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COHOMOLOGIES AND DERIVED BRACKETS OF LEIBNIZ ALGEBRAS

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Abstract

In this thesis, we work on the structure of Leibniz algebras and develop cohomology theories for them. The motivation comes from:

- Roytenberg, Stienon-Xu and Ginot-Grutzmann's work on standard and naive cohomology of Courant algebroids (Courant-Dorfman algebras).
- Kosmann-Schwarzbach, Roytenberg and Alekseev-Xu's constructions of derived brackets for Courant algebroids.
- The classical equivariant cohomology theory and the generalized geometry theory.

This thesis consists of three parts:

1. We introduce standard cohomology and naive cohomology for a Leibniz algebra. We discuss their properties and show that they are isomorphic. By similar methods, we prove a generalization of Ginot-Grutzmann's theorem on transitive Courant algebroids, which was conjectured by Stienon-Xu. The relation between standard complexes of a Leibniz algebra and its corresponding crossed product is also discussed.
2. We observe a canonical 3-cochain in the standard complex of a Leibniz algebra. We construct a bracket on the subspace consisting of so-called representable cochains, and prove that the subspace becomes a graded Poisson algebra. Finally we show that for a fat Leibniz algebra, the Leibniz bracket can be represented as a derived bracket.
3. Inspired by the notion of a Lie algebra action and the idea of generalized geometry, we introduce the notion of a generalized action of a Lie algebra $\mathfrak{g}$ on a smooth manifold M, to be a homomorphism of Leibniz algebras from $\mathfrak{g}$ to the generalized tangent bundle $TM \oplus T^*M$. We define the interior product and Lie derivative so that the standard complex of $TM \oplus T^*M$ becomes a $\mathfrak{g}$

differential algebra, then we discuss its equivariant cohomology. We also study the equivariant cohomology for a subcomplex of a Leibniz complex.

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Contents

Chapter 1

Introduction

A (left) Leibniz algebra L is a vector space over a field $\mathfrak{k}$ ($\mathfrak{k} = \mathbb{R}$ or $\mathbb{C}$) equipped with a bracket $\circ : L \otimes L \to L$, called the Leibniz bracket, satisfying the (left) Leibniz identity:

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z), \qquad \forall x, y, z \in L.$$

A concrete example is the omni Lie algebra $ol(V) \triangleq gl(V) \oplus V$, where V is a vector space. It is first introduced by Weinstein [74]. The Leibniz bracket of $ol(V)$ is given by:

$$(A + v) \circ (B + w) = [A, B] + Aw,$$

for any $A, B \in gl(V),\ v, w \in V$.

Any Lie algebra is a Leibniz algebra. Conversely, any Leibniz algebra whose Leibniz bracket is skew-symmetric is a Lie algebra. So Leibniz algebras can be viewed as non-commutative analogue of Lie algebras.

Such objects of Leibniz algebras date back to the work of "D-algebras" by Bloh [10, 11]. And the notion of Leibniz algebras is due to Loday [49]. In literature, Leibniz algebras are sometimes also called Loday algebras.

In analogue to Lie algebra homology, Loday constructed the so-called Leibniz homology of Lie algebras, which is related to Hochschild homology [21, 48]. Later on, Loday and Pirashvili found that similar constructions apply to Leibniz algebras, leading to the definition of Leibniz homology of Leibniz algebras [50]. Since then, many foundational works on Leibniz algebras are completed [51, 52, 53, 54, 55].

As a weakened version of Lie algebras, Leibniz algebras have been widely studied and used from various aspects. Loday and Pirashvili [50] studied not only Leibniz homology, but also Leibniz cohomology, associated with representations (or corepresentations) of Leibniz algebras. They also studied universal enveloping algebras and PBW theorem for Leibniz algebras.

Some other theorems and properties of Lie algebras are also proved to be valid for Leibniz algebras [7, 8, 58, 5]. For example, Engel's theorem and a weaker version of Levi-Malcev theorem hold for Leibniz algebras.

Nevertheless, many classical methods and results in Lie algebra theory can not be applied and generalized directly to the case of Leibniz algebras. In fact, many interesting questions concerning Leibniz algebras are still open. For example, the generalization of Lie's third theorem for Leibniz algebras as proposed by Loday [49], namely the problem of integrating Leibniz algebras (coquecigrue problem), is not yet answered. Partial answers are achieved by Kinyon [42]and Covez [19, 20].

Leibniz algebras have attracted more interest since the discovery of Courant algebroids, which can be viewed as the geometric realization of Leibniz algebras in certain sense. Courant algebroids are important objects in recent studies of Poisson geometry, symplectic geometry and generalized complex geometry. Here is a short account of Courant algebroids.

In [18], Courant considered a skew-symmetric bracket

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(\iota_X\eta - \iota_Y\xi) \tag{1.0.1}$$

on the generalized tangent bundle $TM \oplus T^*M$, where M is a smooth manifold, $X+\xi$ and $Y+\eta$ are sections in $TM \oplus T^*M$.

A modified version is the twisted bracket (non-skewsymmetric):

$$(X+\xi)\circ(Y+\eta) = [X,Y] + L_X\eta - \iota_Y d\xi \tag{1.0.2}$$

which was independently discovered by Dorfman [23, 22]. The most significant facts are that the bracket 1.0.1 satisfies Jacobi identity up to homotopy, while the bracket 1.0.2 satisfies Leibniz rule. They are now known as "Courant bracket" and "Dorfman bracket" , respectively.

The general notion of Courant algebroids was first introduced by Liu, Weinstein and Xu in [46], as an answer to an earlier question "what kind of object is the double of a Lie bialgebroid". Courant algebroids have various applications in mathematics and physics, e.g. in generalized complex geometry by Hitchin and his school [35, 31], in 3-dimensional topological field theory by Ikeda, Park and many others [40, 41, 57, 38, 63], and in supergravity [56].

In Liu, Weinstein and Xu's original definition, a Courant algebroid is defined in terms of Courant bracket, which is skew-symmetric and does not satisfy Jacobi identity. In a remark, they introduced a non-skewsymmetric "twisted bracket", which is now known as the Dorfman bracket. The twisted bracket is a Leibniz bracket, and the skewsymmetrization of the twisted bracket is exactly the Courant bracket.

Roytenberg [60] proved that Courant algebroids can be equivalently defined in terms of Dorfman bracket subject to five axioms. These axioms can also be found in Severa's emails to Weinstein in 1998 [64]. This definition of Courant algebroids was published by Severa and Weinstein in [65], and by Roytenberg in [62]. Later, it was shown that three of Roytenberg's axioms imply the other two [69, 28, 45]. Here we give the modern definition with only three axioms:

A Courant algebroid consists of a vector bundle $E \to M$, a fibrewise non-degenerate pseudo-metric $(\bullet, \bullet)$, an bundle map $\rho : E \to TM$ called anchor map and a $\mathbb{R}$-bilinear map $\circ : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$ called Dorfman bracket, satisfying the following three conditions:

1). $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)$,

2). $e_1 \circ e_2 + e_2 \circ e_1 = \partial(e_1, e_2)$, where $\partial : C^\infty(M) \to \Gamma(E)$ is defined by $(\partial f, e) \triangleq \rho(e)f$,

3). $\rho(e_1)(e_2, e_3) = (e_1 \circ e_2, e_3) + (e_2, e_1 \circ e_3)$,

for all $e_1, e_2, e_3 \in \Gamma(E)$.

The first condition says that $\Gamma(E)$ is a Leibniz algebra, while the other two tell that Courant algebroids are vector bundle version of quadratic Lie algebras.

One reason why the Dorfman bracket is more convenient than the Courant bracket, is that it is a "derived bracket". In 1998, Kosmann-Schwarzbach demonstrated that the Dorfman bracket on $TM \oplus T^*M$ is a derived bracket in her email to Weinstein (the proof was published much later in [44]). Later in 2002, based on a construction suggested by Weinstein and Severa [64], Roytenberg [62] explicitly showed that the Dorfman bracket for any Courant algebroid is a derived bracket. Here we mention the key steps in this construction:

Given any Courant algebroid E, the pseudo-metric on E turns $E[1]$ into a graded Poisson manifold. Then there is a minimal symplectic realization $(\mathbf{E}, \{\bullet, \bullet\})$ of $E[1]$. The space $\mathbf{E}$ is exactly the symplectic NQ-manifold corresponding to E, with Q-structure given by $Q = \{H, \bullet\}$, where H is a cubic Hamiltonian on $\mathbf{E}$ encoding the Courant algebroid structure of E. The most significant fact is that the Dorfman bracket on E can be given by the derived bracket:

$$(e_1 \circ e_2)^\flat = \{\{H, e_1^\flat\}, e_2^\flat\}.$$

(Another view point of the derived bracket is given by Alekseev and Xu [1], see also Grutzmann, Michel and Xu [30].)

Furthermore, Roytenberg [62] defined the standard cohomology $H^\bullet_{st}(E)$, to be the cohomology of the complex $(A^\bullet, \{H, \bullet\})$, where $A^\bullet$ is the graded algebra of polynomial functions on $\mathbf{E}$. Later in 2008, Stienon and Xu defined the naive cohomology $H^\bullet_{nv}(E)$, to be the cohomology of the complex $(\Gamma(\wedge^\bullet ker\rho), d_{nv})$, where d_{nv} is defined similarly to the Chevalley-Eilenberg differential. Stienon and Xu observed that there is a canonical morphism $\phi : H^\bullet_{nv}(E) \to H^\bullet_{st}(E)$, and conjectured that ϕ is an isomorphism when E is transitive. This conjecture was proved by Ginot and Grutzmann in [26].

There are many other significant developments in the theory of Courant algebroids. In [59], Roytenberg and Weinstein found that Courant algebroids fit into 2-term L_∞-algebras. In [34], Hansen and Strobl introduced H-twisted Courant algebroids, which are further investigated by Sheng and Liu in [67], and by Melchior Grutzmann in [29]. In [71], Vaisman introduced pre-Courant algebroids and Courant vector bundles, which are further studied by Armstrong and Lu in [3], and by Liu, Sheng and Xiaomeng Xu in [47]. In [16], Chen, Liu and Sheng introduced E-Courant algebroids. In [64], Severa defined a cohomology class, now called Severa class, which classifies exact Courant algebroids. In [12], Bressler defined Pontryagin class, which is the obstruction to the existence of a Courant extension, and he related the theory of Courant algebroids to conformal field theory.

This thesis is based on these achievements of Leibniz algebras and Courant algebroids, especially the important role that various cohomologies play in these theories. We wish to explore more general settings in which these cohomologies would still exist, and see how they grasp crucial information of mathematical objects. We shall work with Leibniz algebras and their representations, and develop two cohomology theories: standard cohomology and equivariant cohomology.

As mentioned above, we have the standard complex $(A^\bullet, \{H, \bullet\})$ and the standard cohomology $H^\bullet_{st}(E)$ for a Courant algebroid E. In fact Roytenberg [61] showed that the standard complex $(C^\bullet(\mathcal{E}, R), d)$ and the standard cohomology $H^\bullet_{st}(\mathcal{E})$ can be defined for a Courant-Dorfman algebra $\mathcal{E}$, which is a specific example of Leibniz algebras, as well as an algebraic analogue of Courant algebroids. For any non-degenerate Courant-Dorfman algebra $\mathcal{E}$, he defined a bracket on $C^\bullet(\mathcal{E}, R)$ such that $C^\bullet(\mathcal{E}, R)$ becomes a graded Poisson algebra and the Dorfman bracket of $\mathcal{E}$ can be written as a derived bracket. Furthermore, when $\mathcal{E} = \Gamma(E)$ is the space of sections of a Courant algebroid E, Roytenberg proved that there is an isomorphism of graded Poisson algebras between $C^\bullet(\mathcal{E}, R)$ and $A^\bullet$, and the standard cohomology of $\mathcal{E} = \Gamma(E)$ coincides with the standard cohomology of E.

A natural question is: can these constructions be carried out for Leibniz algebras?

We succeed to give a positive answer, it is divided into two parts:

As the first part of the answer, we shall give the definition of standard complex $C^\bullet_{st}(L, h, R)$, standard cohomology $H^\bullet_{st}(L, h, R)$ and naive cohomology $H^\bullet_{nv}(L, h, R)$ (Definition 3.4 and Definition 3.8), where L is a Leibniz algebra with left center Z, h is an isotropic ideal in L containing Z, and R is a left L-module on which h acts trivially.

Moreover, we shall prove Theorem 3.10 for any Leibniz algebra L:

$$H^\bullet_{st}(L, h, R) \cong H^\bullet_{nv}(L, h, R).$$

Actually $H^\bullet_{st}(L, h, R)$ and $H^\bullet_{nv}(L, h, R)$ can be defined for a general pair (L, h) (see Remark 3.12), in which case they are not necessarily isomorphic. Similarly, for any transitive Courant-Dorfman algebra $\mathcal{E}$, we have Theorem 3.15:

$$H^\bullet_{st}(\mathcal{E}) \cong H^\bullet_{nv}(\mathcal{E}).$$

Note that this theorem doesn't require the symmetric bilinear form on $\mathcal{E}$ to be non-degenerate. It is a generalization of the conjecture for transitive Courant algebroids by Stienon and Xu [68] (proved by Ginot and Grutzmann [26]).

As the second part of the answer, we shall give the construction of derived brackets for (fat) Leibniz algebras. We will construct a bracket for "representable cochains" (Definition 4.3) in the standard complex $C^\bullet_{st}(L, h, S^\bullet(Z))$ ($S^\bullet(Z)$ is the symmetric tensor algebra of Z):

$$\{\omega, \eta\} \triangleq \omega \bullet \eta + \omega \diamond \eta - (-1)^{nm} \eta \diamond \omega.$$

We prove that it is a Poisson bracket (see 4.2.1 and Theorem 4.4 for the detailed construction and proof). Furthermore, we prove that the Leibniz bracket of any fat Leibniz algebra is a derived bracket (Theorem 4.8). We hope these results will lead to new insights into the theory of Leibniz algebras.

Another subject in this thesis is the application of equivariant cohomology theory in Courant algebroids and Leibniz algebras. The motivation comes from the classical equivariant cohomology theory, where one considers the action of a compact Lie group G (or the infinitesimal action of its Lie algebra $\mathfrak{g}$) on a topological space M. Topologically $H^\bullet_G(M)$ is defined to be $H^\bullet((EG \times M)/G)$ [4, 39, 2]. If M is a finite-dimensional differentiable manifold, $H^\bullet_G(M)$ can be alternatively defined using Weil model or Cartan model (Definition 2.38 and 2.39) [15]. When the action of G is free, $H^\bullet_G(M)$ is isomorphic to the cohomology of the quotient space M/G. In fact, equivariant cohomology can be extended to any $\mathfrak{g}$ differential algebra, with $(\Omega^\bullet(M), d_M)$ being a particular example, where M is any manifold with a $\mathfrak{g}$ action. By a $\mathfrak{g}$ differential algebra, we mean a differential graded commutative algebra $(A^\bullet, d)$, equipped with an interior product $\iota : \mathfrak{g} \to Der(A^\bullet)$ of degree -1 and a Lie derivative $L : \mathfrak{g} \to Der(A^\bullet)$ of degree 0 satisfying Cartan calculus [14, 24, 32, 33]. In parallel to the geometric case of a free and proper action, when $A^\bullet$ is of type (C) (Definition 2.35), the equivariant cohomology of $A^\bullet$ is proved to be isomorphic to the cohomology of the basic complex of $A^\bullet$ [14, 15, 24, 32, 33].

In generalized geometry [36, 37], the generalized tangent bundle $TM \oplus T^*M$ plays an important role. In this thesis, we shall introduce the notion of generalized action of a Lie algebra $\mathfrak{g}$ on a manifold M, as a homomorphism of Leibniz algebras from $\mathfrak{g}$ to $\Gamma(TM \oplus T^*M)$. Given such a generalized action, we define the interior product and the Lie derivative on the standard complex of $TM \oplus T^*M$, so that it becomes a $\mathfrak{g}$ differential algebra. Furthermore, this complex $C^\bullet_{st}(TM \oplus T^*M)$ is of type (C) under certain conditions (see Proposition 5.4), then by the classical theorem in [14, 15, 24, 32, 33] the equivariant cohomology of $C^\bullet_{st}(TM \oplus T^*M)$ is isomorphic to the cohomology of the basic complex of $C^\bullet_{st}(TM \oplus T^*M)$. When $M \to N$ is a principal G bundle, we observe that the standard complex of $TN \oplus T^*N$ is a subcomplex of the basic complex $(C^\bullet_{st}(TM \oplus T^*M))_{bas}$. And we conjecture that the inclusion map from $C^\bullet_{st}(TN \oplus T^*N)$ to $(C^\bullet_{st}(TM \oplus T^*M))_{bas}$ is a quasi-isomorphism. The work on this conjecture is still on-going.

Then we focus on equivariant cohomology for Leibniz algebras. The data we need are the following: a Leibniz algebra L with left center Z, an isotropic subalgebra h in L, and a left L-module as well as a commutative algebra R on which L acts as derivations. We define multiplication, interior product and Lie derivative on $(A^\bullet, d_0)$, where $A^\bullet$ is a subcomplex of the Leibniz complex $(Hom(\otimes^\bullet L, R), d_0)$, so that $A^\bullet$ becomes a h differential algebra (Proposition 5.7). Furthermore, we prove that $A^\bullet$ is of type (C) when R is unital and L has a decomposition $h \oplus X$ such that $h \circ X \subseteq X$ (Proposition 5.8).

The structure of this thesis is organized as follows.

In Chapter 2, we provide some basic knowledge of Leibniz algebras, Courant algebroids and equivariant cohomology. In particular, the definition of the natural bilinear product of Leibniz algebras is given in the first section; the definition of standard cohomology and the construction of derived bracket for Courant-Dorfman algebras are listed in the second section.

The main objective of this thesis is to develop cohomology theories for Leibniz algebras, which are divided into the subsequent chapters.

In Chapter 3, we define the standard complex, standard cohomology and naive cohomology for any Leibniz algebra L with left center Z. We prove the isomorphism between standard cohomology and naive cohomology of L. And we use the same methods to prove a similar result for transitive Courant-Dorfman algebras. Moreover, we construct a Courant-Dorfman algebra structure on $S^\bullet(Z) \otimes L$, and prove that under certain conditions the standard complex of L is isomorphic to the standard complex of $S^\bullet(Z) \otimes L$.

In Chapter 4, we focus on the construction of the derived bracket for a Leibniz algebra. We work on the standard complex of a Leibniz algebra throughout this chapter. We define a canonical 3-cochain Θ and construct a bracket $\{\bullet, \bullet\}$ for certain cochains, which we call "representable" cochains. We prove that all representable cochains form a graded Poisson algebra under this bracket. And finally we prove that the Leibniz bracket of any fat Leibniz algebra can be represented as a derived bracket.

Chapter 5 is devoted to the application of equivariant cohomology theory in Courant algebroids and Leibniz algebras. We consider two specific types of $\mathfrak{g}$ differential algebras. The first one is the standard complex of the standard Courant algebroid, based on the generalized action of a Lie algebra on a manifold. The second is a certain subspace of the Leibniz complex. We define interior products ι_ξ and Lie derivatives L_ξ for both of them, so that they become $\mathfrak{g}$ differential algebras. Furthermore, we prove that they are of type (C) under certain assumptions.

We would like to point out other works that might be related to this thesis: Kolesnikov's work [43] on conformal representations of Leibniz algebras; Uchino's work [70] on the derived bracket construction concerning strongly homotopy Leibniz algebras; Benayadi and Hidri's work [9] on quadratic Leibniz algebras; Ginzburg's work [27] on equivariant cohomology for Poisson manifolds; and [13] by Bruzzo, Cirio, Rossi and Rubtsov on equivariant cohomology for Lie algebroids.

Chapter 2

Preliminaries

2.1 Leibniz algebras

In this section we list some basic knowledge about Leibniz algebras. For more details, we refer to [50].

Definition 2.1. A (left) Leibniz algebra is a vector space L over a field $\mathfrak{k}$ ($\mathfrak{k} = \mathbb{R}$ for our main interest), endowed with a bilinear map (called Leibniz bracket) $\circ$: $L \otimes L \to L$, which satisfies (left) Leibniz rule:

$$e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3) \quad \forall e_1, e_2, e_3 \in L$$

It's easily seen that if the Leibniz bracket of L is skew-symmetric, then the Leibniz rule is equivalent to the Jacobi identity, thus L is a Lie algebra. So roughly speaking, a Leibniz algebra is a "weakened" Lie algebra without skew-symmetricity.

A Leibniz subalgebra of L is a vector subspace which is closed under Leibniz bracket.

Given two Leibniz algebras $(L_1, \circ_1)$ and $(L_2, \circ_2)$, a homomorphism φ from L_1 to L_2 is a linear map $\varphi: L_1 \to L_2$ preserving Leibniz brackets, i.e.

$$\varphi(e_1 \circ_1 e_2) = \varphi(e_1) \circ_2 \varphi(e_2), \quad \forall e_1, e_2 \in L_1.$$

Leibniz algebras together with homomorphisms defined above form a category.

Given a Leibniz algebra L, denote by Z the left center of L, i.e.

$$Z \triangleq \{e \in L | e \circ e' = 0, \ \forall e' \in L\}.$$

We have the following:

Proposition 2.2. *Z is an ideal of L, and the Leibniz bracket of L induces a Lie bracket on L/Z.*

Proof. By Leibniz identity

$$(e \circ f) \circ e' = e \circ (f \circ e') - f \circ (e \circ e') = 0, \ \forall e, e' \in L, \ f \in Z,$$

so $e \circ f \in Z$.
Sine $f \circ e = 0$ is also in Z, Z is an ideal of L.
So the Leibniz bracket of L induces a Leibniz bracket on the quotient L/Z:

$$\bar{e_1} \circ \bar{e_2} \triangleq \overline{e_1 \circ e_2},$$

where $\bar{e}$ is the equivalence class of $e \in L$ in L/Z. In order for this bracket to be a Lie bracket, we only need to prove that it is skew-symmetric, i.e.

$$\bar{e_1} \circ \bar{e_2} + \bar{e_2} \circ \bar{e_1} = 0 \in L/Z,$$

or equivalently

$$e_1 \circ e_2 + e_2 \circ e_1 \in Z.$$

This is true because:

$$(e_1 \circ e_2 + e_2 \circ e_1) \circ e = e_1 \circ (e_2 \circ e) - e_2 \circ (e_1 \circ e) + e_2 \circ (e_1 \circ e) - e_1 \circ (e_2 \circ e) = 0, \ \forall e \in L.$$

■

In the proof above, we see that $e_1 \circ e_2 + e_2 \circ e_1 \in Z, \ \forall e_1, e_2 \in L$, this leads to the following definition:

Definition 2.3. With the above notations, we can naturally define the symmetric bilinear product $(\bullet, \bullet) : \ L \otimes L \to Z$ to be:

$$(e_1, e_2) \triangleq e_1 \circ e_2 + e_2 \circ e_1.$$

It induces a map

$$\begin{array}{rcl} (\bullet)^\flat : \ L & \to & Hom(L, Z) \\ e^\flat(e') & \triangleq & (e, e'). \end{array}$$

This product $(\bullet, \bullet)$ will be constantly used in the subsequent chapters.

Proposition 2.4. *The bilinear product* $(\bullet, \bullet)$ *is invariant, i.e.*

$$(e_1, (e_2, e_3)) = (e_1 \circ e_2, e_3) + (e_2, e_1 \circ e_3).$$

Proof.

$$\begin{aligned}
(e_1, (e_2, e_3)) &= e_1 \circ (e_2, e_3) + (e_2, e_3) \circ e_1 \\
&= e_1 \circ (e_2 \circ e_3 + e_3 \circ e_2) \\
&= (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3) + (e_1 \circ e_3) \circ e_2 + e_3 \circ (e_1 \circ e_2) \\
&= (e_1 \circ e_2, e_3) + (e_2, e_1 \circ e_3).
\end{aligned}$$

■

Example 2.5. Given any Lie algebra g and its representation (V, ρ), the semi-direct product $g \ltimes V$ equipped with a bilinear operation $\circ$ defined by

$$(A + v) \circ (B + w) := [A, B] + \rho(A)w, \quad \forall A, B \in g, \ v, w \in V$$

forms a Leibniz algebra.

In particular, $gl(V) \oplus V$ is a Leibniz algebra with Leibniz bracket

$$(A + v) \circ (B + w) = [A, B] + Aw, \quad \forall A, B \in gl(V), \ v, w \in V.$$

It is called an omni Lie algebra, and denoted by $ol(V)$. The notion of omni Lie algebra was introduced by Weinstein in [74] as the linearization of the standard Courant algebroid $T\mathbb{R}^n \oplus T^*\mathbb{R}^n$ (the Leibniz subalgebra consisting of all the sections of linear vector fields and constant 1-forms). The left center of $ol(V)$ is obviously V, so the quotient Lie algebra in Proposition 2.2 is exactly $gl(V)$. The bilinear product of $ol(V)$:

$$(A + v, B + w) = Aw + Bv$$

is exactly the the restriction of the symmetric bilinear form of $T\mathbb{R}^n \oplus T^*\mathbb{R}^n$. See the next section for the definition of Courant algebroids.

Weinstein called $ol(V)$ an omni Lie algebra because all Lie algebra structures on V can be characterized by the Dirac structures in $ol(V)$:

Theorem 2.6. *Let B be a skew-symmetric bilinear operation on V, denote by $ad_B : V \to gl(V)$ the adjoint operator, i.e. $ad_B(v_1)(v_2) \triangleq B(v_1, v_2)$, and denote by $\Gamma_{ad_B} \subseteq ol(V)$ the graph of ad_B, then B satisfies Jacobi identity if and only if Γ_{ad_B} is closed under the Leibniz bracket of $ol(V)$. When this condition is satisfied, the restriction to Γ_{ad_B} of the natural projection from $ol(V)$ to V is an isomorphism of Lie algebras between Γ_{ad_B} and V.*

Next we define representations of Leibniz algebras.

Definition 2.7. A representation of a Leibniz algebra L is a triple (V, l, r), where V is a vector space equipped with two linear maps: left action $l : L \to gl(V)$ and right action $r : L \to gl(V)$ satisfying the following equations:

$$l_{e_1 \circ e_2} = [l_{e_1}, l_{e_2}],\ r_{e_1 \circ e_2} = [l_{e_1}, r_{e_2}],\ r_{e_1} \circ l_{e_2} = -r_{e_1} \circ r_{e_2}, \quad \forall e_1, e_2 \in L, \tag{2.1.1}$$

where the brackets on the right hand side are the commutators in $gl(V)$.

If V is only equipped with left action $l : L \to gl(V)$ which satisfies $l_{e_1 \circ e_2} = [l_{e_1}, l_{e_2}]$, we call (V, l) a left representation of L.

For (V, l, r)(or (V, l)) a representation (or left representation) of L, we call V an L-module (or left L-module).

A homomorphism ϕ from an L-module (V, l, r) to another L-module (V', l', r') is a linear map $\phi : V \to V'$commuting with the left actions as well as the right actions, i.e.

$$\phi(l_e v) = l'_e \phi(v),\ \phi(r_e v) = r'_e \phi(v), \quad \forall e \in L,\ v \in V.$$

L-modules together with homomorphisms defined above form a category.

Given a left representation (V, l), there are two standard ways to extend V to an L-module. One is called symmetric extension, with the right action defined as $r_e = -l_e$; the other is called antisymmetric extension, with the right action defined as $r_e = 0$. It is obvious that both right actions satisfy the second and third equations in 2.1.1.

The left center Z is a natural example of left L-module:

Proposition 2.8. *Leibniz bracket of L induces a left L-module structure on Z.*

Proof. In the proof of Proposition 2.2, we see that $e \circ f \in Z,\ \forall e \in L,\ f \in Z$.

So we can define a map

$$\rho : L \to gl(Z) :\ \rho(e) f \triangleq e \circ f.$$

In order for (Z, ρ) to be a left representation, we only need to prove $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$.

Since

$$\begin{aligned} \rho(e_1 \circ e_2) f &= (e_1 \circ e_2) \circ f, \\ [\rho(e_1), \rho(e_2)] f &= e_1 \circ (e_2 \circ f) - e_2 \circ (e_1 \circ f), \end{aligned}$$

by Leibniz identity, $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$.

The proof is finished. ■

Similar to the case of Lie algebras, we have the following definition of Leibniz cohomology:

Definition 2.9. Given a Leibniz algebra L and an L-module (V, l, r), the Leibniz cohomology of L with coefficients in V is the cohomology of the cochain complex $C^n(L, V) = Hom(\otimes^n L, V)$ $(n \geq 0)$ with the coboundary operator $d_0 : C^n(L, V) \to C^{n+1}(L, V)$ given by:

$$\begin{aligned} & (d_0\alpha)(e_1, \cdots, e_{n+1}) \\ = & \sum_{a=1}^{n}(-1)^{a+1} l_{e_a}\alpha(e_1, \cdots, \hat{e_a}, \cdots, e_{n+1}) + (-1)^{n+1} r_{e_{n+1}}\alpha(e_1, \cdots, e_n) \\ & + \sum_{1\leq a<b\leq n+1}(-1)^a\alpha(e_1, \cdots, \hat{e_a}, \cdots, \hat{e_b}, e_a \circ e_b, \cdots, e_{n+1}) \end{aligned}$$

The resulting cohomology is denoted by $H^\bullet(L; V, l, r)$, or simply $H^\bullet(L, V)$ if it causes no confusion.

Example 2.10. (1). $(\mathbb{R}, 0, 0)$ is a representation of L, called the trivial representation. The corresponding Leibniz cochain complex is denoted by $C^\bullet(L; \mathbb{R})$, and Leibniz cohomology by $H^\bullet(L; \mathbb{R})$.

(2). L itself is an L-module, with left action $ad_L(e)(e') := e \circ e'$ and right action $ad_R(e)(e') := e' \circ e$, $\forall e, e' \in L$. (L, ad_L, ad_R) is called the adjoint representation of L, and the corresponding Leibniz cochain complex is denoted by $C^\bullet(L; ad_L, ad_R)$, and Leibniz cohomology by $H^\bullet(L; ad_L, ad_R)$.

The graded vector space $\oplus_n C^n(g, g)$ is a graded Lie algebra with the standard commutator as Lie bracket:

$$[\alpha, \beta] = \alpha \circ \beta + (-1)^{ab+1}\beta \circ \alpha, \quad \forall \alpha \in C^{a+1}(L, L),\ \beta \in C^{b+1}(L, L),$$

where $\alpha \circ \beta \in C^{a+b+1}(L, L)$ is defined by:

$$\begin{aligned} & (\alpha \circ \beta)(e_1, \cdots, e_{a+b+1}) \\ = & \sum_{i=0}^{a}(-1)^{ib}\sum_{\sigma\in sh(i,b)}(-1)^\sigma \\ & \alpha(e_{\sigma(1)}, \cdots, e_{\sigma(i)}, \beta(e_{\sigma(i+1)}, \cdots, e_{\sigma(i+b)}, e_{i+b+1}), e_{i+b+2}, \cdots, e_{a+b+1}) \\ & \forall e_1, \cdots, e_{a+b+1} \in L. \end{aligned}$$

(Refer to [6, 25] for more details.)

In particular, when $\alpha \in C^2(L, L)$,

$$[\alpha, \alpha](e_1, e_2, e_3) = 2(\alpha\circ\alpha)(e_1, e_2, e_3) = 2(\alpha(\alpha(e_1, e_2), e_3) - \alpha(e_1, \alpha(e_2, e_3)) + \alpha(e_2, \alpha(e_1, e_3))).$$

We see that α defines a Leibniz bracket on L iff $[\alpha, \alpha] = 0$.

2.2 Courant algebroids and Courant-Dorfman algebras

Definition 2.11. A Courant algebroid consists of a vector bundle $E \to M$, a fibrewise non-degenerate psedo-metric $(\bullet, \bullet)$, a bundle map $\rho : E \to TM$ called anchor map and a $\mathbb{R}$-bilinear operation $\circ$ on $\Gamma(E)$ called Dorfman bracket, which satisfy the following relations for all $f \in C^\infty(M),\ e, e_1, e_2, e_3 \in \Gamma(E)$:

1). $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)$
2). $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$
3). $e_1 \circ (fe_2) = (\rho(e_1)f)e_2 + f(e_1 \circ e_2)$
4). $e_1 \circ e_2 + e_2 \circ e_1 = \partial(e_1, e_2)$
5). $\partial f \circ e = 0$
6). $\rho(e_1)(e_2, e_3) = (e_1 \circ e_2, e_3) + (e_2, e_1 \circ e_3)$
where $\partial : C^\infty(M) \to \Gamma(E)$ is the $\mathbb{R}$-linear map defined by $(\partial f, e) = \rho(e)f$.

This definition is an equivalent version of the original one in [46]. The Courant bracket there is the skew-symmetric part $[e_1, e_2] = \frac{1}{2}(e_1 \circ e_2 - e_2 \circ e_1)$ of the Dorfman bracket.

Example 2.12. 1). If M is a point, then the anchor map ρ (and thus ∂) is trivial, and E is just a vector space. The properties 2), 3), 5) above are trivial. 1) implies that E is a Leibniz algebra, and 4) implies that the Leibniz bracket is skew-symmetric, thus E is a Lie algebra. 6) says that E is endowed with an invariant inner product. As a conclusion, E is a quadratic Lie algebra.

2). Given a manifold M, $TM \oplus T^*M$ with anchor map given by the projection, with pseudo-metric given by

$$(X + \xi, Y + \eta) \triangleq \langle X, \eta \rangle + \langle Y, \xi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing of TM and T^*M, and with Dorfman bracket given by

$$(X + \xi) \circ (Y + \eta) := [X, Y] + L_X\eta - \iota_Y d_M \xi, \quad \forall X, Y \in \mathfrak{X}(M),\ \xi, \eta \in \Omega^1(M),$$

becomes a Courant algebroid. This is called the standard Courant algebroid. The skew-symmetric part

$$[X + \xi, Y + \eta] = [X, Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(\eta(X) - \xi(Y))$$

is the original bracket introduced by Courant in [18].

Given a Courant algebroid $E \to M$, E^* (or $\Gamma(E^*)$) is isomorphic to E (or $\Gamma(E)$ resp.) by pseudo-metric. The isomorphism maps are denoted by

$$(\cdot)^\sharp : E^* \to E$$

and

$$(\cdot)^\flat : E \to E^*.$$

Sometimes we simply omit these two notations and identify E^* with E.

The dual of the anchor

$$\rho^* : \ T^*M \to E^*$$

is called the coanchor map. Since E^* can be identified with E by pseudo-metric, the coanchor map can be viewed as a map

$$\rho^* : \ T^*M \to E$$

such that

$$(\rho^*\alpha, e) = \alpha(\rho(e)).$$

Property 5) and 6) in Definition 2.11 implies that

$$\rho(\partial f) = 0, \quad \forall f \in C^\infty(M).$$

It follows that

$$\rho \circ \rho^* = 0.$$

Identifying E^* with E by pseudo-metric, it is easily seen that $(ker\rho)^\perp$, the subbundle of E orthogonal to $ker\rho$, coincides with $\rho^*(T^*M)$, the subbundle of E generated by the image of $\partial : C^\infty(M) \to \Gamma(E)$. Thus any $\alpha \in \Gamma((ker\rho)^\perp)$ can be written as $\sum_i f_i \partial g_i$, $f_i, g_i \in C^\infty(M)$. Since $\rho \circ \partial = 0$, it follows that $ker\rho$ is coisotropic, i.e. $(ker\rho)^\perp \subset ker\rho$.

A Courant algebroid E is called exact iff

$$0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \to 0$$

is an exact sequence of vector bundles.

The standard Courant algebroid is the prototypical example of an exact Courant algebroid.

Severa in [64] gave a classification of exact Courant algebroids by cohomology classes in $H^3(M, \mathbb{R})$, called Severa classes. In particular, the Severa class for the standard Courant algebroid is $0 \in H^3(M, \mathbb{R})$.

