1}}(\mathcal{A}+\mathcal{B}), \quad \frac{I^4}{a_4} = \overline{\mathcal{H}} + \frac{2}{\sqrt{-1}}(\overline{\mathcal{A}}+\overline{\mathcal{B}}).$$

On the other hand, directly by definition, we have

$$\begin{aligned}
\mathcal{H} &= \int_X \left[\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\right]\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)+\int_X\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\right]\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X \partial\left[\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\right]\frac{\partial\psi}{\partial t}+\int_X\overline{\partial}\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\right]\frac{\partial\psi}{\partial t}\\
&= \int_X \frac{\partial\psi}{\partial t}\left[\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi+\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\right]\\
&+ \int_X \frac{\partial\psi}{\partial t}\left[\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi-\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\right]\\
&= \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi-\int_X\frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\\
&= \int_X \overline{\partial}\left[\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\right]\wedge\partial\psi-\int_X\frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\\
&= \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\psi-\int_X\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\partial\psi\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\\
&= \int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&+ \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\overline{\partial}\omega\wedge\partial\psi+\int_X\frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\overline{\partial}\omega\wedge\overline{\partial}\psi
\end{aligned}$$

Taking the complex conjugate gives

$$\begin{aligned}
\overline{\mathcal{H}} &= \int_X \overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\overline{\partial}\omega\wedge\overline{\partial}\psi-\int_X\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\overline{\partial}\omega\wedge\partial\psi.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{H}+\overline{\mathcal{H}} &= \int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)+\int_X\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\\
&= \frac{\mathcal{A}+\mathcal{B}}{-\sqrt{-1}}+\frac{\overline{\mathcal{A}}+\overline{\mathcal{B}}}{\sqrt{-1}} \quad = \quad \frac{(\mathcal{A}+\mathcal{B})-(\overline{\mathcal{A}}+\overline{\mathcal{B}})}{-\sqrt{-1}}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\frac{2I^0}{6\sqrt{-1}} &= \frac{I^1}{a_1}-\frac{I^2}{a_2}+(\mathcal{A}+\mathcal{B})-(\overline{\mathcal{A}}+\overline{\mathcal{B}}),\\
\frac{I^3}{a_3}+\frac{I^4}{a_4} &= \mathcal{H}+\overline{\mathcal{H}}+\frac{2}{-\sqrt{-1}}\left[(\mathcal{A}+\mathcal{B})-(\overline{\mathcal{A}}+\overline{\mathcal{B}})\right] \quad = \quad 3\sqrt{-1}[(\mathcal{A}+\mathcal{B})-(\overline{\mathcal{A}}+\overline{\mathcal{B}})]
\end{aligned}$$

it follows that

$$\frac{I^0}{3\sqrt{-1}}=\frac{I^0}{a_1}-\frac{I^2}{a_2}+\frac{1}{3\sqrt{-1}}\left(\frac{I^3}{a_3}+\frac{I^4}{a_4}\right).$$

or

$$I^0=\frac{3\sqrt{-1}}{a_1}I^1-\frac{3\sqrt{-1}}{a_2}I^2+\frac{I^3}{a_3}+\frac{I^4}{a_4}.$$

Selecting

$$a_1 = -3\sqrt{-1}, \quad a_2 = 3\sqrt{-1}, \quad a_3 = a_4 = -1$$

we deduce

$$I^0 + I^1 + I^2 + I^3 + I^4 = 0. \tag{2.16}$$

Theorem 2.1. *The functional*

$$\begin{aligned}
\mathcal{L}^{\mathrm{M}}_\omega(\varphi', \varphi'') &= \frac{1}{V_\omega}\int_0^1\int_X \dot{\varphi}_t \omega^3_{\varphi_t} dt \\
&- \frac{1}{V_\omega}\int_0^1\int_X 3\sqrt{-1}\partial\omega \wedge (\overline{\partial}\dot{\varphi}_t \cdot \varphi_t) \wedge \omega_{\varphi_t} dt \\
&+ \frac{1}{V_\omega}\int_0^1\int_X 3\sqrt{-1}\overline{\partial}\omega \wedge (\partial\dot{\varphi}_t \cdot \varphi_t) \wedge \omega_{\varphi_t} dt \qquad (2.17)\\
&- \frac{1}{V_\omega}\int_0^1\int_X \partial\varphi_t \wedge \overline{\partial}\varphi_t \wedge \partial\omega \wedge \overline{\partial}\dot{\varphi}_t - \frac{1}{V_\omega}\int_0^1\int_X \overline{\partial}\varphi_t \wedge \partial\varphi_t \wedge \overline{\partial}\omega \wedge \partial\dot{\varphi}_t
\end{aligned}$$

is independent of the choice of the smooth path $\{\varphi_t\}_{0\le t\le 1}$ in $\mathcal{P}_\omega$, where $\varphi_0 = \varphi'$ and $\varphi_1 = \varphi''$.

Proof. It immediately follows from (2.16). □

Corollary 2.2. *For any $\varphi \in \mathcal{P}_\omega$ one has*

$$\begin{aligned}
\mathcal{L}^{\mathrm{M}}_\omega(\varphi) := \mathcal{L}^{\mathrm{M}}_\omega(0,\varphi) &= \frac{1}{4V_\omega}\sum_{i=0}^{3}\int_X \varphi\omega^i_\varphi \wedge \omega^{3-i} \\
&- \sum_{i=0}^{1}\frac{i+1}{2V_\omega}\int_X \varphi\omega^i_\varphi \wedge \omega^{1-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \qquad (2.18)\\
&+ \sum_{i=0}^{1}\frac{i+1}{2V_\omega}\int_X \varphi\omega^i_\varphi \wedge \omega^{1-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
\end{aligned}$$

Proof. In Theorem 2.1 we take $\varphi_t = t\cdot\varphi$, then the last two terms vanish and

$$\begin{aligned}
\mathcal{L}^{\mathrm{M}}_\omega(\varphi) &= \frac{1}{V_\omega}\int_0^1\int_X \varphi\omega^3_{t\varphi} dt - \frac{1}{V_\omega}\int_0^1\int_X 3\sqrt{-1}\partial\omega \wedge (\overline{\partial}\varphi \cdot t\varphi) \wedge \omega_{t\varphi} dt \\
&+ \frac{1}{V_\omega}\int_0^1\int_X 3\sqrt{-1}\overline{\partial}\omega \wedge (\partial\varphi \cdot t\varphi) \wedge \omega_{t\varphi} dt := J_0 + J_1 + J_2.
\end{aligned}$$