A slightly more general notion is a transitive Courant algebroid: a Courant algebroid E is called transitive if the anchor map ρ is surjective.

A further more general notion is a regular Courant algebroid: a Courant algebroid E is called regular if $F := im\rho$ has constant rank, in which case F is an integrable distribution on M. Moreover, if E is regular, then $ker\rho$ and $(ker\rho)^{\perp}$ are smooth constant rank subbundles of E and the quotients $E/ker\rho$ and $E/(ker\rho)^{\perp}$ are Lie algebroids. It is obvious that $E/ker\rho$ and F are canonically isomorphic. $(ker\rho)^{\perp}$ and F^* are also isomorphic. We call $E/(ker\rho)^{\perp}$ the ample Lie algebroid associated to E. Let $\mathcal{G} \triangleq ker\rho/(ker\rho)^{\perp}$, and denote by π the projection map $ker\rho \to \mathcal{G}$. The Dorfman bracket on $\Gamma(E)$ induces a $C^{\infty}(M)$-bilinear and skew-symmetric bracket on $\Gamma(\mathcal{G})$ as follows:

$$[\pi(e_1), \pi(e_2)]_{\mathcal{G}} \triangleq \pi(e_1 \circ e_2), \quad \forall e_1, e_2 \in \Gamma(ker\rho).$$

$\mathcal{G}$ becomes a bundle of Lie algebras under the bracket $[\bullet, \bullet]_{\mathcal{G}}$. Moreover, the map $(\bullet, \bullet)_{\mathcal{G}}$ defined by

$$(\pi(e_1), \pi(e_2))_{\mathcal{G}} \triangleq (e_1, e_2), \quad \forall e_1, e_2 \in \Gamma(ker\rho)$$

is a well-defined non-degenerate symmetric and ad-invariant pseudo-metric on $\mathcal{G}$, turning $\mathcal{G}$ into a bundle of quadratic Lie algebras. Chen, Stienon and Xu in [17] gave the following:

Definition 2.13. Let E be a regular Courant algebroid with characteristic distribution F and bundle of quadratic Lie algebras $\mathcal{G}$. A dissection of E is an isomorphism of vector bundles

$$\Psi : \; F^* \oplus \mathcal{G} \oplus F \to E$$

such that $\forall \xi, \eta \in \Gamma(F^*), \; r, s \in \Gamma(\mathcal{G}), \; x, y \in \Gamma(F)$,

$$(\Psi(\xi + r + x), \Psi(\eta + s + y)) = \langle \xi, y \rangle + \langle \eta, x \rangle + (r, s)_{\mathcal{G}},$$

where $\langle \bullet, \bullet \rangle$ on the right hand side is the natural pairing of F and F^*, and $(\bullet, \bullet)_{\mathcal{G}}$ is the symmetric bilinear form of $\mathcal{G}$.

A standard dissection of E is a dissection satisfying two more conditions:

1). $\rho(\Psi(\xi + r + x)) = x$;

2). $Pr_{\mathcal{G}}(\Psi^{-1}(\Psi(r) \circ \Psi(s))) = [r, s]_{\mathcal{G}}$.

The Courant algebroid structure of E can be transported to $F^* \oplus \mathcal{G} \oplus F$ through Ψ, turning Ψ into an isomorphism of Courant algebroids.

Chen, Stienon and Xu proved that standard dissections always exist for regular Courant algebroids.

Each dissection of E induces three canonical maps (Pr is the projection map):
1). $\nabla : \Gamma(F) \otimes \Gamma(\mathcal{G}) \to \Gamma(\mathcal{G})$:

$$\nabla_x r := Pr_{\mathcal{G}}(x \circ r), \quad \forall x \in \Gamma(F),\ r \in \Gamma(\mathcal{G});$$

2). $R : \Gamma(F) \otimes \Gamma(F) \to \Gamma(\mathcal{G})$:

$$R(x, y) := Pr_{\mathcal{G}}(x \circ y), \quad \forall x, y \in \Gamma(F);$$

3). $H : \Gamma(F) \otimes \Gamma(F) \otimes \Gamma(F) \to C^\infty(M)$:

$$H(x, y, z) := \langle Pr_{F^*}(x \circ y), z\rangle, \quad \forall x, y, z \in \Gamma(F).$$

These maps (∇, R, H) satisfy the following properties:

Proposition 2.14. *1). ∇ is a covariant derivative:*

$$\begin{aligned} \nabla_{fx} r &= f \nabla_x r \\ \nabla_x (fr) &= f \nabla_x r + (x(f))r \end{aligned}$$
$$\forall x \in \Gamma(F),\ r \in \Gamma(\mathcal{G}),\ f \in C^\infty(M);$$

2). R is skew-symmetric and $C^\infty(M)$-bilinear. It can thus be regarded as a bundle map $\wedge^2 F \to \mathcal{G}$;

3). H is skew-symmetric and $C^\infty(M)$-trilinear. It can thus be regarded as a section of $\wedge^3 F^*$.

Chen, Stienon and Xu proved that the Dorfman bracket on $\Gamma(F^* \oplus \mathcal{G} \oplus F)$ can be recovered from (∇, R, H):

Lemma 2.15. *Let $P : \Gamma(\mathcal{G}) \otimes \Gamma(\mathcal{G}) \to \Gamma(F^*)$ and $Q : \Gamma(F) \otimes \Gamma(\mathcal{G}) \to \Gamma(F^*)$ be the maps defined by:*

$$\begin{aligned} \langle P(r, s), x\rangle &:= (s, \nabla_x r)_{\mathcal{G}} \\ \langle Q(x, r), y\rangle &:= (r, R(x, y))_{\mathcal{G}}. \end{aligned}$$

Then we have

$$\begin{aligned} x \circ y &= H(x, y, \cdot) + R(x, y) + [x, y] \\ r \circ s &= P(r, s) + [r, s]_{\mathcal{G}} \\ \xi \circ r &= r \circ \xi = 0 \\ \xi \circ \eta &= 0 \\ x \circ \xi &= L_x \xi \\ \xi \circ x &= -L_x \xi + d_M \langle \xi, x\rangle \\ x \circ r &= -r \circ x = -Q(x, r) + \nabla_x r \end{aligned}$$
$$\forall x, y \in \Gamma(F),\ r, s \in \Gamma(\mathcal{G}),\ \xi, \eta \in \Gamma(F^*).$$

Remark 2.16. Suppose F is an integrable subbundle of TM and $\mathcal{G}$ is a bundle of quadratic Lie algebras over M. Given (∇, R, H) satisfying the properties in 2.14, it is easy to prove that $F^* \oplus \mathcal{G} \oplus F$ with anchor map

$$\rho(\xi + r + x) = x,$$

pseudo-metric

$$(\xi + r + x, \eta + s + y) = \langle \xi, y \rangle + \langle \eta, x \rangle + (r, s)_{\mathcal{G}},$$

and Dorfman bracket defined as in 2.15 is a (regular) Courant algebroid. And $F^* \oplus \mathcal{G} \oplus F$ is a standard dissection of itself.

Extracting the properties of $\Gamma(E)$, we have the following definition of algebraic version (given by D. Roytenberg in [61]):

Definition 2.17. A Courant-Dorfman algebra $(\mathcal{E}, R, (\bullet, \bullet), \partial, \circ)$ consists of the following data:

a commutative algebra R (over a commutative ring $\mathbb{K}$ which contains $\frac{1}{2}$, or $\mathbb{R}$ for our main interest);

an R-module $\mathcal{E}$;

a symmetric bilinear form $(\bullet, \bullet): \ \mathcal{E} \otimes_R \mathcal{E} \to R$;

a derivation $\partial : \ R \to \mathcal{E}$;

a Dorfman bracket $\circ : \ \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$.

These data are required to satisfy the following conditions for any $e, e_1, e_2, e_3 \in \mathcal{E}$ and $f, g \in R$:

(1). $e_1 \circ (f e_2) = f(e_1 \circ e_2) + (e_1, \partial f) e_2$;

(2). $(e_1, \partial(e_2, e_3)) = (e_1 \circ e_2, e_3) + (e_2, e_1 \circ e_3)$

(3). $e_1 \circ e_2 + e_2 \circ e_1 = \partial(e_1, e_2)$;

(4). $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)$;

(5). $\partial f \circ e = 0$;

(6). $(\partial f, \partial g) = 0$.

Given any Courant algebroid $E \to M$, $\Gamma(E)$ is obviously a Courant-Dorfman algebra with the commutative algebra $R = C^\infty(M)$. And it has an additional property that the symmetric bilinear form on $\mathcal{E} = \Gamma(E)$ induces an isomorphism

$$\mathcal{E} \to \mathcal{E}^\vee = Hom_R(\mathcal{E}, R).$$

We call the symmetric bilinear form with this property strongly non-degenerate and the corresponding Courant-Dorfman algebra non-degenerate. Similarly the isomorphism maps can be denoted by

$$(\cdot)^\sharp : \mathcal{E}^\vee \to \mathcal{E}$$

and

$$(\cdot)^\flat : \mathcal{E} \to \mathcal{E}^\vee.$$

Given a Courant-Dorfman algebra $\mathcal{E}$, we can recover the anchor map

$$\rho : \ \mathcal{E} \to \mathfrak{X}^1 = Der(R, R)$$

from the derivation ∂ by setting:

$$\rho(e) \cdot f \triangleq (e, \partial f). \tag{2.2.1}$$

Let Ω^1 be the R-module of Kahler differentials with the universal derivation $d_R : R \to \Omega^1$. By the universal property of Ω^1, there is a unique homomorphism of R-modules $\rho^* : \Omega^1 \to \mathcal{E}$ such that

$$\rho^*(d_R f) \triangleq \partial f. \tag{2.2.2}$$

ρ^* is called the coanchor map of $\mathcal{E}$. When $\mathcal{E}$ is non-degenerate, analogously to the case of Courant algebroids, ρ^* can be equivalently defined by

$$(\rho^* \alpha, e) = (\alpha, \rho(e)).$$

From the conditions in Definition 2.17, it is easily proved that $\rho^*(\Omega^1) = R\partial R$ is an isotropic ideal in $\mathcal{E}$. So the Dorfman bracket of $\mathcal{E}$ induces a Leibniz bracket on the quotient $\mathcal{E}/\rho^*(\Omega^1)$. The induced bracket is skew-symmetric since

$$e_1 \circ e_2 + e_2 \circ e_1 = \partial(e_1, e_2) \in \rho^*(\Omega^1), \quad \forall e_1, e_2 \in \mathcal{E}.$$

Actually $\mathcal{E}/\rho^*(\Omega^1)$ is a Lie-Rinehart algebra with anchor map given by Equation 2.2.1.

Remark 2.18. From the fourth condition in Definition 2.17, we see that any Courant-Dorfman algebra is a Leibniz algebra. Conversely, given any Leibniz algebra L with left center Z, if we replace in Definition 2.17 $\mathcal{E}$ with L, R with Z, symmetric bilinear form of $\mathcal{E}$ with the naturally defined bilinear product $(\bullet, \bullet)$, Dorfman bracket with Leibniz bracket of L, and the derivation map $\partial : R \to \mathcal{E}$ with the inclusion map $Z \hookrightarrow L$, it is obvious that the last four conditions in 2.17 are still satisfied.

Condition (3): $e_1 \circ e_2 + e_2 \circ e_1 = (e_1, e_2), \ \forall e_1, e_2 \in L.$

This is the definition of bilinear product $(\bullet, \bullet)$.

Condition (4): $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3), \ \forall e_1, e_2, e_3 \in L.$

This is the Leibniz identity.

Condition (5): $f \circ e = 0, \ \forall f \in Z, e \in L.$

This is true by definition of Z.

Condition (6): $(f, g) = 0,\ \forall f, g \in Z$.
This is true by definition of Z and $(\bullet, \bullet)$.
And Condition (2) also holds by Proposition 2.4.
In Chapter 3, we will construct a Courant-Dorfman algebra structure on the tensor product $S^\bullet(Z) \otimes L$, check 3.21 for details.

The Courant bracket of a Courant algebroid can be realized as a derived bracket on a degree 2 graded symplectic manifold, and the standard cohomology is defined to be the cohomology of the complex of graded polynomial functions on this graded manifold [62]. These constructions concerning graded manifolds are included in the appendix. In the following, we focus on the corresponding algebraic construction for a Courant-Dorfman algebra.

The following construction of standard complex is due to Roytenberg [61]:

Given a Courant-Dorfman algebra $\mathcal{E}$, denote by $C^n(\mathcal{E}, R)$ the space of all sequences $\omega = (\omega_0, \cdots, \omega_{[\frac{n}{2}]})$, where ω_k is a linear map from $\otimes^{n-2k}\mathcal{E} \otimes \odot^k R$ to R, $\forall k$, satisfying the following conditions:

1). Derivation: ω_k is a derivation in each argument in R.

2). Weak skew-symmetricity (in arguments in $\mathcal{E}$): ω_k is weakly skew-symmetric up to ω_{k+1}:

$$\begin{aligned} & \omega_k(e_1, \cdots, e_a, e_{a+1}, \cdots, e_{n-2k}; f_1, \cdots, f_k) + \omega_k(e_1, \cdots, e_{a+1}, e_a, \cdots e_{n-2k}; f_1, \cdots, f_k) \\ = & -\omega_{k+1}(e_1, \cdots, \widehat{e_a}, \widehat{e_{a+1}}, \cdots, e_{n-2k}; (e_a, e_{a+1}), f_1, \cdots, f_k) \\ & \forall k,\ \forall e_a \in \mathcal{E},\ \forall f_i \in R \end{aligned}$$

3). Weak R-linearity (in arguments in $\mathcal{E}$): ω_k is weakly-R-linear up to ω_{k+1}:

$$\begin{aligned} & \omega_k(e_1, \cdots, fe_a, \cdots, e_{n-2k}; f_1, \cdots, f_k) \\ = & f\omega_k((e_1, \cdots, e_a, \cdots, e_{n-2k}; f_1, \cdots, f_k) \\ & + \sum_{b>a} (-1)^{b-a} (e_a, e_b) \omega_{k+1}(\cdots, \widehat{e_a}, \cdots, \widehat{e_b}, \cdots; f, f_1, \cdots, f_k) \\ & \forall k,\ \forall e_a \in \mathcal{E},\ \forall f, f_i \in R. \end{aligned}$$

The third condition above implies the property that ω_k is R-linear in the last argument in $\mathcal{E}$. Conversely, this property together with the second condition induce the third condition. So the third condition can be replaced by this property.

$C^\bullet(\mathcal{E}, R)$ is an algebra with multiplication map given by:

$$\begin{aligned} & (\omega \cdot \eta)_k(e_1, \cdots, e_{n+m-2k}; f_1, \cdots f_k) \\ = & \sum_{a+b=k} \sum_{\sigma \in sh(n-2a, m-2b)} \sum_{\tau \in sh(a,b)} (-1)^\sigma \\ & \omega_a(e_{\sigma(1)}, \cdots, e_{\sigma(n-2a)}; f_{\tau(1)}, \cdots, f_{\tau(a)}) \cdot \eta_b(e_{\sigma(n-2a+1)}, \cdots, e_{\sigma(n+m-2k)}; f_{\tau(a+1)}, \cdots, f_{\tau(k)}) \\ & \forall \omega \in C^n(\mathcal{E}, R),\ \eta \in C^m(\mathcal{E}, R),\ \forall k,\ \forall e_a \in \mathcal{E},\ \forall f_i \in R. \end{aligned}$$

For any $\omega \in C^n(\mathcal{E}, R)$, define $d\omega = ((d\omega)_0, \cdots, (d\omega)_{[\frac{n+1}{2}]})$ by setting

$$\begin{aligned} & (d\omega)_k(e_1, \cdots, e_{n+1-2k}; f_1, \cdots, f_k) \\ = \ & \sum_a (-1)^{a+1}\rho(e_a)\omega_k(\cdots, \widehat{e_a}, \cdots; f_1, \cdots, f_k) \\ & + \sum_{a<b} (-1)^a \omega_k(\cdots, \widehat{e_a}, \cdots, \widehat{e_b}, e_a \circ e_b, \cdots; \cdots) \\ & + \sum_i \omega_{k-1}(\partial f_i, e_1, \cdots, e_{n+1-2k}; f_1, \cdots, \widehat{f_i}, \cdots, f_k) \\ & \forall k, \ \forall e_a \in \mathcal{E}, \ \forall f_i \in R \end{aligned}$$

The first two terms in the equation above is just the Leibniz cohomology differential d_0 with $\mathcal{E}$ viewed as Leibniz algebra and R viewed as its module

$$l_e f = \rho(e) \cdot f = -r_e f.$$

To be more precise, the Leibniz module involved here is actually

$$\{\alpha \in Hom(\odot^k R, R) | \alpha \text{ is a derivation in each argument}\}.$$

Denoting by $(\delta\omega)_k$ the last term, the equation above can be summarized as

$$d = d_0 + \delta.$$

We have the following:

Proposition 2.19. *The operator d is a derivation of the algebra $C^\bullet(\mathcal{E}, R)$ of degree +1, and it squares to zero.*

Definition 2.20. The cochain complex $(C^\bullet(\mathcal{E}, R), d)$ is called the standard (cochain) complex of Courant-Dorfman algebra $\mathcal{E}$, the resulting cohomology is called the standard cohomology of $\mathcal{E}$, and denoted by $H^\bullet_{st}(\mathcal{E})$.

Remark 2.21. Actually Roytenberg [61] described an alternative point of view of the standard complex and standard cohomology.

The new standard complex is denoted by $(\bar{C}^\bullet(\mathcal{E}, R), \bar{d})$. An n-cochain in $\bar{C}^\bullet(\mathcal{E}, R)$ is a sequence $\bar{\omega} = (\bar{\omega}_0, \cdots, \bar{\omega}_{[\frac{n}{2}]})$, where $\bar{\omega}_k$ is a linear map from $\otimes^{n-2k}\mathcal{E} \otimes \odot^k \Omega^1$ to R, satisfying the following conditions:

1). R-linearity: $\bar{\omega}_k$ is R-linear in each argument in Ω^1,

2). Weak skew-symmetricity (in arguments in $\mathcal{E}$),

3). Weak R-linearity (in arguments in $\mathcal{E}$).

And the codifferential $\bar{d}\bar{\omega} = ((\bar{d}\bar{\omega})_0, \cdots, (\bar{d}\bar{\omega})_{[\frac{n+1}{2}]})$ is given by:

$$\begin{aligned} & (\bar{d}\bar{\omega})_k(e_1, \cdots, e_{n+1-2k}; \alpha_1, \cdots, \alpha_k) \\ = & \sum_a (-1)^{a+1} \rho(e_a) \bar{\omega}_k(\cdots, \widehat{e_a}, \cdots; \cdots) \\ & + \sum_{a<b} (-1)^a \bar{\omega}_k(\cdots, \widehat{e_a}, \cdots, \widehat{e_b}, e_a \circ e_b, \cdots; \cdots) \\ & + \sum_i \bar{\omega}_{k-1}(\rho^* \alpha_i, e_1, \cdots; \cdots, \widehat{\alpha_i}, \cdots) \\ & + \sum_{a,i} (-1)^a \bar{\omega}_k(\cdots, \widehat{e_a}, \cdots; \cdots, \widehat{\alpha_i}, \iota_{\rho(e_a)} d_R \alpha_i, \cdots) \\ & \forall e_a \in \mathcal{E}, \ \forall \alpha_i \in \Omega^1. \end{aligned}$$

Denoting by $(d'\bar{\omega})_k$ the last term in the equation above, then we have

$$\bar{d} = d_0 + \delta + d'.$$

There is a 1-1 correspondence between $\omega \in C^\bullet(\mathcal{E}, R)$ and $\bar{\omega} \in \bar{C}^\bullet(\mathcal{E}, R)$:

$$\bar{\omega}_k(\cdots; d_R f_1, \cdots, d_R f_k) = \omega_k(\cdots; f_1, \cdots, f_k),$$

which induces an isomorphism of complexes.

Therefore, the standard cohomology $H^\bullet_{st}(\mathcal{E})$ can be equivalently defined as the cohomology of $(\bar{C}(\mathcal{E}, R), \bar{d})$.

Now suppose $\mathcal{E}$ is a non-degenerate Courant-Dorfman algebra. Any $e \in \mathcal{E}$ can be identified with $e^\flat \in C^1(\mathcal{E}, R)$ by pseudo-metric. In the following we will construct a 3-cocycle Θ and a bracket $\{\bullet, \bullet\}$ on $C^\bullet(\mathcal{E}, R)$ so that the Dorfman bracket of $\mathcal{E}$ is realized as a derived bracket:

$$(e_1 \circ e_2)^\flat = -\{\{\Theta, e_1^\flat\}, e_2^\flat\}.$$

The construction is also due to Roytenberg [61]:

First, $\Theta = (\Theta_0, \Theta_1)$ is defined as follows:

$$\begin{aligned} \Theta_0(e_1, e_2, e_3) &= (e_1 \circ e_2, e_3) \\ \Theta_1(e; f) &= -(e, \partial f) \end{aligned}$$

Proposition 2.22. *[61]Θ is a cocycle in $C^3(\mathcal{E}, R)$.*

We call Θ the canonical cocycle of $\mathcal{E}$, and its class $[\Theta] \in H^3_{st}(\mathcal{E})$ the canonical class of $\mathcal{E}$.

The bracket $\{\bullet, \bullet\}$ on $C^\bullet(\mathcal{E}, R)$ for a non-degenerate Courant-Dorfman algebra $\mathcal{E}$ is constructed as follows (actually the construction applies to any R-module with strongly non-degenerate bilinear form):

$\forall \omega \in C^n(\mathcal{E}, R)$,

$$\omega_k : \otimes^{n-2k}\mathcal{E} \otimes \odot^k R \to R$$

is R-linear in the $(n-2k)$-th argument, so it gives rise to a linear map

$$\tilde{\omega}_k : \otimes^{n-2k-1}\mathcal{E} \to Der(\odot^k R, \mathcal{E}^\vee),$$

where any element in $Der(\odot^k R, \mathcal{E}^\vee) \cong Hom_R(S^k\Omega^1, \mathcal{E}^\vee)$ is a linear map $\odot^k R \to \mathcal{E}^\vee$ which is a derivation in each argument R.

$\tilde{\omega}_k$ is defined as follows:

$$\tilde{\omega}_k(e_1 \cdots e_{n-2k-1})(f_1 \cdots f_k)(e) = \omega_k(e_1 \cdots e_{n-2k-1}, e; f_1 \cdots f_k).$$

Composing $(\cdot)^\sharp : \mathcal{E}^\vee \to \mathcal{E}$ with $\tilde{\omega}_k$, the resulting map is denoted by

$$\omega_k^\sharp = (\tilde{\omega}_k)^\sharp : \otimes^{n-2k-1}\mathcal{E} \to Der(\odot^k R, \mathcal{E}).$$

$\forall \alpha \in Der(\odot^i R, \mathcal{E}),\ \beta \in Der(\odot^j R, \mathcal{E})$, define $\langle \alpha \cdot \beta \rangle \in Der(\odot^{i+j} R, R)$ as follows:

$$\begin{aligned} &\langle \alpha \cdot \beta \rangle (f_1, \cdots, f_{i+j}) \\ = &\sum_{\sigma \in sh(i,j)} (\alpha(f_{\sigma(1)}, \cdots, f_{\sigma(i)}), \beta(f_{\sigma(i+1)}, \cdots, f_{\sigma(i+j)})) \end{aligned}$$

$\forall \gamma \in Der(\odot^i R, R),\ \delta \in Der(\odot^j R, R)$, define $\gamma \circ \delta \in Der(\odot^{i+j-1} R, R)$ as follows:

$$\begin{aligned} &(\gamma \circ \delta)(f_1, \cdots, f_{i+j-1}) \\ = &\sum_{\sigma \in sh(j,i-1)} \gamma(\delta(f_{\sigma(1)}, \cdots, f_{\sigma(j)}), f_{\sigma(j+1)}, \cdots, f_{\sigma(i+j-1)}) \end{aligned}$$

Finally given $\omega \in C^n(\mathcal{E}, R),\ \eta \in C^m(\mathcal{E}, R)$, the bracket $\{\omega, \eta\} \in C^{n+m-2}(\mathcal{E}, R)$ is defined by:

$$\{\omega, \eta\} = \omega \bullet \eta + \omega \diamond \eta - (-1)^{nm} \eta \diamond \omega$$

where $\omega \bullet \eta = ((\omega \bullet \eta)_0, (\omega \bullet \eta)_1, \cdots)$ with $(\omega \bullet \eta)_k : \otimes^{n+m-2-2k}\mathcal{E} \to Der(\odot^k R, R)$ defined by:

$$\begin{aligned} &(\omega \bullet \eta)_k(e_1, \cdots, e_{n+m-2-2k}) \\ = &(-1)^{m-1} \sum_{i+j=k} \sum_{\sigma \in sh(n-2i-1, m-2j-1)} (-1)^\sigma \\ &\langle \omega_i^\sharp(e_{\sigma(1)}, \cdots, e_{\sigma(n-2i-1)}) \cdot \eta_j^\sharp(e_{\sigma(n-2i)}, \cdots, e_{\sigma(n+m-2-2k)}) \rangle, \end{aligned}$$

and $\omega \diamond \eta = ((\omega \diamond \eta)_0, (\omega \diamond \eta)_1, \cdots)$ with $(\omega \diamond \eta)_k : \otimes^{n+m-2-2k}\mathcal{E} \to Der(\odot^k R, R)$ defined by:

$$\begin{aligned} & (\omega \diamond \eta)_k(e_1, \cdots, e_{n+m-2-2k}) \\ = & \sum_{i+j=k} \sum_{\sigma \in sh(n-2i-2, m-2j)} (-1)^\sigma \\ & \omega_{i+1}(e_{\sigma(1)}, \cdots, e_{\sigma(n-2i-2)}) \circ \eta_j(e_{\sigma(n-2i-1)}, \cdots, e_{\sigma(n+m-2-2k)}). \end{aligned}$$

Θ and $\{\bullet, \bullet\}$ constructed above satisfy the following:

Theorem 2.23. *([61]) With the above notations, we have:*

(1). $\{\cdot, \cdot\}$ is a non-degenerate Poisson bracket on the algebra $C^\bullet(\mathcal{E}, R)$ of degree -2;

(2). $\{\Theta, \Theta\} = 0$;

(3). $d\omega = -\{\Theta, \omega\}, \ \forall \omega \in C^\bullet(\mathcal{E}, R)$;

and

Theorem 2.24. *([61]) With the above notations, we have*

$$(e_1 \circ e_2)^\flat = -\{\{\Theta, e_1^\flat\}, e_2^\flat\}, \quad \forall e_1, e_2 \in \mathcal{E},$$

which implies that the Dorfman bracket of $\mathcal{E}$ is a derived bracket.

The following theorem asserts the equivalence between the standard cohomology of E (Definition A.8) and the standard cohomology of $\mathcal{E} = \Gamma(E)$:

Theorem 2.25. *([61])Suppose $E \to M$ is a Courant algebroid, $\mathcal{E} = \Gamma(E)$, $R = C^\infty(M)$. Then the map*

$$\Phi : \ (C^\bullet_{st}(E) := A^\bullet, \{H, \cdot\}) \to (C^\bullet(\mathcal{E}, R), d)$$

given, for any $\omega \in C^n_{st}(E)$, by

$$\begin{aligned} & (\Phi\omega)_k(e_1, \cdots, e_{n-2k}; f_1, \cdots, f_k) \\ = & (-1)^{\frac{(n-2k)(n-2k-1)}{2}} \{\cdots\{\omega, e_1^\flat\}, \cdots\}, e_{n-2k}^\flat\}, f_1\}, \cdots f_k\} \end{aligned}$$

is an isomorphism of graded Poisson algebras.

It is easily checked that the image of the cubic Hamiltonian H under the isomorphism map Φ is $-\Theta$. So the equations in Theorem 2.23 and Theorem 2.24 correspond to the facts that

$$\begin{aligned} \{H, H\} &= 0, \\ Q &= \{H, \cdot\}, \\ (e_1 \circ e_2)^\flat &= \{\{H, e_1^\flat\}, e_2^\flat\}. \end{aligned}$$

Next, we direct our attention to the definition of naive cohomologies, for both Courant algebroids and Courant-Dorfman algebras.

Given a Courant algebroid E, mimicking the definition of Chevalley-Eilenberg differential, we can consider the operator

$$d_{nv} : \Gamma(\wedge^{\bullet} ker\rho) \to \Gamma(\wedge^{\bullet+1} E)$$

defined by:

$$\begin{aligned} & (d_{nv}\omega, e_1 \wedge \cdots \wedge e_{n+1}) \\ \triangleq \; & \sum_{1 \le a \le n+1} (-1)^{a+1} \rho(e_a)(\omega, e_1 \wedge \cdots \wedge \widehat{e_a} \cdots \wedge e_{n+1}) \\ & + \sum_{1 \le a < b \le n+1} (-1)^{a+b} (\omega, (e_a \circ e_b) \wedge e_1 \wedge \cdots \wedge \widehat{e_a} \cdots \wedge \widehat{e_b} \cdots \wedge e_{n+1}) \\ & \forall \omega \in \Gamma(\wedge^n ker\rho), \; \forall e_a \in \Gamma(E). \end{aligned}$$

Note that we identify $\wedge^{\bullet} E$ and $\wedge^{\bullet} E^*$ by pseudo-metric in the above.

Stienon-Xu [68] proved the following:

Lemma 2.26. *$d_{nv}\Gamma(\wedge^n ker\rho) \subset \Gamma(\wedge^{n+1} ker\rho)$, and $(\Gamma(\wedge^{\bullet} ker\rho), d_{nv})$ is a cochain complex.*

And they gave the following definition:

Definition 2.27. The cohomology of $(\Gamma(\wedge^{\bullet} ker\rho), d_{nv})$ is called the naive cohomology of E, and denoted by $H^{\bullet}_{nv}(E)$.

Elements of $\Gamma(\wedge^{\bullet} E)$ can be viewed as super functions on $E[1]$ via the pseudo-metric, and can further be pulled back to the minimal symplectic realization $\mathbf{E}$. So we can identify $\Gamma(\wedge^{\bullet} ker\rho)$ with a subalgebra of $A^{\bullet}$ (see appendix for the construction of $\mathbf{E}$ and $A^{\bullet}$). Stienon-Xu [68] proved the following:

Lemma 2.28. *1). If $\omega \in \Gamma(\wedge^n ker\rho)$, then $Q\omega = d_{nv}\omega$.*

2). If $\omega \in \Gamma(\wedge^n E)$ satisfies $Q\omega = 0$, then $\omega \in \Gamma(\wedge^n ker\rho)$ and $d_{nv}\omega = 0$.

Thus the naive complex $(\Gamma(\wedge^{\bullet} ker\rho), d_{nv})$ could be viewed as a subcomplex of the standard complex $(A^{\bullet}, Q)$, and we have a homomorphism

$$\phi : H^{\bullet}_{nv}(E) \to H^{\bullet}_{st}(E).$$

The following theorem is conjectured by Stienon-Xu [68], and proved by Ginot-Grutzmann in [26] using the tool of spectral sequence:

Theorem 2.29. *If E is a transitive Courant algebroid, ϕ is an isomorphism.*

Actually Ginot-Grutzmann [26] computed the standard cohomology for more general case: a Courant algebroid $E \to M$ is said to have split base iff $M \cong L \times N$ and $im\rho \cong TL \times N \subset TM$.

Theorem 2.30. *The standard cohomology of a Courant algebroid E with split base is given by*

$$H^n_{st}(E) \cong \bigoplus_{l+m=n} H^l_{nv}(E)/(T_3) \otimes \mathfrak{X}^{kil,m}(N),$$

where (T_3) is the ideal in $H^\bullet_{nv}(E)$ generated by the image of

$$\begin{array}{rcl} T_3 : \mathcal{X}(N) & \to & H^3_{nv}(E) \\ T_3(q) & \triangleq & [\{H, q\}] \end{array}$$

($q \in \mathcal{X}(N)$ is viewed as a degree 2 graded function), and $\mathfrak{X}^{kil,m}(N)$ is the space of "symmetric Killing multivector fields" $S^{m/2}_{C^\infty(N)}(\mathcal{X}^{kil}(N))$ with the convention that $\mathfrak{X}^{kil,m}(N) = \{0\}$ for odd m, $\mathcal{X}^{kil}(N)$ is the kernel of T_3 (elements of it are called Killing vector fields).

Remark 2.31. When E is a regular Courant algebroid, it is easily seen that the naive complex $(\Gamma(\wedge^\bullet ker\rho), d_{nv})$ coincides with the Chevalley-Eilenberg complex of the ample Lie algebroid $E/(ker\rho)^\perp$. Thus the naive cohomology of E is isomorphic to the Lie algebroid cohomology of $E/(ker\rho)^\perp$.

2.3 Equivariant cohomology theory

In this section, we give the definitions of $\mathfrak{g}$-differential complex, Weil algebra, equivariant cohomology, etc, and then we introduce the equivariant de Rham theorem. For more details, we refer to [15, 24, 32, 33, 27].

Throughout this section, let G be a Lie group, $\mathfrak{g}$ be its Lie algebra, $\{\xi_i\}_{1\leq i\leq n}$ be a basis of $\mathfrak{g}$ and M be a manifold acted on by G.

We consider the de Rham complex $(\Omega^\bullet(M), d)$. The G action on M naturally induces an action ρ of G on $\Omega^\bullet(M)$. The infinitesimal action of $\mathfrak{g}$ on $\Omega^\bullet(M)$ is defined by

$$L_\xi \omega := \frac{d}{dt}|_{t=0}\rho(exp(t\xi))\omega = \frac{d}{dt}|_{t=0}(exp(-t\xi))^*\omega.$$

Denote by $\hat{\xi}$ the vector field

$$\hat{\xi}(x) \triangleq \frac{d}{dt}|_{t=0}(exp(-t\xi))(x), \quad \forall x \in M,$$

we see that L_ξ is simply the Lie derivative along $\hat{\xi}$. We call $\hat{\xi}$ the vector field corresponding to ξ on M. Denote by ι_ξ the interior product by the vector field $\hat{\xi}$. It is a well-known result that, $\rho(a)$ ($\forall a \in G$) acts on $\Omega^\bullet(M)$ as automorphism, L_ξ, ι_ξ, d ($\forall \xi \in \mathfrak{g}$) act on $\Omega^\bullet(M)$ as derivations, and they satisfy the following equations:

$$\begin{aligned} \rho(a) \circ d \circ \rho(a^{-1}) &= d \\ \rho(a) \circ \iota_\xi \circ \rho(a^{-1}) &= \iota_{Ad_a \xi} \\ \rho(a) \circ L_\xi \circ \rho(a^{-1}) &= L_{Ad_a \xi} \\ L_\xi &= d \circ \iota_\xi + \iota_\xi \circ d \\ L_\xi \circ d &= d \circ L_\xi \\ \iota_\xi \circ \iota_\eta &= -\iota_\eta \iota_\xi \\ L_{[\xi,\eta]} &= [L_\xi, L_\eta] \\ \iota_{[\xi,\eta]} &= [L_\xi, \iota_\eta]. \end{aligned} \tag{2.3.1}$$

Motivated by the example above, we have the following:

Definition 2.32. A G differential complex (or G^* module by some authors) is a complex $(A^\bullet, d)$ equipped with a G action $\rho : G \to Gl(A^\bullet)$ of degree 0, and a linear map $\iota : \mathfrak{g} \to gl(A^\bullet)$ of degree -1, such that the following conditions hold for any $a \in G,\ \xi, \eta \in \mathfrak{g}$:

1). $\rho(a) \circ d \circ \rho(a^{-1}) = d$;

2). $\rho(a) \circ \iota_\xi \circ \rho(a^{-1}) = \iota_{Ad_a \xi}$;

3). $L_\xi = \iota_\xi \circ d + d \circ \iota_\xi$;

($L_\xi := \frac{d}{dt}|_{t=0} \rho(exp(t\xi))$ is the infinitesimal action of $\mathfrak{g}$ on $A^\bullet$)

4). $\iota_\xi \circ \iota_\eta + \iota_\eta \circ \iota_\xi = 0$.