Three terms are computed as follows by using the identity $\omega_{t\varphi} = t\omega_\varphi + (1-t)\omega$:

$$\begin{aligned}
J_0 &= \frac{1}{V_\omega}\int_0^1\int_X \varphi[t\omega_\varphi + (1-t)\omega]^3 dt \\
&= \frac{1}{V_\omega}\int_0^1\int_X \varphi\sum_{i=0}^{3}\omega^i_\varphi \wedge \omega^{3-i}\binom{3}{i}t^i(1-t)^{3-i} dt \\
&= \frac{1}{V_\omega}\sum_{i=0}^{3}\int_X \varphi\omega^i_\varphi \wedge \omega^{3-i} \cdot \int_0^1 \binom{3}{i}t^i(1-t)^{3-i} dt = \frac{1}{4V_\omega}\sum_{i=0}^{3}\int_X \varphi\omega^i_\varphi \wedge \omega^{3-i}.
\end{aligned}$$

For J_1, we have

$$\begin{aligned} J_1 &= -\frac{1}{V_\omega}\int_0^1\int_X 3\sqrt{-1}\partial\omega\wedge(\overline{\partial}\varphi\cdot t\varphi)\wedge\omega_{t\varphi}dt \\ &= -\frac{1}{V_\omega}\int_0^1\int_X 3\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\cdot t\varphi\wedge[t\omega_\varphi+(1-t)\omega]dt \\ &= -\frac{1}{V_\omega}\int_X 3\sqrt{-1}\varphi\cdot\partial\omega\wedge\overline{\partial}\varphi\wedge\omega_\varphi\cdot\int_0^1 t^2dt \\ &- \frac{1}{V_\omega}\int_X 3\sqrt{-1}\varphi\cdot\partial\omega\wedge\overline{\partial}\varphi\wedge\omega\cdot\int_0^1 t(1-t)dt \\ &= -\frac{1}{V_\omega}\int_X \sqrt{-1}\varphi\cdot\partial\omega\wedge\overline{\partial}\varphi\wedge\omega_\varphi-\frac{1}{V_\omega}\int_X\frac{\sqrt{-1}}{2}\varphi\cdot\partial\omega\wedge\overline{\partial}\varphi\wedge\omega. \end{aligned}$$

Taking the complex conjugate gives the third term J_2. □

Remark 2.3. *(1) When (X,g) is a compact Kähler three-fold, the functional (2.1) or (2.18) coincides with the original one.*
(2) The last two terms in (2.17) may not be zero since $\overline{\partial}\varphi_t\wedge\overline{\partial}\dot{\varphi}_t$ is not identically zero in general. For instance, take $\varphi_t=t\varphi''+(1-t)\varphi'$; then

$$\begin{aligned} \overline{\partial}\varphi_t\wedge\overline{\partial}\dot{\varphi}_t &= \overline{\partial}\varphi_t\wedge\frac{d}{dt}\overline{\partial}\varphi_t \\ &= (t\overline{\partial}\varphi''+(1-t)\overline{\varphi}\varphi')\wedge(\overline{\partial}\varphi''-\overline{\partial}\varphi') \\ &= t\overline{\partial}\varphi'\wedge\overline{\partial}\varphi''+(1-t)\overline{\partial}\varphi'\wedge\overline{\partial}\varphi'' \quad = \quad \overline{\partial}\varphi'\wedge\overline{\partial}\varphi''. \end{aligned}$$

If $\varphi'=0$, then $\overline{\partial}\varphi_t\wedge\overline{\partial}\dot{\varphi}_t=0$ and hence, by taking the complex conjugate, $\partial\varphi_t\wedge\partial\dot{\varphi}_t=0$. This is a reason why in the Corollary 2.2 there are only three terms.

Let S be a non-empty set and A an additive group. A mapping $\mathcal{N}:S\times S\to A$ is said to satisfy the 1-cocycle condition if

(i) $\mathcal{N}(\sigma_1,\sigma_2)+\mathcal{N}(\sigma_2,\sigma_1)=0$;
(ii) $\mathcal{N}(\sigma_1,\sigma_2)+\mathcal{N}(\sigma_2,\sigma_3)+\mathcal{N}(\sigma_3,\sigma_1)=0$.

Corollary 2.4. *(1) The functional $\mathcal{L}^{\mathrm{M}}_\omega$ satisfies the 1-cocycle condition.*
(2) For any $\varphi\in\mathcal{P}_\omega$ and any constant $C\in\mathbb{R}$, we have

$$\mathcal{L}^{\mathrm{M}}_\omega(\varphi,\varphi+C)=C\cdot\left(1-\frac{\mathrm{Err}_\omega(\varphi)}{V_\omega}\right). \tag{2.19}$$

In particular, if $\partial\overline{\partial}\omega=\partial\omega\wedge\overline{\partial}\omega=0$, then $\mathcal{L}^{\mathrm{M}}_\omega(\varphi,\varphi+C)=C$.
(3) For any $\varphi_1,\varphi_2\in\mathcal{P}_\omega$ and any constant $C\in\mathbb{R}$, we have

$$\mathcal{L}^{\mathrm{M}}_\omega(\varphi_1,\varphi_2+C)=\mathcal{L}^{\mathrm{M}}_\omega(\varphi_1,\varphi_2)+C\cdot\left(1-\frac{\mathrm{Err}_\omega(\varphi_2)}{V_\omega}\right). \tag{2.20}$$

In particular, if $\partial\overline{\partial}\omega=\partial\omega\wedge\overline{\partial}\omega=0$, then $\mathcal{L}^{\mathrm{M}}_\omega(\varphi_1,\varphi_2+C)=\mathcal{L}^{\mathrm{M}}_\omega(\varphi_1,\varphi_2)+C$.