A G differential algebra (or G^* algebra by some authors) is a differential graded commutative algebra $(A^\bullet, d)$ which is a G differential complex with the additional condition that G acts as algebra automorphisms and ι_ξ acts on $A^\bullet$ as derivation for any $\xi \in \mathfrak{g}$ (thus L_ξ also acts as derivation).

From the conditions in the definition above, it is easy to deduce the other equations in 2.3.1.

If instead of a G action, $(A^\bullet, d)$ carries only an infinitesimal action of $\mathfrak{g}$, we have the following:

Definition 2.33. A $\mathfrak{g}$ differential complex is a complex $(A^\bullet, d)$ equipped with a $\mathfrak{g}$-action $L : \mathfrak{g} \to gl(A^\bullet)$ of degree 0, and a linear map $\iota : \mathfrak{g} \to gl(A^\bullet)$ of degree -1, such that the following conditions hold for any $\xi, \eta \in \mathfrak{g}$:

1). $L_\xi \circ d = d \circ L_\xi$;

2). $[L_\xi, \iota_\eta] = \iota_{[\xi,\eta]}$;
3). $L_\xi = d \circ \iota_\xi + \iota_\xi \circ d$;
4). $\iota_\xi \circ \iota_\eta + \iota_\eta \circ \iota_\xi = 0$.

A $\mathfrak{g}$ differential algebra is a differential graded commutative algebra $(A^\bullet, d)$ which is a $\mathfrak{g}$ differential complex with the additional condition that ι_ξ acts on $A^\bullet$ as derivation for any $\xi \in \mathfrak{g}$ (thus $L_\xi$also acts as derivation).

$(\Omega^\bullet(M), d)$ is obviously a $G(\mathfrak{g})$ differential algebra, with L_ξ being the Lie derivative along the corresponding vector field $\hat{\xi}$, and ι_ξ being the interior product by $\hat{\xi}$.

Remark 2.34. Actually the data of a $\mathfrak{g}$ differential complex $(A^\bullet, d)$ can be encoded in an action of the graded Lie algebra $\tilde{\mathfrak{g}} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ on $A^\bullet$, where $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$ are copies of $\mathfrak{g}$ as vector spaces, $\mathfrak{g}_1$ is one-dimensional vector space with distinguished generator d. $\forall \xi \in \mathfrak{g}$, we denote the corresponding element in $\mathfrak{g}_{-1}$ by ι_ξ and the corresponding element in $\mathfrak{g}_0$ by L_ξ. The Lie brackets of $\tilde{\mathfrak{g}}$ are given by the Cartan commutation relations:

$$\begin{aligned} [\iota_\xi, \iota_\eta] &= 0 \\ [L_\xi, L_\eta] &= L_{[\xi,\eta]} \\ [d, d] &= 0 \\ [L_\xi, \iota_\eta] &= \iota_{[\xi,\eta]} \\ [d, \iota_\xi] &= L_\xi \\ [L_\xi, d] &= 0 \end{aligned}$$

Let's return to the prototypical example. If G is a compact Lie group, and it acts freely on a manifold M, then the quotient M/G is also a manifold, so we have the de Rham cohomology $H^\bullet_{dR}(M/G) = H^\bullet(\Omega^\bullet(M/G), d)$. For general case, when the action is not free, the quotient M/G is no longer a manifold, but we can take $(M \times EG)/G$ as a substitute, where EG is the universal bundle of G. EG is contractible and G acts freely on EG, so $M \times EG$ is homotopy equivalent to M and the diagonal action of G on $M \times EG$ is also free, thus $(M \times EG)/G$ is a manifold. We define the equivariant cohomology $H^\bullet_{eq}(M)$ to be the de Rham cohomology $H^\bullet_{dR}((M \times EG)/G)$. This is the geometric version of equivariant cohomology.

For a G(or $\mathfrak{g}$) differential complex $(A^\bullet, d)$, we want to find the algebraic substitutes for "free action", "quotient" and the universal bundle EG, and define the equivariant cohomology $H^\bullet_{eq}(A^\bullet)$ analogously.

We know that an action of G on M is said to be locally free, iff the corresponding infinitesimal action of $\mathfrak{g}$ is free, i.e. the map

$$\begin{array}{rcl}\mathfrak{g} & \to & \mathfrak{X}(M) \\ \xi & \mapsto & \hat{\xi}\end{array}$$

is injective. Let $\{\xi_i\}_{1\leq i\leq n}$ be a basis of $\mathfrak{g}$, the duals of the corresponding vector fields $\hat{\xi}_i$, denoted by θ^i $(\in \Omega^1(M))$, satisfy $\iota_{\xi_i}\theta^j = \delta_i^j$. Moreover if G is compact, by averaging over the group if necessary, we can arrange that $\{\theta^i\}$ transform like the coadjoint representation, i.e. $L_{\xi_i}\theta^j = -C_{ik}^j\theta^k$, where C_{ik}^js are the structure constants of $\mathfrak{g}$. Conversely, if there are differential 1 forms $\{\theta^i\}_{1\leq i\leq n}$ such that $\iota_{\xi_i}\theta^j = \delta_i^j,\ \forall i, j$, the action is obviously locally free. These lead to the following:

Definition 2.35. A G(or $\mathfrak{g}$) differential complex $(A^\bullet, d)$ is said to be locally free, if there exist $\{\theta^i\}_{1\leq i\leq n} \in A^1$, such that

$$\iota_{\xi_i}\theta^j = \delta_i^j, \quad \forall i, j.$$

If moreover the space spanned by $\{\theta^i\}$ is invariant under G(or $\mathfrak{g}$), we say that $(A^\bullet, d)$ is of type (C).

Next we consider the substitute for "quotient". Suppose G acts freely on M, denote by π the projection map $M \to M/G$. It is easily proved that the image of

$$\pi^* : \Omega^\bullet(M/G) \to \Omega^\bullet(M)$$

consists of all forms $\omega \in \Omega^\bullet(M)$ satisfying

$$\begin{array}{rcl}\iota_\xi\omega & = & 0 \\ L_\xi\omega & = & 0,\end{array}$$

$\forall \xi \in \mathfrak{g}$. This leads to the following:

Definition 2.36. An element ω in a G(or $\mathfrak{g}$) differential complex $(A^\bullet, d)$ is called a basis element if it satisfies

$$\begin{array}{rcl}\iota_\xi\omega & = & 0 \\ L_\xi\omega & = & 0,\end{array}$$

$\forall \xi \in \mathfrak{g}$. All basic elements form a subcomplex of $(A^\bullet, d)$, called the basic complex, and denoted by $A^\bullet_{bas}$.

Finally we consider the substitute for the universal bundle EG. Since EG is contractible and G acts freely on EG, an ideal substitute must be a G differential algebra $E^\bullet$ which is acyclic and of type (C), and then we can define the equivariant cohomology of a G(or $\mathfrak{g}$) differential complex $(A^\bullet, d)$ to be $H^\bullet((A^\bullet \otimes E^\bullet)_{bas})$: the cohomology of the basic complex of $A^\bullet \otimes E^\bullet$. The following theorem gives an example for such a substitute:

Theorem 2.37. *Weil algebra $(W^\bullet(\mathfrak{g}), d_W)$ is an acyclic G differential algebra of type (C).*

Weil algebra $W^\bullet(\mathfrak{g})$ (or $W^\bullet$ for simplicity) in the above theorem is constructed as follows:

Let

$$W^n := \bigoplus_{p+2q=n} \wedge^p \mathfrak{g}^* \otimes S^q(\mathfrak{g}^*).$$

The dual basis of $\{\xi_i\}$ in $\wedge^1 \mathfrak{g}^* \otimes S^0(\mathfrak{g}^*)$ is denoted by $\{\theta^i\}$, and the dual basis of $\{\xi_i\}$ in $\wedge^0 \mathfrak{g}^* \otimes S^1(\mathfrak{g}^*)$ is denoted by $\{\mu^i\}$. Obviously θ^i, μ^j generate $W^\bullet$ as an algebra.

Let

$$\begin{aligned} d_W \theta^k &= -\frac{1}{2} C^k_{ij} \theta^i \theta^j + \mu^k \\ d_W \mu^k &= -C^k_{ij} \theta^i \mu^j \end{aligned}$$

then $(W^\bullet, d_W)$ form a differential graded algebra.

Moreover, let G acts on $W^\bullet$ via the coadjoint representation, and let the interior product be taken on the $\wedge^\bullet \mathfrak{g}^*$ component, i.e.

$$\begin{aligned} \iota_{\xi_i} \theta^j &= \delta^j_i \\ \iota_{\xi_i} \mu^j &= 0 \\ L_{\xi_i} \theta^j &= -C^j_{ik} \theta^k \\ L_{\xi_i} \mu^j &= -C^j_{ik} \mu^k \end{aligned}$$

then $(W^\bullet, d_W)$ become a G differential algebra.

Definition 2.38. The equivariant cohomology of a G(or $\mathfrak{g}$) differential complex $(A^\bullet, d)$ is defined to be

$$H^\bullet_{eq}(A^\bullet) \triangleq H^\bullet((A^\bullet \otimes W^\bullet)_{bas}).$$

This definition is also called the Weil model for equivariant cohomology. There is an equivalent definition: the Cartan model for equivariant cohomology.

Definition 2.39. For any G(or $\mathfrak{g}$) differential complex $(A^\bullet, d)$, let

$$C^n_{\mathfrak{g}}(A) = \bigoplus_{p+2q=n} (A^p \otimes S^q(\mathfrak{g}^*))^{\mathfrak{g}},$$

the $\mathfrak{g}$ invariant elements in $A^p \otimes S^q(\mathfrak{g}^*)$(with diagonal action).

Let $d_{\mathfrak{g}} : C^\bullet_{\mathfrak{g}}(A) \to C^{\bullet+1}_{\mathfrak{g}}(A)$ be the map:

$$d_{\mathfrak{g}} = d \otimes 1 - \iota_{\xi_i} \otimes \mu^i,$$

then $(C^\bullet_{\mathfrak{g}}(A), d_{\mathfrak{g}})$ is a complex.

The resulting cohomology is called (the Cartan model for) the equivariant cohomology of $A^\bullet$, denoted by $H^\bullet_{\mathfrak{g}}(A^\bullet)$.

Actually any $\omega \in C^\bullet_{\mathfrak{g}}(A)$ can be viewed as an $A^\bullet$-valued polynomial function on $\mathfrak{g}$, and the differential $d_{\mathfrak{g}}$ is interpreted as

$$(d_{\mathfrak{g}}\omega)(\xi) = d(\omega(\xi)) - \iota_\xi \omega(\xi), \ \forall \xi \in \mathfrak{g}.$$

The following theorem tells the equivalence of the above two definitions:

Theorem 2.40. $H^\bullet((A^\bullet \otimes W^\bullet)_{bas}) \cong H^\bullet_{\mathfrak{g}}(A^\bullet)$.

The proof of this theorem uses Mathai-Quillen isomorphism, for details we refer to [32, 33].

We know that when the action of a compact Lie group G on M is free, the equivariant cohomology $H^\bullet_{eq}(M)$ is isomorphic to the de Rham cohomology of the quotient manifold M/G. Analogously we have the following:

Theorem 2.41. *If $(A^\bullet, d)$ is a G(or $\mathfrak{g}$) differential algebra of type (C), then*

$$H^\bullet_{eq}(A^\bullet) \cong H^\bullet(A^\bullet_{bas}).$$

This theorem can further be generalized to:

Theorem 2.42. *If $(A^\bullet, d_A)$ is a G(or $\mathfrak{g}$) differential algebra of type (C), and $(B^\bullet, d_B)$ is a G(or $\mathfrak{g}$) differential $A^\bullet$-module , then*

$$H^\bullet_{eq}(B^\bullet) \cong H^\bullet(B^\bullet_{bas}).$$

A G(or $\mathfrak{g}$) differential $A^\bullet$-module means a differential graded $A^\bullet$-module $(B^\bullet, d_B)$ with action map $A^\bullet \otimes B^\bullet \to B^\bullet$ being a homomorphism of G(or $\mathfrak{g}$) differential complexes (i.e. commuting with $\iota_\xi, L_\xi, d \ \forall \xi \in \mathfrak{g}$).

Chapter 3

Standard Cohomology

In this chapter, we define standard cohomology and naive cohomology of Leibniz algebra analogously to the case of Courant-Dorfman algebra. And we prove a similar result to proposition 2.29, asserting the isomorphism between standard and naive cohomologies of Leibniz algebra. We also discuss the relation between standard cohomology of Leibniz algebra and Courant-Dorfman algebra.

3.1 Definitions and properties

Given a Leibniz algebra L with left center Z and bilinear product $(\bullet, \bullet)$. Let $h \supseteq Z$ be an isotropic ideal in L. Let R be a left module of L on which h acts trivially. For example, Z is such an module with left action ρ induced by the Leibniz bracket of L (see Proposition 2.8). By abuse of notation, we still denote by ρ the left action of L on R.

Denote by $C^\bullet(L, h, R)$ (or $C^\bullet(L)$ for short if it causes no confusion) the graded vector space with degree n subspace defined as:

$$C^n(L) = \bigoplus_{p+2q=n} Hom(\otimes^p L \otimes \odot^q h, R) \cong \bigoplus_{p+2q=n} Hom(\otimes^p L, R) \otimes S^q(h^*).$$

Any $\omega \in C^n(L)$ can be regarded as a sequence $(\omega_0, \omega_1, \cdots \omega_{[\frac{n}{2}]})$, where ω_k is a linear map

$$(\otimes^{n-2k} L) \otimes (\odot^k h) \to R.$$

$\omega(\omega_k$ respectively) can also be viewed as a polynomial map from h to $Hom(\otimes^\bullet L, R)(Hom(\otimes^{n-2k} L, R)$ respectively). Taking the symmetric extension of the left action, R becomes an L-module. We denote by d_0 the coboundary differential of the corresponding Leibniz cohomology. Obviously $C^\bullet(L)$ becomes a cochain complex under the coboundary differential $d_0 \otimes id$. We will abuse the notation and simply write $d_0 \otimes id$ as d_0 from now on.

Now consider a graded subspace $C^\bullet_{st}(L,h,R)$ of $C^\bullet(L)$ defined as follows:

$$\begin{aligned} & C^n_{st}(L,h,R) \\ = \ & \{\omega \in C^n(L) | \omega_k(e_1,\cdots,e_i,e_{i+1},\cdots,e_{n-2k};f_1,\cdots,f_k) + \omega_k(\cdots,e_{i+1},e_i,\cdots;\cdots) \\ & \quad = -\omega_{k+1}(\cdots,\widehat{e_i},\widehat{e_{i+1}},\cdots;(e_i,e_{i+1}),\cdots), \ \forall k\}. \end{aligned}$$

$\forall \omega = (\omega_0,\cdots,\omega_k,\omega_{k+1},\cdots) \in C^n_{st}(L,h,R)$, we say that ω_k is "weakly skew-symmetric" up to ω_{k+1}.

We'll write $C^\bullet_{st}(L,h,R)$ simply as $C^\bullet_{st}(L)$ if it causes no confusion.

$C^\bullet_{st}(L)$ is not a subcomplex of $(C^\bullet(L),d_0)$ because it is not closed under d_0, but we have the following:

Theorem 3.1. *$C^\bullet_{st}(L)$ is a cochain complex with coboundary differential $d = d_0 + d' + \delta$,*

where $\delta: \ C^\bullet(L) \to C^{\bullet+1}(L)$ is the operator defined by:

$$\begin{aligned} & (\delta\omega)_k(e_1,\cdots,e_{n+1-2k};f_1,\cdots,f_k) \\ = \ & \sum_{1\le j\le k} \omega_{k-1}(f_j,e_1,\cdots,e_{n+1-2k};f_1,\cdots,\hat{f_j},\cdots f_k) \\ & \forall n, \ \forall \omega \in C^n(L), \ \forall k \le [\frac{n+1}{2}] \end{aligned}$$

($(\delta\omega)_0$ is defined to be 0),

and $d': \ C^\bullet(L) \to C^{\bullet+1}(L)$ is the operator defined by:

$$\begin{aligned} & (d'\omega)_k(e_1,\cdots,e_{n+1-2k};f_1,\cdots,f_k) \\ = \ & \sum_{1\le a\le n+1-2k} \sum_{1\le j\le k} (-1)^{a+1}\omega_k(e_1,\cdots\hat{e_a},\cdots e_{n+1-2k};f_1,\cdots\hat{f_j},f_j\circ e_a,\cdots f_k) \\ & \forall n, \ \forall \omega \in C^n(L), \ \forall k \le [\frac{n+1}{2}]. \end{aligned}$$

First, we prove two lemmas.

Lemma 3.2. *$C^\bullet_{st}(L)$ is closed under $d_0 + \delta$.*

Proof. We need to prove:

$$\begin{aligned} & ((d_0+\delta)\omega)_k(e_1\cdots e_i,e_{i+1}\cdots e_{n+1-2k};f_1\cdots f_k) + ((d_0+\delta)\omega)_k(\cdots e_{i+1},e_i\cdots;\cdots) \\ + \ & ((d_0+\delta)\omega)_{k+1}(\cdots,\widehat{e_i},\widehat{e_{i+1}},\cdots;(e_i,e_{i+1}),\cdots) = 0 \end{aligned}$$

The d_0 part equals:

$$\begin{aligned}
&\sum_{a\neq i,i+1}(-1)^{a+1}\rho(e_a)(\omega_k(\cdots\widehat{e_a}\cdots e_i,e_{i+1}\cdots;\cdots)+\omega_k(\cdots\widehat{e_a}\cdots e_{i+1},e_i\cdots;\cdots)\\
&+\omega_{k+1}(\cdots,\widehat{e_a},\cdots,\widehat{e}_i,\widehat{e_{i+1}},\cdots;(e_i,e_{i+1}),\cdots))\\
&+(-1)^{i+1}\rho(e_i)\omega_k(\cdots,\widehat{e}_i,e_{i+1},\cdots;\cdots)+(-1)^i\rho(e_{i+1})\omega_k(\cdots,e_i,\widehat{e_{i+1}},\cdots;\cdots)\\
&+(-1)^{i+1}\rho(e_{i+1})\omega_k(\cdots,\widehat{e_{i+1}},e_i,\cdots;\cdots)+(-1)^i\rho(e_i)\omega_k(\cdots,e_{i+1},\widehat{e}_i,\cdots;\cdots)\\
&+\sum_{(a<b)\neq i,i+1}(-1)^a\Big(\omega_k(\cdots,\widehat{e_a},\cdots,e_a\circ e_b,\cdots,e_i,e_{i+1},\cdots;\cdots)\\
&\quad+\omega_k(\cdots,\widehat{e_a},\cdots,e_a\circ e_b,\cdots,e_{i+1},e_i,\cdots;\cdots)\\
&\quad+\omega_{k+1}(\cdots,\widehat{e_a},\cdots,e_a\circ e_b,\cdots,\widehat{e}_i,\widehat{e_{i+1}},\cdots;(e_i,e_{i+1}),\cdots)\Big)\\
&+\sum_{b>i+1}((-1)^i+(-1)^{i+1})\omega_k(\cdots,\widehat{e}_i,e_{i+1},\cdots,e_i\circ e_b\cdots;\cdots)\\
&+\sum_{b>i+1}((-1)^{i+1}+(-1)^i)\omega_k(\cdots,e_i,\widehat{e_{i+1}},\cdots,e_{i+1}\circ e_b\cdots;\cdots)\\
&+\sum_{a<i}(-1)^a\Big(\omega_k(\cdots\widehat{e_a}\cdots e_a\circ e_i,e_{i+1}\cdots;\cdots)+\omega_k(\cdots\widehat{e_a}\cdots e_{i+1},e_a\circ e_i\cdots;\cdots)\Big)\\
&+\sum_{a<i}(-1)^a\Big(\omega_k(\cdots\widehat{e_a}\cdots e_i,e_a\circ e_{i+1}\cdots;\cdots)+\omega_k(\cdots\widehat{e_a}\cdots e_a\circ e_{i+1},e_i\cdots;\cdots)\Big)\\
&+(-1)^i\omega_k(\cdots,e_i\circ e_{i+1},\cdots;\cdots)+(-1)^i\omega_k(\cdots,e_{i+1}\circ e_i,\cdots;\cdots)\\
=\ &\sum_{a<i}(-1)^{a+1}\omega_{k+1}(\cdots,\widehat{e_a},\cdots,\widehat{e}_i,\widehat{e_{i+1}},\cdots;(e_a\circ e_i,e_{i+1})+(e_i,e_a\circ e_{i+1}),\cdots)\\
&+(-1)^i\omega_k(\cdots\widehat{e}_i,\widehat{e_{i+1}},(e_i,e_{i+1}),\cdots;\cdots)\\
=\ &\sum_{a<i}(-1)^{a+1}\omega_{k+1}(\cdots,\widehat{e_a},\cdots,\widehat{e}_i,\widehat{e_{i+1}},\cdots;(e_a,(e_i,e_{i+1})),\cdots)\\
&+(-1)^i\omega_k(\cdots\widehat{e}_i,\widehat{e_{i+1}},(e_i,e_{i+1}),\cdots;\cdots)\\
=\ &\sum_{a<i}(-1)^a\Big(\omega_k(\cdots,e_a,(e_i,e_{i+1}),e_{a+1},\cdots,\widehat{e}_i,\widehat{e_{i+1}},\cdots;\cdots)\\
&\quad+\omega_k(\cdots,e_{a-1},(e_i,e_{i+1}),e_a,\cdots,\widehat{e}_i,\widehat{e_{i+1}},\cdots;\cdots)\Big)\\
&+(-1)^i\omega_k(\cdots\widehat{e}_i,\widehat{e_{i+1}},(e_i,e_{i+1}),\cdots;\cdots)\\
=\ &-\omega_k((e_i,e_{i+1}),e_1,\cdots,\widehat{e}_i,\widehat{e_{i+1}},\cdots;\cdots) \qquad (3.1.1)
\end{aligned}$$

The δ part equals:

$$\begin{aligned}
&\sum_j\Big(\omega_{k-1}(f_j,e_1\cdots e_i,e_{i+1}\cdots;\cdots\hat{f}_j\cdots)+\omega_{k-1}(f_j,e_1\cdots e_{i+1},e_i\cdots;\cdots\hat{f}_j\cdots)\Big)\\
&+\sum_j\omega_k(f_j,e_1,\cdots,\widehat{e}_i,\widehat{e_{i+1}},\cdots;(e_i,e_{i+1}),\cdots,\hat{f}_j,\cdots)
\end{aligned}$$

$$
\begin{aligned}
&+\omega_k((e_i,e_{i+1}),e_1,\cdots,\widehat{e_i},\widehat{e_{i+1}},\cdots;\cdots)\\
=\ &\omega_k((e_i,e_{i+1}),e_1,\cdots,\widehat{e_i},\widehat{e_{i+1}},\cdots;\cdots)
\end{aligned}
$$

So their sum is 0, the lemma is proved. ∎

Lemma 3.3. *$C^n_{st}(L)$ is closed under d'.*

Proof.

$$
\begin{aligned}
&(d'\omega)_k(e_1\cdots e_i,e_{i+1}\cdots e_{n+1-2k};f_1\cdots f_k)+(d'\omega)_k(e_1\cdots e_{i+1},e_i\cdots e_{n+1-2k};\cdots)\\
=\ &\sum_{j,a\neq i,i+1}(-1)^{a+1}\Big(\omega_k(\cdots,\hat{e_a},\cdots,e_i,e_{i+1},\cdots;\cdots,f_j\circ e_a,\cdots)\\
&\quad+\omega_k(\cdots,\hat{e_a},\cdots,e_{i+1},e_i,\cdots;\cdots,f_j\circ e_a,\cdots)\Big)\\
&+\sum_j(-1)^{i+1}\omega_k(\cdots,\hat{e_i},e_{i+1},\cdots;\cdots,f_j\circ e_i,\cdots)\\
&+\sum_j(-1)^{i}\omega_k(\cdots,e_i,\widehat{e_{i+1}},\cdots;\cdots,f_j\circ e_{i+1},\cdots)\\
&+\sum_j(-1)^{i+1}\omega_k(\cdots,\widehat{e_{i+1}},e_i,\cdots;\cdots,f_j\circ e_{i+1},\cdots)\\
&+\sum_j(-1)^{i}\omega_k(\cdots,e_{i+1},\hat{e_i},\cdots;\cdots,f_j\circ e_i,\cdots)\\
=\ &\sum_{j,a\neq i,i+1}(-1)^{a}\omega_{k+1}(\cdots,\hat{e_a},\cdots,\hat{e_i},\widehat{e_{i+1}},\cdots;(e_i,e_{i+1}),\cdots,f_j\circ e_a,\cdots)\\
=\ &-(d'\omega)_{k+1}(\cdots,\hat{e_i},\widehat{e_{i+1}},\cdots;(e_i,e_{i+1}),\cdots)
\end{aligned}
\tag{3.1.2}
$$

the lemma is proved. ∎

Proof of Theorem 3.1:

Proof. From the lemmas above we see that $C^\bullet_{st}(L)$ is closed under $d=d_0+\delta+d'$. So we only need to prove that $d^2=0$.

d^2 can be divided into six parts

$$d^2=d_0^2+\delta^2+(d_0\circ\delta+\delta\circ d_0)+(d_0\circ d'+d'\circ d_0)+(\delta\circ d'+d'\circ\delta)+d'^2.$$

The first part equals 0, so we only need to compute the other five parts:

$$\begin{aligned}
& (\delta^2\omega)_k(e_1,\cdots,e_{n+2-2k};f_1,\cdots,f_k) \\
= & \sum_i (\delta\omega)_{k-1}(f_i,e_1,\cdots,e_{n+2-2k};\cdots,\hat{f}_i,\cdots) \\
= & \sum_i \sum_{j\neq i} \omega_{k-2}(f_j,f_i,\cdots;\cdots,\hat{f}_i,\cdots,\hat{f}_j,\cdots) \\
= & \sum_{i<j} \Big(\omega_{k-2}(f_j,f_i,\cdots;\cdots,\hat{f}_i,\cdots,\hat{f}_j,\cdots)+\omega_{k-2}(f_i,f_j,\cdots;\cdots,\hat{f}_i,\cdots,\hat{f}_j,\cdots)\Big) \\
= & 0
\end{aligned}$$

$$\begin{aligned}
& ((d_0\circ\delta+\delta\circ d_0)\omega)_k(e_1,\cdots e_{n+2-2k};f_1,\cdots f_k) \\
= & \sum_a (-1)^{a+1}\rho(e_a)(\delta\omega)_k(e_1\cdots\hat{e_a}\cdots e_{n+2-2k};\cdots) \\
& +\sum_{a<b}(-1)^a(\delta\omega)_k(e_1\cdots\hat{e_a}\cdots e_a\circ e_b,\cdots e_{n+2-2k};\cdots) \\
& +\sum_i (d_0\omega)_{k-1}(f_i,e_1\cdots e_{n+2-2k};\cdots,\hat{f}_i,\cdots) \\
= & \sum_{a,i}(-1)^{a+1}\rho(e_a)\omega_{k-1}(f_i,e_1\cdots\hat{e_a},\cdots e_{n+2-2k};\cdots\hat{f}_i,\cdots) \\
& +\sum_{i,a<b}(-1)^a\omega_{k-1}(f_i,e_1\cdots,\hat{e_a},\cdots e_a\circ e_b,\cdots e_{n+2-2k};\cdots\hat{f}_i,\cdots) \\
& +\sum_{i,a}(-1)^a\rho(e_a)\omega_{k-1}(f_i,e_1\cdots\hat{e_a}\cdots e_{n+2-2k};\cdots\hat{f}_i\cdots) \\
& +\sum_{i,a}(-1)\omega_{k-1}(e_1\cdots,\widehat{e_a},f_i\circ e_a\cdots e_{n+2-2k};\cdots\hat{f}_i\cdots) \\
& +\sum_{i,a<b}(-1)^{a+1}\omega_{k-1}(f_i,e_1\cdots,\hat{e_a},\cdots e_a\circ e_b,\cdots e_{n+2-2k};\cdots,\hat{f}_i,\cdots) \\
= & -\sum_{i,a}\omega_{k-1}(e_1,\cdots,\widehat{e_a},f_i\circ e_a,\cdots e_{n+2-2k};\cdots,\hat{f}_i,\cdots)
\end{aligned}$$

$$\begin{aligned}
& ((d_0\circ d'+d'\circ d_0)\omega)_k(e_1,\cdots e_{n+2-2k};f_1,\cdots f_k) \\
= & \sum_a(-1)^{a+1}\rho(e_a)(d'\omega)_k(\cdots\hat{e_a}\cdots;\cdots)+\sum_{a<c}(-1)^a(d'\omega)_k(\cdots\hat{e_a}\cdots e_a\circ e_c\cdots;\cdots) \\
& +\sum_{b,j}(-1)^{b+1}(d_0\omega)_k(\cdots,\hat{e_b},\cdots;\cdots,f_j\circ e_b,\cdots) \\
= & \sum_{j,b<a}(-1)^{a+1}\rho(e_a)(-1)^{b+1}\omega_k(\cdots,\hat{e_b},\cdots,\hat{e_a},\cdots;\cdots,f_j\circ e_b,\cdots)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j,b>a}(-1)^{a+1}\rho(e_a)(-1)^b\omega_k(\cdots,\hat{e_a},\cdots,\hat{e_b},\cdots;\cdots,f_j\circ e_b,\cdots)\\
&+\sum_{j,b<a<c}(-1)^{a+b+1}\omega_k(\cdots,\hat{e_b},\cdots\hat{e_a},\cdots,e_a\circ e_c,\cdots;\cdots,f_j\circ e_b,\cdots)\\
&+\sum_{j,a<b<c}(-1)^{a+b}\omega_k(\cdots,\hat{e_a},\cdots\hat{e_b},\cdots,e_a\circ e_c,\cdots;\cdots,f_j\circ e_b,\cdots)\\
&+\sum_{j,a<c}(-1)^{a+c}\omega_k(\cdots,\hat{e_a},\cdots\hat{e_c},\cdots;\cdots,f_j\circ(e_a\circ e_c),\cdots)\\
&+\sum_{j,a<c<b}(-1)^{a+b}\omega_k(\cdots,\hat{e_a},\cdots,e_a\circ e_c,\cdots,\hat{e_b},\cdots;\cdots,f_j\circ e_b,\cdots)\\
&+\sum_{j,a<b}(-1)^{a+1+b+1}\rho(e_a)\omega_k(\cdots,\hat{e_a},\cdots,\hat{e_b},\cdots;\cdots,f_j\circ e_b,\cdots)\\
&+\sum_{j,a>b}(-1)^{a+b+1}\rho(e_a)\omega_k(\cdots,\hat{e_b},\cdots\hat{e_a},\cdots;\cdots,f_j\circ e_b,\cdots)\\
&+\sum_{j,a<c<b}(-1)^{b+1+a}\omega_k(\cdots,\hat{e_a},\cdots e_a\circ e_c,\cdots\hat{e_b},\cdots;\cdots,f_j\circ e_b,\cdots)\\
&+\sum_{j,a<b<c}(-1)^{b+1+a}\omega_k(\cdots,\hat{e_a},\cdots,\hat{e_b},\cdots e_a\circ e_c,\cdots;\cdots,f_j\circ e_b,\cdots)\\
&+\sum_{j,b<a<c}(-1)^{b+1+a+1}\omega_k(\cdots,\hat{e_b},\cdots,\hat{e_a},\cdots e_a\circ e_c,\cdots;\cdots,f_j\circ e_b,\cdots)\\
=&\sum_{j,a<c}(-1)^{a+c}\omega_k(\cdots,\hat{e_a},\cdots\hat{e_c},\cdots;\cdots,f_j\circ(e_a\circ e_c),\cdots)
\end{aligned}
$$

$$
\begin{aligned}
&((\delta\circ d'+d'\circ\delta)\omega)_k(e_1,\cdots e_{n+2-2k};f_1,\cdots f_k)\\
=&\sum_i(d'\omega)_{k-1}(f_i,e_1\cdots e_{n+2-2k};\cdots,\hat{f}_i,\cdots)\\
&+\sum_{j,a}(-1)^{a+1}(\delta\omega)_k(e_1\cdots\hat{e_a}\cdots e_{n+2-2k};\cdots\widehat{f_j},f_j\circ e_a,\cdots)\\
=&\sum_{j\neq i}(-1)^{1+1}\omega_{k-1}(\cdots;\cdots,\hat{f}_i,\cdots,f_j\circ f_i,\cdots)\\
&+\sum_{j\neq i,a}(-1)^a\omega_{k-1}(f_i,\cdots,\hat{e_a},\cdots;\cdots,\hat{f}_i,\cdots,f_j\circ e_a,\cdots)\\
&+\sum_{i\neq j,a}(-1)^{a+1}\omega_{k-1}(f_i,\cdots,\hat{e_a},\cdots;\cdots,\hat{f}_i,\cdots,f_j\circ e_a,\cdots)\\
&+\sum_{j,a}(-1)^{a+1}\omega_{k-1}(f_j\circ e_a,\cdots,\hat{e_a},\cdots;\cdots,\hat{f}_j,\cdots)\\
=&\sum_{j,a}(-1)^{a+1}\omega_{k-1}(f_j\circ e_a,\cdots,\hat{e_a},\cdots;\cdots,\hat{f}_j,\cdots)
\end{aligned}
$$

$$
\begin{aligned}
& (d'^2\omega)_k(e_1, \cdots e_{n+2-2k}; f_1, \cdots f_k) \\
= & \sum_{a,i}(-1)^{a+1}(d'\omega)_k(\cdots, \hat{e_a}, \cdots; \cdots, f_i \circ e_a, \cdots) \\
= & \sum_{b<a, j\neq i}(-1)^{a+1+b+1}\omega_k(\cdots, \hat{e_b}, \cdots, \hat{e_a}, \cdots; \cdots, f_i \circ e_a, \cdots, f_j \circ e_b, \cdots) \\
& + \sum_{b>a, j\neq i}(-1)^{a+1+b}\omega_k(\cdots, \hat{e_a}, \cdots, \hat{e_b}, \cdots; \cdots, f_i \circ e_a, \cdots, f_j \circ e_b, \cdots) \\
& + \sum_{b<a,i}(-1)^{a+1+b+1}\omega_k(\cdots, \hat{e_b}, \cdots, \hat{e_a}, \cdots; \cdots, (f_i \circ e_a) \circ e_b, \cdots) \\
& + \sum_{b>a,i}(-1)^{a+1+b}\omega_k(\cdots, \hat{e_a}, \cdots, \hat{e_b}, \cdots; \cdots, (f_i \circ e_a) \circ e_b, \cdots) \\
= & \sum_{a<b,i}(-1)^{a+b}\omega_k(\cdots \hat{e_a} \cdots \hat{e_b} \cdots; \cdots \widehat{f_i}, (f_i \circ e_b) \circ e_a - (f_i \circ e_a) \circ e_b, \cdots)
\end{aligned}
$$

So the sum of the above parts is:

$$
\begin{aligned}
& (d^2\omega)_k(e_1, \cdots e_{n+2-2k}; f_1, \cdots f_k) \\
= & \sum_{j,a}(-1)^{a+1}\omega_{k-1}(f_j \circ e_a, \cdots, \hat{e_a}, \cdots; \cdots, \hat{f_j}, \cdots) \\
& - \sum_{i,a}\omega_{k-1}(\cdots, f_i \circ e_a, \cdots; \cdots, \hat{f_i}, \cdots) \\
& + \sum_{j,a<b}(-1)^{a+b}\omega_k(\cdots, \hat{e_a}, \cdots \hat{e_b}, \cdots; \cdots, f_j \circ (e_a \circ e_b), \cdots) \\
& + \sum_{a<b,i}(-1)^{a+b}\omega_k(\cdots, \hat{e_a}, \cdots, \hat{e_b}, \cdots; \cdots, (f_i \circ e_b) \circ e_a - (f_i \circ e_a) \circ e_b, \cdots) \\
= & \sum_{i,b<a}(-1)^{a+b}\Big(\omega_{k-1}(\cdots, f_i \circ e_a, e_b, \cdots \hat{e_a}, \cdots; \cdots, \hat{f_i}, \cdots) \\
& \quad + \omega_{k-1}(\cdots e_b, f_i \circ e_a, \cdots \hat{e_a}, \cdots; \cdots, \hat{f_i}, \cdots)\Big) \\
& + \sum_{i,a<b}(-1)^{a+b}\omega_k(\cdots, \hat{e_a}, \cdots, \hat{e_b}, \cdots; \cdots, (e_a, f_i \circ e_b), \cdots) \\
= & \ 0
\end{aligned}
$$

Thus $C^\bullet_{st}(L)$ becomes a cochain complex with coboundary differential d. ■

Definition 3.4. $(C^\bullet_{st}(L, h, R), d)$ is called the standard complex of L with respect to the ideal h and module R. And the resulting cohomology $H^\bullet_{st}(L, h, R) = H^\bullet(C^\bullet_{st}(L, h, R), d)$ is called the standard cohomology of L with respect to h and R.