Proof. The proof is similar to that given in [4]. □

3. Aubin-Yau functionals on compact complex three-folds

Let (X, g) be a compact complex manifold of the complex dimension 3 and ω be its associated real $(1,1)$-form. We recall some notation in [4]. For any $\varphi \in \mathcal{P}_\omega$ we set

$$\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) = \frac{1}{V_\omega}\int_X \varphi(\omega^3 - \omega^3_\varphi), \tag{3.1}$$

$$\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) = \int_0^1 \frac{\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(s\cdot\varphi)}{s}ds = \frac{1}{V_\omega}\int_0^1\int_X \varphi(\omega^3 - \omega^3_{s\cdot\varphi})ds. \tag{3.2}$$

Two relations showed in [4] are

$$\frac{3}{4}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) = \frac{1}{V_\omega}\int_X \varphi(-\sqrt{-1}\partial\overline{\partial}\varphi)\wedge\sum_{j=1}^{2}\frac{j}{4}\omega^{2-j}\wedge\omega^j_\varphi, \tag{3.3}$$

$$4\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) = \frac{1}{V_\omega}\int_X \varphi(-\sqrt{-1}\partial\overline{\partial}\varphi)\wedge\sum_{j=0}^{1}(2-j)\omega^{2-j}\wedge\omega^j_\varphi. \tag{3.4}$$

According to the expression of $\mathcal{L}^{\mathrm{M}}_\omega(\varphi)$, we set

$$\mathcal{A}_\omega(\varphi) := \sum_{i=0}^{1}\frac{i+1}{2V_\omega}\int_X \varphi\omega^i_\varphi\wedge\omega^{1-i}\wedge -\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi, \tag{3.5}$$

$$\mathcal{B}_\omega(\varphi) := \sum_{i=0}^{1}\frac{i+1}{2V_\omega}\int_X \varphi\omega^i_\varphi\wedge\omega^{1-i}\wedge\sqrt{-1}\partial\omega\wedge\partial\varphi. \tag{3.6}$$

Using (3.3) we obtain

$$\begin{aligned}
&\frac{3}{4}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)\\
= &\ \frac{1}{V_\omega}\int_X\left(\varphi\sum_{j=1}^{2}\frac{j}{4}\omega^{2-j}\wedge\omega^j_\varphi\right)\wedge(-\sqrt{-1}\partial\overline{\partial}\varphi)\\
= &\ \frac{1}{V_\omega}\int_X\sqrt{-1}\partial\left(\varphi\sum_{j=1}^{2}\frac{j}{4}\omega^{2-j}\wedge\omega^j_\varphi\right)\wedge\overline{\partial}\varphi\\
= &\ \frac{1}{V_\omega}\int_X\sqrt{-1}\left(\partial\varphi\wedge\sum_{j=1}^{2}\frac{j}{4}\omega^{2-j}\wedge\omega^j_\varphi\right)\wedge\overline{\partial}\varphi\\
+ &\ \frac{1}{V_\omega}\int_X\sqrt{-1}\varphi\sum_{j=1}^{2}\frac{j}{4}[(2-j)\omega^{1-j}\wedge\partial\omega\wedge\omega^j_\varphi + \omega^{2-j}\wedge j\omega^{j-1}_\varphi\wedge\partial\omega]\wedge\overline{\partial}\varphi;
\end{aligned}$$

setting $i=j-1$ in the third term gives

$$
\begin{aligned}
&\frac{3}{4}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)-\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)\\
=\;&\frac{1}{V_\omega}\sum_{j=1}^{2}\frac{j}{4}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{2-j}\wedge\omega_\varphi^{j}\\
+\;&\frac{1}{V_\omega}\sum_{j=1}^{2}\frac{j(2-j)}{4}\int_X\sqrt{-1}\varphi\omega^{1-j}\wedge\partial\omega\wedge\omega_\varphi^{j}\wedge\overline{\partial}\varphi\\
+\;&\frac{1}{V_\omega}\sum_{j=1}^{2}\frac{j^2}{4}\int_X\sqrt{-1}\varphi\omega^{2-j}\wedge\omega_\varphi^{j-1}\wedge\partial\omega\wedge\overline{\partial}\varphi\\
=\;&\frac{1}{V_\omega}\sum_{i=1}^{2}\frac{i}{4}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{2-i}\wedge\omega_\varphi^{i}+\frac{1}{4V_\omega}\int_X\varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
+\;&\frac{1}{V_\omega}\sum_{i=0}^{1}\frac{(i+1)^2}{4}\int_X\varphi\omega_\varphi^{i}\wedge\omega^{1-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
=\;&\frac{1}{V_\omega}\sum_{i=1}^{2}\frac{i}{4}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^{i}\wedge\omega^{2-i}\\
+\;&\frac{5}{4V_\omega}\int_X\varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi+\frac{1}{4V_\omega}\int_X\varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi.
\end{aligned}
$$

To simplify the notation, we set

$$
\mathcal{C}_\omega(\varphi):=\frac{3}{4V_\omega}\int_X\varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi. \tag{3.7}
$$

Therefore

$$
\begin{aligned}
&\frac{3}{4}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)-\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)\\
=\;&\frac{1}{V_\omega}\sum_{i=1}^{2}\frac{i}{4}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^{i}\wedge\omega^{2-i}-\frac{1}{2}\mathcal{A}_\omega(\varphi)+\mathcal{C}_\omega(\varphi).
\end{aligned} \tag{3.8}
$$

On the other hand, using the slightly different method, we obtain (see A.1)

$$
\begin{aligned}
&\frac{3}{4}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)-\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)\\
=\;&\frac{1}{V_\omega}\sum_{i=1}^{2}\frac{i}{4}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^{i}\wedge\omega^{2-i}-\frac{1}{2}\mathcal{B}_\omega(\varphi)+\mathcal{D}_\omega(\varphi)
\end{aligned} \tag{3.9}
$$

where

$$
\mathcal{D}_\omega(\varphi)=\frac{3}{4V_\omega}\int_X\varphi\omega_\varphi\wedge-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi. \tag{3.10}
$$

Equations (3.8) and (3.9) implies

$$
\begin{aligned}
\frac{3}{4}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)-\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)\;&=\;\frac{1}{V_\omega}\sum_{i=1}^{2}\frac{i}{4}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^{i}\wedge\omega^{2-i}\\
&-\;\frac{\mathcal{A}_\omega(\varphi)+\mathcal{B}_\omega(\varphi)}{4}+\frac{\mathcal{C}_\omega(\varphi)+\mathcal{D}_\omega(\varphi)}{2}.
\end{aligned} \tag{3.11}
$$

By the definition we have

$$\begin{aligned}\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\int_0^1\int_X(\varphi\omega^3-\varphi\omega^3_{s\varphi})ds = \frac{1}{V_\omega}\int_X\varphi\omega^3-\frac{1}{V_\omega}\int_0^1\int_X\varphi\omega^3_{t\varphi}dt\\ &= \frac{1}{V_\omega}\int_X\varphi\omega^3-(\mathcal{L}^{\mathrm{M}}_\omega(\varphi)-\mathcal{A}_\omega(\varphi)-\mathcal{B}_\omega(\varphi))\\ &= \frac{1}{V_\omega}\int_X\varphi\omega^3-\mathcal{L}^{\mathrm{M}}_\omega(\varphi)+\mathcal{A}_\omega(\varphi)+\mathcal{B}_\omega(\varphi).\end{aligned}$$

If we define

$$\mathcal{E}_\omega(\varphi) = \frac{9}{V_\omega}\int_X\varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi, \tag{3.12}$$

$$\mathcal{A}^1_\omega(\varphi) = \frac{1}{2V_\omega}\int_X\varphi\omega\wedge-\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \tag{3.13}$$

$$\mathcal{A}^2_\omega(\varphi) = \frac{1}{V_\omega}\int_X\varphi\omega_\varphi\wedge-\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi, \tag{3.14}$$

then $\mathcal{A}^1_\omega(\varphi)+\mathcal{A}^2_\omega(\varphi)=\mathcal{A}_\omega(\varphi)$ and it follows that (see A.1)

$$\begin{aligned}4\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}-\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=0}^{1}(2-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{2-i}\\ &+ \mathcal{E}_\omega(\varphi)+8\mathcal{A}^1_\omega(\varphi)-\mathcal{A}^2_\omega(\varphi).\end{aligned} \tag{3.15}$$

Introduce

$$\mathcal{F}_\omega(\varphi) = \frac{9}{V_\omega}\int_X\varphi\omega\wedge-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi, \tag{3.16}$$

$$\mathcal{B}^1_\omega(\varphi) = \frac{1}{2V_\omega}\int_X\varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi \tag{3.17}$$

$$\mathcal{B}^2_\omega(\varphi) = \frac{1}{V_\omega}\int_X\varphi\omega_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi. \tag{3.18}$$

Then $\mathcal{B}^1_\omega(\varphi)+\mathcal{B}^2_\omega(\varphi)=\mathcal{B}_\omega(\varphi)$ and hence (see A.3)

$$\begin{aligned}4\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)-\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=0}^{1}(2-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{2-i}\\ &+ \mathcal{F}_\omega(\varphi)+8\mathcal{B}^1_\omega(\varphi)-\mathcal{B}^2_\omega(\varphi).\end{aligned} \tag{3.19}$$

(3.15) and (3.19) together gives

$$\begin{aligned}4\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)-\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=0}^{1}(2-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{n-1-i}\\ &+ \frac{\mathcal{E}_\omega(\varphi)+\mathcal{F}_\omega(\varphi)}{2}+4[\mathcal{A}^1_\omega(\varphi)+\mathcal{B}^1_\omega(\varphi)]\\ &- \frac{\mathcal{A}^2_\omega(\varphi)+\mathcal{B}^2_\omega(\varphi)}{2}.\end{aligned} \tag{3.20}$$

Now, we define Aubin-Yau functionals over any compact complex manifolds as follows:

$$
\begin{aligned}
\mathcal{I}^{\rm AY}_{\omega}(\varphi) &:= \mathcal{I}^{\rm AY}_{\omega|\bullet}(\varphi)\\
&+ a^1_1\mathcal{A}^1_{\omega}(\varphi)+a^2_1\mathcal{A}^2_{\omega}(\varphi)+b^1_1\mathcal{B}^1_{\omega}(\varphi)+b^2_1\mathcal{B}^2_{\omega}(\varphi)\\
&+ c_1\mathcal{C}_{\omega}(\varphi)+d_1\mathcal{D}_{\omega}(\varphi)+e_1\mathcal{E}_{\omega}(\varphi)+f_1\mathcal{F}_{\omega}(\varphi), \qquad (3.21)\\
\mathcal{J}^{\rm AY}_{\omega}(\varphi) &:= -\mathcal{L}^{\rm M}_{\omega}(\varphi)+\frac{1}{V_{\omega}}\int_X \varphi\omega^3\\
&+ a^1_2\mathcal{A}^1_{\omega}(\varphi)+a^2_2\mathcal{A}^2_{\omega}(\varphi)+b^1_2\mathcal{B}^1_{\omega}(\varphi)+b^2_2\mathcal{B}^2_{\omega}(\varphi)\\
&+ c_2\mathcal{C}_{\omega}(\varphi)+d_2\mathcal{D}_{\omega}(\varphi)+e_2\mathcal{E}_{\omega}(\varphi)+f_2\mathcal{F}_{\omega}(\varphi),\\
&= \mathcal{J}^{\rm AY}_{\omega|\bullet}(\varphi)\\
&+ (a^1_2-1)\mathcal{A}^1_{\omega}(\varphi)+(a^2_2-1)\mathcal{A}^2_{\omega}(\varphi)+(b^1_2-1)\mathcal{B}^1_{\omega}(\varphi)+(b^2_2-1)\mathcal{B}^2_{\omega}(\varphi)\\
&+ c_2\mathcal{C}_{\omega}(\varphi)+d_2\mathcal{D}_{\omega}(\varphi)+e_2\mathcal{E}_{\omega}(\varphi)+f_2\mathcal{F}_{\omega}(\varphi). \qquad (3.22)
\end{aligned}
$$