We'll write $H^\bullet_{st}(L,h,R)$ simply as $H^\bullet_{st}(L)$ if it causes no confusion.

Remark 3.5. Given any Courant-Dorfman algebra $(\mathcal{E}, R, (\bullet,\bullet), \partial, \circ)$, the standard complex $C^\bullet(\mathcal{E}, R)$ and standard cohomology $H^\bullet_{st}(\mathcal{E})$ is defined as in Definition 2.20. If we view $\mathcal{E}$ as a Leibniz algebra, and take $h = \rho^*(\Omega^1) = R\partial R$, then we also have the standard complex $C^\bullet_{st}(\mathcal{E}, h, R)$ and standard cohomology $H^\bullet_{st}(\mathcal{E}, h, R)$ as defined in Definition 3.4. We see that $C^\bullet_{st}(\mathcal{E}, h, R)$ are different from $C^\bullet(\mathcal{E}, R)$. The differences rely on the R-module structure of $\mathcal{E}$: cochains in $C^\bullet(\mathcal{E}, R)$ are required to be weakly R-linear in each argument of $\mathcal{E}$ and be a derivation in each argument of R, while cochains in $C^\bullet_{st}(\mathcal{E}, h, R)$ have no such requirements.

Let's consider the standard cohomology in lower degrees:

Degree 0:

$H^0_{st}(L)$ is the submodule of R consisting of all invariants, i.e.

$$H^0_{st}(L) = \{r \in R | \rho(e)r = 0,\ \forall e \in L\}.$$

Degree 1:

A cocycle ω in $C^1_{st}(L)$ is a map $\omega_0 : L \to R$ satisfying:

$$\omega_0(e_1 \circ e_2) = \rho(e_1)\omega_0(e_2) - \rho(e_2)\omega_0(e_1), \quad \forall e_1, e_2 \in L$$

and

$$\omega_0(f) = 0, \quad \forall f \in h.$$

The first equation above tells that ω_0 is a derivation from L to R, while the second equation tells that ω_0 induces a map from L/h to R.

$\eta \in C^1_{st}(L)$ is a coboundary iff there exists $\alpha \in R$ such that:

$$\eta_0(e) = \rho(e)\alpha, \quad \forall e \in L,$$

i.e. η_0 is an inner derivation from L to R.

Thus $H^1_{st}(L)$ is the space of "outer derivations": $\{\text{derivations}\}/\{\text{inner derivations}\}$ from L to R acting trivially on h. Or equivalently, $H^1_{st}(L)$ is the space of outer derivations from L/h to R.

Degree 2:

$\omega = (\omega_0, \omega_1) \in C^2_{st}(L)$ is a 2-cocycle iff:

$$\begin{aligned} &\rho(e_1)\omega_0(e_2,e_3) - \rho(e_2)\omega_0(e_1,e_3) + \rho(e_3)\omega_0(e_1,e_2) \\ &-\omega_0(e_1\circ e_2, e_3) - \omega_0(e_2, e_1\circ e_3) + \omega_0(e_1, e_2\circ e_3) \quad = \quad 0 \quad \forall e_1,e_2,e_3 \in L \end{aligned}$$

and

$$\rho(e)\omega_1(f) + \omega_0(f,e) + \omega_1(f\circ e) = 0, \quad \forall e \in L, f \in h.$$

The first equation above holds iff the bracket on $\bar{L} \triangleq L \oplus R$ defined for any $e_1, e_2 \in L$, $r_1, r_2 \in R$ by:

$$(e_1 + r_1) \circ (e_2 + r_2) \triangleq e_1 \circ e_2 + \Big(\rho(e_1)r_2 - \rho(e_2)r_1 + \omega_0(e_1, e_2)\Big)$$

is a Leibniz bracket, while the second equation above tells that

$$\bar{h} \triangleq \{f - \omega_1(f) | f \in h\}$$

is an ideal of $\bar{L}$. So a 2-cocycle ω induces a Leibniz bracket (actually it is a Lie bracket) on $\bar{L}/\bar{h} \cong (L/h) \oplus R$.

In other words, 2-cocycles are in 1-1 correspondence with abelian extensions of the Leibniz algebra L by R:

$$0 \to R \to \bar{L} \to L \to 0$$

such that $\bar{h}$ is an ideal of $\bar{L}$.

$\omega = (\omega_0, \omega_1) \in C^2_{st}(L)$ is a 2-coboundary iff there exists $\alpha \in C^1_{st}(L)$ such that:

$$\omega_0(e_1, e_2) = \rho(e_1)\alpha_0(e_2) - \rho(e_2)\alpha_0(e_1) - \alpha_0(e_1 \circ e_2), \quad \forall e_1, e_2 \in L$$

and

$$\omega_1(f) = \alpha_0(f), \quad \forall f \in h.$$

So 2-coboundaries are in 1-1 correspondence with abelian extensions of L by R such that

$$0 \to R \to \bar{L} \to L \to 0$$

is split in the category of Leibniz algebras.

Therefore, $H^2_{st}(L)$ classifies the equivalence classes of abelian extensions of L by R satisfying that $\bar{h}$ is an ideal of $\bar{L}$. (By Theorem 3.10 in the next section, we see that $H^2_{st}(L)$ actually classifies the equivalence classes of abelian extensions of the Lie algebra L/h by R.)

Proposition 3.6. *If R is endowed with an algebra structure, then we can define a multiplication on $C^\bullet_{st}(L)$ so that $C^\bullet_{st}(L)$ becomes a differential graded algebra. Furthermore if R is commutative, $C^\bullet_{st}(L)$ is graded-commutative.*

Proof. $\forall \omega \in C^n_{st}(L),\ \eta \in C^m_{st}(L)$, define the multiplication $\omega \cdot \eta$ as:

$$\begin{aligned} & (\omega \cdot \eta)_k(e_1, \cdots, e_{n+m-2k}; f_1, \cdots, f_k) \\ = \ & \sum_{i+j=k} \sum_{\sigma \in sh(n-2i, m-2j)} \sum_{\tau \in sh(i,j)} (-1)^\sigma \\ & \omega_i(e_{\sigma(1)} \cdots e_{\sigma(n-2i)}; f_{\tau(1)} \cdots f_{\tau(i)}) \eta_j(e_{\sigma(n-2i+1)} \cdots e_{\sigma(n+m-2k)}; f_{\tau(i+1)} \cdots f_{\tau(k)}) \end{aligned}$$

We give the proof in 4 steps.
Step 1:
$C^{\bullet}_{st}(L)$ is closed under the multiplication, i.e. $\omega\cdot\eta\in C^{n+m}_{st}(L)$:

$$
\begin{aligned}
& (\omega\cdot\eta)_k(\cdots,e_a,e_{a+1},\cdots;f_1,\cdots,f_k)+(\omega\cdot\eta)_k(\cdots,e_{a+1},e_a,\cdots;f_1,\cdots,f_k)\\
= & \sum_{i+j=k}\ \sum_{\sigma\in sh(n-2i,m-2j),\sigma^{-1}(a),\sigma^{-1}(a+1)\le n-2i}\ \sum_{\tau\in sh(i,j)}(-1)^{\sigma}\\
& \Big(\omega_i(\cdots,e_a,e_{a+1},\cdots;\cdots)+\omega_i(\cdots,e_{a+1},e_a,\cdots;\cdots)\Big)\eta_j(\cdots)\\
& +\sum_{i+j=k}\ \sum_{\sigma\in sh(n-2i,m-2j),\sigma^{-1}(a),\sigma^{-1}(a+1)>n-2i}\ \sum_{\tau\in sh(i,j)}(-1)^{\sigma}\\
& \omega_i(\cdots)\Big(\eta_j(\cdots,e_a,e_{a+1},\cdots;\cdots)+\eta_i(\cdots,e_{a+1},e_a,\cdots;\cdots)\Big)\\
& +\sum_{i+j=k}\ \sum_{\sigma\in sh(n-2i,m-2j),\sigma^{-1}(a)\le n-2i<\sigma^{-1}(a+1)}\ \sum_{\tau\in sh(i,j)}(-1)^{\sigma}\\
& \Big(\omega_i(\cdots e_a\cdots;\cdots)\eta_j(\cdots e_{a+1}\cdots;\cdots)+\omega_i(\cdots e_{a+1}\cdots;\cdots)\eta_j(\cdots e_a\cdots;\cdots)\Big)\\
& +\sum_{i+j=k}\ \sum_{\sigma\in sh(n-2i,m-2j),\sigma^{-1}(a+1)\le n-2i<\sigma^{-1}(a)}\ \sum_{\tau\in sh(i,j)}(-1)^{\sigma}\\
& \Big(\omega_i(\cdots e_{a+1}\cdots;\cdots)\eta_j(\cdots e_a\cdots;\cdots)+\omega_i(\cdots e_a\cdots;\cdots)\eta_j(\cdots e_{a+1}\cdots;\cdots)\Big)\\
& (\textit{note that the same sequence } (\cdots,e_a,\cdots,e_{a+1},\cdots) \textit{ viewed as permutations}\\
& \textit{of } (\cdots,e_a,e_{a+1},\cdots) \textit{ and } (\cdots,e_{a+1},e_a,\cdots) \textit{ have opposite signs})\\
= & \sum_{i+j=k}\ \sum_{\sigma\in sh(n-2i,m-2j),\sigma^{-1}(a),\sigma^{-1}(a+1)\le n-2i}\ \sum_{\tau\in sh(i,j)}(-1)^{\sigma+1}\\
& \omega_{i+1}(\cdots,\hat{e_a},\widehat{e_{a+1}},\cdots;(e_a,e_{a+1}),\cdots)\eta_j(\cdots)\\
& +\sum_{i+j=k}\ \sum_{\sigma\in sh(n-2i,m-2j),\sigma^{-1}(a),\sigma^{-1}(a+1)>n-2i}\ \sum_{\tau\in sh(i,j)}(-1)^{\sigma+1}\\
& \omega_i(\cdots)\eta_{j+1}(\cdots,\hat{e_a},\widehat{e_{a+1}},\cdots;(e_a,e_{a+1}),\cdots)\\
= & \sum_{l+j=k+1}\ \sum_{\sigma\in sh(n-2l,m-2j)}\ \sum_{\tau\in sh(l,j),\tau^{-1}((e_a,e_{a+1}))\le l}(-1)^{\sigma+1}\omega_l(\cdots;(e_a,e_{a+1}),\cdots)\eta_j(\cdots)\\
& +\sum_{i+l=k+1}\ \sum_{\sigma\in sh(n-2i,m-2l)}\ \sum_{\tau\in sh(i,l),\tau^{-1}((e_a,e_{a+1}))>i}(-1)^{\sigma+1}\omega_i(\cdots)\eta_l(\cdots;(e_a,e_{a+1}),\cdots)\\
= & -(\omega\cdot\eta)_{k+1}(e_1,\cdots,\widehat{e_a},\widehat{e_{a+1}},\cdots;(e_a,e_{a+1}),\cdots)
\end{aligned}
$$

Step 2:
The multiplication is associative:
$\forall\omega\in C^n_{st}(L),\ \eta\in C^m_{st}(L),\ \lambda\in C^l_{st}(L)$, by definition it is an easy calculation that, $((\omega\cdot\eta)\cdot\lambda)_k(e_1,\cdots,e_{n+m+l-2k};f_1,\cdots,f_k)$ and $(\omega\cdot(\eta\cdot\lambda))_k(e_1,\cdots,e_{n+m+l-2k};f_1,\cdots,f_k)$

both equal to:

$$\sum_{a+b+c=k}\sum_{\sigma\in shuffle(n-2a,m-2b,l-2c)}\sum_{\tau\in shuffle(a,b,c)}(-1)^{\sigma}\omega_a(\cdots)\eta_b(\cdots)\lambda_c(\cdots)$$

Step 3:

In order for $C^{\bullet}_{st}(L)$ to be a differential graded algebra, we need to prove the following:

$$d(\omega\cdot\eta)=(d\omega)\cdot\eta+(-1)^n\omega\cdot(d\eta),\ \forall\omega\in C^n_{st}(L),\ \eta\in C^m_{st}(L).$$

Since $d=d_0+d'+\delta$, it suffices to prove the equation for d_0,d',δ respectively.

For d_0, we only give the proof for the case of degree 0 here, since the proof is almost the same for cases of higher degrees (the only difference is that the sum should be taken over permutations of the arguments in h as well).

$$\begin{aligned}
& (d_0(\omega\cdot\eta))_0(e_1,\cdots,e_{n+m+1})\\
= & \sum_a(-1)^{a+1}\rho(e_a)(\omega\cdot\eta)_0(\cdots\hat{e_a}\cdots)+\sum_{a<b}(-1)^a(\omega\cdot\eta)_0(\cdots\hat{e_a}\cdots\hat{e_b},e_a\circ e_b\cdots)\\
= & \sum_a(-1)^{a+1}\rho(e_a)\Big(\sum_{\sigma\in sh(n,m)\{\cdots,\hat{a},\cdots\}}(-1)^{\sigma}\omega_0(e_{\sigma(1)}\cdots e_{\sigma(n)})\eta_0(e_{\sigma(n+1)}\cdots e_{\sigma(n+m)})\Big)\\
& +\sum_{a<b}(-1)^a\sum_{\sigma\in sh(n,m)\{\cdots\hat{a},\cdots\},\sigma^{-1}(b)<n+1}(-1)^{\sigma}\\
& \omega_0(e_{\sigma(1)},\cdots,\hat{e_b},e_a\circ e_b,\cdots e_{\sigma(n)})\eta_0(e_{\sigma(n+1)},\cdots e_{\sigma(n+m+1)})\\
& +\sum_{a<b}(-1)^a\sum_{\sigma\in sh(n,m)\{\cdots\hat{a},\cdots\},\sigma^{-1}(b)>n}(-1)^{\sigma}\\
& \omega_0(e_{\sigma(1)},\cdots e_{\sigma(n)})\eta_0(e_{\sigma(n+1)},\cdots,\hat{e_b},e_a\circ e_b,\cdots e_{\sigma(n+m+1)})\\
& (letting\ \sigma_1\ be\ the\ permutation\ adding\ a\ to\ \sigma\ in\ front,\\
& \ \sigma_2\ be\ the\ permutation\ adding\ a\ to\ \sigma\ at\ back)\\
= & \sum_a\sum_{\sigma_1\in sh(n+1,m)}(-1)^{a+1}(-1)^{\sigma_1+\sigma_1^{-1}(a)-a}\\
& \Big(\rho(e_a)\omega_0(e_{\sigma_1(1)}\cdots\hat{e_a},e_{\sigma_1(\sigma_1^{-1}(a)+1)}\cdots e_{\sigma_1(n+1)})\Big)\eta_0(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)})\\
+ & \sum_a\sum_{\sigma_2\in sh(n,m+1)}(-1)^{a+1}(-1)^{\sigma_2+\sigma_2^{-1}(a)-a}\\
& \omega_0(e_{\sigma_2(1)}\cdots e_{\sigma_2(n)})\Big(\rho(e_a)\eta_0(e_{\sigma_2(n+1)}\cdots\hat{e_a},e_{\sigma_2(\sigma_2^{-1}(a)+1)}\cdots e_{\sigma_2(n+m+1)})\Big)\\
& +\sum_{a<b}\sum_{\sigma_1\in sh(n+1,m),\ \sigma_1^{-1}(b)<n+2}(-1)^a(-1)^{\sigma_1+\sigma_1^{-1}(a)-a}
\end{aligned}$$

$$
\begin{aligned}
& \omega_0(e_{\sigma_1(1)}\cdots\hat{e_a}, e_{\sigma_1(\sigma_1^{-1}(a)+1)}\cdots\hat{e_b}, e_a\circ e_b\cdots)\eta_0(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)}) \\
+ & \sum_{a<b}\sum_{\sigma_2\in sh(n,m+1),\ \sigma_2^{-1}(b)>n+1}(-1)^a(-1)^{\sigma_2+\sigma_2^{-1}(a)-a} \\
& \omega_0(e_{\sigma_2(1)}\cdots)\eta_0(e_{\sigma_2(n+1)}\cdots\hat{e_a}, e_{\sigma_2(\sigma_2^{-1}(a)+1)}\cdots\hat{e_b}, e_a\circ e_b\cdots e_{\sigma_2(n+m+1)}) \\
= & \sum_{\sigma_1}(-1)^{\sigma_1}\sum_{a_1:=\sigma_1^{-1}(a)<n+2}(-1)^{a_1+1} \\
& \Big(\rho(e_{\sigma_1(a_1)})\omega_0(e_{\sigma_1(1)}\cdots\widehat{e_{\sigma_1(a_1)}}\cdots e_{\sigma_1(n+1)})\Big)\eta_0(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)}) \\
& +\sum_{\sigma_1}(-1)^{\sigma_1}\sum_{a_1:=\sigma_1^{-1}(a)<b_1:=\sigma_1^{-1}(b)<n+2}(-1)^{a_1} \\
& \omega_0(e_{\sigma_1(1)}\cdots\widehat{e_{\sigma_1(a_1)}}\cdots\widehat{e_{\sigma_1(b_1)}}, e_{\sigma_1(a_1)}\circ e_{\sigma_1(b_1)}\cdots)\eta_0(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)}) \\
& +\sum_{\sigma_2}(-1)^{\sigma_2+n}\omega_0(e_{\sigma_2(1)},\cdots,e_{\sigma_2(n)})\cdot\sum_{a_2:=\sigma_2^{-1}(a)} \\
& (-1)^{a_2-n+1}\rho(e_{\sigma_2(a_2)})\eta_0(e_{\sigma_2(n+1)},\cdots,\widehat{e_{\sigma_2(a_2)}},\cdots,e_{\sigma_2(n+m+1)}) \\
& +\sum_{\sigma_2}(-1)^{\sigma_2+n}\omega_0(e_{\sigma_2(1)},\cdots,e_{\sigma_2(n)})\cdot\sum_{n<a_2:=\sigma_2^{-1}(a)<b_2:=\sigma_2^{-1}(b)} \\
& (-1)^{a_2-n}\eta_0(e_{\sigma_2(n+1)}\cdots\widehat{e_{\sigma_2(a_2)}}\cdots\widehat{e_{\sigma_2(b_2)}}, e_{\sigma_2(a_2)}\circ e_{\sigma_2(b_2)}\cdots e_{\sigma_2(n+m+1)}) \\
= & \Big((d_0\omega)\cdot\eta+(-1)^n\omega\cdot(d_0\eta)\Big)_0(e_1,\cdots,e_{n+m+1})
\end{aligned}
$$

For d',

$$
\begin{aligned}
& (d'(\omega\cdot\eta))_k(e_1,\cdots,e_{n+m+1-2k};f_1,\cdots,f_k) \\
= & \sum_{i,j}(-1)^{i+1}(\omega\cdot\eta)_k(\cdots,\hat{e}_i,\cdots;\cdots,\hat{f}_j,f_j\circ e_i,\cdots) \\
= & \sum_{i,j}(-1)^{i+1}\sum_{a+b=k}\sum_{\sigma\in sh(n-2a,m-2b)}\sum_{\tau\in sh(a,b),\tau^{-1}(f_j\circ e_i)\le a}(-1)^\sigma \\
& \omega_a(e_{\sigma(1)},\cdots,\hat{e}_i,\cdots,e_{\sigma(n-2a)};\cdots,\hat{f}_j,f_j\circ e_i,\cdots)\eta_b(\cdots) \\
& +\sum_{i,j}(-1)^{i+1}\sum_{a+b=k}\sum_{\sigma\in sh(n-2a,m-2b)}\sum_{\tau\in sh(a,b),\tau^{-1}(f_j\circ e_i)>a}(-1)^\sigma \\
& \omega_a(\cdots)\eta_b(e_{\sigma(n-2a+1)},\cdots,\hat{e}_i,\cdots,e_{\sigma(n+m-2k)};\cdots,\hat{f}_j,f_j\circ e_i,\cdots) \\
& (insert\ e_i\ at\ front\ and\ at\ back\ to\ \sigma\ respectively) \\
= & \sum_{a+b=k}\sum_{\sigma\in sh(n-2a+1,m-2b)}\sum_{\tau\in sh(a,b)}\sum_{\sigma^{-1}(e_i)\le n-2a+1,\tau^{-1}(f_j\circ e_i)\le a}(-1)^{i+1}(-1)^{\sigma+\sigma^{-1}(i)-i} \\
& \omega_a(e_{\sigma(1)},\cdots,\widehat{e_{\sigma(\sigma^{-1}(i))}},\cdots,e_{\sigma(n-2a)};\cdots,\hat{f}_j,f_j\circ e_i,\cdots)\eta_b(\cdots) \\
& +\sum_{a+b=k}\sum_{\sigma\in sh(n-2a,m-2b+1)}\sum_{\tau\in sh(a,b)}\sum_{\sigma^{-1}(e_i)>n-2a,\tau^{-1}(f_j\circ e_i)>a}(-1)^{i+1}(-1)^{\sigma+\sigma^{-1}(i)-i}
\end{aligned}
$$

$$
\begin{aligned}
& \omega_a(\cdots)\eta_b(e_{\sigma(n-2a+1)},\cdots,\widehat{e_{\sigma(\sigma^{-1}(i))}},\cdots,e_{\sigma(n+m+1-2k)};\cdots,\hat{f}_j,f_j\circ e_i,\cdots) \\
= & \sum_{a+b=k}\sum_{\sigma\in sh(n-2a+1,m-2b)}\sum_{\tau\in sh(a,b)}(-1)^{\sigma}(d'\omega)_a(\cdots)\eta_b(\cdots) \\
& +\sum_{a+b=k}\sum_{\sigma\in sh(n-2a,m-2b+1)}\sum_{\tau\in sh(a,b)}(-1)^{\sigma+n}\omega_a(\cdots)(d'\eta)_b(\cdots) \\
= & ((d'\omega)\cdot\eta)_k(\cdots)+(-1)^n(\omega\cdot(d'\eta))_k(\cdots)
\end{aligned}
$$

For δ,

$$
\begin{aligned}
& (\delta(\omega\cdot\eta))_k(e_1,\cdots,e_{n+m+1-2k};f_1,\cdots,f_k) \\
= & \sum_i(\omega\cdot\eta)_{k-1}(f_i,e_1,\cdots,e_{n+m+1-2k};\cdots,\hat{f}_i,\cdots) \\
= & \sum_i\sum_{a+b=k-1}\sum_{\sigma\in sh(n-2a,m-2b),\sigma^{-1}(f_i)\le n-2a}\sum_{\tau\in sh(a,b)}(-1)^{\sigma}\omega_a(f_i,\cdots;\cdots\hat{f}_i\cdots)\eta_b(\cdots) \\
& +\sum_i\sum_{a+b=k-1}\sum_{\sigma\in sh(n-2a,m-2b),\sigma^{-1}(f_i)>n-2a}\sum_{\tau\in sh(a,b)}(-1)^{\sigma}\omega_a(\cdots)\eta_b(f_i,\cdots;\cdots\hat{f}_i\cdots) \\
& (removing\ f_i\ from\ \sigma,\ adding\ f_i\ to\ \tau\ in\ front\ and\ at\ back\ respectively) \\
= & \sum_{a+b=k}\sum_{\sigma\in sh(n+1-2a,m-2b)}\sum_{\tau\in sh(a,b)}(-1)^{\sigma} \\
& \quad \sum_{\tau^{-1}(i)\le a}\omega_{a-1}(f_i,e_{\sigma(1)},\cdots;\cdots,\widehat{f_{\tau(\tau^{-1}(i))}},\cdots)\eta_b(\cdots) \\
& +\sum_{a+b=k}\sum_{\sigma\in sh(n-2a,m+1-2b)}\sum_{\tau\in sh(a,b)}(-1)^{\sigma+n} \\
& \quad \sum_{\tau^{-1}(i)>a}\omega_a(\cdots)\eta_{b-1}(f_i,e_{\sigma(n-2a+1)},\cdots;\cdots,\widehat{f_{\tau(\tau^{-1}(i))}},\cdots) \\
= & ((\delta\omega)\cdot\eta)_k(\cdots)+(-1)^n(\omega\cdot(\delta\eta))_k(\cdots)
\end{aligned}
$$

Step 4:
If R is commutative, $C^{\bullet}_{st}(L)$ is graded-commutative:

$$
\begin{aligned}
& (\eta\cdot\omega)_k(e_1,\cdots,e_{n+m-2k};f_1,\cdots,f_k) \\
= & \sum_{a+b=k}\sum_{\sigma\in sh(n-2a,m-2b)}\sum_{\tau\in sh(a,b)}(-1)^{\sigma}\eta_a(\cdots)\omega_b(\cdots) \\
= & \sum_{b+a=k}\sum_{\sigma\in sh(m-2b,n-2a)}\sum_{\tau\in sh(b,a)}(-1)^{\sigma+nm}\omega_b(\cdots)\eta_a(\cdots) \\
= & (-1)^{nm}(\omega\cdot\eta)_k(e_1,\cdots,e_{n+m-2k};f_1,\cdots,f_k)
\end{aligned}
$$

Thus the proposition is proved. ■

Next, we consider a graded subspace of $C^\bullet_{st}(L)$:

$$C^\bullet_{nv}(L) \triangleq \{\omega \in C^\bullet_{st}(L) | \omega_k = 0,\ \forall k > 0\ \&\ \iota_f\omega_0 = 0,\ \forall f \in h\},$$

i.e. $C^\bullet_{nv}(L)$ consists of all $\beta \in Hom(\wedge^\bullet L,\ R),\ \iota_f\beta = 0,\ \forall f \in h$.

Proposition 3.7. $C^\bullet_{nv}(L)$ is a subcomplex of $C^\bullet_{st}(L)$.

Proof. It's easily seen from the definitions that δ and d'equals 0 on $C^\bullet_{nv}(L)$, so $d = d_0$. $\forall \beta \in C^n_{nv}(L)$,

$$\begin{aligned}
& (d_0\beta)(e_1, \cdots, e_i, e_{i+1}, \cdots e_{n+1}) + (d_0\beta)(e_1, \cdots, e_{i+1}, e_i, \cdots e_{n+1}) \\
= & \sum_{a \neq i, i+1} (-1)^{a+1} \rho(e_a)\Big(\beta(\cdots, \hat{e_a}, \cdots, e_i, e_{i+1}, \cdots) + \beta(\cdots, \hat{e_a}, \cdots, e_{i+1}, e_i, \cdots)\Big) \\
& + (-1)^{i+1}\rho(e_i)\beta(\cdots, \hat{e_i}, e_{i+1}, \cdots) + (-1)^i \rho(e_{i+1})\beta(\cdots, e_i, \widehat{e_{i+1}}, \cdots) \\
& + (-1)^{i+1}\rho(e_{i+1})\beta(\cdots, \widehat{e_{i+1}}, e_i, \cdots) + (-1)^i \rho(e_i)\beta(\cdots, e_{i+1}, \hat{e_i}, \cdots) \\
& + \sum_{a < b \neq i, i+1} (-1)^a \Big(\beta(\cdots, \hat{e_a}, \cdots e_a \circ e_b, \cdots, e_i, e_{i+1}, \cdots) \\
& \qquad + \beta(\cdots, \hat{e_a}, \cdots e_a \circ e_b, \cdots, e_{i+1}, e_i, \cdots)\Big) \\
& + \sum_{a < i} (-1)^a \Big(\beta(\cdots, \hat{e_a}, \cdots e_a \circ e_i, e_{i+1}, \cdots) + \beta(\cdots, \hat{e_a}, \cdots, e_{i+1}, e_a \circ e_i, \cdots)\Big) \\
& + \sum_{a < i} (-1)^a \Big(\beta(\cdots, \hat{e_a}, \cdots e_i, e_a \circ e_{i+1}, \cdots) + \beta(\cdots, \hat{e_a}, \cdots, e_a \circ e_{i+1}, e_i, \cdots)\Big) \\
& + \sum_{b > i+1} ((-1)^i + (-1)^{i+1})\beta(\cdots, \hat{e_i}, e_{i+1}, \cdots e_i \circ e_b, \cdots) \\
& + \sum_{b > i+1} ((-1)^{i+1} + (-1)^i)\beta(\cdots, e_i, \widehat{e_{i+1}}, \cdots e_{i+1} \circ e_b, \cdots) \\
& + (-1)^i \beta(\cdots, e_i \circ e_{i+1}, \cdots) + (-1)^i \beta(\cdots, e_{i+1} \circ e_i, \cdots) \\
= & \ 0
\end{aligned}$$

$$\begin{aligned}
& \iota_f(d_0\beta)(e_1, \cdots, e_n) \\
= & \ (d_0\beta)(f, e_1, \cdots e_n) \\
= & \ \rho(f)\beta(\cdots) + \sum_a (-1)^a \rho(e_a)\beta(f, \cdots, \hat{e_a}, \cdots) \\
& + \sum_a (-1)^1 \beta(\cdots, f \circ e_a, \cdots) + \sum_{a<b} (-1)^{a+1}\beta(f, \cdots, \hat{e_a}, \cdots, e_a \circ e_b, \cdots) \\
= & \ 0
\end{aligned}$$

So $C^\bullet_{nv}(L)$ is a subcomplex of $C^\bullet_{st}(L)$. ■

Definition 3.8. $(C^\bullet_{nv}(L), d_0)$ is called the naive complex of L with respect to the ideal h and module R. The resulting cohomology $H^\bullet_{nv}(L) = H^\bullet(C^\bullet_{nv}(L), d)$ is called the naive cohomology of L with respect to h and R.

This definition is analogous to the naive cohomology of Courant algebroid. And similar to that case, we have the following:

Proposition 3.9. *With the above notations, we have*

$$H^\bullet_{nv}(L) \cong H^n(C^\bullet_{CE}(L/h, R)),$$

where $C^\bullet_{CE}(L/h, R)$ is the Chevalley-Eilenberg cochain complex of Lie algebra L/h with coefficients in the module R.

Proof. Given any $\beta \in C^n_{nv}(L)$, we define $\varphi(\beta) \in C^n_{CE}(L/h, R)$ to be:

$$\varphi(\beta)([e_1], \cdots [e_n]) := \beta(e_1, \cdots, e_n).$$

Conversely, given any $\alpha \in C^n_{CE}(L/h, R)$, we define $\phi(\alpha) \in C^n_{nv}(L)$ to be:

$$\phi(\alpha)(e_1, \cdots e_n) := \alpha([e_1], \cdots [e_n]).$$

It's easily checked that φ, ϕ are well-defined cochain maps and are invertible to each other, so they induce isomorphisms on cohomology. ■

3.2 Isomorphism theorems

First, we prove an isomorphism theorem for Leibniz algebras.

Theorem 3.10. *Suppose L is a Leibniz algebra with left center Z, $h \supseteq Z$ is an isotropic ideal in L, and R is a left L-module on which h acts trivially, then we have:*

$$H^n_{st}(L) \cong H^n_{nv}(L).$$

First we prove the following key lemma:

Lemma 3.11. *For any $\omega \in C^n_{st}(L)$ which satisfies $(d\omega)_k = 0,\ \forall k > 0$, there exists $\beta \in C^n_{nv}(L)$ and $\lambda \in C^{n-1}_{st}(L)$ such that $\omega = \beta + d\lambda$.*

Proof. Given any vector space decomposition: $L = h \oplus X$, we will give an inductive construction of λ and β. The construction below depends on the decomposition, but the cohomology class of β doesn't depend on the decomposition.

Suppose $n = 2m$ *or* $2m-1$, we will define $\lambda_{m-1}, \lambda_{m-2}, \cdots, \lambda_0$ one by one, so that each $\lambda_p : \otimes^{n-1-2p} L \otimes \odot^p h \to R,\ 0 \leq p \leq m-1$ satisfies the following conditions, which we call "Lambda Conditions":

1). λ_p is weakly skew-symmetric up to λ_{p+1},

2). $\omega_{p+1} = (d\lambda)_{p+1}$,

3). $\sum_i (\omega_p - d_0\lambda_p - d'\lambda_p)(f_i, e_1, \cdots, e_{n-1-2p}; f_1, \cdots, \hat{f}_i, \cdots f_{p+1}) = 0, \forall f_j \in h, e_a \in L$.

The construction of $\lambda_{m-1}, \lambda_{m-2}, \cdots, \lambda_0$ is done in the following four steps.

Step 1:

Construction of λ_{m-1}:

When $n = 2m-1$ is odd, let

$$\lambda_{m-1}(f_1, \cdots, f_{m-1}) \equiv 0,\ \forall f_i \in h.$$

When $n = 2m$ is even, let

$$\lambda_{m-1}(g; f_1, \cdots, f_{m-1}) = \frac{1}{m}\omega_m(g, f_1, \cdots, f_{m-1}),\ \forall g, f_i \in h$$

and

$$\lambda_{m-1}(x; f_1, \cdots f_{m-1}) \equiv 0,\ \forall x \in X, f_i \in h.$$

It is obvious that λ_{m-1} defined above satisfies Lambda Conditions 1) and 2). So we only need to prove Condition 3).

When $n = 2m-1$, the left hand side in condition 3) equals

$$\sum_i \omega_{m-1}(f_i; \cdots, \hat{f}_i, \cdots) = (d\omega)_m(f_1, \cdots f_m) = 0.$$

When $n = 2m$, the left hand side in condition 3) equals

$$\begin{aligned}
& \sum_i (\omega_{m-1} - d_0\lambda_{m-1} - d'\lambda_{m-1})(f_i, e; \cdots, \hat{f}_i, \cdots) \\
= \ & (\delta\omega)_m(e; f_1, \cdots, f_m) \\
& + \sum_i \rho(e)\lambda_{m-1}(f_i; \cdots, \hat{f}_i, \cdots) + \sum_i \lambda_{m-1}(f_i \circ e; \cdots, \hat{f}_i, \cdots) \\
& + \sum_{j \neq i} (-1)\lambda_{m-1}(e; \cdots, \hat{f}_i, \cdots, f_j \circ f_i, \cdots) \\
& + \sum_{j \neq i} \lambda_{m-1}(f_i; \cdots, \hat{f}_i, \cdots, f_j \circ e, \cdots) \\
= \ & (\delta\omega)_m(e; f_1, \cdots, f_m) + \rho(e)\omega_m(f_1, \cdots, f_m)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{m}\sum_i \omega_m(\cdots, f_i \circ e, \cdots) \\
& +\sum_{i<j}(-1)\lambda_{m-1}(e; f_i \circ f_j + f_j \circ f_i, \cdots, \hat{f}_i, \cdots, \hat{f}_j, \cdots) \\
& +\frac{1}{m}\sum_j \sum_{i\neq j} \omega_m(\cdots, f_j \circ e, \cdots) \\
= \;& (\delta\omega)_m(e; f_1, \cdots, f_m) + (d_0\omega)_m(e; f_1, \cdots, f_m) \\
& +(\frac{1}{m} + \frac{m-1}{m})\sum_i \omega_m(\cdots, f_i \circ e, \cdots) \\
= \;& (d\omega)_m(e; f_1, \cdots, f_m) \\
= \;& 0
\end{aligned}
$$

Step 2:

Suppose $\lambda_{m-1}, \cdots, \lambda_k (k > 0)$ are already defined so that they satisfy Lambda Conditions, we will construct λ_{k-1}, so that it also satisfies Lambda Conditions.