Plugging (3.21) and (3.22) into (3.20) and (3.11), we obtain

$$
\frac{3}{4}\mathcal{I}^{\rm AY}_{\omega}(\varphi)-\mathcal{J}^{\rm AY}_{\omega}(\varphi)=\frac{1}{V_{\omega}}\sum^{2}_{i=1}\frac{i}{4}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_{\varphi}\wedge\omega^{2-i}\geq 0 \qquad (3.23)
$$

and

$$
4\mathcal{J}^{\rm AY}_{\omega}(\varphi)-\mathcal{I}^{\rm AY}_{\omega}(\varphi)=\sum^{2}_{i=0}\frac{2-i}{V_{\omega}}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_{\varphi}\wedge\omega^{2-i}\geq 0 \qquad (3.24)
$$

where we require that constants satisfy the following linear equations system

$$
\begin{aligned}
\frac{3}{4}a^1_1-(a^1_2-1) &= \frac{1}{4}, \quad \frac{3}{4}a^2_1-(a^2_2-1) = \frac{1}{4}, \qquad (3.25)\\
\frac{3}{4}b^1_1-(b^1_2-1) &= \frac{1}{4}, \quad \frac{3}{4}b^2_1-(b^2_2-1) = \frac{1}{4}, \qquad (3.26)\\
\frac{3}{4}c_1-c_2 &= -\frac{1}{2}, \quad \frac{3}{4}d_1-d_2 = -\frac{1}{2}, \qquad (3.27)\\
\frac{3}{4}e_1-e_2 &= 0, \quad \frac{3}{4}f_1-f_2 = 0, \qquad (3.28)\\
4(a^1_2-1)-a^1_1 &= -4, \quad 4(a^2_2-1)-a^2_1 = \frac{1}{2}, \qquad (3.29)\\
4(b^1_2-1)-b^1_1 &= -4, \quad 4(b^2_2-1)-b^2_1 = \frac{1}{2}, \qquad (3.30)\\
4c_2-c_1 &= 0, \quad 4d_2-d_1 = 0, \qquad (3.31)\\
4e_2-e_1 &= -\frac{1}{2}, \quad 4f_2-f_1 = -\frac{1}{2}. \qquad (3.32)
\end{aligned}
$$

The constants $a_i^j, b_i^j, c_i, d_i, e_i$ and f_i , calculated in Appendix B, are

$$a_1^1 = b_1^1 = -\frac{3}{2}, \quad a_2^1 = b_2^1 = -\frac{3}{8}, \tag{3.33}$$

$$a_1^2 = b_1^2 = \frac{3}{4}, \quad a_2^2 = b_2^2 = \frac{3}{4}\left(1+\frac{3}{4}\right) = \frac{21}{16}, \tag{3.34}$$

$$c_1 = d_1 = -1, \quad e_2 = f_2 = -\frac{3}{16}, \tag{3.35}$$

$$c_2 = d_2 = e_1 = f_1 = -\frac{1}{4}. \tag{3.36}$$

The explicit formulas for $\mathcal{I}_\omega^{\rm AY}(\varphi)$ and $\mathcal{J}_\omega^{\rm AY}(\varphi)$ are given in Proposition C.1 and C.2 respectively. Namely,

$$\begin{aligned} \mathcal{I}_\omega^{\rm AY}(\varphi) &= \frac{1}{V_\omega}\int_X \varphi(\omega^3-\omega_\varphi^3) \\ &- \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi - \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\ &+ \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi, \end{aligned} \tag{3.37}$$

$$\begin{aligned} \mathcal{J}_\omega^{\rm AY}(\varphi) &= -\mathcal{L}_\omega^{\rm M}(\varphi) + \frac{1}{V_\omega}\int_X \varphi\omega^3 \\ &- \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi - \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\ &+ \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi. \end{aligned} \tag{3.38}$$

From (3.23), (3.24), (3.31) and (3.38), we deduce the following

Theorem 3.1. *For any $\varphi\in\mathcal{P}_\omega$, one has*

$$\frac{3}{4}\mathcal{I}_\omega^{\rm AY}(\varphi) - \mathcal{J}_\omega^{\rm AY}(\varphi) \geq 0, \tag{3.39}$$

$$4\mathcal{J}_\omega^{\rm AY}(\varphi) - \mathcal{I}_\omega^{\rm AY}(\varphi) \geq 0. \tag{3.40}$$

In particular

$$\begin{aligned} \frac{1}{4}\mathcal{I}_\omega^{\rm AY}(\varphi) &\leq \mathcal{J}_\omega^{\rm AY}(\varphi) \leq \frac{3}{4}\mathcal{I}_\omega^{\rm AY}(\varphi), \\ \frac{3}{4}\mathcal{J}_\omega^{\rm AY}(\varphi) &\leq \mathcal{I}_\omega^{\rm AY}(\varphi) \leq 4\mathcal{J}_\omega^{\rm AY}(\varphi), \\ \frac{1}{3}\mathcal{J}_\omega^{\rm AY}(\varphi) \leq \frac{1}{4}\mathcal{J}_\omega^{\rm AY}(\varphi) &\leq \mathcal{I}_\omega^{\rm AY}(\varphi) - \mathcal{J}_\omega^{\rm AY}(\varphi) \\ &\leq \frac{3}{4}\mathcal{I}_\omega^{\rm AY}(\varphi) \leq n\mathcal{J}_\omega^{\rm AY}(\varphi). \end{aligned}$$

4. Volume estimates

Let (X,g) be a compact Hermitian manifold of the complex dimension n and ω be its associated real $(1,1)$-form. Define

$$\mathcal{P}_\omega := \{\varphi\in C^\infty(X)_{\mathbb{R}} | \omega_\varphi := \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0\}, \tag{4.1}$$

and $\mathcal{P}^0_\omega := \left\{\varphi\in\mathcal{P}_\omega \middle| \sup_X \varphi = 0\right\}$. Consider the quantity

$$\mathrm{InfErr}_\omega := \inf_{\varphi\in\mathcal{P}^0_\omega} \mathrm{Err}_\omega(\varphi), \tag{4.2}$$

where

$$\mathrm{Err}_\omega(\varphi) := \int_X \omega^n - \int_X \omega^n_\varphi. \tag{4.3}$$

It's clear that $\mathrm{InfErr}_\omega \le 0$. But we don't know whether the quantity InfErr_ω is finite. For $n=2$, V. Tosatti and B. Weinkove [9] showed that InfErr_ω is always bounded from below, using the existence of Ganduchon metrics on any compact Hermitian manifolds. If ω satisfies the condition (see [3], [9])

$$\partial\overline{\partial}(\omega^k) = 0, \quad k = 1, 2, \tag{4.4}$$

one can show that the quantity $\mathrm{InfErr}_\omega = 0$ (see [1], [4], or [9]).