To determine λ_{k-1}, first we let

$$
\begin{aligned}
& \lambda_{k-1}(g_1, \cdots, g_l, x_1, \cdots, x_{n+1-2k-l}; f_1, \cdots f_{k-1}) \\
\triangleq \;& \frac{1}{k+l-1}\sum_{1\leq j\leq l}(-1)^{j+1}(\omega_k - d_0\lambda_k - d'\lambda_k)(g_1, \cdots, \hat{g}_j, \cdots, g_l, \cdots; g_j, \cdots) \\
& \forall g_r, f_s \in h, x_a \in X
\end{aligned}
\tag{3.2.1}
$$

(We call $(g, \cdots, g, x, \cdots, x)$ a regular permutation.)

Note that if $l = 0$, the sum above has no summand, and we simply let

$$\lambda_{k-1}(x_1, \cdots, x_{n+1-2k}; f_1, \cdots, f_{k-1}) = 0.$$

Then for any permutation σ of $(g_1 \cdots g_l, x_1 \cdots x_{n+1-2k-l})$, $\lambda_{k-1}(\sigma; f_1, \cdots, f_{k-1})$ is defined as follows:

First, move the last element in X of σ to the last position by weakly skew-symmetric property (i.e. switch the last element in X of σ with rearward elements one by one, each switch brings in a λ_k).

Next, move the last element but one in X of σ to the last position but one by weakly skew-symmetric property.

⋮

Finally, we will get a $\lambda_{k-1}(\bar{\sigma}; f_1, \cdots, f_{k-1})$ with $\bar{\sigma}$ being a regular permutation, this value is already defined by 3.2.1. Thus the value of $\lambda_{k-1}(\sigma; f_1, \cdots, f_{k-1})$ is uniquely determined.

As a summary, the extension could be written as a formula: $\lambda_{k-1}(\sigma; f_1, \cdots, f_{k-1}) = (\pm 1)\lambda_{k-1}(\bar{\sigma}; \cdots) + \sum(\pm 1)\lambda_k(\bullet; \bullet)$. We observe that, for different k, if we do exactly the same switches, then the extension formulas should be similar (each term has the same sign, with the subscripts modified correspondingly). For example, if we have an extension formula for k: $\lambda_k(\sigma; f_1, \cdots, f_k) = (\pm 1)\lambda_k(\bar{\sigma}; \cdots) + \sum(\pm 1)\lambda_{k+1}(\bullet; \bullet)$, then for $k-1$, we have similar formula: $\lambda_{k-1}(g', \sigma, x'; \cdots) = (\pm 1)\lambda_{k-1}(g', \bar{\sigma}, x'; \cdots) + \sum(\pm 1)\lambda_k(g', \bullet, x'; \bullet)$.

Step 3:

We need to prove that λ_{k-1} constructed above satisfies Lambda Conditions:

Proof of Lambda Condition 1):

First we prove that λ_{k-1} for regular permutations is weakly skew-symmetric up to λ_k for the arguments in h and X respectively.

When the number of arguments in h is 0, the result is obvious.

Otherwise, for the arguments in h,

$$
\begin{aligned}
& \lambda_{k-1}(g_1, \cdots g_r, g_{r+1}, \cdots x_1, \cdots; f_1, \cdots f_{k-1}) + \lambda_{k-1}(g_1, \cdots g_{r+1}, g_r, \cdots x_1, \cdots; \cdots) \\
= & \frac{1}{k+l-1}((-1)^{r+1} + (-1)^r)(\omega_k - d_0\lambda_k - d'\lambda_k)(\cdots, \hat{g}_r, g_{r+1} \cdots; g_r, \cdots) \\
& + \frac{1}{k+l-1}((-1)^r + (-1)^{r+1})(\omega_k - d_0\lambda_k - d'\lambda_k)(\cdots, g_r, \widehat{g_{r+1}}, \cdots; g_{r+1}, \cdots) \\
& + \frac{1}{k+l-1}\sum_{j\neq r, r+1}(-1)^{j+1}\{(\omega_k - d_0\lambda_k - d'\lambda_k)(\cdots \hat{g}_j \cdots g_r, g_{r+1} \cdots; g_j, \cdots) \\
& + (\omega_k - d_0\lambda_k - d'\lambda_k)(\cdots, \hat{g}_j, \cdots, g_{r+1}, g_r, \cdots; g_j, \cdots)\} \\
= & \frac{1}{k+l-1}\sum_{j\neq r, r+1}(-1)^j \\
& \{(d_0\lambda_k)(\cdots \hat{g}_j \cdots g_r, g_{r+1} \cdots; g_j, \cdots) + (d_0\lambda_k)(\cdots \hat{g}_j \cdots g_{r+1}, g_r \cdots; g_j, \cdots) \\
& + (d'\lambda_k)(\cdots \hat{g}_j \cdots g_r, g_{r+1} \cdots; g_j, \cdots) + (d'\lambda_k)(\cdots \hat{g}_j \cdots g_{r+1}, g_r \cdots; g_j, \cdots)\} \\
& \text{(by equation 3.1.1 and 3.1.2)} \\
= & \frac{1}{k+l-1}\sum_{j\neq r, r+1}(-1)^j \\
& \{-d_0\lambda_{k+1}(\cdots \hat{g}_j \cdots \hat{g}_r, \widehat{g_{r+1}} \cdots; (g_r, g_{r+1}), g_j, \cdots) \\
& -\lambda_k((g_r, g_{r+1}), g_1, \cdots \hat{g}_j \cdots \hat{g}_r, \widehat{g_{r+1}} \cdots; g_j, \cdots) \\
& -d'\lambda_{k+1}(\cdots \hat{g}_j \cdots \hat{g}_r, \widehat{g_{r+1}} \cdots; (g_r, g_{r+1}), g_j, \cdots)\} \\
= & 0
\end{aligned}
$$

For the arguments in X,

$$
\begin{aligned}
& \lambda_{k-1}(g_1 \cdots g_l, x_1 \cdots x_a, x_{a+1} \cdots ; \cdots) + \lambda_{k-1}(g_1 \cdots g_l, x_1 \cdots x_{a+1}, x_a \cdots ; \cdots) \\
& + \lambda_k(g_1, \cdots, g_l, x_1, \cdots, \hat{x_a}, x_{a+1}\hat{}, \cdots ; (x_a, x_{a+1}), \cdots) \\
= \; & \frac{1}{k+l-1} \sum_{1 \le j \le l} (-1)^{j+1} \{(\omega_k - d_0\lambda_k - d'\lambda_k)(\cdots \hat{g_j} \cdots x_a, x_{a+1} \cdots ; g_j, \cdots) \\
& \quad + (\omega_k - d_0\lambda_k - d'\lambda_k)(\cdots, \hat{g_j}, \cdots x_1, \cdots, x_{a+1}, x_a, \cdots ; g_j, \cdots)\} \\
& + \frac{1}{k+l} \sum_{1 \le j \le l} (-1)^{j+1} \\
& \quad (\omega_{k+1} - d_0\lambda_{k+1} - d'\lambda_{k+1})(\cdots \hat{g_j} \cdots \hat{x_a}, \widehat{x_{a+1}} \cdots ; g_j, (x_a, x_{a+1}), \cdots) \\
& \text{(by equation 3.1.1 and 3.1.2)} \\
= \; & \frac{1}{k+l-1} \sum_{1 \le j \le l} (-1)^{j+1} \\
& \{-\omega_{k+1}(\cdots, \hat{g_j}, \cdots, \hat{x_a}, \widehat{x_{a+1}}, \cdots ; g_j, (x_a, x_{a+1}), \cdots) \\
& + d_0\lambda_{k+1}(\cdots, \hat{g_j}, \cdots, \hat{x_a}, \widehat{x_{a+1}}, \cdots ; g_j, (x_a, x_{a+1}), \cdots) \\
& + \lambda_k((x_a, x_{a+1}), g_1, \cdots \hat{g_j}, \cdots, \hat{x_a}, \widehat{x_{a+1}}, \cdots ; g_j, \cdots) \\
& + d'\lambda_{k+1}(\cdots, \hat{g_j}, \cdots, \hat{x_a}, \widehat{x_{a+1}}, \cdots ; g_j, (x_a, x_{a+1}), \cdots)\} \\
& + \frac{1}{k+l} \sum_{1 \le j \le l} (-1)^{j+1} \\
& \quad (\omega_{k+1} - d_0\lambda_{k+1} - d'\lambda_{k+1})(\cdots \hat{g_j} \cdots \hat{x_a}, \widehat{x_{a+1}} \cdots ; g_j, (x_a, x_{a+1}), \cdots) \\
= \; & \frac{1}{(k+l-1)(k+l)} \sum_{1 \le j \le l} (-1)^j \\
& \quad (\omega_{k+1} - d_0\lambda_{k+1} - d'\lambda_{k+1})(\cdots \hat{g_j} \cdots \hat{x_a}, \widehat{x_{a+1}} \cdots ; g_j, (x_a, x_{a+1}), \cdots) \\
& + \frac{1}{k+l-1} \sum_{1 \le j \le l} (-1)^{j+1} \lambda_k((x_a, x_{a+1}), g_1, \cdots \hat{g_j}, \cdots, \hat{x_a}, \widehat{x_{a+1}}, \cdots ; g_j, \cdots) \\
= \; & \frac{1}{(k+l-1)(k+l)} \sum_{1 \le j \le l} (-1)^{j+1} \sum_{i<j} (-1)^i \\
& \quad (\omega_{k+1} - d_0\lambda_{k+1} - d'\lambda_{k+1})((x_a, x_{a+1}), \cdots \hat{g_i} \cdots \hat{g_j} \cdots \hat{x_a}, \widehat{x_{a+1}} \cdots ; g_i, g_j \cdots) \\
& + \frac{1}{(k+l-1)(k+l)} \sum_{1 \le j \le l} (-1)^{j+1} \sum_{i>j} (-1)^{i+1} \\
& \quad (\omega_{k+1} - d_0\lambda_{k+1} - d'\lambda_{k+1})((x_a, x_{a+1}), \cdots \hat{g_j} \cdots \hat{g_i} \cdots \hat{x_a}, \widehat{x_{a+1}} \cdots ; g_j, g_i \cdots) \\
= \; & \frac{1}{(k+l-1)(k+l)} \sum_{1 \le i < j \le l} ((-1)^{i+j+1} + (-1)^{i+j}) \\
& \quad (\omega_{k+1} - d_0\lambda_{k+1} - d'\lambda_{k+1})((x_a, x_{a+1}), \cdots \hat{g_i} \cdots \hat{g_j} \cdots \hat{x_a}, \widehat{x_{a+1}} \cdots ; g_i, g_j \cdots)
\end{aligned}
$$

$$= \quad 0$$

Next, for general permutation σ, we give the proof in the following three cases:
(1). $\lambda_{k-1}(\sigma_1, g_1, g_2, \sigma_2; \cdots) + \lambda_{k-1}(\sigma_1, g_2, g_1, \sigma_2; \cdots) = 0, \ \forall g_1, g_2 \in h$
If every element in σ_1 is in h, then

$$\begin{aligned}
& \lambda_{k-1}(\sigma_1, g_1, g_2, \sigma_2; \cdots) + \lambda_{k-1}(\sigma_1, g_2, g_1, \sigma_2; \cdots) \\
= \ & (\pm 1)\lambda_{k-1}(\sigma_1, g_1, g_2, \bar{\sigma_2}; \cdots) + \sum (\pm 1)\lambda_k(\sigma_1, g_1, g_2, \bullet; \bullet) \\
& + (\pm 1)\lambda_{k-1}(\sigma_1, g_2, g_1, \bar{\sigma_2}; \cdots) + \sum (\pm 1)\lambda_k(\sigma_1, g_2, g_1, \bullet; \bullet) \\
= \ & (\pm 1)\Big(\lambda_{k-1}(\sigma_1, g_1, g_2, \bar{\sigma_2}; \cdots) + \lambda_{k-1}(\sigma_1, g_2, g_1, \bar{\sigma_2}; \cdots)\Big) \\
& + \sum (\pm 1)\Big(\lambda_k(\sigma_1, g_1, g_2, \bullet; \bullet) + \lambda_k(\sigma_1, g_2, g_1, \bullet; \bullet)\Big) \\
= \ & 0
\end{aligned}$$

Now suppose (1) holds for σ_1 containing at most m elements in X, consider the case when σ_1 contains $m+1$ elements in X, suppose x is the last element of them, move x to the last position and denote the elements in front of x as $\tilde{\sigma_1}$, $\tilde{\sigma_1}$ contains m elements in X.

$$\begin{aligned}
& \lambda_{k-1}(\sigma_1, g_1, g_2, \sigma_2; \cdots) + \lambda_{k-1}(\sigma_1, g_2, g_1, \sigma_2; \cdots) \\
= \ & (\pm 1)\lambda_{k-1}(\sigma_1, g_1, g_2, \bar{\sigma_2}; \cdots) + \sum (\pm 1)\lambda_k(\sigma_1, g_1, g_2, \bullet; \bullet) \\
& + (\pm 1)\lambda_{k-1}(\sigma_1, g_2, g_1, \bar{\sigma_2}; \cdots) + \sum (\pm 1)\lambda_k(\sigma_1, g_2, g_1, \bullet; \bullet) \\
= \ & (\pm 1)\Big(\lambda_{k-1}(\sigma_1, g_1, g_2, \bar{\sigma_2}; \cdots) + \lambda_{k-1}(\sigma_1, g_2, g_1, \bar{\sigma_2}; \cdots)\Big) \\
= \ & (\pm 1)\Big((\pm 1)\lambda_{k-1}(\tilde{\sigma_1}, x, g_1, g_2, \bar{\sigma_2}; \cdots) + \sum (\pm 1)\lambda_k(\bullet, g_1, g_2, \bar{\sigma_2}; \bullet) \\
& \quad + (\pm 1)\lambda_{k-1}(\tilde{\sigma_1}, x, g_2, g_1, \bar{\sigma_2}; \cdots) + \sum (\pm 1)\lambda_k(\bullet, g_2, g_1, \bar{\sigma_2}; \bullet)\Big) \\
= \ & (\pm 1)\Big(\lambda_{k-1}(\tilde{\sigma_1}, x, g_1, g_2, \bar{\sigma_2}; \cdots) + \lambda_{k-1}(\tilde{\sigma_1}, x, g_2, g_1, \bar{\sigma_2}; \cdots)\Big) \\
= \ & (\pm 1)\Big(- \lambda_k(\tilde{\sigma_1}, g_2, \bar{\sigma_2}; (x, g_1), \cdots) + \lambda_k(\tilde{\sigma_1}, g_1, \bar{\sigma_2}; (x, g_2), \cdots) \\
& \quad + \lambda_{k-1}(\tilde{\sigma_1}, g_1, g_2, x, \bar{\sigma_2}; \cdots) - \lambda_k(\tilde{\sigma_1}, g_1, \bar{\sigma_2}; (x, g_2), \cdots) \\
& \quad + \lambda_k(\tilde{\sigma_1}, g_2, \bar{\sigma_2}; (x, g_1), \cdots) + \lambda_{k-1}(\tilde{\sigma_1}, g_2, g_1, x, \bar{\sigma_2}; \cdots)\Big) \\
= \ & (\pm 1)\Big(\lambda_{k-1}(\tilde{\sigma_1}, g_1, g_2, x, \bar{\sigma_2}; \cdots) + \lambda_{k-1}(\tilde{\sigma_1}, g_2, g_1, x, \bar{\sigma_2}; \cdots)\Big) \\
= \ & 0
\end{aligned}$$

By induction, (1) is proved.
(2). $\lambda_{k-1}(\sigma_1, g, y, \sigma_2; \cdots) + \lambda_{k-1}(\sigma_1, y, g, \sigma_2; \cdots) = -\lambda_k(\sigma_1, \sigma_2; (g, y), \cdots), \ \forall g \in h, \ y \in X$

$$
\begin{aligned}
& \lambda_{k-1}(\sigma_1, g, y, \sigma_2; \cdots) + \lambda_{k-1}(\sigma_1, y, g, \sigma_2; \cdots) \\
= \; & (\pm 1)\lambda_{k-1}(\sigma_1, g, y, \bar{\sigma_2}; \cdots) + \sum (\pm 1)\lambda_k(\sigma_1, g, y, \bullet; \bullet) \\
& + (\pm 1)\lambda_{k-1}(\sigma_1, y, g, \bar{\sigma_2}; \cdots) + \sum (\pm 1)\lambda_k(\sigma_1, y, g, \bullet; \bullet) \\
= \; & (\pm 1)\Big(\lambda_{k-1}(\sigma_1, g, y, \bar{\sigma_2}; \cdots) + \lambda_{k-1}(\sigma_1, y, g, \bar{\sigma_2}; \cdots)\Big) \\
& + \sum (\pm 1)\Big(\lambda_k(\sigma_1, g, y, \bullet; \bullet) + \lambda_k(\sigma_1, y, g, \bullet; \bullet)\Big) \\
= \; & (\pm 1)\Big(\lambda_{k-1}(\sigma_1, g, y, \bar{\sigma_2}; \cdots) + (-\lambda_k(\sigma_1, \bar{\sigma_2}; (y, g), \cdots) - \lambda_{k-1}(\sigma_1, g, y, \bar{\sigma_2}; \cdots))\Big) \\
& - \sum (\pm 1)\lambda_{k+1}(\sigma_1, \bullet; (y, g), \bullet) \\
= \; & -\Big((\pm 1)\lambda_k(\sigma_1, \bar{\sigma_2}; (y, g), \cdots) + \sum (\pm 1)\lambda_{k+1}(\sigma_1, \bullet; (y, g), \bullet)\Big) \\
& (\textit{by the observation above}) \\
= \; & -\lambda_k(\sigma_1, \sigma_2; (y, g), \cdots)
\end{aligned}
$$

(3). $\lambda_{k-1}(\sigma_1, y_1, y_2, \sigma_2; \cdots) + \lambda_{k-1}(\sigma_1, y_2, y_1, \sigma_2; \cdots) = -\lambda_k(\sigma_1, \sigma_2; (y_1, y_2), \cdots)$, $\forall y_1, y_2 \in X$.

Suppose $\bar{\sigma_2} = (g_1, \cdots, g_a, x_1, \cdots, x_b)$, then

$$
\begin{aligned}
& \lambda_{k-1}(\sigma_1, y_1, y_2, \bar{\sigma_2}; \cdots) + \lambda_{k-1}(\sigma_1, y_2, y_1, \bar{\sigma_2}; \cdots) \\
= \; & \Big(\sum_{1 \le i \le a} (-1)^i \lambda_k(\sigma_1, y_1, \hat{y_2}, g_1, \cdots, \hat{g_i}, \cdots, g_a, x_1, \cdots, x_b; (y_2, g_i), \cdots) \\
& \quad + (-1)^a \lambda_{k-1}(\sigma_1, y_1, g_1, \cdots, g_a, y_2, x_1, \cdots, x_b; \cdots)\Big) \\
& + \Big(\sum_{1 \le j \le a} (-1)^j \lambda_k(\sigma_1, y_2, \hat{y_1}, g_1, \cdots, \hat{g_j}, \cdots, g_a, x_1, \cdots, x_b; (y_1, g_j), \cdots) \\
& \quad + (-1)^a \lambda_{k-1}(\sigma_1, y_2, g_1, \cdots, g_a, y_1, x_1, \cdots, x_b; \cdots)\Big) \\
= \; & \sum_i (-1)^i \Big(\sum_{1 \le j < i} (-1)^j \lambda_{k+1}(\sigma_1, \hat{y_1}, \hat{y_2}, g_1 \cdots \hat{g_j} \cdots \hat{g_i} \cdots g_a, x_1 \cdots; (y_1, g_j), (y_2, g_i) \cdots) \\
& \quad + \sum_{j > i} (-1)^{j+1} \lambda_{k+1}(\sigma_1, \hat{y_1}, \hat{y_2}, g_1 \cdots \hat{g_i} \cdots \hat{g_j} \cdots g_a, x_1 \cdots x_b; (y_1, g_j), (y_2, g_i) \cdots) \\
& \quad + (-1)^{a+1} \lambda_k(\sigma_1, g_1, \cdots, \hat{g_i}, \cdots, g_a, y_1, x_1, \cdots, x_b; (y_2, g_i), \cdots)\Big) \\
& + (-1)^a \Big(\sum_{1 \le j \le a} (-1)^j \lambda_k(\sigma_1, g_1, \cdots, \hat{g_j}, \cdots, g_a, y_2, x_1, \cdots, x_b; (y_1, g_j), \cdots) \\
& \quad + (-1)^a \lambda_{k-1}(\sigma_1, g_1, \cdots, g_a, y_1, y_2, x_1, \cdots, x_b; \cdots)\Big) \\
& + \sum_j (-1)^j \Big(\sum_{1 \le i < j} (-1)^i \lambda_{k+1}(\sigma_1, \hat{y_2}, \hat{y_1}, g_1 \cdots \hat{g_i} \cdots \hat{g_j} \cdots x_1 \cdots; (y_1, g_j), (y_2, g_i) \cdots)
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\sum_{i>j}(-1)^{i+1}\lambda_{k+1}(\sigma_1,\hat{y_2},\hat{y_1},g_1\cdots\hat{g_j}\cdots\hat{g_i}\cdots g_a,x_1\cdots x_b;(y_1,g_j),(y_2,g_i)\cdots)\\
&\quad +(-1)^{a+1}\lambda_k(\sigma_1,g_1,\cdots,\hat{g_j},\cdots,g_a,y_2,x_1,\cdots,x_b;(y_1,g_j),\cdots)\Big)\\
&+(-1)^a\Big(\sum_{1\le i\le a}(-1)^i\lambda_k(\sigma_1,g_1,\cdots,\hat{g_i},\cdots,g_a,y_1,x_1,\cdots,x_b;(y_2,g_i),\cdots)\\
&\quad +(-1)^a\lambda_{k-1}(\sigma_1,g_1,\cdots,g_a,y_2,y_1,x_1,\cdots,x_b;\cdots)\Big)\\
=\ &\lambda_{k-1}(\sigma_1,g_1\cdots g_a,y_1,y_2,x_1\cdots x_b;\cdots)+\lambda_{k-1}(\sigma_1,g_1\cdots g_a,y_2,y_1,x_1\cdots x_b;\cdots)
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\lambda_{k-1}(\sigma_1,y_1,y_2,\sigma_2;\cdots)+\lambda_{k-1}(\sigma_1,y_2,y_1,\sigma_2;\cdots)\\
=\ &(\pm1)\lambda_{k-1}(\sigma_1,y_1,y_2,\bar{\sigma_2};\cdots)+\sum(\pm1)\lambda_k(\sigma_1,y_1,y_2,\bullet;\bullet)\\
&+(\pm1)\lambda_{k-1}(\sigma_1,y_2,y_1,\bar{\sigma_2};\cdots)+\sum(\pm1)\lambda_k(\sigma_1,y_2,y_1,\bullet;\bullet)\\
=\ &(\pm1)\Big(\lambda_{k-1}(\sigma_1,y_1,y_2,\bar{\sigma_2};\cdots)+\lambda_{k-1}(\sigma_1,y_2,y_1,\bar{\sigma_2};\cdots)\Big)\\
&+\sum(\pm1)\Big(\lambda_k(\sigma_1,y_1,y_2,\bullet;\bullet)+\lambda_k(\sigma_1,y_2,y_1,\bullet;\bullet)\Big)\\
=\ &(\pm1)\Big(\lambda_{k-1}(\sigma_1,g_1,\cdots,g_a,y_1,y_2,x_1,\cdots,x_b;\cdots)\\
&\quad +\lambda_{k-1}(\sigma_1,g_1,\cdots,g_a,y_2,y_1,x_1,\cdots,x_b;\cdots)\Big)\\
&-\sum(\pm1)\lambda_{k+1}(\sigma_1,\hat{y_1},\hat{y_2},\bullet;(y_1,y_2),\bullet)\\
&(\textit{denote by } \tau \textit{ the permutation } (\sigma_1,g_1,\cdots,g_a))\\
=\ &(\pm1)\Big((\pm1)\lambda_{k-1}(\bar{\tau},y_1,y_2,x_1,\cdots,x_b;\cdots)+\sum(\pm1)\lambda_k(\bullet,y_1,y_2,x_1,\cdots,x_b;\bullet)\\
&\quad +(\pm1)\lambda_{k-1}(\bar{\tau},y_2,y_1,x_1,\cdots,x_b;\cdots)+\sum(\pm1)\lambda_k(\bullet,y_2,y_1,x_1,\cdots,x_b;\bullet)\Big)\\
&-\sum(\pm1)\lambda_{k+1}(\sigma_1,\hat{y_1},\hat{y_2},\bullet;(y_1,y_2),\bullet)\\
=\ &-(\pm1)\Big((\pm1)\lambda_k(\bar{\tau},\hat{y_1},\hat{y_2},x_1,\cdots,x_b;(y_1,y_2),\cdots)\\
&\quad +\sum(\pm1)\lambda_{k+1}(\bullet,\hat{y_1},\hat{y_2},x_1,\cdots,x_b;(y_1,y_2),\bullet)\Big)\\
&-\sum(\pm1)\lambda_{k+1}(\sigma_1,\hat{y_1},\hat{y_2},\bullet;(y_1,y_2),\bullet)\\
=\ &-(\pm1)\lambda_k(\tau,\hat{y_1},\hat{y_2},x_1,\cdots,x_b;(y_1,y_2),\cdots)-\sum(\pm1)\lambda_{k+1}(\sigma_1,\hat{y_1},\hat{y_2},\bullet;(y_1,y_2),\bullet)\\
=\ &-\Big((\pm1)\lambda_k(\sigma_1,\hat{y_1},\hat{y_2},\bar{\sigma_2};(y_1,y_2),\cdots)+\sum(\pm1)\lambda_{k+1}(\sigma_1,\hat{y_1},\hat{y_2},\bullet;(y_1,y_2),\bullet)\Big)\\
&(\textit{by the observation above})\\
=\ &-\lambda_k(\sigma_1,\hat{y_1},\hat{y_2},\sigma_2;(y_1,y_2),\cdots)
\end{aligned}
$$

Proof of Lambda Condition 2):

For regular permutations:

$$
\begin{aligned}
& (\delta\lambda)_k(g_1,\cdots,g_l,x_1,\cdots,x_{n-2k-l};f_1,\cdots f_k) \\
= & \sum_{1\le i\le k} \lambda_{k-1}(f_i,g_1,\cdots,g_l,x_1,\cdots,x_{n-2k-l};\cdots,\hat{f}_i,\cdots) \\
= & \frac{1}{k+l}\sum_{1\le i\le k}(\omega_k-d_0\lambda_k-d'\lambda_k)(g_1,\cdots,g_l,\cdots;f_1,\cdots,f_k) \\
& +\frac{1}{k+l}\sum_{1\le i\le k}\sum_{1\le j\le l}(-1)^j(\omega_k-d_0\lambda_k-d'\lambda_k)(f_i,g_1\cdots\hat{g}_j\cdots g_l\cdots;g_j\cdots\hat{f}_i\cdots) \\
= & \frac{k}{k+l}(\omega_k-d_0\lambda_k-d'\lambda_k)(g_1,\cdots,g_l,\cdots;f_1,\cdots,f_k) \\
& +\frac{1}{k+l}\sum_{1\le j\le l}(-1)^j(-1)(\omega_k-d_0\lambda_k-d'\lambda_k)(g_j,g_1,\cdots,\hat{g}_j,\cdots,g_l,\cdots;f_1,\cdots,f_k) \\
= & \frac{k+l}{k+l}(\omega_k-d_0\lambda_k-d'\lambda_k)(g_1,\cdots,g_l,\cdots;f_1,\cdots,f_k)
\end{aligned}
$$

So $\omega_k=(d\lambda)_k$ on $(g_1,\cdots,g_l,x_1,\cdots,x_{n-2k-l})$.

For general permutations, since both sides in the above equation are weakly skew-symmetric up to $\omega_{k+1}=(d\lambda)_{k+1}$, so $\omega_k=(d\lambda)_k$ still holds.

Proof of Lambda Condition 3):

Equivalently, we will prove the following:

$$
\begin{aligned}
& \sum_i(\omega_{k-1}-d_0\lambda_{k-1}-d'\lambda_{k-1})(f_i,e_1,\cdots,e_{n+1-2k};f_1,\cdots,\hat{f}_i,\cdots f_k) \\
= & (d\omega)_k(e_1,\cdots,e_{n+1-2k};f_1,\cdots,f_k)
\end{aligned}
$$

that is

$$
\begin{aligned}
& (d_0\omega_k+d'\omega_k)(e_1,\cdots,e_{n+1-2k};f_1,\cdots,f_k) \\
& +\sum_i(d_0\lambda_{k-1}+d'\lambda_{k-1})(f_i,e_1,\cdots,e_{n+1-2k};f_1,\cdots,\hat{f}_i,\cdots f_k) \\
= & (d_0\omega_k+d'\omega_k)(e_1,\cdots,e_{n+1-2k};f_1,\cdots,f_k) \\
& +\sum_i\sum_a(-1)^a\rho(e_a)\lambda_{k-1}(f_i,\cdots,\hat{e_a},\cdots;\cdots,\hat{f}_i,\cdots) \\
& +\sum_i\sum_{a<b}(-1)^{a+1}\lambda_{k-1}(f_i,\cdots,\hat{e_a},\cdots,e_a\circ e_b,\cdots;\cdots,\hat{f}_i,\cdots) \\
& +\sum_i\sum_a(-1)\lambda_{k-1}(e_1,\cdots,f_i\circ e_a,\cdots;\cdots,\hat{f}_i,\cdots)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_i\sum_{j\neq i}\lambda_{k-1}(e_1,\cdots,e_{n+1-2k};\cdots,\hat{f_i},\cdots,f_j\circ f_i,\cdots)\\
&+\sum_i\sum_{a,j\neq i}(-1)^a\lambda_{k-1}(f_i,\cdots,\hat{e_a},\cdots;\cdots,\hat{f_i},\cdots,f_j\circ e_a,\cdots)\\
=\ &(d_0\omega_k+d'\omega_k)(e_1,\cdots,e_{n+1-2k};f_1,\cdots,f_k)\\
&+\sum_a(-1)^a\rho(e_a)(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots,\hat{e_a},\cdots;f_1,\cdots,f_k)\\
&+\sum_{a<b}(-1)^{a+1}(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots,\hat{e_a},\cdots,e_a\circ e_b,\cdots;f_1,\cdots,f_k)\\
&+\sum_{a,i}(-1)\lambda_{k-1}(e_1,\cdots,f_i\circ e_a,\cdots;\cdots,\hat{f_i},\cdots)\\
&+\sum_{i<j}\lambda_{k-1}(e_1,\cdots,e_{n+1-2k};f_i\circ f_j+f_j\circ f_i,\cdots,\hat{f_i},\cdots,\hat{f_j},\cdots)\\
&+\Big(\sum_{a,j}(-1)^a(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots,\hat{e_a},\cdots;\cdots,f_j\circ e_a,\cdots)\\
&\quad-\sum_{a,j}(-1)^a\lambda_{k-1}(f_j\circ e_a,\cdots,\hat{e_a},\cdots;\cdots,\hat{f_j},\cdots)\Big)\\
=\ &((d_0^2+d_0\circ d')\lambda)_k(e_1,\cdots,e_{n+1-2k};f_1,\cdots,f_k)\\
&+\sum_{a,i}(-1)\lambda_{k-1}(e_1,\cdots,f_i\circ e_a,\cdots;\cdots,\hat{f_i},\cdots)\\
&+((d'\circ d_0+d'^2)\lambda)_k(e_1,\cdots,e_{n+1-2k};f_1,\cdots,f_k)\\
&+\sum_{a,j}(-1)^{a+1}\lambda_{k-1}(f_j\circ e_a,\cdots,\hat{e_a},\cdots;\cdots,\hat{f_j},\cdots)\\
=\ &((d_0+d'+\delta)^2\lambda)_k(e_1,\cdots,e_{n+1-2k};f_1,\cdots,f_k)\\
=\ &0
\end{aligned}
$$

Thus λ_{k-1} satisfies Lambda Conditions.

Step 4:

By mathematical induction, finally we obtain $(\lambda_0,\cdots,\lambda_{m-1})$ with each λ_p satisfying Lambda Conditions.

Let $\lambda\triangleq(\lambda_0,\cdots,\lambda_{m-1})$, Lambda Condition 1) implies that λ is a cochain in $C^{n-1}_{st}(L)$.

Let $\beta\triangleq\omega-d\lambda\in C^n_{st}(L)$.

By Lambda Condition 2), $\beta=(\omega_0-(d\lambda)_0,0,\cdots,0)=(\omega_0-d_0\lambda_0-d'\lambda_0,0,\cdots,0)$.

By Lambda Condition 3), $(\omega_0-d_0\lambda_0-d'\lambda_0)(f,e_1,\cdots,e_{n-1})=0,\ \forall f\in h,\ e_i\in L$.

So $\iota_f\beta_0=\iota_f(\omega_0-d_0\lambda_0-d'\lambda_0)=0,\ \forall f\in h$, which implies that $\beta\in C^n_{nv}(L)$.

Thus $\omega=\beta+d\lambda,\ \beta\in C^n_{nv}(L),\ \lambda\in C^{n-1}_{st}(L)$, the proof is finished. ■

Now we go to the proof of Theorem 3.10:

Proof. As explained in the previous section, $C^\bullet_{nv}(L)$ is a subcomplex of $C^\bullet_{st}(L)$. The inclusion map ψ from $C^n_{nv}(L)$ to $C^n_{st}(L)$ which sends β to $(\beta, 0, \cdots, 0)$ is a cochain map since $d(\beta, 0, \cdots, 0) = (d_0\beta, 0, \cdots, 0)$, so it induces a map $\psi_\star$ on cohomology. We need to check that $\psi_\star$ is an isomorphism.

1). $\psi_\star$ is surjective.

Given any $[\omega] \in H^n_{st}(L)$, since ω is closed, by Lemma 3.11, there exists $\beta \in C^n_{nv}(L)$ and $\lambda \in C^{n-1}_{st}(L)$ such that $\omega = \beta + d\lambda$. We see that $\beta = \omega - d\lambda$ is itself a cocycle, thus $\psi_\star([\beta]) = [\omega]$.

2). $\psi_\star$ is injective.

Given $\alpha \in C^n_{nv}(L)$, suppose $\psi\alpha = (\alpha, 0, \cdots, 0)$ is exact in $C^n_{st}(L)$, i.e. $\exists \omega \in C^{n-1}_{st}(L)$, $d\omega = (\alpha, 0, \cdots, 0)$, we need to prove that α itself is exact in $C^n_{nv}(L)$.

Since $(d\omega)_k = 0$, $\forall k > 0$, by Lemma 3.11, there exists $\beta \in C^{n-1}_{nv}(L)$ and $\lambda \in C^{n-2}_{st}(L)$ such that $\omega = \beta + d\lambda$. Thus $d\omega = (\alpha, 0, \cdots, 0) = d\beta = (d_0\beta, 0, \cdots, 0)$, which implies that $\alpha = d_0\beta$ is exact in $C^n_{nv}(L)$.

Thus the theorem is proved. ■

Remark 3.12. In our definition of the standard complex $(C^\bullet_{st}(L, h, R), d)$ (3.4), we assume that h is an isotropic ideal in L, i.e. $h \to L$ is an injective homomorphism. Such a pair (L, h) can be viewed as a "transitive" pair of Leibniz algebras, and Theorem 3.10 holds for a transitive pair (L, h).