When $n=2$, the condition (4.4) reduces to

$$\partial\overline{\partial}\omega = 0, \tag{4.5}$$

which is a Ganduchon metric; however, if $n=3$, we can show that InfErr_ω is bounded from below under this condition.

Theorem 4.1. *Suppose that (X, g) is a compact Hermitian manifold of the complex dimension 3 and ω is its associated real $(1,1)$-form. If $\partial\overline{\partial}\omega = 0$, then InfErr_ω is bounded from below. More precisely, we have*

$$\mathrm{InfErr}_\omega \ge 3\left(1 - e^{2\cdot \mathrm{osc}(u)}\right)\cdot\int_X \omega^3. \tag{4.6}$$

Here u is a real-valued smooth function on X such that $\omega_G = e^u\cdot\omega$ is a Gauduchon metric, i.e., $\partial\overline{\partial}(\omega_G^2) = 0$.

Proof. As in [9], page 21, we compute

$$\begin{aligned}\int_X \omega^3_\varphi &= \int_X (\omega^3 + 3\omega^2\wedge\sqrt{-1}\partial\overline{\partial}\varphi + 3\omega\wedge(\sqrt{-1}\partial\overline{\partial}\varphi)^2) \\ &= \int_X (-2\omega^3 + 3\omega^2\wedge(\omega+\sqrt{-1}\partial\overline{\partial}\varphi) + 3\omega\wedge(\sqrt{-1}\partial\overline{\partial}\varphi)^2).\end{aligned}$$

Since $\partial\overline{\partial}\omega = 0$, the last integral vanishes, and hence

$$\begin{aligned}\int_X \omega^3_\varphi &\le \int_X -2\omega^3 + \int_X 3\left(e^{u-\inf_X(u)}\omega\right)^2\wedge(\omega+\sqrt{-1}\partial\overline{\partial}\varphi) \\ &= \int_X -2\omega^3 + 3e^{2(\sup_X(u)-\inf_X(u))}\int_X \omega^3 \\ &+ \int_X 3e^{-2\inf_X(u)}\cdot\omega_G^2\wedge\sqrt{-1}\partial\overline{\partial}\varphi \\ &= \int_X -2\omega^3 + 3e^{2\cdot\mathrm{osc}(u)}\int_X\omega^3 = \left(3e^{2\cdot\mathrm{osc}(u)} - 2\right)\int_X \omega^3.\end{aligned}$$

From the definition of InfErr_ω, we immediately obtain

$$\mathrm{InfErr}_\omega \ge \int_X \omega^3 - \left(3e^{2\cdot\mathrm{osc}(u)} - 2\right)\int_X\omega^3 = 3\left(1 - e^{2\cdot\mathrm{osc}(u)}\right)\cdot\int_X\omega^3$$

where $\mathrm{osc}(u) := \sup_X(u) - \inf_X(u)$. □

Theorem 4.2. *Suppose that (X, g) is a compact Hermitian manifold of the complex dimension 3 and ω is its associated real $(1,1)$-form. We select a real-valued smooth function u on X so that $e^u \cdot \omega$ is a Gauduchon metric. If*

$$\operatorname{osc}(u) = \sup_X(u) - \inf_X(u) \leq \frac{1}{2} \cdot \ln\frac{3}{2}, \quad \partial\overline{\partial}\omega = 0,$$

then

$$\inf_{\varphi \in \mathcal{P}^0_\omega} \int_X \omega^3_\varphi \geq \int_X \omega^3 > 0. \tag{4.7}$$

Proof. Using the similar procedure, we deduce

$$\int_X \omega^3_\varphi \geq \left(3 \cdot e^{2(\inf_X(u) - \sup_X(u))} - 2\right) \int_X \omega^3.$$

Since $\sup_X u - \inf_X u \leq \frac{1}{2} \cdot \ln\frac{3}{2}$, it follows that $3 \cdot e^{2(\inf_X u - \sup_X u)} - 2 \geq 1$. □

Appendix A. Proof the identities (3.8), (3.15) and (3.19)

In Appendix A we verify the identities (3.8), (3.15) and (3.19).

$$\begin{aligned}
&\frac{3}{4}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \\
= \;& \frac{1}{V_\omega}\int_X -\sqrt{-1}\partial\left(\varphi\sum_{j=1}^{2}\frac{j}{4}\omega^{2-j}\wedge\omega^j_\varphi\right)\wedge\partial\varphi \\
= \;& \frac{1}{V_\omega}\int_X -\sqrt{-1}\left(\overline{\partial}\varphi\wedge\sum_{j=1}^{2}\frac{j}{4}\omega^{2-j}\wedge\omega^j_\varphi\right)\wedge\partial\varphi \\
+ \;& \frac{1}{V_\omega}\int_X -\sqrt{-1}\varphi\sum_{j=1}^{2}\frac{j}{4}[(2-j)\omega^{1-j}\wedge\overline{\partial}\omega\wedge\omega^j_\varphi + \omega^{2-j}\wedge j\omega^{j-1}_\varphi\wedge\overline{\partial}\omega]\wedge\partial\varphi \\
= \;& \frac{1}{V_\omega}\sum_{i=1}^{2}\frac{i}{4}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{2-i} \\
+ \;& \frac{1}{V_\omega}\sum_{j=1}^{2}\frac{j(2-j)}{4}\int_X -\sqrt{-1}\varphi\omega^{1-j}\wedge\omega^j_\varphi\wedge\overline{\partial}\omega\wedge\partial\varphi \\
+ \;& \frac{1}{V_\omega}\sum_{j=1}^{2}\frac{j^2}{4}\int_X -\sqrt{-1}\varphi\omega^{2-j}\wedge\omega^{j-1}_\varphi\wedge\overline{\partial}\omega\wedge\partial\varphi \\
= \;& \frac{1}{V_\omega}\sum_{i=1}^{2}\frac{i}{4}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{2-i}\wedge\omega^i_\varphi + \frac{1}{4V_\omega}\int_X -\sqrt{-1}\varphi\omega_\varphi\wedge\overline{\partial}\omega\wedge\partial\varphi \\
+ \;& \frac{1}{V_\omega}\sum_{i=0}^{1}\frac{(i+1)^2}{4}\int_X -\sqrt{-1}\varphi\omega^{1-i}\wedge\omega^i_\varphi\wedge\overline{\partial}\omega\wedge\partial\varphi \\
= \;& \frac{1}{V_\omega}\sum_{i=1}^{2}\frac{i}{4}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{2-i}\wedge\omega^i_\varphi - \frac{1}{2}\mathcal{B}_\omega(\varphi) + \mathcal{D}_\omega(\varphi)
\end{aligned} \tag{A.1}$$