Actually the standard complex and naive complex can be similarly defined for a general pair (L, h) of Leibniz algebras: h is required to be a Leibniz algebra fitting into a sequence $Z \overset{\psi}{\hookrightarrow} h \overset{\varphi}{\to} L$, where φ is a homomorphism of Leibniz algebras, and there is required to be a map $\phi : h \otimes L \to h$ satisfying certain conditions. In this general case, the standard cohomology $H^\bullet_{st}(L, h, R)$ is not necessarily isomorphic to the naive cohomology $H^\bullet_{nv}(L, h, R)$.

In fact, the methods above also apply to the case of transitive Courant-Dorfman algebras:

Definition 3.13. A Courant-Dorfman algebra $(\mathcal{E}, R, (\bullet, \bullet), \partial, \circ)$ is called transitive, if the coanchor map $\rho^* : \Omega^1 \to \mathcal{E}$ as defined by Equation 2.2.2 is injective.

We see that when $\mathcal{E} = \Gamma(E)$ is the space of sections of a Courant algebroid E, $\mathcal{E}$ is transitive iff E is transitive.

Next, in parallel to Definition 2.27, the naive cohomology of $\mathcal{E}$ is defined as follows:

Definition 3.14. Given a Courant-Dorfman algebra $(\mathcal{E}, R, (\bullet, \bullet), \partial, \circ)$, the standard complex $(C^\bullet(\mathcal{E}, R), d)$ is defined as in Definition 2.20. Let

$$C^\bullet_{nv}(\mathcal{E}, R) \triangleq \{\omega \in C^\bullet(\mathcal{E}, R) | \omega_k = 0,\ \forall k \geq 1,\ \iota_{\partial f}\omega_0 = 0,\ \forall f \in R\}.$$

Obviously $(C^\bullet_{nv}(\mathcal{E}, R), d_0)$ is a subcomplex of $(C^\bullet(\mathcal{E}, R), d)$, and is called the naive complex of $\mathcal{E}$. The resulting cohomology, denoted by $H^\bullet_{nv}(\mathcal{E})$, is called the naive cohomology of $\mathcal{E}$.

It is easily seen that the naive complex $C^\bullet_{nv}(\mathcal{E}, R)$ defined above is isomorphic to the Chevalley-Eilenberg complex of the Lie-Rinehart algebra $\mathcal{E}/\rho^*(\Omega^1)$ with coefficients in its module R, so we have

$$H^\bullet_{nv}(\mathcal{E}) \cong H^\bullet_{CE}(\mathcal{E}/\rho^*(\Omega^1), R).$$

If $\mathcal{E} = \Gamma(E)$ is the space of sections of a Courant algebroid E, then $\mathcal{E}$ is non-degenerate. Since $(e, \partial f) = \rho(e) \cdot f$, any $\alpha \in \Gamma(\wedge^\bullet ker\rho)$ can be characterized as an element $\tilde{\alpha} \in \Gamma(\wedge^\bullet E^*) = Hom_R(\wedge^\bullet_R \mathcal{E}, R)$ such that $\iota_{\partial f}\tilde{\alpha} = 0,\ \forall f \in R$ (identification of $\mathcal{E}$ and $\mathcal{E}^\vee$ applied here), i.e. $\tilde{\alpha} \in C^\bullet_{nv}(\mathcal{E}, R)$. So the map $\alpha \mapsto \tilde{\alpha}$ is an isomorphism from the naive complex $(\Gamma(\wedge^\bullet ker\rho), d_{nv})$ of E to the naive complex $(C^\bullet_{nv}(\mathcal{E}, R), d_0)$ of $\mathcal{E}$. Therefore, Definition 3.14 recovers Definition 2.27 when $\mathcal{E} = \Gamma(E)$.

Now we can describe the isomorphism theorem for transitive Courant-Dorfman algebras:

Theorem 3.15. *If $\mathcal{E}$ is a transitive Courant-Dorfman algebra, the inclusion map of complexes*

$$C^\bullet_{nv}(\mathcal{E}, R) \hookrightarrow C^\bullet(\mathcal{E}, R)$$

induces an isomorphism between $H^\bullet_{nv}(\mathcal{E})$ and $H^\bullet_{st}(\mathcal{E})$.

Similar to the proof of Theorem 3.10, we need to prove the following lemma first:

Lemma 3.16. *$\forall \omega \in C^n(\mathcal{E}, R)$, if $(d\omega)_k = 0,\ \forall k > 0$, there exists $\beta \in C^n_{nv}(\mathcal{E}, R)$ and $\lambda \in C^{n-1}(\mathcal{E}, R)$ such that $\omega = \beta + d\lambda$.*

Proof. The proof is quite similar to that of Lemma 3.11. We list the key steps, as well as the different settings here. Most details will be omitted.

Suppose $\mathcal{E} = \rho^*(\Omega^1) \oplus X$ as R-module.

Suppose $n = 2m$ or $2m - 1$.

We will define $\lambda_{m-1}, \lambda_{m-1}, \cdots, \lambda_0$ one by one, so that each $\lambda_p : \otimes^{n-1-2p}\mathcal{E} \otimes \odot^p R \to R$ satisfies the following so-called "Lambda Conditions":

1). λ_p is weakly skew-symmetric up to λ_{p+1},

2). $\omega_{p+1} = (d\lambda)_{p+1}$,
3). $\sum_i (\omega_p - d_0\lambda_p)(\partial f_i, e_1, \cdots, e_{n-1-2p}; f_1, \cdots, \hat{f}_i, \cdots f_{p+1}) = 0$, $\forall f_j \in R, e_a \in \mathcal{E}$,
4). λ_p is weakly R-linear in each argument of $\mathcal{E}$,
5). λ_p is a derivation in each argument of R.
The construction of $\lambda_{m-1}, \cdots, \lambda_0$ is done in the following four steps.
Step 1:
Construction of λ_{m-1}:
When $n = 2m-1$, let

$$\lambda_{m-1}(f_1, \cdots, f_{m-1}) = 0, \quad \forall f_i \in R.$$

When $n = 2m$, let

$$\lambda_{m-1}(g'\partial g; f_1, \cdots, f_{m-1}) = \frac{g'}{m}\omega_m(g, f_1, \cdots, f_{m-1}),$$

and

$$\lambda_{m-1}(x; f_1, \cdots, f_{m-1}) = 0,$$

$\forall g', g, f_i \in R,\ x \in X$.

It is easily checked that λ_{m-1} satisfies Lambda Conditions.

Step 2:

Suppose $\lambda_{m-1}, \cdots, \lambda_k (k > 0)$ are already defined so that they satisfy Lambda Conditions, we will construct λ_{k-1} so that it also satisfies Lambda Conditions.

The value of λ_{k-1} for regular permutation $(\partial g, \cdots \partial g, x, \cdots x)$ is determined by:

$$\begin{aligned} & \lambda_{k-1}(\partial g_1, \cdots, \partial g_l, x_1, \cdots, x_{n+1-2k-l}; f_1, \cdots f_{k-1}) \\ \triangleq\ & \frac{1}{k+l-1}\sum_{1\le j\le l}(-1)^{j+1}(\omega_k - d_0\lambda_k)(\partial g_1, \cdots \widehat{\partial g_j}, \cdots \partial g_l, x_1, \cdots; g_j, f_1, \cdots) \\ & \forall f_i, g_j \in R,\ x_a \in X. \end{aligned}$$

Then by weak R-linearity in arguments of $\mathcal{E}$, the value of λ_{k-1} for regular permutation $(g'\partial g, \cdots g'\partial g, x, \cdots x)$ can be determined.

Here in order for λ_{k-1} to be well-defined, we need to check that

$$\begin{aligned} & \lambda_{k-1}(\partial g_1, \cdots \partial(g_l g_l'), x, \cdots; f, \cdots) \\ =\ & \lambda_{k-1}(\partial g_1, \cdots g_l \partial g_l', x, \cdots; f, \cdots) + \lambda_{k-1}(\partial g_1, \cdots g_l' \partial g_l, x, \cdots; f, \cdots). \end{aligned}$$

Finally by weak skew-symmetricity in arguments of $\mathcal{E}$, the value of λ_{k-1} for general permutation can be determined.

Step 3:

We need to prove that λ_{k-1} defined above satisfies Lambda Conditions.

Step 4:

By mathematical induction, we will eventually obtain $(\lambda_0, \cdots, \lambda_{m-1})$ with each λ_p satisfying Lambda Conditions.

Let $\lambda \triangleq (\lambda_0, \cdots, \lambda_{m-1})$, Lambda Condition 1), 4) and 5) implies that λ is a cochain in $C^{n-1}(\mathcal{E}, R)$.

Let $\beta \triangleq \omega - d\lambda \in C^n(\mathcal{E}, R)$.

By Lambda Condition 2), $\beta = (\omega_0 - (d\lambda)_0, 0, \cdots, 0) = (\omega_0 - d_0\lambda_0, 0, \cdots, 0)$.

By Lambda Condition 3), $(\omega_0 - d_0\lambda_0)(\partial f, e_1, \cdots, e_{n-1}) = 0, \ \forall f \in R, \ e_i \in \mathcal{E}$.

So $\iota_{\partial f}\beta_0 = \iota_{\partial f}(\omega_0 - d_0\lambda_0) = 0, \ \forall f \in R$, which implies that $\beta \in C^n_{nv}(\mathcal{E}, R)$.

Thus $\omega = \beta + d\lambda, \ \beta \in C^n_{nv}(\mathcal{E}, R), \ \lambda \in C^{n-1}(\mathcal{E}, R)$, the proof is finished. ■

Applying the lemma above, the proof of Theorem 3.15 can be done almost the same with that of Theorem 3.10. We omit it here.

Theorem 3.15 is a generalization of Theorem 2.29. When $\mathcal{E} = \Gamma(E)$ is the space of sections of a transitive Courant algebroid E, they give the same result. Note that Theorem 3.15 holds even if the symmetric bilinear form of $\mathcal{E}$ is degenerate.

In the last of this section, we compute standard cohomology for some examples of Leibniz algebras.

First we consider the omni Lie algebra $ol(V)$. The left center is V. Given any left module R of $ol(V)$, since any $v \in V$ equals to $e_1 \circ e_2 + e_2 \circ e_1$ for some $e_1, e_2 \in ol(V)$, it is easily deduced that V acts trivially on R, so Theorem 3.10 tells that

$$H^\bullet_{st}(ol(V), V, R) \cong H^\bullet_{CE}(gl(V), R).$$

If $R = V$ with Leibniz module structure given by $\rho(A+v)w = Aw$, it is a well-known result that

$$H^n_{CE}(gl(V), V) = 0, \quad \forall n.$$

So we have the following:

Proposition 3.17. $H^n_{st}(ol(V), V, V) = 0, \ \forall n.$

Similarly, based on the standard Courant algebroid $T\mathbb{R}^n \oplus T^*\mathbb{R}^n$, if we take the sections of all linear vector fields and linear 1-forms, we can also obtain a Leibniz algebra

$$ol^1(V) \triangleq gl(V) \oplus \Omega^1(V),$$

where $\Omega^1(V)$ denotes the linear 1-forms. Suppose $\{x^i\}_{1 \le i \le n}$ is a chart of $\mathbb{R}^n$, then any $\xi \in \Omega^1(V)$ can be uniquely written as $\xi = \xi_{ij}x^i dx^j$, so $\Omega^1(V) \cong gl(V)$ as vector spaces. (Any $A = (A_i^j) \in gl(V)$ viewed as linear vector field is $A_i^j x^i \frac{\partial}{\partial x^j}$.) $\forall A, B \in gl(V),\ \xi, \eta \in \Omega^1(V)$, the Courant bracket

$$\begin{aligned}
& (A+\xi)\circ(B+\eta) \\
= \ & [A,B] + L_A\eta - \iota_B d\xi \\
= \ & [A,B] + d(\iota_A\eta) + \iota_A(d\eta) - \iota_B d\xi \\
= \ & [A,B] + d((A_i^j x^i \frac{\partial}{\partial x^j}, \eta_{kj}x^k dx^j)) + \iota_A(\eta_{ij}dx^i \wedge dx^j) - \iota_B(\xi_{ij}dx^i \wedge dx^j) \\
= \ & [A,B] + A_i^j\eta_{kj}x^i dx^k + A_i^j\eta_{kj}x^k dx^i + \eta_{ij}(\iota_A dx^i)dx^j - \eta_{ij}(\iota_A dx^j)dx^i \\
& -\xi_{ij}(\iota_B dx^i)dx^j + \xi_{ij}(\iota_B dx^j)dx^i \\
= \ & [A,B] + A_i^k\eta_{jk}x^i dx^j + A_j^k\eta_{ik}x^i dx^j + A_k^i\eta_{ij}x^k dx^j - A_k^j\eta_{ij}x^k dx^i \\
& -B_k^i\xi_{ij}x^k dx^j + B_k^j\xi_{ij}x^k dx^i \\
= \ & [A,B] + (A_i^k\eta_{jk} + A_j^k\eta_{ik} + A_i^k\eta_{kj} - A_i^k\eta_{jk})x^i dx^j - (B_i^k\xi_{kj} - B_i^k\xi_{jk})x^i dx^j \\
= \ & [A,B] + (\eta_{ik}A_j^k + A_i^k\eta_{kj} - B_i^k\xi_{kj} + B_i^k\xi_{jk})x^i dx^j
\end{aligned}$$

So the Leibniz bracket of $ol^1(V)$ is given by

$$(A+\xi)\circ(B+\eta) = [A,B] + \eta A^T + A\eta - B\xi + B\xi^T,\ \forall A, B \in gl(V),\ \xi, \eta \in \Omega^1(V).$$

Proposition 3.18. *The left center of $ol^1(V)$ is $sym(V) \triangleq \{\xi \in \Omega^1(V) | \xi = \xi^T\}$.*

Proof. Suppose $A + \xi$ is in the left center.

First let $\eta = 0$, then $(A+\xi)\circ B = [A,B] + B(\xi^T - \xi) = 0,\ \forall B$ implies that $[A,B] = 0,\ \forall B$ and $\xi^T = \xi$, so $A = aI$ for some $a \in \mathbb{R}$.

Next let $B = 0$, we have $(A+\xi)\circ\eta = \eta A^T + A\eta = 2a\eta = 0,\ \forall \eta$, so $a = 0$, thus $A = 0$. ■

$$ol^1(V)/sym(V) = gl(V) \oplus (\Omega^1(V)/sym(V)) \cong gl(V) \oplus asym(V)$$

is a Lie algebra with Lie bracket:

$$\begin{aligned}
& [A+\alpha, B+\beta] \\
= \ & \overline{(A+\alpha)\circ(B+\beta)} \\
= \ & [A,B] + \overline{(\beta A^T + A\beta - B\alpha + B\alpha^T)} \\
= \ & [A,B] + \frac{1}{2}\{(\beta A^T + A\beta - B\alpha + B\alpha^T) - (\beta A^T + A\beta - B\alpha + B\alpha^T)^T\} \\
= \ & [A,B] + A\beta + \beta A^T - B\alpha - \alpha B^T \\
& \forall A, B \in gl(V),\ \forall \alpha, \beta \in asym(V)
\end{aligned}$$

Let $R = sym(V)$, with Leibniz module structure given by

$$\rho(A+\xi)\eta = A\eta + \eta A^T, \quad \forall A \in gl(V),\ \xi \in \Omega^1(V),\ \eta \in sym(V),$$

then 3.10 tells that

Proposition 3.19. $H^\bullet_{st}(ol^1(V), sym(V), sym(V)) \cong H^\bullet_{CE}(gl(V) \oplus asym(V), sym(V))$.

In the following, we compute the degree 0 and degree 1 cohomology:
Degree 0:
Suppose $\omega \in C^0_{CE}((gl(V) \oplus asym(V), sym(V))$ is a cocycle, i.e.

$$d\omega(A+\alpha) = \rho(A+\alpha)\omega = A\omega + \omega A^T = 0,\ \forall A \in gl(V),\ \alpha \in asym(V).$$

Let $A = I$, it follows that $\omega = 0$. So degree 0 cohomology is trivial:

$$H^0_{CE}(gl(V) \oplus asym(V), sym(V)) = 0.$$

Degree 1:
Suppose ω is a 1-cocycle, i.e.

$$\begin{aligned} & d\omega(A+\alpha, B+\beta) \\ = & \ \rho(A+\alpha)\omega(B+\beta) - \rho(B+\beta)\omega(A+\alpha) - \omega([A+\alpha, B+\beta]) \\ = & \ 0 \\ & \forall A, B \in gl(V),\ \alpha, \beta \in asym(V) \end{aligned}$$

Let $B = I,\ \beta = 0$, we have

$$\rho(I)\omega(A+\alpha) = 2\omega(A+\alpha) = \rho(A+\alpha)\omega(I) - \omega([A+\alpha, I]) = \rho(A)\omega(I) + 2\omega(\alpha).$$

It follows that

$$\omega(A) = \frac{1}{2}\rho(A)\omega(I) = (d\lambda)(A),$$

where $\lambda \triangleq \frac{1}{2}\omega(I) \in C^0_{CE}((gl(V) \oplus asym(V), sym(V))$. In order for $\omega(A+\alpha) = (d\lambda)(A+\alpha) + \omega(\alpha)$ to be closed, it suffices that

$$\rho(A)\omega(\beta) - \rho(B)\omega(\alpha) - \omega(A\beta + \beta A^T - B\alpha - \alpha B^T) = 0,\ \forall A, B \in gl(V),\ \alpha, \beta \in asym(V),$$

or equivalently

$$\rho(A)\omega(\beta) = \omega(A\beta + \beta A^T) = \omega(\rho(A)\beta),\ \forall A \in gl(V),\ \beta \in asym(V).$$

(Note that $asym(V)$ is also a $ol^1(V)$ module with the same action map $\rho(A+\xi)\beta = A\beta + \beta A^T$.)

Thus

$$\begin{aligned} & H^1_{CE}(gl(V) \oplus asym(V), sym(V)) \\ = & \{f \in Hom(asym(V), sym(V)) |\ f \text{ is equivariant w.r.t. } ol^1(V)\}. \end{aligned}$$

Using the same methods, we can obtain Leibniz algebras

$$ol^k(V) \triangleq gl(V) \oplus \Omega^k(V),$$

where $\Omega^k(V)$ is the set of 1-forms of homogeneous degree k

$$\Omega^k(V) \triangleq \{\sum f_i(x)dx^i | f_i(x) \text{ is a degree } k \text{ polynomial}\},$$

and Leibniz algebras

$$ol^{\leq k}(V) \triangleq gl(V) \oplus \Omega^{\leq k}(V),$$

with Leibniz brackets induced by the standard Courant algebroid. Since $\Omega(\mathbb{R}^n)$ is an isotropic ideal of $\Gamma(T\mathbb{R}^n \oplus T^*\mathbb{R}^n)$, $\Omega^k(V)$ (or $\Omega^{\leq k}(V)$) is an isotropic ideal of $ol^k(V)$ (or $ol^{\leq k}(V)$ resp.). For $R = V$ with the usual module structure, applying Theorem 3.10, we have the following

Proposition 3.20. $\forall k, n \in \mathbb{N}$, *we have*

$$H^n_{st}(ol^k(V), \Omega^k(V), V) = 0$$

and

$$H^n_{st}(ol^{\leq k}(V), \Omega^{\leq k}(V), V) = 0.$$

3.3 Crossed products of Leibniz algebras

Given a Leibniz algebra L with left center Z, let $S^\bullet(Z)$ be the algebra of symmetric tensors of Z. We construct a Courant-Dorfman algebra structure on the tensor product $S^\bullet(Z) \otimes L$ as follows:

the associated commutative algebra is taken to be $S^\bullet(Z)$;

the $S^\bullet(Z)$-module structure of $S^\bullet(Z) \otimes L$ is given by multiplication of $S^\bullet(Z)$, i.e.

$$f_1 \cdot (f_2 \otimes e) \triangleq (f_1 f_2) \otimes e;$$

(For simplicity, we will write $f \otimes e$ ($f \in S^\bullet(Z),\ e \in L$) as fe from now on.)

the symmetric bilinear form of $S^\bullet(Z)\otimes L$, still denoted by $(\bullet,\bullet)$, is the $S^\bullet(Z)$-bilinear extension of the bilinear product of L, i.e.

$$(f_1e_1, f_2e_2)\triangleq f_1f_2(e_1,e_2);$$

the derivation $\partial:\ S^\bullet(Z)\to S^\bullet(Z)\otimes L$ is the extension of the inclusion map $Z\hookrightarrow L$ by Leibniz rule, i.e.

$$\partial(f_1\cdots f_k)\triangleq\sum_{1\le i\le k}(f_1\cdots\hat{f}_i\cdots f_k)\partial f_i;$$

the Dorfman bracket on $S^\bullet(Z)\otimes L$, still denoted by $\circ$, is the extension of the Leibniz bracket of L:

$$f_1e_1\circ f_2e_2\triangleq f_1f_2(e_1\circ e_2)+(e_1,e_2)f_2\partial f_1+(e_1,\partial f_2)f_1e_2-(e_2,\partial f_1)f_2e_1.$$

Proposition 3.21. *With the above notations,* $(S^\bullet(Z)\otimes L, S^\bullet(Z),(\bullet,\bullet),\partial,\circ)$ *becomes a Courant-Dorfman algebra (called the crossed product of* L*).*

Proof. We need to check all the six conditions of Courant-Dorfman algebra.

1). $f_1e_1\circ f(f_2e_2)=f(f_1e_1\circ f_2e_2)+(f_1e_1,\partial f)f_2e_2$

$$\begin{aligned}&\quad The\ LHS\\ &= ff_1f_2(e_1\circ e_2)+(e_1,e_2)ff_2\partial f_1+(e_1,\partial(ff_2))f_1e_2-(e_2,\partial f_1)ff_2e_1\\ &= ff_1f_2(e_1\circ e_2)+(e_1,e_2)ff_2\partial f_1+f(e_1,\partial f_2)f_1e_2+f_2(e_1,\partial f)f_1e_2-(e_2,\partial f_1)ff_2e_1\end{aligned}$$

$$\begin{aligned}&\quad The\ RHS\\ &= ff_1f_2(e_1\circ e_2)+f(e_1,e_2)f_2\partial f_1+f(e_1,\partial f_2)f_1e_2-f(e_2,\partial f_1)f_2e_1+f_1(e_1,\partial f)f_2e_2\end{aligned}$$

Thus the equation holds.

2). $(f_1e_1,\partial(f_2e_2,f_3e_3))=(f_1e_1\circ f_2e_2,f_3e_3)+(f_2e_2,f_1e_1\circ f_3e_3)$

$$\begin{aligned}&\quad The\ LHS\\ &= f_1(e_1,\partial(f_2f_3(e_2,e_3)))\\ &= f_1f_2f_3(e_1,\partial(e_2,e_3))+f_1f_2(e_2,e_3)(e_1,\partial f_3)+f_1f_3(e_2,e_3)(e_1,\partial f_2)\\ &= f_1f_2f_3\big((e_1\circ e_2,e_3)+(e_2,e_1\circ e_3)\big)+f_1f_2(e_2,e_3)(e_1,\partial f_3)+f_1f_3(e_2,e_3)(e_1,\partial f_2)\end{aligned}$$

$$\begin{aligned}
& \quad The\ RHS \\
= & \ \Big(f_1f_2(e_1\circ e_2)+(e_1,e_2)f_2\partial f_1+(e_1,\partial f_2)f_1e_2-(e_2,\partial f_1)f_2e_1, f_3e_3\Big) \\
& +\Big(f_2e_2, f_1f_3(e_1\circ e_3)+(e_1,e_3)f_3\partial f_1+(e_1,\partial f_3)f_1e_3-(e_3,\partial f_1)f_3e_1\Big) \\
= & \ f_1f_2f_3(e_1\circ e_2,e_3)+f_2f_3(e_1,e_2)(e_3,\partial f_1) \\
& +f_1f_3(e_2,e_3)(e_1,\partial f_2)-f_2f_3(e_1,e_3)(e_2,\partial f_1) \\
& +f_1f_2f_3(e_2,e_1\circ e_3)+f_2f_3(e_1,e_3)(e_2,\partial f_1) \\
& +f_1f_2(e_2,e_3)(e_1,\partial f_3)-f_2f_3(e_1,e_2)(e_3,\partial f_1) \\
= & \ f_1f_2f_3\Big((e_1\circ e_2,e_3)+(e_2,e_1\circ e_3)\Big)+f_1f_2(e_2,e_3)(e_1,\partial f_3)+f_1f_3(e_2,e_3)(e_1,\partial f_2) \\
= & \ The\ LHS.
\end{aligned}$$

3). $f_1e_1\circ f_2e_2+f_2e_2\circ f_1e_1=\partial(f_1e_1,f_2e_2)$

$$\begin{aligned}
& \quad The\ LHS \\
= & \ f_1f_2(e_1\circ e_2)+(e_1,e_2)f_2\partial f_1+(e_1,\partial f_2)f_1e_2-(e_2,\partial f_1)f_2e_1 \\
& +f_1f_2(e_2\circ e_1)+(e_1,e_2)f_1\partial f_2+(e_2,\partial f_1)f_2e_1-(e_1,\partial f_2)f_1e_2 \\
= & \ f_1f_2\partial(e_1,e_2)+(e_1,e_2)f_2\partial f_1+(e_1,e_2)f_1\partial f_2 \\
= & \ The\ RHS.
\end{aligned}$$

Combining 1) and 3), we get the following:

$$\begin{aligned}
& \quad f(f_1e_1)\circ f_2e_2 \\
=(& \ f(f_1e_1)\circ f_2e_2+f_2e_2\circ f(f_1e_1))-f_2e_2\circ f(f_1e_1) \\
= & \ \partial(f(f_1e_1),f_2e_2)-(f(f_2e_2\circ f_1e_1)+(f_2e_2,\partial f)f_1e_1 \\
= & \ (f_1e_1,f_2e_2)\partial f+f\partial(f_1e_1,f_2e_2)-f(f_2e_2\circ f_1e_1)-(f_2e_2,\partial f)f_1e_1 \\
= & \ f(f_1e_1\circ f_2e_2)+(f_1e_1,f_2e_2)\partial f-(f_2e_2,\partial f)f_1e_1
\end{aligned}$$

4). $(\partial f,\partial g)=0.$

We only need to consider the case of monomials: suppose $f=f_1f_2\cdots f_k,\ g=g_1g_2\cdots g_l,\ f_i,g_j\in Z$, then

$$\begin{aligned}
& \quad (\partial f,\partial g) \\
= & \ (\sum_i(f_1\cdots\hat{f_i}\cdots f_k)\partial f_i,\sum_j(g_1\cdots\hat{g_j}\cdots g_l)\partial g_j) \\
= & \ \sum_{i,j}(f_1\cdots\hat{f_i}\cdots f_kg_1\cdots\hat{g_j}\cdots g_l)(\partial f_i\circ\partial g_j+\partial g_j\circ\partial f_i) \\
= & \ 0
\end{aligned}$$

5). $\partial f \circ (ge) = 0$

First we prove that $\partial f \circ e = 0,\ \forall f \in S^\bullet(Z),\ e \in L$.

We only need to consider the case of monomials: suppose $f = f_1 f_2 \cdots f_k,\ f_i \in Z$.

When $k = 1$, i.e. $f = f_1 \in Z$, the equation is trivial.

Now suppose the equation holds for any $k \leq m$, let's consider the case of $k = m+1$.

$$\begin{aligned} & \partial(f_1 f_2 \cdots f_{m+1}) \circ e \\ = & \Big((f_1 \cdots f_m)\partial f_{m+1} + f_{m+1}\partial(f_1 \cdots f_m)\Big) \circ e \\ = & (f_1 \cdots f_m)(\partial f_{m+1} \circ e) + (\partial f_{m+1}, e)\partial(f_1 \cdots f_m) - (e, \partial(f_1 \cdots f_m))\partial f_{m+1} \\ & + f_{m+1}(\partial(f_1 \cdots f_m) \circ e) + (\partial(f_1 \cdots f_m), e)\partial f_{m+1} - (e, \partial f_{m+1})\partial(f_1 \cdots f_m) \\ = & 0 \end{aligned}$$

Thus by induction, the equation holds for any k.

Combining with 1) and 4), we have

$$\partial f \circ (ge) = g(\partial f \circ e) + (\partial f, \partial g)e = 0$$

6). $f_1 e_1 \circ (f_2 e_2 \circ f_3 e_3) = (f_1 e_1 \circ f_2 e_2) \circ f_3 e_3 + f_2 e_2 \circ (f_1 e_1 \circ f_3 e_3)$

First we prove the equation for the case $f_2 = f_3 = 1$:

$$f_1 e_1 \circ (e_2 \circ e_3) = (f_1 e_1 \circ e_2) \circ e_3 + e_2 \circ (f_1 e_1 \circ e_3)$$

$$\textit{The LHS} = f_1(e_1 \circ (e_2 \circ e_3)) + (e_1, e_2 \circ e_3)\partial f_1 - (e_2 \circ e_3, \partial f_1)e_1$$

$$\begin{aligned} & \textit{The RHS} \\ = & \Big(f_1(e_1 \circ e_2) + (e_1, e_2)\partial f_1 - (e_2, \partial f_1)e_1\Big) \circ e_3 \\ & + e_2 \circ \Big(f_1(e_1 \circ e_3) + (e_1, e_3)\partial f_1 - (e_3, \partial f_1)e_1\Big) \\ = & f_1((e_1 \circ e_2) \circ e_3) + (e_1 \circ e_2, e_3)\partial f_1 - (e_3, \partial f_1)(e_1 \circ e_2) \\ & + (e_1, e_2)(\partial f_1 \circ e_3) + (\partial f_1, e_3)\partial(e_1, e_2) - (e_3, \partial(e_1, e_2))\partial f_1 \\ & - \Big((e_2, \partial f_1)(e_1 \circ e_3) + (e_1, e_3)\partial(e_2, \partial f_1) - (e_3, \partial(e_2, \partial f_1))e_1\Big) \\ & + f_1(e_2 \circ (e_1 \circ e_3) + (e_2, \partial f_1)(e_1 \circ e_3) \end{aligned}$$

$$
\begin{aligned}
&+(e_1,e_3)(e_2\circ\partial f_1)+(e_2,\partial(e_1,e_3))\partial f_1\\
&-((e_3,\partial f_1)(e_2\circ e_1)+(e_2,\partial(e_3,\partial f_1))e_1)\\
=\;& f_1((e_1\circ e_2)\circ e_3)+f_1(e_2\circ(e_1\circ e_3)\\
&+\Big((e_1\circ e_2,e_3)-(e_3,\partial(e_1,e_2))+(e_2,\partial(e_1,e_3))\Big)\partial f_1\\
&+\Big((e_3,\partial(e_2,\partial f_1))-(e_2,\partial(e_3,\partial f_1))\Big)e_1\\
&+(e_1,e_3)\Big(e_2\circ\partial f_1-\partial(e_2,\partial f_1)\Big)\\
&+(e_3,\partial f_1)\Big(\partial(e_1,e_2)-e_1\circ e_2-e_2\circ e_1\Big)\\
&+(e_1,e_2)(\partial f_1\circ e_3)+(e_2,\partial f_1)(e_1\circ e_3)-(e_2,\partial f_1)(e_1\circ e_3)\\
=\;& f_1(e_1\circ(e_2\circ e_3))+\Big((e_2,\partial(e_1,e_3))-(e_2\circ e_1,e_3)\Big)\partial f_1-(e_2\circ e_3,\partial f_1)e_1\\
=\;& \textit{The LHS}
\end{aligned}
$$

Then we prove the equation for the case only $f_3=1$:

$$f_1e_1\circ(f_2e_2\circ e_3)=(f_1e_1\circ f_2e_2)\circ e_3+f_2e_2\circ(f_1e_1\circ e_3)$$

For simplicity, we write f_1e_1 as $x_1\in S^\bullet(Z)\otimes L$.

$$
\begin{aligned}
&\textit{The LHS}\\
=\;& x_1\circ\Big(f_2(e_2\circ e_3)+(e_2,e_3)\partial f_2-(e_3,\partial f_2)e_2\Big)\\
=\;& f_2(x_1\circ(e_2\circ e_3))+(x_1,\partial f_2)(e_2\circ e_3)\\
&+(e_2,e_3)(x_1\circ\partial f_2)+(x_1,\partial(e_2,e_3))\partial f_2\\
&-\Big((e_3,\partial f_2)(x_1\circ e_2)+(x_1,\partial(e_3,\partial f_2))e_2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\textit{The RHS}\\
=\;& \Big(f_2(x_1\circ e_2)+(x_1,\partial f_2)e_2\Big)\circ e_3\\
&+f_2(e_2\circ(x_1\circ e_3))+(e_2,x_1\circ e_3)\partial f_2-(x_1\circ e_3,\partial f_2)e_2\\
=\;& f_2((x_1\circ e_2)\circ e_3)+(x_1\circ e_2,e_3)\partial f_2-(e_3,\partial f_2)(x_1\circ e_2)\\
&+(x_1,\partial f_2)(e_2\circ e_3)+(e_2,e_3)\partial(x_1,\partial f_2)-(e_3,\partial(x_1,\partial f_2))e_2\\
&+f_2(e_2\circ(x_1\circ e_3))+(e_2,x_1\circ e_3)\partial f_2-(x_1\circ e_3,\partial f_2)e_2\\
=\;& f_2((x_1\circ e_2)\circ e_3)+f_2(e_2\circ(x_1\circ e_3))+(x_1,\partial f_2)(e_2\circ e_3)\\
&+(e_2,e_3)\partial(x_1,\partial f_2)+\Big((x_1\circ e_2,e_3)+(e_2,x_1\circ e_3)\Big)\partial f_2\\
&-\Big((e_3,\partial f_2)(x_1\circ e_2)+((e_3,\partial(x_1,\partial f_2))+(x_1\circ e_3,\partial f_2))e_2\Big)\\
=\;& \textit{The LHS.}
\end{aligned}
$$

Finally, write f_2e_2 as x_2, we will prove the following:

$$x_1 \circ (x_2 \circ f_3e_3) = (x_1 \circ x_2) \circ f_3e_3 + x_2 \circ (x_1 \circ f_3e_3)$$

$$\begin{aligned} & The\ LHS \\ = \ & x_1 \circ \Big(f_3(x_2 \circ e_3) + (x_2, \partial f_3)e_3\Big) \\ = \ & f(x_1 \circ (x_2 \circ e_3)) + (x_1, \partial f_3)(x_2 \circ e_3) + (x_2, \partial f_3)(x_1 \circ e_3) + (x_1, \partial(x_2, \partial f_3))e_3 \end{aligned}$$

$$\begin{aligned} & The\ RHS \\ = \ & f((x_1 \circ x_2) \circ e_3) + (x_1 \circ x_2, \partial f_3)e_3 \\ & + f(x_2 \circ (x_1 \circ e_3)) + (x_2, \partial f_3)(x_1 \circ e_3) + (x_1, \partial f_3)(x_2 \circ e_3) + (x_2, \partial(x_1, \partial f_3))e_3 \\ = \ & The\ LHS. \end{aligned}$$

Thus the proposition is proved. ■

Obviously, when the bilinear product of L is non-degenerate, the induced symmetric bilinear form of $S^\bullet(Z) \otimes L$ is also non-degenerate. But it is not strongly non-degenerate in general.

Suppose $h \supset Z$ is an isotropic ideal in L, R is a left L-module on which h acts trivially, then we have the following

Proposition 3.22. *With the above notations,*

1). $S^\bullet(Z) \otimes L$ is transitive.

2). $S^\bullet(Z) \otimes h$ is an isotropic ideal in $S^\bullet(Z) \otimes L$

3). $S^\bullet(Z) \otimes (L/h) \cong (S^\bullet(Z) \otimes L)/(S^\bullet(Z) \otimes h)$ is a Lie-Rinehart algebra with anchor $\tau :\ S^\bullet(Z) \otimes (L/h) \to Der(S^\bullet(Z), S^\bullet(Z))$ defined as:

$$\tau(f[e])(f_1 \cdots f_k) \triangleq f \sum_{1 \le i \le k} f_1 \cdots \hat{f_i} \cdots f_k(e \circ f_i),\ \forall e \in L,\ f \in S^\bullet(Z),\ f_j \in Z$$

and bracket induced by the Dorfman bracket of $S^\bullet(Z) \otimes L$.

4). The left L-module structure on R can be extended to a left $S^\bullet(Z) \otimes L$-module structure (module of Leibniz algebra) on $S^\bullet(Z) \otimes R$ as following:

$$\rho(f_1e)(f_2r) \triangleq f_1\Big(\rho(e)(f_2r)\Big) \triangleq f_1\Big((e, \partial f_2)r + f_2(\rho(e)r)\Big),\ \forall f_1, f_2 \in S^\bullet(Z),\ e \in L,\ r \in R$$

Furthermore, it induces a Lie-Rinehart module structure of $S^\bullet(Z) \otimes (L/h)$ on $S^\bullet(Z) \otimes R$.

Proof. 1). We need to prove that the coanchor map ρ^* of $S^\bullet(Z)\otimes L$ is injective. Since $\rho^*(d_{S^\bullet(Z)}f)=\partial f$ (see 2.2.2), it suffices to prove $ker\partial = S^0(Z)(=\mathbb{R})$, i.e. we need to prove that $\partial f\neq 0,\ \forall f\in S^{\bullet\geq 1}(Z)$.

Suppose $\{f_i\}_{1\leq i\leq m}$ is a basis of Z.

Since $\partial:\ Z\to L$ is injective and ∂ is extended by Leibniz rule, any nonzero monomial f in $S^{\bullet\geq 1}(Z)$ satisfies $\partial f\neq 0$.

Now suppose there is a nonzero polynomial $g\in S^{\bullet\geq 1}(Z)$ satisfies $\partial g=0$. If $g_1,g_2\in S^{\bullet\geq 1}(Z)$ contain different powers of f_1, e.g. $g_1=f_1^{l_1}g_1',\ g_2=f_1^{l_2}g_2',\ l_1\neq l_2,\ g_1',g_2'\in S^\bullet(Z)$ don't contain f_1, it's obvious that $\partial(g_1+g_2)=0$ implies $\partial g_1=0$ & $\partial g_2=0$. So without loss of generality, we can assume that every monomial in g contains the same power of f_1. Conducting the same discussions for $f_2,f_3,\cdots,f_m$, finally we can obtain a nonzero monomial $f\in S^{\bullet\geq 1}(Z),\ \partial f=0$, this is a contradiction. Thus 1). is proved.

2). Since the bilinear form of $S^\bullet(Z)\otimes L$ is just the linear extension of that of L, $S^\bullet(Z)\otimes h$ is isotropic in $S^\bullet(Z)\otimes L$. Then from the definition of the bracket of $S^\bullet(Z)\otimes L$:

$$f_1e_1\circ f_2e_2=f_1f_2(e_1\circ e_2)+(e_1,e_2)f_2\partial f_1+(e_1,\partial f_2)f_1e_2-(e_2,\partial f_1)f_2e_1$$

we can see that $S^\bullet(Z)\otimes h$ is an ideal in $S^\bullet(Z)\otimes L$(note that $(e,\partial f)=0,\ \forall e\in h,\ f\in S^\bullet(Z)$).

3). Due to 2), the bracket on $S^\bullet(Z)\otimes L$ induces one on $S^\bullet(Z)\otimes L/h$. We need to check the following:

(1). $x_1\circ x_2=-x_2\circ x_1\ \forall x_1,x_2\in S^\bullet(Z)\otimes L/h$

Suppose $x_i=[y_i],\ y_i\in S^\bullet(Z)\otimes L$, then

$$x_1\circ x_2+x_2\circ x_1=[y_1\circ y_2+y_2\circ y_1]=[\partial(y_1,y_2)]=0.$$

(2). $x_1\circ(fx_2)=f(x_1\circ x_2)+(\tau(x_1)f)x_2,\ \forall x_1,x_2\in S^\bullet(Z)\otimes L/h,\ \forall f\in S^\bullet(Z)$

From the definition of τ, it's easily seen that $\tau(x)f=(x,\partial f),\ \forall x\in S^\bullet(Z)\otimes L/h,\ f\in S^\bullet(Z)$, so (2) can be deduced from the first condition in Definition 2.17, since $S^\bullet(Z)\otimes L$ is a Courant-Dorfman algebra.

(3). $\tau(x_1\circ x_2)=[\tau(x_1),\tau(x_2)],\ \forall x_1,x_2\in S^\bullet(Z)\otimes L/h$

We need to prove that $\tau(x_1\circ x_2)f=\tau(x_1)\tau(x_2)f-\tau(x_2)\tau(x_1)f,\ \forall f\in S^\bullet(Z)$:

$$\begin{aligned}
&\textit{The RHS}\\
=\ &(x_1,\partial(x_2,\partial f))-(x_2,\partial(x_1,\partial f))\\
=\ &(x_1\circ x_2,\partial f)+(x_2,x_1\circ\partial f)-(x_2,x_1\circ\partial f+\partial f\circ x_1)\\
=\ &(x_1\circ x_2,\partial f)\\
=\ &\textit{The LHS}
\end{aligned}$$

4). In order for $S^\bullet(Z)\otimes R$ to become a (left) $S^\bullet(Z)\otimes L$-module, we need to prove that

$$\rho(f_1e_1\circ f_2e_2)=[\rho(f_1e_1),\rho(f_2e_2)] \tag{3.3.1}$$

i.e.

$$\rho(f_1e_1\circ f_2e_2)(fr)=\rho(f_1e_1)(\rho(f_2e_2)(fr))-\rho(f_2e_2)(\rho(f_1e_1)(fr))$$

$$\begin{aligned}
& \textit{The LHS}\\
= \; & \rho(f_1f_2(e_1\circ e_2)+(e_1,e_2)f_2\partial f_1+(e_1,\partial f_2)f_1e_2-(e_2,\partial f_1)f_2e_1)(fr)\\
= \; & f_1f_2\Big((e_1\circ e_2,\partial f)r+f\rho(e_1\circ e_2)r\Big)+(e_1,e_2)f_2\Big((\partial f_1,\partial f)r+f\rho(\partial f_1)r\Big)\\
& +(e_1,\partial f_2)f_1\Big((e_2,\partial f)r+f\rho(e_2)r\Big)-(e_2,\partial f_1)f_2\Big((e_1,\partial f)r+f\rho(e_1)r\Big)
\end{aligned}$$

$$\begin{aligned}
& \textit{The RHS}\\
= \; & \rho(f_1e_1)\Big(f_2(e_2,\partial f)r+f_2f\rho(e_2)r\Big)-\rho(f_2e_2)\Big(f_1(e_1,\partial f)r+f_1f\rho(e_1)r\Big)\\
= \; & \Big(f_1(e_1,\partial(f_2(e_2,\partial f)))r+f_1f_2(e_2,\partial f)\rho(e_1)r\\
& +f_1(e_1,\partial(f_2f))\rho(e_2)r+f_1f_2f\rho(e_1)\rho(e_2)r\Big)\\
& -\Big(f_2(e_2,\partial(f_1(e_1,\partial f)))r+f_2f_1(e_1,\partial f)\rho(e_2)r\\
& +f_2(e_2,\partial(f_1f))\rho(e_1)r+f_2f_1f\rho(e_2)\rho(e_1)r\Big)\\
= \; & f_1f_2\Big((e_1,\partial(e_2,\partial f))-(e_2,\partial(e_1,\partial f))\Big)r\\
& +f_1f_2f(\rho(e_1)\rho(e_2)r-\rho(e_2)\rho(e_1)r)\\
& +f_1(e_2,\partial f)(e_1,\partial f_2)r-f_2(e_1,\partial f)(e_2,\partial f_1)r\\
& +f_1f(e_1,\partial f_2)\rho(e_2)r-f_2f(e_2,\partial f_1)\rho(e_1)r\\
= \; & \textit{The LHS}
\end{aligned}$$

It's easily seen that the action of $S^\bullet(Z)\otimes h$ on $S^\bullet(Z)\otimes R$ is trivial, so ρ induces a (left) $S^\bullet(Z)\otimes L/h$-module structure (module of Leibniz algebra) on $S^\bullet(Z)\otimes R$. In order for $S^\bullet(Z)\otimes R$ to become a Lie-Rinehart module of $S^\bullet(Z)\otimes(L/h)$, we need to check the following:

(1). $\rho(x)(f(gr)) = f\rho(x)(gr)+(\tau(x)f)(gr),\ \forall x\in S^\bullet(Z)\otimes(L/h),\ f,g\in S^\bullet(Z),\ r\in R$

$$\begin{aligned}
& \textit{The LHS}\\
= \; & (\tau(x)(fg))r+fg\rho(x)r\\
= \; & f((\tau(x)g)r+g\rho(x)r)+(\tau(x)f)gr\\
= \; & \textit{The RHS}
\end{aligned}$$

(2). $\rho(x_1 \circ x_2) = [\rho(x_1), \rho(x_2)],\ \forall x_1, x_2 \in S^\bullet(Z) \otimes (L/h)$
This is a direct inference of 3.3.1.
The proof is finished. ■

Proposition 3.23. *If $h = Z,\ R = S^\bullet(Z)$, the standard cochain complex of Leibniz algebra L is isomorphic to the standard cochain complex of Courant-Dorfman algebra $S^\bullet(Z) \otimes L$.*

Proof. Denote by $(C_1^\bullet, d_1)$ and $(C_2^\bullet, d_2)$ the standard cochain complex of Leibniz algebra L and Courant-Dorfman algebra $S^\bullet(Z) \otimes L$ respectively. Given $\eta \in C_2^n$, obviously we can obtain an associated cochain in C_1^n by restriction, denote this restriction map by ψ. ψ is a cochain map:

$$\begin{aligned}
& (\psi(d_2\eta))_k(e_1, \cdots, e_{n+1-2k}; f_1, \cdots, f_k) \\
= \ & (d_2\eta)_k(e_1, \cdots, e_{n+1-2k}; f_1, \cdots, f_k) \\
= \ & \sum_a (-1)^{a+1}\rho(e_a)\eta_k(\cdots, \hat{e_a}, \cdots; f_1, \cdots, f_k) \\
& + \sum_{a<b} (-1)^a \eta_k(\cdots, \hat{e_a}, \cdots, e_a \circ e_b, \cdots; f_1, \cdots, f_k) \\
& + \sum_i \eta_{k-1}(\partial f_i, e_1, \cdots, e_{n+1-2k}; f_1, \cdots, \hat{f_i}, \cdots f_k) \\
= \ & \sum_a (-1)^{a+1}\rho(e_a)(\psi\eta)_k((\cdots, \hat{e_a}, \cdots; f_1, \cdots, f_k) \\
& + \sum_{a<b} (-1)^a (\psi\eta)_k(\cdots, \hat{e_a}, \cdots, e_a \circ e_b, \cdots; f_1, \cdots, f_k) \\
& + \sum_i (\psi\eta)_{k-1}(\partial f_i, e_1, \cdots, e_{n+1-2k}; f_1, \cdots, \hat{f_i}, \cdots f_k) \\
= \ & (d_1(\psi\eta))_k(e_1, \cdots, e_{n+1-2k}; f_1, \cdots, f_k) \\
& \forall k,\ \forall e_a \in L,\ f_i \in Z
\end{aligned}$$

Next, given $\omega \in C_1^n$, we extend it to a cochain $\varphi\omega \in C_2^n$ as follows:
for the degree 2 arguments, extend ω by Leibniz rule;
for the degree 1 arguments, first extend the last argument from L to $S^\bullet(Z) \otimes L$ linearly, then applying the following property we can extend the last argument but one from L to $S^\bullet(Z) \otimes L$, then the last but two, $\cdots$, finally we obtain a unique $\varphi\omega$:

$$\begin{aligned}
& \eta_k(e_1, \cdots, e_{i-1}, ge_i, x_{i+1}, \cdots, x_{n-2k}; g_1, \cdots, g_k) \\
= \ & g\eta_k(e_1, \cdots, e_{i-1}, e_i, x_{i+1}, \cdots, x_{n-2k}; g_1, \cdots, g_k) \\
& + \sum_{a>i} (-1)^{a-i}(e_i, x_a)\eta_{k+1}(e_1, \cdots, e_{i-1}, \hat{e_i}, x_{i+1}, \cdots, \hat{x_a}, \cdots, x_{n-2k}; g, g_1, \cdots, g_k) \\
& \forall k, i\ \forall e_b \in L,\ x_a \in S^\bullet(Z) \otimes L,\ g_j \in S^\bullet(Z)
\end{aligned}$$

The proof that $\varphi\omega$ is really a cochain in C_2^n is left to the lemma below 3.24.

Obviously, $\psi \circ \varphi = id_{C_1^\bullet},\ \varphi \circ \psi = id_{C_2^\bullet}$. And we can easily prove thatφ is also a cochain map:

$$\varphi(d_1\omega) = \varphi(d_1(\psi(\varphi\omega))) = \varphi(\psi(d_2(\varphi\omega))) = d_2(\varphi\omega),\ \forall \omega \in C_1^\bullet$$

Thus, $C_1^\bullet$ and $C_2^\bullet$ are isomorphic as cochain complex. ∎

Lemma 3.24. *$\eta := \varphi\omega$ as constructed above is a cochain in C_2^n.*

Proof. We need to prove that:

$$\begin{aligned} & \eta_k(x_1 \cdots x_a, x_{a+1} \cdots x_{n-2k}; g_1 \cdots g_k) + \eta_k(x_1 \cdots x_{a+1}, x_a \cdots x_{n-2k}; \cdots) \\ = \ & -\eta_{k+1}(x_1, \cdots, \hat{x_a}, \hat{x_{a+1}}, \cdots, x_{n-2k}; (x_a, x_{a+1}), g_1, \cdots, g_k) \\ & \forall k, a\ \forall x_i \in S^\bullet(Z) \otimes L,\ g_j \in S^\bullet(Z) \end{aligned} \tag{3.3.2}$$

First we prove:

$$\begin{aligned} & \eta_k(e_1 \cdots e_{a-1}, x_a, x_{a+1} \cdots x_{n-2k}; \cdots) + \eta_k(e_1 \cdots e_{a-1}, x_{a+1}, x_a \cdots x_{n-2k}; \cdots) \\ = \ & -\eta_{k+1}(e_1, \cdots, e_{a-1}, \hat{x_a}, \hat{x_{a+1}}, \cdots, x_{n-2k}; (x_a, x_{a+1}), g_1, \cdots, g_k) \\ & \forall k, a\ \forall e_i \in L,\ x_j \in S^\bullet(Z) \otimes L,\ g_l \in S^\bullet(Z) \end{aligned} \tag{3.3.3}$$

$$\begin{aligned} & \eta_k(e_1 \cdots e_{a-1}, x_a, x_{a+1} \cdots x_{n-2k}; \cdots) + \eta_k(\cdots x_{a+1}, x_a \cdots x_{n-2k}; \cdots) \\ = \ & f_a\eta_k(e_1 \cdots e_a, f_{a+1}e_{a+1}, \cdots x_{n-2k}; \cdots) \\ & -(e_a, f_{a+1}e_{a+1})\eta_{k+1}(\cdots, \hat{e_a}, \hat{e_{a+1}}, \cdots; f_a, \cdots) \\ & + \sum_{b>a+1} (-1)^{b+a}(e_a, x_b)\eta_{k+1}(\cdots, \hat{e_a}, f_{a+1}e_{a+1}, \cdots, \hat{x_b}, \cdots; f_a, \cdots) \\ & + f_{a+1}\eta_k(\cdots, e_{a+1}, f_a e_a, \cdots; \cdots) \\ & -(e_{a+1}, f_a e_a)\eta_{k+1}(\cdots, \hat{e_{a+1}}, \hat{e_a}, \cdots; f_{a+1}, \cdots) \\ & + \sum_{b>a+1} (-1)^{b+a}(e_{a+1}, x_b)\eta_{k+1}(\cdots, \hat{e_{a+1}}, f_a e_a, \cdots, \hat{x_b}, \cdots; f_{a+1}, \cdots) \\ = \ & f_a f_{a+1}\eta_k(\ldots, e_a, e_{a+1}, \cdots; \cdots) \\ & + f_a \sum_{b>a+1} (-1)^{b+a+1}(e_{a+1}, x_b)\eta_{k+1}(\cdots, e_a, \hat{e_{a+1}}, \cdots, \hat{x_b}, \cdots; f_{a+1}, \cdots) \\ & + f_{a+1} \sum_{b>a+1} (-1)^{b+a}(e_a, x_b)\eta_{k+1}(\cdots, \hat{e_a}, e_{a+1}, \cdots, \hat{x_b}, \cdots; f_a, \cdots) \\ & + \sum_{a+1<b<c} (-1)^{c+a}(-1)^{b+a+1}(e_a, x_c)(e_{a+1}, x_b) \end{aligned}$$

$$
\begin{aligned}
& \eta_{k+2}(\cdots, \hat{e_a}, \hat{e_{a+1}}, \cdots, \hat{x_b}, \cdots, \hat{x_c}, \cdots; f_a, f_{a+1}, \cdots) \\
& + \sum_{a+1<c<b} (-1)^{c+a}(-1)^{b+a}(e_a, x_c)(e_{a+1}, x_b) \\
& \eta_{k+2}(\cdots, \hat{e_a}, \hat{e_{a+1}}, \cdots, \hat{x_c}, \cdots, \hat{x_b}, \cdots; f_a, f_{a+1}, \cdots) \\
& + f_a f_{a+1} \eta_k(\ldots, e_{a+1}, e_a, \cdots; \cdots) \\
& + f_{a+1} \sum_{b>a+1} (-1)^{b+a+1}(e_a, x_b)\eta_{k+1}(\cdots, e_{a+1}, \hat{e_a}, \cdots, \hat{x_b}, \cdots; f_a, \cdots) \\
& + f_a \sum_{b>a+1} (-1)^{b+a}(e_{a+1}, x_b)\eta_{k+1}(\cdots, \hat{e_{a+1}}, e_a, \cdots, \hat{x_b}, \cdots; f_{a+1}, \cdots) \\
& + \sum_{a+1<b<c} (-1)^{c+a}(-1)^{b+a+1}(e_{a+1}, x_c)(e_a, x_b) \\
& \eta_{k+2}(\cdots, \hat{e_{a+1}}, \hat{e_a}, \cdots, \hat{x_b}, \cdots, \hat{x_c}, \cdots; f_a, f_{a+1}, \cdots) \\
& + \sum_{a+1<c<b} (-1)^{c+a}(-1)^{b+a}(e_{a+1}, x_c)(e_a, x_b) \\
& \eta_{k+2}(\cdots, \hat{e_{a+1}}, \hat{e_a}, \cdots, \hat{x_c}, \cdots, \hat{x_b}, \cdots; f_a, f_{a+1}, \cdots) \\
& -\Big(f_{a+1}(e_a, e_{a+1})\eta_{k+1}(\cdots, \hat{e_a}, \hat{e_{a+1}}, \cdots; f_a, \cdots) \\
& \quad + f_a(e_{a+1}, e_a)\eta_{k+1}(\cdots, \hat{e_{a+1}}, \hat{e_a}, \cdots; f_{a+1}, \cdots)\Big) \\
= \; & f_a f_{a+1}\Big(\eta_k(\ldots, e_a, e_{a+1}, \cdots; \cdots) + \eta_k(\ldots, e_{a+1}, e_a, \cdots; \cdots)\Big) \\
& -\Big(f_{a+1}(e_a, e_{a+1})\eta_{k+1}(\cdots, \hat{e_a}, \hat{e_{a+1}}, \cdots; f_a, \cdots) \\
& \quad + f_a(e_{a+1}, e_a)\eta_{k+1}(\cdots, \hat{e_{a+1}}, \hat{e_a}, \cdots; f_{a+1}, \cdots)\Big) \\
= \; & -\eta_{k+1}(\cdots, \hat{e_a}, \hat{e_{a+1}}, \cdots, x_{n-2k}; f_a f_{a+1}(e_a, e_{a+1}), \cdots)
\end{aligned}
$$

If $n = 2l$ is even, when $k = l-1$, due to 3.3.3, 3.3.2 holds.
If $n = 2l+1$ is odd, when $k = l-1$, 3.3.2 holds due to 3.3.3 and the following

$$
\begin{aligned}
& \eta_{l-1}(f_1e_1, f_2e_2, f_3e_3; g_1, \cdots, g_{l-1}) + \eta_{l-1}(f_1e_1, f_3e_3, f_2e_2; g_1, \cdots, g_{l-1}) \\
= \; & f_1\eta_{l-1}(e_1, f_2e_2, f_3e_3; \cdots) - (e_1, f_2e_2)\eta_l(f_3e_3; f_1, \cdots) + (e_1, f_3e_3)\eta_l(f_2e_2; f_1, \cdots) \\
& + f_1\eta_{l-1}(e_1, f_3e_3, f_2e_2; \cdots) - (e_1, f_3e_3)\eta_l(f_2e_2; f_1, \cdots) + (e_1, f_2e_2)\eta_l(f_3e_3; f_1, \cdots) \\
= \; & -f_1\eta_l(e_1; (f_2e_2, f_3e_3), \cdots) \\
= \; & -\eta_l(f_1e_1; (f_2e_2, f_3e_3), \cdots)
\end{aligned}
$$

Now suppose 3.3.2 holds for $k > m$, consider the case when $k = m$. Due to 3.3.3, we can further suppose that 3.3.2 holds for $x_1, \cdots x_i \in L, \; i < a$. We will prove 3.3.2

for the case when $k = m$ and $x_1, \cdots, x_{i-1} \in L$:

$$\begin{aligned}
& \eta_m(e_1 \cdots e_{i-1}, f_i e_i \cdots f_a e_a, f_{a+1} e_{a+1} \cdots ; \cdots) + \eta_m(\cdots f_i e_i \cdots f_{a+1} e_{a+1}, f_a e_a \cdots ; \cdots) \\
= \; & f_i \eta_m(\cdots, e_i, \cdots f_a e_a, f_{a+1} e_{a+1}, \cdots ; \cdots) \\
& + \sum_{b>i} (-1)^{b+i} (e_i, f_b e_b) \eta_{m+1}(\cdots, \hat{e_i}, \cdots, \hat{e_b}, \cdots f_a e_a, f_{a+1} e_{a+1}, \cdots ; f_i \cdots) \\
& + f_i \eta_m(\cdots, e_i, \cdots f_{a+1} e_{a+1}, f_a e_a, \cdots ; \cdots) \\
& + \sum_{b>i} (-1)^{b+i} (e_i, f_b e_b) \eta_{m+1}(\cdots, \hat{e_i}, \cdots, \hat{e_b}, \cdots f_{a+1} e_{a+1}, f_a e_a, \cdots ; f_i \cdots) \\
= \; & f_i \Big(\eta_m(\cdots, e_i, \cdots f_a e_a, f_{a+1} e_{a+1}, \cdots ; \cdots) + \eta_m(\cdots, e_i, \cdots f_{a+1} e_{a+1}, f_a e_a, \cdots ; \cdots) \Big) \\
& + \Big(\sum_{b>i, b \neq a, a+1} (-1)^{b+i} (e_i, f_b e_b) \eta_{m+1}(\cdots, \hat{e_i}, \cdots, \hat{e_b}, \cdots, f_a e_a, f_{a+1} e_{a+1}, \cdots ; f_i \cdots) \\
& \quad + \sum_{b>i, b \neq a, a+1} (-1)^{b+i} (e_i, f_b e_b) \eta_{m+1}(\cdots, \hat{e_i}, \cdots, \hat{e_b}, \cdots f_{a+1} e_{a+1}, f_a e_a, \cdots ; f_i \cdots) \Big) \\
& + \Big((-1)^{a+i} (e_i, f_a e_a) \eta_{m+1}(\cdots, \hat{e_i}, \cdots, \hat{e_a}, f_{a+1} e_{a+1}, \cdots ; f_i \cdots) \\
& \quad + (-1)^{a+1+i} (e_i, f_{a+1} e_{a+1}) \eta_{m+1}(\cdots, \hat{e_i}, \cdots, f_a e_a, \hat{e_{a+1}} \cdots ; f_i \cdots) \Big) \\
& + \Big((-1)^{a+i} (e_i, f_{a+1} e_{a+1}) \eta_{m+1}(\cdots, \hat{e_i}, \cdots, \hat{e_{a+1}}, f_a e_a \cdots ; f_i \cdots) \\
& \quad + (-1)^{a+1+i} (e_i, f_a e_a) \eta_{m+1}(\cdots, \hat{e_i}, \cdots, f_{a+1} e_{a+1}, , \hat{e_a} \cdots ; f_i \cdots) \Big) \\
= \; & -f_i \eta_{m+1}(\cdots, e_i, \cdots, \hat{e_a}, \hat{e_{a+1}}, \cdots ; (f_a e_a, f_{a+1} e_{a+1}), \cdots) \\
& - \sum_{b>i, b \neq a, a+1} (-1)^{b+i} (e_i, f_b e_b) \eta_{m+2}(\cdots \hat{e_i} \cdots \hat{e_b} \cdots \hat{e_a}, \hat{e_{a+1}} \cdots ; (f_a e_a, f_{a+1} e_{a+1}), f_i \cdots) \\
= \; & -\eta_{m+1}(e_1, \cdots, e_{i-1}, f_i e_i, \cdots, \hat{e_a}, \hat{e_{a+1}}, \cdots ; (f_a e_a, f_{a+1} e_{a+1}), \cdots)
\end{aligned}$$

By induction, 3.3.2 is proved.
Next, by the construction of φ, it suffices to prove

$$\begin{aligned}
& \eta_k(x_1, \cdots, x_{i-1}, g x_i, \cdots, x_{n-2k}; g_1, \cdots, g_k) \\
= \; & g \eta_k(x_1, \cdots, x_{i-1}, x_i, \cdots, x_{n-2k}; g_1, \cdots, g_k) \qquad (3.3.4) \\
& + \sum_{a>i} (-1)^{a-i} (x_i, x_a) \eta_{k+1}(x_1, \cdots, x_{i-1}, \hat{x_i}, \cdots, \hat{x_a}, \cdots, x_{n-2k}; g, g_1, \cdots, g_k) \\
& \forall k, i \; \forall e_b \in L, \; x_a \in S^\bullet(Z) \otimes L, \; g_j \in S^\bullet(Z)
\end{aligned}$$

When $x_1, \cdots, x_{i-1} \in L$,3.3.4 holds:

$$\begin{aligned}
& \eta_k(e_1, \cdots, e_{i-1}, g f_i e_i, \cdots ; \cdots) \\
= \; & g f_i \eta_k(\cdots, e_i, \cdots ; \cdots) + \sum_{a>i} (-1)^{a+i} (e_i, x_a) \eta_{k+1}(\cdots, \hat{e_i}, \cdots, \hat{x_a}, \cdots ; g f_i, \cdots)
\end{aligned}$$

$$\begin{aligned}
&= g(\eta_k(\cdots, f_i e_i, \cdots; \cdots) - \sum_{a>i}(-1)^{a+i}(e_i, x_a)\eta_{k+1}(\cdots, \hat{e}_i, \cdots, \hat{x}_a, \cdots; f_i, \cdots)) \\
&\quad + \sum_{a>i}(-1)^{a+i}(e_i, x_a)\eta_{k+1}(\cdots, \hat{e}_i, \cdots, \hat{x}_a, \cdots; g f_i, \cdots) \\
&= g\eta_k(\cdots, f_i e_i, \cdots; \cdots) + \sum_{a>i}(-1)^{a+i}(f_i e_i, x_a)\eta_{k+1}(\cdots, \hat{e}_i, \cdots, \hat{x}_a, \cdots; g, \cdots)
\end{aligned}$$

Now suppose 3.3.4 holds for any $k > m$, and for the case when $x_1, \cdots, x_j \in L(j < i)$, $k = m$ as well. Consider the case when $x_1, \cdots, x_{j-1} \in L,\ x_j = f_j e_j,\ k = m$:

$$\begin{aligned}
&\eta_k(e_1, \cdots, e_{j-1}, f_j e_j, \cdots, g x_i, \cdots; \cdots) \\
&= f_j \eta_k(\cdots e_j, \cdots g x_i, \cdots; \cdots) + (-1)^{i+j}(e_j, g x_i)\eta_{k+1}(\cdots \hat{e}_j, \cdots \hat{x}_i, \cdots; f_j, \cdots) \\
&\quad + \sum_{b>j, b\neq i}(-1)^{b+j}(e_j, x_b)\eta_{k+1}(\cdots, \hat{e}_j, \cdots, \hat{x}_b, \cdots, g x_i, \cdots; f_j, \cdots) \\
&= f_j\Big(g\eta_k(\cdots, e_j, \cdots, x_i, \cdots; \cdots) \\
&\qquad + \sum_{a>i}(-1)^{a+i}(x_i, x_a)\eta_{k+1}(\cdots, e_j, \cdots, \hat{x}_i, \cdots, \hat{x}_a, \cdots; g, \cdots)\Big) \\
&\quad + \sum_{j<b<i}(-1)^{b+j}(e_j, x_b)\Big(g\eta_{k+1}(\cdots, \hat{e}_j, \cdots, \hat{x}_b, \cdots, x_i, \cdots; f_j, \cdots) \\
&\qquad + \sum_{a>i}(-1)^{a+i}(x_i, x_a)\eta_{k+2}(\cdots, \hat{e}_j, \cdots, \hat{x}_b, \cdots, \hat{x}_i, \cdots, \hat{x}_a, \cdots; g, f_j, \cdots)\Big) \\
&\quad + \sum_{b>i}(-1)^{b+j}(e_j, x_b)\Big(g\eta_{k+1}(\cdots, \hat{e}_j, \cdots, x_i, \cdots, \hat{x}_b, \cdots; f_j, \cdots) \\
&\qquad + \sum_{i<a<b}(-1)^{a+i}(x_i, x_a)\eta_{k+2}(\cdots, \hat{e}_j, \cdots, \hat{x}_i, \cdots, \hat{x}_a, \cdots, \hat{x}_b, \cdots; g, f_j, \cdots) \\
&\qquad + \sum_{b<a}(-1)^{a+i+1}(x_i, x_a)\eta_{k+2}(\cdots, \hat{e}_j, \cdots, \hat{x}_i, \cdots, \hat{x}_b, \cdots, \hat{x}_a, \cdots; g, f_j, \cdots)\Big) \\
&\quad + (-1)^{i+j}(e_j, g x_i)\eta_{k+1}(\cdots, \hat{e}_j, \cdots, \hat{x}_i, \cdots; f_j, \cdots)
\end{aligned}$$

$$\begin{aligned}
&g\eta_k(\cdots f_j e_j \cdots x_i \cdots; \cdots) \\
&+ \sum_{a>i}(-1)^{a+i}(x_i, x_a)\eta_{k+1}(\cdots f_j e_j \cdots \hat{x}_i \cdots \hat{x}_a \cdots; g \cdots) \\
= \;& g\Big(f_j\eta_k(\cdots, e_j, \cdots, x_i, \cdots; \cdots) \\
&\quad + (-1)^{i+j}(e_j, x_i)\eta_{k+1}(\cdots, \hat{e}_j, \cdots, \hat{x}_i, \cdots; f_j, \cdots) \\
&\quad + \sum_{j<b<i}(-1)^{b+j}(e_j, x_b)\eta_{k+1}(\cdots, \hat{e}_j, \cdots, \hat{x}_b, \cdots, x_i, \cdots; f_j, \cdots) \\
&\quad + \sum_{b>i}(-1)^{b+j}(e_j, x_b)\eta_{k+1}(\cdots, \hat{e}_j, \cdots, x_i, \cdots, \hat{x}_b, \cdots; f_j, \cdots)\Big)
\end{aligned}$$

$$+\sum_{a>i}(-1)^{a+i}(x_i,x_a)\Big(f_j\eta_{k+1}(\cdots,e_j,\cdots,\hat{x}_i\cdots,\hat{x}_a,\cdots;g,\cdots)$$
$$+\sum_{j<b<i}(-1)^{b+j}(e_j,x_b)\eta_{k+2}(\cdots,e_j,\cdots\hat{x}_b,\cdots,\hat{x}_i,\cdots x_a,\cdots;g,f_j,\cdots)$$
$$+\sum_{i<b<a}(-1)^{b+j+1}(e_j,x_b)\eta_{k+2}(\cdots\hat{e}_j,\cdots\hat{x}_i,\cdots\hat{x}_b,\cdots\hat{x}_a,\cdots;g,f_j,\cdots)$$
$$+\sum_{b>a}(-1)^{b+j}(e_j,x_b)\eta_{k+2}(\cdots,\hat{e}_j,\cdots,\hat{x}_i,\cdots,\hat{x}_a,\cdots,\hat{x}_b,\cdots;g,f_j,\cdots)\Big)$$

The above two equals. Thus by induction, 3.3.4 is proved. ■

Using the same methods as above, we can prove the following:

Proposition 3.25. $C^{\bullet}_{CE}(L/h,\ S^{\bullet}(Z)\otimes R)\cong C^{\bullet}_{CE}(S^{\bullet}(Z)\otimes L/h,\ S^{\bullet}(Z)\otimes R)$

Remark 3.26. When $h=Z,\ R=S^{\bullet}(Z)$, due to Proposition 3.23 and 3.25, Theorem 3.10 can be rephrased as:

$$H^{\bullet}_{st}(S^{\bullet}(Z)\otimes L)\cong H^{\bullet}_{CE}(L/h,\ S^{\bullet}(Z))\cong H^{\bullet}_{CE}(S^{\bullet}(Z)\otimes L/h,\ S^{\bullet}(Z))\cong H^{\bullet}_{nv}(S^{\bullet}(Z)\otimes L).$$

So in this case, Theorem 3.10 recovers Theorem 3.15 for the specific Courant-Dorfman algebra $S^{\bullet}(Z)\otimes L$.

For general case when $h\neq Z$, using the same methods as above we can prove that the standard cochain complex of Leibniz algebra L w.r.t. the module $S^{\bullet}(Z)\otimes R$ is isomorphic to the following cochain complex $D^{\bullet}$ of Courant-Dorfman algebra $S^{\bullet}(Z)\otimes L$:

A cochain $\omega\in D^n$ is a sequence $(\omega_0,\omega_1,\cdots,\omega_{[n/2]})$ *with* $\omega_k:\ \otimes^{n-2k}(S^{\bullet}(Z)\otimes L)\otimes\odot^k(S^{\bullet}(Z)\otimes h)\to S^{\bullet}(Z)\otimes R$ satisfying:

$$\begin{aligned}1).\quad & \omega_k(x_1,\cdots,x_i,x_{i+1},\cdots;\cdots)+\omega_k(x_1,\cdots,x_{i+1},x_i,\cdots;\cdots)\\ = & -\omega_{k+1}(x_1,\cdots,\hat{x}_i,\hat{x_{i+1}},\cdots;\partial(x_i,x_{i+1}),\cdots)\\ & \forall x_j\in S^{\bullet}(Z)\otimes L\end{aligned}$$

$$\begin{aligned}2).\quad & \omega_k(x_1,\cdots,fx_i,x_{i+1},\cdots;\cdots)\\ = & f\omega_k(\cdots x_i,x_{i+1},\cdots;\cdots)+\sum_{j>i}(-1)^{j-i}(x_i,x_j)\omega_{k+1}(\cdots\hat{x}_i,\cdots\hat{x}_j,\cdots;\partial f,\cdots)\\ & \forall x_l\in S^{\bullet}(Z)\otimes L,\ f\in S^{\bullet}(Z)\end{aligned}$$

3). $\omega_k(\cdots;f_1g_1,\cdots,f_kg_k)=f_1\cdots f_k\omega_k(\cdots;g_1,\cdots,g_k)$, $\forall f_i\in S^{\bullet}(Z),\ g_i\in S^{\bullet}(Z)\otimes h$

The coboundary differential is defined as following:

$$\begin{aligned} & (d\omega)_k(x_1, \cdots, x_{n+1-2k}; g_1, \cdots, g_k) \\ = \ & \sum_a (-1)^{a+1} \rho(x_a) \omega_k(\cdots \hat{x_a}, \cdots; \cdots) + \sum_{a<b} (-1)^a \omega_k(\cdots \hat{x_a}, \cdots x_a \circ x_b, \cdots; \cdots) \\ & + \sum_{a,i} (-1)^{a+1} \omega_k(\cdots, \hat{x_a}, \cdots; \cdots, \hat{f_i}, f_i \circ x_a, \cdots) \\ & + \sum_i \omega_{k-1}(g_i, x_1, \cdots; \cdots, \hat{g}_i, \cdots) \end{aligned}$$

Combining 3.25, 3.10 can be rephrased as

$$H^\bullet(D^\bullet) \cong H^\bullet_{CE}(S^\bullet(Z) \otimes L/h, S^\bullet(Z) \otimes R).$$

In fact the cochain complex $D^\bullet$ can be defined similarly for any triple $(\mathcal{E}, \mathcal{H}, \mathcal{R})$, where $\mathcal{E}$ is a Courant-Dorfman algebra, $\mathcal{H}$ is an isotropic ideal of $\mathcal{E}$ containing the left center, $\mathcal{R}$ is an $\mathcal{E}$-module (module of Leibniz algebra) on which $\mathcal{H}$ acts trivially. Then we have the following conjecture, as a further generalization of Theorem 3.15:

Conjecture 3.27. *$H^\bullet(D^\bullet(\mathcal{E}, \mathcal{H}, \mathcal{R})) \cong H^\bullet_{CE}(\mathcal{E}/\mathcal{H}, \mathcal{R})$.*

Chapter 4

Derived brackets for fat Leibniz algebras

Throughout this chapter, let L be a Leibniz algebra with left center Z, and $h \supseteq Z$ be an isotropic ideal of L. And we take $C^\bullet_{st}(L)$ to be short for $C^\bullet_{st}(L, h, S^\bullet(Z))$, where $S^\bullet(Z)$ is the left module of L extended from Z by Leibniz rule.