which gives (3.9). Calculate

$$
\begin{aligned}
& 4\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet} - \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \\
= & \frac{1}{V_\omega}\int_X \sqrt{-1}\partial\left(\varphi\sum_{j=0}^{2}(2-j)\omega^{2-j}\wedge\omega_\varphi^j\right)\wedge\overline{\partial}\varphi \\
= & \frac{1}{V_\omega}\int_X \sqrt{-1}\partial\varphi\wedge\sum_{j=0}^{2}(2-j)\omega^{2-j}\wedge\omega_\varphi^j\wedge\overline{\partial}\varphi \\
+ & \frac{1}{V_\omega}\int_X \sqrt{-1}\varphi\sum_{j=0}^{2}[(2-j)^2\omega^{1-j}\wedge\partial\omega\wedge\omega_\varphi^j + (2-j)j\omega^{2-j}\wedge\omega_\varphi^{j-1}\wedge\partial\omega]\wedge\overline{\partial}\varphi \\
= & \frac{1}{V_\omega}\sum_{i=0}^{2}(2-i)\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^i\wedge\omega^{2-i} \\
+ & \frac{1}{V_\omega}\sum_{j=0}^{1}(2-j)^2\int_X \varphi\omega^{1-j}\wedge\omega_\varphi^j\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{1}{V_\omega}\int_X\varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\
= & \frac{1}{V_\omega}\sum_{i=0}^{2}(2-i)\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^i\wedge\omega^{2-i} \\
+ & \frac{5}{V_\omega}\int_X\varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{1}{V_\omega}\int_X\varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi.
\end{aligned}
$$

Using the definitions of $\mathcal{E}_\omega(\varphi), \mathcal{A}^1_\omega(\varphi)$, $\mathcal{A}^2_\omega(\varphi)$, we have $\mathcal{A}^1_\omega(\varphi)+\mathcal{A}^2_\omega(\varphi) = \mathcal{A}_\omega(\varphi)$ and hence (3.15) holds. Similarly, we have $\mathcal{B}^1_\omega(\varphi)+\mathcal{B}^2_\omega(\varphi)=\mathcal{B}_\omega(\varphi)$ and

$$
\begin{aligned}
& 4\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \\
= & \frac{1}{V_\omega}\int_X -\sqrt{-1}\overline{\partial}\left(\varphi\sum_{j=0}^{2}(2-j)\omega^{2-j}\wedge\omega_\varphi^j\right)\wedge\partial\varphi \\
= & \frac{1}{V_\omega}\int_X -\sqrt{-1}\overline{\partial}\varphi\wedge\sum_{j=0}^{2}(2-j)\omega^{2-j}\wedge\omega_\varphi^j\wedge\partial\varphi \\
+ & \frac{1}{V_\omega}\int_X -\sqrt{-1}\varphi\sum_{j=0}^{2}(2-j)[(2-j)\omega^{1-j}\wedge\overline{\partial}\omega\wedge\omega_\varphi^j + j\omega^{2-j}\wedge\omega_\varphi^{j-1}\wedge\overline{\partial}\omega]\wedge\partial\varphi \\
= & \frac{1}{V_\omega}\sum_{i=0}^{2}(2-i)\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{2-i}\wedge\omega_\varphi^i \\
+ & \frac{1}{V_\omega}\sum_{i=0}^{1}(2-i)^2\int_X \varphi\omega^{1-i}\wedge\omega_\varphi^i\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) + \frac{1}{V_\omega}\int_X\varphi\omega\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) \\
= & \frac{1}{V_\omega}\sum_{i=0}^{2}(2-i)\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{2-i}\wedge\omega_\varphi^i \\
+ & \frac{5}{V_\omega}\int_X\varphi\omega\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) + \frac{1}{V_\omega}\int_X\varphi\omega_\varphi\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi).
\end{aligned}
$$

and hence

$$\begin{aligned} 4\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=0}^{2}(2-i)\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{2-i} \\ &+ \mathcal{F}_\omega(\varphi) + 8\mathcal{B}^1_\omega(\varphi) - \mathcal{B}^2_\omega(\varphi). \end{aligned} \tag{A.2}$$

Therefore (3.15) and (3.19) together gives

$$\begin{aligned} 4\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=0}^{2}(2-i)\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{2-i} \\ &+ \frac{\mathcal{E}_\omega(\varphi)+\mathcal{F}_\omega(\varphi)}{2} + 4(\mathcal{A}^1_\omega(\varphi)+\mathcal{B}^1_\omega(\varphi)) - \frac{\mathcal{A}^2_\omega(\varphi)+\mathcal{B}^2_\omega(\varphi)}{2}. \end{aligned} \tag{A.3}$$

Appendix B. Solve the linear equations system

In this section we try to solve the linear equations system (3.25)-(3.32). Firstly we solve (3.25) and (3.29) as follows: (3.25) and (3.29) gives us the following equations

$$\frac{3}{4}a^1_1 - \frac{1}{4} = a^1_2 - 1, \quad 4(a^1_2-1)+4 = a^1_1, \tag{B.1}$$

$$\frac{3}{4}a^2_1 - \frac{1}{4} = a^2_2 - 1, \quad 4(a^2_2-1)-\frac{1}{2} = a^2_1. \tag{B.2}$$

Plugging the first equation into second equation in (B.1), we have

$$4\left(\frac{3}{4}a^1_1 - \frac{1}{4}\right) + 4 = a^1_1$$

which implies

$$a^1_1 = -\frac{3}{2}, \quad a^1_2 = -\frac{3}{8}. \tag{B.3}$$

Similarly,

$$4\left(\frac{3}{4}a^2_1 - \frac{1}{4}\right) - \frac{1}{2} = a^2_1,$$

therefore

$$a^2_1 = \frac{3}{4}, \quad a^2_2 = \frac{3}{4}\left(1+\frac{3}{4}\right) = \frac{21}{16}. \tag{B.4}$$

Secondly, (3.26) and (3.30) implies

$$\frac{3}{4}b^1_1 - \frac{1}{4} = b^1_2 - 1, \quad 4(b^1_2-1) = b^1_1 - 4, \tag{B.5}$$

$$\frac{3}{4}b^2_1 - \frac{1}{4} = b^2_2 - 1, \quad 4(b^2_2-1) = b^2_1 + \frac{1}{2}. \tag{B.6}$$

The above linear equations system gives

$$4\left(\frac{3}{4}b^1_1 - \frac{1}{4}\right) = b^1_1 - 4, \quad 4\left(\frac{3}{4}b^2_1 - \frac{1}{4}\right) = b^2_1 + \frac{1}{2},$$

respectively. Hence

$$b^1_1 = -\frac{3}{2}, \quad b^1_2 = -\frac{3}{8}, \tag{B.7}$$

$$b^2_1 = \frac{3}{4}, \quad b^2_2 = \frac{3}{4}\left(1+\frac{3}{4}\right) = \frac{21}{16}. \tag{B.8}$$

Continuously, equations (3.27) and (3.31) shows that

$$\begin{aligned} \frac{3}{4}c_1 - c_2 &= -\frac{1}{2}, \quad 4c_2 - c_1 = 0, \\ \frac{3}{4}d_1 - d_2 &= -\frac{1}{2}, \quad 4d_2 - d_1 = 0. \end{aligned}$$

Eliminating c_2 and d_2 respectively, we have

$$4\left(\frac{3}{4}c_1 + \frac{1}{2}\right) - c_1 = 0, \quad 4\left(\frac{3}{4}d_1 + \frac{1}{2}\right) - d_1 = 0.$$

Thus

$$c_1 = -1, \quad c_2 = -\frac{1}{4} \tag{B.9}$$

$$d_1 = -1, \quad d_2 = -\frac{1}{4}. \tag{B.10}$$

Similarly, from (3.28) and (3.32) we obtain

$$\begin{aligned} \frac{3}{4}e_1 - e_2 = 0, \quad 4e_2 - e_1 &= -\frac{1}{2}, \\ \frac{3}{4}f_1 - f_2 = 0, \quad 4f_2 - f_1 &= -\frac{1}{2}, \end{aligned}$$

and hence

$$e_1 = f_1 = -\frac{1}{4}, \tag{B.11}$$

$$e_2 = f_2 = -\frac{3}{16}. \tag{B.12}$$

Appendix C. Explicit formulas of $\mathcal{I}^{\rm AY}_\omega(\varphi)$ and $\mathcal{J}^{\rm AY}_\omega(\varphi)$

In this section we give the explicit formulas of $\mathcal{I}^{\rm AY}_\omega(\varphi)$ and $\mathcal{J}^{\rm AY}_\omega(\varphi)$. Using the constants determined in Appendix B, we have

$$\begin{aligned} \mathcal{I}^{\rm AY}_\omega(\varphi) &= \frac{1}{V_\omega}\int_X \varphi(\omega^3 - \omega^3_\varphi) \\ &+ \frac{3}{4V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi - \frac{3}{4V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\ &- \frac{3}{4V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi + \frac{3}{4V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi \\ &- \frac{3}{4V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{3}{4V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi \\ &- \frac{9}{4V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{9}{4V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi \\ &= \frac{1}{V_\omega}\int_X \varphi(\omega^3 - \omega^3_\varphi) \\ &- \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi \\ &- \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi. \end{aligned}$$

Thus

Proposition C.1. *One has*

$$\begin{aligned}\mathcal{I}^{\mathrm{AY}}_{\omega}(\varphi) &= \frac{1}{V_\omega}\int_X \varphi(\omega^3-\omega^3_\varphi)\\ &- \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi - \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\ &+ \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) &= -\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi) + \frac{1}{V_\omega}\int_X \varphi\omega^3\\ &+ \frac{3}{8}\frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi - \frac{3}{4}\left(2+\frac{3}{2}\right)\frac{1}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\ &- \frac{3}{8}\frac{1}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi + \frac{3}{4}\left(2+\frac{3}{2}\right)\frac{1}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\ &- \frac{3}{8}\frac{1}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{3}{8}\frac{1}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\ &- \frac{27}{8}\frac{1}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{27}{8}\frac{1}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\ &= -\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi) + \frac{1}{V_\omega}\int_X \varphi\omega^3\\ &- \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\ &- \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\end{aligned}$$

So

Proposition C.2. *One has*

$$\begin{aligned}\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) &= -\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi) + \frac{1}{V_\omega}\int_X \varphi\omega^3\\ &- \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi - \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\ &+ \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.\end{aligned}$$

References

1. Dinew, S., Kolodziej, S., *Pluripotential estimates on comapct Hermitian manifolds*, preprint, arXiv: 0910.3937, 2009.
2. Gauduchon, P., *Sur la 1-forme de torsion d'une variètè hermitienne compacte*, Math. Ann. **267** (1984), 495–518.
3. Guan, B., Li, Q., *Complex Monge-Ampère equations on Hermitian manifold*, preprint, arXiv: 0906.3548, 2009.
4. Li, Y., *Mabuchi and Aubin-Yau functionals over complex surfaces*, preprint, arXiv: math.DG/1002.3411, 2010.
5. Li, Y., *Mabuchi and Aubin-Yau functionals over complex manifolds*, in preparation.
6. Li, Y., Tosatti, V., *Private communications*, March, 2010.
7. Mabuchi, T., *K-energy maps integrating Futaki invatiants*, Tohoku Math. Journ., **38**(1986), 575–593.

8. Phong, D.H., Sturm, J., *Lectures on stability and constant scalar curvature*, preprint, arXiv: 0801.4179, 2008.
9. Tosatti, V., Weinkove, B., *Estimates for the complex Monge-Ampère equation on Hermitian and balanced manifolds*, preprint, arXiv: 0909.4496v1, 2009.
10. Tosatti, V., Weinkove, B., *The complex Monge-ampère equation on compact Hermitian manifolds*, preprint, arXiv: 0910.1390, 2009.
11. Yau, S.-T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure. Appl. Math., **31** (1978), no.3, 339–411.
12. Yau, S.-T., *Review of geometry and analysis*, Kodaira's issue. Asian J. Math. **4**(2000), no.1, 235–278

Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: `yili@math.harvard.edu`