By proposition 3.23, Roytenberg's theorems 2.23 and 2.24 can be stated in the language of Leibniz algebra:

Theorem. *Suppose $h = Z$, and the symmetric bilinear form on $S^\bullet(Z) \otimes L$ is strongly non-degenerate, then there is a bracket $\{\bullet, \bullet\}$ defined on $C^\bullet_{st}(L, Z, S^\bullet(Z))$, satisfying:*

1). $\{\bullet, \bullet\}$ is a non-degenerate Poisson bracket on the algebra $C^\bullet_{st}(L, Z, S^\bullet(Z))$ of degree -2;

2). $\{\Theta, \Theta\} = 0$;

3). $d\omega = -\{\Theta, \omega\}, \ \forall \omega \in C^\bullet_{st}(L, Z, S^\bullet(Z))$;

and

Theorem. *With the above assumptions,*

$$(e_1 \circ e_2)^\flat = -\{\{\Theta, e_1^\flat\}, e_2^\flat\}, \ \forall e_1, e_2 \in L,$$

where $e^\flat := (e, \bullet) \in C^1_{st}(L, Z, S^\bullet(Z))$.

Since $(\bullet)^\flat$ is an isomorphism, the second theorem above implies that Leibniz bracket of L can be represented as a derived bracket.

However, the assumptions in the theorems above are too strong. Even if the bilinear product of L is non-degenerate, the induced symmetric bilinear form on $S^\bullet(Z) \otimes L$ is not strongly non-degenerate in general.

A natural question is: what is the case for general h and weaker requirement on the symmetric bilinear form? Can Leibniz bracket of L still be represented as a derived bracket?

The main objective of this chapter is to answer this question.

4.1 The canonical 3-cochain

Define $\Theta_0 : L \otimes L \otimes L \to S^\bullet(Z)$ as:

$$\Theta_0(e_1, e_2, e_3) = (e_1 \circ e_2, e_3),$$

and $\Theta_1 : L \otimes h \to S^\bullet(Z)$ as:

$$\Theta_1(e; f) = -(e, f). \tag{4.1.1}$$

We see that Θ_0 has the same definition with Cartan 3-form of Lie algebra (but the bilinear products are quite different). Consider the Leibniz cohomology of L with coefficients in $S^\bullet(Z)$ (or Z), we have the following:

Proposition 4.1. *Θ_0 is a 3-cocycle in $C^3(L, S^\bullet(Z))$ (or $C^3(L, Z)$).*

Proof.

$$\begin{aligned}
& (d_0\Theta_0)(e_1, e_2, e_3, e_4) \\
= \; & \rho(e_1)\Theta_0(e_2, e_3, e_4) - \rho(e_2)\Theta_0(e_1, e_3, e_4) + \rho(e_3)\Theta_0(e_1, e_2, e_4) - \rho(e_4)\Theta_0(e_1, e_2, e_3) \\
& -\Theta_0(e_1 \circ e_2, e_3, e_4) - \Theta_0(e_2, e_1 \circ e_3, e_4) - \Theta_0(e_2, e_3, e_1 \circ e_4) \\
& +\Theta_0(e_1, e_2 \circ e_3, e_4) + \Theta_0(e_1, e_3, e_2 \circ e_4) - \Theta_0(e_1, e_2, e_3 \circ e_4) \\
= \; & \rho(e_1)(e_2 \circ e_3, e_4) - \rho(e_2)(e_1 \circ e_3, e_4) + \rho(e_3)(e_1 \circ e_2, e_4) - \rho(e_4)(e_1 \circ e_2, e_3) \\
& -((e_1 \circ e_2) \circ e_3, e_4) - (e_2 \circ (e_1 \circ e_3), e_4) - (e_2 \circ e_3, e_1 \circ e_4) \\
& +(e_1 \circ (e_2 \circ e_3), e_4) + (e_1 \circ e_3, e_2 \circ e_4) - (e_1 \circ e_2, e_3 \circ e_4) \\
= \; & \big((e_1 \circ (e_2 \circ e_3), e_4) + (e_2 \circ e_3, e_1 \circ e_4)\big) \\
& -\big((e_2 \circ (e_1 \circ e_3), e_4) + (e_1 \circ e_3, e_2 \circ e_4)\big) \\
& +\big((e_3 \circ (e_1 \circ e_2), e_4) + (e_1 \circ e_2, e_3 \circ e_4)\big) \\
& -\big((e_4, (e_1 \circ e_2) \circ e_3) + (e_4, e_3 \circ (e_1 \circ e_2))\big) \\
& -((e_1 \circ e_2) \circ e_3, e_4) - (e_2 \circ (e_1 \circ e_3), e_4) + (e_1 \circ (e_2 \circ e_3), e_4) \\
& -(e_2 \circ e_3, e_1 \circ e_4) + (e_1 \circ e_3, e_2 \circ e_4) - (e_1 \circ e_2, e_3 \circ e_4) \\
= \; & 2(e_1 \circ (e_2 \circ e_3), e_4) - 2(e_2 \circ (e_1 \circ e_3), e_4) - 2((e_1 \circ e_2) \circ e_3, e_4)
\end{aligned}$$

$$\begin{aligned} &= 2(e_1 \circ (e_2 \circ e_3) - (e_1 \circ e_2) \circ e_3 - e_2 \circ (e_1 \circ e_3), e_4) \\ &= 0 \\ &\quad \forall e_1, e_2, e_3, e_4 \in L \end{aligned}$$

The proposition is proved. ■

Now consider $\Theta = (\Theta_0, \Theta_1)$ as a whole.

Proposition 4.2. *Θ is a cochain in $C^3_{st}(L)$. When $h = Z$, Θ is a 3-cocycle.*

Proof. $\forall e_1, e_2, e_3 \in L$,

$$\begin{aligned} & \Theta_0(e_1, e_2, e_3) + \Theta_0(e_2, e_1, e_3) \\ = & (e_1 \circ e_2, e_3) + (e_2 \circ e_1, e_3) \\ = & ((e_1, e_2), e_3) \\ = & -\Theta_1(e_3; (e_1, e_2)) \end{aligned}$$

$$\begin{aligned} & \Theta_0(e_1, e_2, e_3) + \Theta_0(e_1, e_3, e_2) \\ = & (e_1 \circ e_2, e_3) + (e_1 \circ e_3, e_2) \\ = & \rho(e_1)(e_2, e_3) \\ = & -\Theta_1(e_1; (e_2, e_3)) \end{aligned}$$

So Θ is a cochain in $C^3_{st}(L)$.
When $h = Z$, we need to prove that $d\Theta = ((d\Theta)_0, (d\Theta)_1, (d\Theta)_2) = 0$:
For $(d\Theta)_0$, by Proposition 4.1, $(d\Theta)_0 = d_0\Theta_0 = 0$.
For $(d\Theta)_1$,

$$\begin{aligned} & (d\Theta)_1(e_1, e_2; f) \\ = & -\rho(e_1)(e_2, f) + \rho(e_2)(e_1, f) + (e_1 \circ e_2, f) - (f \circ e_1, e_2) + (f \circ e_2, e_1) + (f \circ e_1, e_2) \\ & (f \in h = Z \ \Rightarrow \ f \circ e_1 = 0 \ \& \ f \circ e_2 = 0) \\ = & -((e_1 \circ e_2, f) + (e_2, e_1 \circ f)) + (e_2, e_1 \circ f) + (e_1 \circ e_2, f) \\ = & 0 \end{aligned}$$

For $(d\Theta)_2$,

$$\begin{aligned} & (d\Theta)_2(f_1, f_2) \\ = & \Theta_1(f_2; f_1) + \Theta_1(f_1; f_2) \\ = & -(f_2, f_1) - (f_1, f_2) \\ = & 0 \end{aligned}$$

Thus the proof is finished. ■

In fact when $h = Z$, by proposition 3.23, Θ defined above is exactly the restriction of the canonical 3-cocycle of the Courant-Dorfman algebra $S^\bullet(Z) \otimes L$. This is the reason why we use the same notation Θ.

4.2 The Poisson algebra structure on $\tilde{C}^\bullet_{st}(L)$

In this section, we define a bracket for certain cochains in $C^\bullet_{st}(L)$. The construction is analogous to the case of Courant-Dorfman algebra.

Let $L^\vee \triangleq Hom(L,\ S^\bullet(Z))$.

$\forall \omega \in C^n_{st}(L)$, ω_k gives rise to a map $\bar{\omega_k}:\ L^{\otimes n-2k-1} \otimes S^k(h) \to L^\vee$:

$$\begin{aligned} & \bar{\omega_k}(e_1, \cdots, e_{n-2k-1}; f_1, \cdots, f_k)(e) \\ \triangleq\ & (\iota_{f_k} \cdots \iota_{f_1} \iota_{e_{n-2k-1}} \cdots \iota_{e_1} \omega_k)(e) \\ =\ & \omega_k(e_1, \cdots, e_{n-2k-1}, e; f_1, \cdots, f_k). \end{aligned}$$

The bilinear product of L induces a map

$$(\bullet,\ ) :\ S^\bullet(Z) \otimes L \to L^\vee.$$

In fact it is the $S^\bullet(Z)$-linear extension of $(\bullet)^\flat : L \to Hom(L, Z)$ (2.3).

Definition 4.3. $\forall \omega \in C^n_{st}(L)$, if the image of $\bar{\omega_k}$ falls into the image of $(\bullet,\ )$, $\forall k$, we call ω a "representable" cochain. The graded subspace of $C^\bullet_{st}(L)$ consisting of all representable cochains is denoted by $\tilde{C}^\bullet_{st}(L)$.

Obviously, Θ and $e^\flat$ are representable cochains.

Given $\omega \in \tilde{C}^n_{st}(L)$, ω_k can induce a map

$$\tilde{\omega_k} :\ L^{\otimes n-2k-1} \to Hom(S^k(h),\ S^\bullet(Z) \otimes L),$$

which is defined by

$$\begin{aligned} & \tilde{\omega_k}(e_1, \cdots, e_{n-2k-1})(f_1, \cdots, f_k) \\ \triangleq\ & (\bullet,\ )^{-1}(\bar{\omega_k}(e_1, \cdots, e_{n-2k-1}; f_1, \cdots, f_k)). \end{aligned}$$

Note that $\tilde{\omega_k}$ depends on the choices of preimage of $\bar{\omega_k}(e_1, \cdots, e_{n-2k-1}; f_1, \cdots, f_k)$, so it is not uniquely determined unless $(\bullet,\ )$ is injective (i.e. the bilinear product of L is non-degenerate).

$\forall \alpha \in Hom(S^k(h),\ S^\bullet(Z) \otimes L)$, $\beta \in Hom(S^l(h),\ S^\bullet(Z) \otimes L)$, define $\langle \alpha \cdot \beta \rangle \in Hom(S^{k+l}(h),\ S^\bullet(Z))$ as

$$\langle \alpha \cdot \beta \rangle (f_1, \cdots, f_{k+l}) \triangleq \sum_{\sigma \in sh(k,l)} (\alpha(f_{\sigma(1)}, \cdots, f_{\sigma(k)}),\ \beta(f_{\sigma(k+1)}, \cdots, f_{\sigma(k+l)})).$$

$\forall \gamma \in Hom(S^k(h),\ S^{\bullet}(Z)),\ \delta \in Hom(S^l(h),\ S^{\bullet}(Z))$, define $\gamma\circ\delta \in Hom(S^{k+l-1}(h),\ S^{\bullet}(Z))$ as

$$\gamma \circ \delta(f_1, \cdots, f_{k+l-1}) \triangleq \sum_{\sigma\in sh(l,k-1)} \check{\gamma}(\delta(f_{\sigma(1)}, \cdots, f_{\sigma(l)}), f_{\sigma(l+1)}, \cdots, f_{\sigma(l+k-1)}),$$

where $\check{\gamma}:\ S^{\bullet}(Z)\otimes S^{k-1}(h) \to S^{\bullet}(Z)$ is the extension of γ by Leibniz rule w.r.t the first argument.

Now given $\omega \in \tilde{C}^n_{st}(L),\ \eta \in \tilde{C}^m_{st}(L)$, we define the bracket $\{\omega, \eta\}$ as follows:

$$\{\omega, \eta\} \triangleq \omega \bullet \eta + \omega \diamond \eta - (-1)^{nm}\eta \diamond \omega, \tag{4.2.1}$$

where $\omega\bullet\eta = ((\omega\bullet\eta)_0, (\omega\bullet\eta)_1, \cdots)$, with $(\omega\bullet\eta)_k:\ \otimes^{n+m-2-2k}L \to Hom(S^k(h),\ S^{\bullet}(Z))$ defined by

$$\begin{aligned}
& (\omega \bullet \eta)_k(e_1, \cdots, e_{n+m-2-2k}) \\
\triangleq\ & (-1)^{m-1} \sum_{i+j=k} \sum_{\sigma\in sh(n-2i-1,m-2j-1)} (-1)^{\sigma} \\
& \langle \tilde{\omega_i}(e_{\sigma(1)}, \cdots, e_{\sigma(n-2i-1)}) \cdot \tilde{\eta_j}(e_{\sigma(n-2i)}, \cdots, e_{\sigma(n+m-2-2k)})\rangle,
\end{aligned}$$

(obviously the value does not depend on the choices of $\tilde{\omega_i}$ and $\tilde{\eta_j}$, so it is well-defined)

and $\omega\diamond\eta = ((\omega\diamond\eta)_0, (\omega\diamond\eta)_1, \cdots)$, with $(\omega\diamond\eta)_k:\ \otimes^{n+m-2-2k}L \to Hom(S^k(h),\ S^{\bullet}(Z))$ defined by

$$\begin{aligned}
& (\omega \diamond \eta)_k(e_1, \cdots, e_{n+m-2-2k}) \\
\triangleq\ & \sum_{i+j=k} \sum_{\sigma\in sh(n-2i-2,m-2j)} (-1)^{\sigma}\omega_{i+1}(e_{\sigma(1)} \cdots e_{\sigma(n-2i-2)}) \circ \eta_j(e_{\sigma(n-2i-1)} \cdots e_{\sigma(n+m-2-2k)}).
\end{aligned}$$

Analogously to theorem 2.23, we have the following:

Theorem 4.4. *With the above notations,*

1). $\tilde{C}^{\bullet}_{st}(L)$ form a graded-commutative Poisson algebra under the bracket $\{\bullet, \bullet\}$.

2). $\{\Theta, \eta\} = -(d_0 + \delta)\eta$.

First we prove a lemma.

Lemma 4.5. $\omega \bullet \eta,\ \omega \diamond \eta,\ \{\omega, \eta\}$ *are all cochains in* $C^{n+m-2}_{st}(L)$.

Proof. $\omega \bullet \eta$ is a cochain in $C_{st}^{n+m-2}(L)$ because:

$$
\begin{aligned}
& (\omega\bullet\eta)_k(e_1\cdots e_a, e_{a+1}\cdots e_{n+m-2-2k}; f_1\cdots f_k) + (\omega\bullet\eta)_k(\cdots e_{a+1}, e_a\cdots;\cdots) \\
= & (-1)^{m-1}\sum_{i+j=k} \\
& \{\sum_{\sigma, a\in\omega, a+1\in\eta} (-1)^\sigma \langle\tilde{\omega_i}(e_{\sigma(1)}\cdots e_a\cdots)\cdot\tilde{\eta_j}(e_{\sigma(n-2i)}\cdots e_{a+1}\cdots e_{\sigma(n+m-2-2k)})\rangle \\
& + \sum_{\sigma, a\in\eta, a+1\in\omega} (-1)^\sigma \langle\tilde{\omega_i}(e_{\sigma(1)}\cdots e_a\cdots)\cdot\tilde{\eta_j}(e_{\sigma(n-2i)}\cdots e_{a+1}\cdots e_{\sigma(n+m-2-2k)})\rangle\} \\
& +(-1)^{m-1}\sum_{i+j=k} \\
& \{\sum_{\sigma, a\in\eta, a+1\in\omega} (-1)^\sigma \langle\tilde{\omega_i}(e_{\sigma(1)}\cdots e_{a+1},\cdots)\cdot\tilde{\eta_j}(e_{\sigma(n-2i)}\cdots, e_a\cdots e_{\sigma(n+m-2-2k)})\rangle \\
& + \sum_{\sigma, a\in\omega, a+1\in\eta} (-1)^\sigma \langle\tilde{\omega_i}(e_{\sigma(1)}\cdots e_{a+1}\cdots)\cdot\tilde{\eta_j}(e_{\sigma(n-2i)}\cdots e_a\cdots e_{\sigma(n+m-2-2k)})\rangle\} \\
& +(-1)^{m-1}\sum_{i+j=k} \\
& \{\sum_{\sigma, a\in\omega, a+1\in\omega} (-1)^\sigma \langle\tilde{\omega_i}(e_{\sigma(1)}\cdots e_a, e_{a+1}\cdots)\cdot\tilde{\eta_j}(e_{\sigma(n-2i)}\cdots e_{\sigma(n+m-2-2k)})\rangle \\
& + \sum_{\sigma, a\in\omega, a+1\in\omega} (-1)^\sigma \langle\tilde{\omega_i}(e_{\sigma(1)}\cdots e_{a+1}, e_a\cdots)\cdot\tilde{\eta_j}(e_{\sigma(n-2i)}\cdots e_{\sigma(n+m-2-2k)})\rangle\} \\
& +(-1)^{m-1}\sum_{i+j=k} \\
& \{\sum_{\sigma, a\in\eta, a+1\in\eta} (-1)^\sigma \langle\tilde{\omega_i}(e_{\sigma(1)}\cdots)\cdot\tilde{\eta_j}(e_{\sigma(n-2i)}\cdots e_a, e_{a+1}\cdots e_{\sigma(n+m-2-2k)})\rangle \\
& + \sum_{\sigma, a\in\eta, a+1\in\eta} (-1)^\sigma \langle\tilde{\omega_i}(e_{\sigma(1)}\cdots)\cdot\tilde{\eta_j}(e_{\sigma(n-2i)}\cdots e_{a+1}, e_a\cdots)\rangle\} \\
= & (-1)^{m-1}\sum_{i+j=k}\sum_{\sigma\in sh(n-2i-1,m-2j-1), a\in\omega, a+1\in\omega}(-1)^\sigma(-1) \\
& \langle\tilde{\omega_{i+1}}(e_{\sigma(1)}\cdots\hat{e_a}, \hat{e_{a+1}}\cdots e_{\sigma(n-2i-1)}; (e_a, e_{a+1}))\cdot\tilde{\eta_j}(e_{\sigma(n-2i)}\cdots)\rangle \\
& +(-1)^{m-1}\sum_{i+j=k}\sum_{\sigma\in sh(n-2i-1,m-2j-1), a\in\omega, a+1\in\omega}(-1)^\sigma(-1) \\
& \langle\tilde{\omega_i}(e_{\sigma(1)}\cdots)\cdot\tilde{\eta_{j+1}}(e_{\sigma(n-2i)}\cdots\hat{e_a}, \hat{e_{a+1}}\cdots e_{\sigma(n+m-2-2k)}; (e_a, e_{a+1}))\rangle \\
= & (-1)^m\sum_{i'+j=k+1}\sum_{\sigma'\in sh(n-2i'-1,m-2j-1)}(-1)^{\sigma'}\sum_{\tau\in sh(i',j), (e_a,e_{a+1})\in\omega} \\
& (\tilde{\omega_{i'}}(e_{\sigma'(1)}\cdots)((e_a, e_{a+1}), f_{\tau(1)}\cdots), \tilde{\eta_j}(e_{\sigma'(n-2i')}\cdots)(f_{\tau(i')}\cdots f_{\tau(k)})) \\
& +(-1)^m\sum_{i+j'=k+1}\sum_{\sigma'\in sh(n-2i-1,m-2j'-1)}(-1)^{\sigma'}\sum_{\tau\in sh(i,j'), (e_a,e_{a+1})\in\eta}
\end{aligned}
$$

$$
\begin{aligned}
& (\tilde{\omega_i}(e_{\sigma'(1)}\cdots)(f_{\tau(1)}\cdots), \tilde{\eta_{j'}}(e_{\sigma'(n-2i)}\cdots)((e_a, e_{a+1}), f_{\tau(i+1)}\cdots f_{\tau(k)})) \\
= \quad & -(\omega \bullet \eta)_{k+1}(e_1, \cdots, \hat{e_a}, \hat{e_{a+1}}, \cdots, e_{n+m-2-2k}; (e_a, e_{a+1}), f_1, \cdots, f_k).
\end{aligned}
$$

$\omega \diamond \eta$ is a cochain in $C^{n+m-2}_{st}(L)$ because:

$$
\begin{aligned}
& (\omega \diamond \eta)_k(\cdots, e_a, e_{a+1}\cdots; f_1 \cdots f_k) + (\omega \diamond \eta)_k(\cdots e_{a+1}, e_a \cdots; \cdots) \\
= \quad & \sum_{i+j=k}\{\sum_{\sigma, a\in\omega, a+1\in\eta} (-1)^\sigma \omega_{i+1}(e_{\sigma(1)}\cdots e_a\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_{a+1}\cdots) \\
& \quad + \sum_{\sigma, a\in\eta, a+1\in\omega} (-1)^\sigma \omega_{i+1}(e_{\sigma(1)}\cdots e_a\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_{a+1}\cdots)\} \\
& + \sum_{i+j=k}\{\sum_{\sigma, a\in\eta, a+1\in\omega} (-1)^\sigma \omega_{i+1}(e_{\sigma(1)}\cdots e_{a+1}\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_a\cdots) \\
& \quad + \sum_{\sigma, a\in\omega, a+1\in\eta} (-1)^\sigma \omega_{i+1}(e_{\sigma(1)}\cdots e_{a+1}\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_a\cdots)\} \\
& + \sum_{i+j=k}\{\sum_{\sigma, a\in\omega, a+1\in\omega} (-1)^\sigma \omega_{i+1}(e_{\sigma(1)}\cdots e_a, e_{a+1}\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots) \\
& \quad + \sum_{\sigma, a\in\omega, a+1\in\omega} (-1)^\sigma \omega_{i+1}(e_{\sigma(1)}\cdots e_{a+1}, e_a\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots)\} \\
& + \sum_{i+j=k}\{\sum_{\sigma, a\in\eta, a+1\in\eta} (-1)^\sigma \omega_{i+1}(e_{\sigma(1)}\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_a, e_{a+1}\cdots) \\
& \quad + \sum_{\sigma, a\in\eta, a+1\in\eta} (-1)^\sigma \omega_{i+1}(e_{\sigma(1)}\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_{a+1}, e_a\cdots)\} \\
= \quad & \sum_{i+j=k}\ \sum_{\sigma\in sh(n-2i-2,m-2j), a\in\omega, a+1\in\omega} (-1)^\sigma(-1) \\
& \quad \omega_{i+2}(e_{\sigma(1)}\cdots\hat{e_a}, \hat{e_{a+1}}\cdots e_{\sigma(n-2i-2)}; (e_a, e_{a+1}))\circ\eta_j(e_{\sigma(n-2i-1)}\cdots) \\
& + \sum_{i+j=k}\ \sum_{\sigma\in sh(n-2i-2,m-2j), a\in\eta, a+1\in\eta} (-1)^\sigma(-1) \\
& \quad \omega_{i+1}(e_{\sigma(1)}\cdots e_{\sigma(n-2i-2)})\circ\eta_{j+1}(e_{\sigma(n-2i-1)}\cdots\hat{e_{a+1}}, \hat{e_a}\cdots; (e_a, e_{a+1})) \\
= \quad & \sum_{i'+j=k+1}\ \sum_{\sigma'\in sh(n-2i'-2,m-2j)} (-1)^{\sigma'} \sum_{\tau\in sh(j,i'-1),(e_a,e_{a+1})\in\omega} \\
& \quad \omega_{i'+1}(e_{\sigma'(1)}\cdots; \eta_j(e_{\sigma'(n-2i'-1)}\cdots; f_{\tau(1)}\cdots), (e_a, e_{a+1}), f_{\tau(j+1)}\cdots f_{\tau(k)}) \\
& + \sum_{i+j'=k+1}\ \sum_{\sigma'\in sh(n-2i-2,m-2j')} (-1)^{\sigma'} \sum_{\tau\in sh(j'-1,i),(e_a,e_{a+1})\in\eta} \\
& \quad \omega_{i+1}(e_{\sigma'(1)}\cdots; \eta_{j'}(e_{\sigma'(n-2i-1)}\cdots; (e_a, e_{a+1}), f_{\tau(1)}\cdots), f_{\tau(j')}\cdots f_{\tau(k)}) \\
= \quad & -(\omega\diamond\eta)_{k+1}(e_1, \cdots, \hat{e_a}, \hat{e_{a+1}}, \cdots, e_{n+m-2-2k}; (e_a, e_{a+1}), f_1, \cdots, f_k).
\end{aligned}
$$

So $\{\omega, \eta\} = \omega \bullet \eta + \omega \diamond \eta - (-1)^{nm}\eta \diamond \omega$ is also a cochain in $C^{n+m-2}_{st}(L)$. ■

Proof of theorem 4.4:

Proof. 1) Due to the lemma above, in order for { , } to be a graded-commutative Poisson bracket, we need to prove the following three properties:

(1). For any two representable cochains ω, η, $\{\omega, \eta\} = -(-1)^{nm}\{\eta, \omega\}$,

(2). If $\eta \in \tilde{C}^m_{st}(L), \lambda \in \tilde{C}^l_{st}(L)$, then $\eta\lambda \in \tilde{C}^{m+l-2}_{st}(L)$, and

$$\{\omega, \eta\lambda\} = \{\omega, \eta\}\lambda + (-1)^{nm}\eta\{\omega, \lambda\},$$

(3). The bracket of any two representable cochains is still a representable cochain, and

$$\{\omega, \{\eta, \lambda\} = \{\{\omega, \eta\}, \lambda\} + (-1)^{nm}\{\eta, \{\omega, \lambda\}\}.$$

For (1), it suffices to prove $\omega \bullet \eta = -(-1)^{nm}\eta \bullet \omega$.

$\forall \sigma \in sh(n-2i-1, m-2j-1)$, switching the first $n-2i-1$ arguments with the last $m-2j-1$ arguments results in a sign difference $(-1)^{(n-1)(m-1)}$, so by definition there is merely a sign difference between $\omega \bullet \eta$ and $\eta \bullet \omega$ of $(-1)^{n-m+(n-1)(m-1)} = (-1)^{nm+1}$.

Thus (1) is proved.

For (2), in order for $\eta\lambda$ to be a representable cochain, we need to check that

$$(\eta\lambda)_k(e_1, \cdots, e_{m+l-2k}; f_1, \cdots, f_k) = (e_{m+l-2k}, \bullet), \ \forall k$$

$$\begin{aligned}
& (\eta\lambda)_k(e_1, \cdots, e_{m+l-2k}; f_1, \cdots, f_k) \\
= & \sum_{i+j=k} \sum_{\sigma \in sh(m-2i,l-2j)} \sum_{\tau \in sh(i,j)} (-1)^{\sigma} \\
& \eta_i(e_{\sigma(1)} \cdots ; f_{\tau(1)} \cdots f_{\tau(i)})\lambda_j(e_{\sigma(m-2i+1)} \cdots e_{\sigma(m+l-2k)}; f_{\tau(i+1)} \cdots f_{\tau(k)}) \\
= & \sum_{i+j=k} \sum_{\sigma \in sh(m-2i,l-2j), \sigma^{-1}(m+l-2k)=m-2i} \sum_{\tau \in sh(i,j)} (-1)^{\sigma} \\
& (\tilde{\eta_i}(\cdots, e_{m+\hat{l}-2k}; \cdots), e_{m+l-2k})\lambda_j(\cdots ; \cdots) \\
& + \sum_{i+j=k} \sum_{\sigma \in sh(m-2i,l-2j), \sigma^{-1}(m+l-2k)=m+l-2k} \sum_{\tau \in sh(i,j)} (-1)^{\sigma} \\
& \eta_i(\cdots ; \cdots)(\tilde{\lambda_j}(\cdots, e_{m+\hat{l}-2k}; \cdots), e_{m+l-2k}) \\
= & (e_{m+l-2k}, \\
& \{ \sum_{i+j=k} \sum_{\bar{\sigma} \in sh(m-2i-1,l-2j)} \sum_{\tau \in sh(i,j)} (-1)^{\bar{\sigma}+l} \\
& \tilde{\eta_i}(e_{\bar{\sigma}(1)}, \cdots ; f_{\tau(1)}, \cdots)\lambda_j(e_{\bar{\sigma}(m-2i)} \cdots ; f_{\tau(i+1)}, \cdots) \\
& + \sum_{i+j=k} \sum_{\bar{\sigma} \in sh(m-2i,l-2j-1)} \sum_{\tau \in sh(i,j)} (-1)^{\bar{\sigma}} \\
& \eta_i(e_{\bar{\sigma}(1)} \cdots ; f_{\tau(1)}, \cdots)\tilde{\lambda_j}(e_{\bar{\sigma}(m-2i+1)} \cdots ; f_{\tau(i+1)}, \cdots)\} \\
&)
\end{aligned}$$

Next, we prove that $\{\omega, \bullet\}$ is a graded derivative.

$$\begin{aligned} & \{\omega, \eta\lambda\}_k(e_1, \cdots, e_{n+m+l-2-2k}; f_1, \cdots, f_k) \\ = \; & (\omega \bullet \eta\lambda)_k(\cdots) + (\omega \diamond \eta\lambda)_k(\cdots) + (-1)^{n(m+l)+1}(\eta\lambda \diamond \omega)_k(\cdots) \end{aligned}$$

We calculate the three parts above respectively:

$$\begin{aligned} & (\omega \bullet \eta\lambda)_k(e_1, \cdots, e_{n+m+l-2-2k}; f_1, \cdots, f_k) \\ = \; & (-1)^{m+l+1} \sum_{a+b=k} \sum_{\sigma \in sh(n-2a-1, m+l-2b-1)} \sum_{\tau \in sh(a,b)} \\ & (\tilde{\omega_a}(e_{\sigma(1)}, \cdots; f_{\tau(1)}, \cdots, f_{\tau(a)}), \widetilde{(\eta\lambda)}_b(e_{\sigma(n-2a)}, \cdots, e_{\sigma(n+m+l-2-2k)}; \cdots)) \\ = \; & (-1)^{m+l+1} \sum_{a+b+c=k} \sum_{\sigma \in sh(n-2a-1, m-2b-1, l-2c)} \sum_{\tau \in sh(a,b,c)} (-1)^{\sigma+l} \\ & (\tilde{\omega}_a(e_{\sigma(1)} \cdots; \cdots), \tilde{\eta_b}(e_{\sigma(n-2a)} \cdots; \cdots)\lambda_c(e_{n+m-2a-2b-1} \cdots; \cdots)) \\ & +(-1)^{m+l+1} \sum_{a+b+c=k} \sum_{\sigma \in sh(n-2a-1, m-2b, l-2c-1)} \sum_{\tau \in sh(a,b,c)} (-1)^{\sigma} \\ & (\tilde{\omega_a}(e_{\sigma(1)} \cdots; \cdots), \eta_b(e_{\sigma(n-2a)} \cdots; \cdots)\tilde{\lambda}_c(e_{n+m-2a-2b} \cdots; \cdots)) \\ = \; & \sum_{a+c=k} \sum_{\sigma \in sh(n+m-2a-2, l-2c)} \sum_{\tau \in sh(a,c)} (-1)^{\sigma} \\ & (\omega \bullet \eta)_a(e_{\sigma(1)}, \cdots; f_{\tau(1)}, \cdots)\lambda_c(e_{\sigma(n+m-2a-1)}, \cdots; f_{\tau(a+1)}, \cdots) \\ & + \sum_{b+a=k} \sum_{\sigma \in sh(m-2b, n+l-2a-2)} \sum_{\tau \in sh(b,a)} (-1)^{\sigma+(n-1)m}(-1)^m \\ & \eta_b(e_{\sigma(1)}, \cdots; f_{\tau(1)}, \cdots)(\omega \bullet \lambda)_a(e_{\sigma(m-2b+1)}, \cdots; f_{\tau(b+1)}, \cdots) \\ = \; & \Big((\omega \bullet \eta) \cdot \lambda\Big)_k(\cdots) + (-1)^{nm}\Big(\eta \cdot (\omega \bullet \lambda)\Big)_k(\cdots) \end{aligned}$$

$$\begin{aligned} & (\omega \diamond \eta\lambda)_k(e_1, \cdots, e_{n+m+l-2-2k}; f_1, \cdots, f_k) \\ = \; & \sum_{a+b=k} \sum_{\sigma \in sh(n-2a-2, m+l-2b)} \sum_{\tau \in sh(b,a)} (-1)^{\sigma} \\ & \omega_{a+1}(e_{\sigma(1)} \cdots e_{\sigma(n-2a-2)}; (\eta\lambda)_b(e_{\sigma(n-2a-1)} \cdots; f_{\tau(1)} \cdots), f_{\tau(b+1)} \cdots) \\ = \; & \sum_{a+b+c=k} \sum_{\sigma \in sh(n-2a-2, m-2b, l-2c)} \sum_{\tau \in sh(b,c,a)} (-1)^{\sigma} \\ & \omega_{a+1}(e_{\sigma(1)} \cdots; \eta_b(e_{\sigma(n-2a-1)} \cdots; \cdots)\lambda_c(e_{\sigma(n+m-2a-2b-1)} \cdots; \cdots), \cdots) \\ = \; & \sum_{a+b+c=k} \sum_{\sigma \in sh(n-2a-2, m-2b, l-2c)} \sum_{\tau \in sh(b,a,c)} (-1)^{\sigma} \end{aligned}$$

$$
\begin{aligned}
& \omega_{a+1}(e_{\sigma(1)}\cdots;\eta_b(e_{\sigma(n-2a-1)}\cdots;\cdots),\cdots)\lambda_c(e_{\sigma(n+m-2a-2b-1)}\cdots;\cdots) \\
& + \sum_{a+b+c=k}\ \sum_{\sigma\in sh(m-2b,n-2a-2,l-2c)}\ \sum_{\tau\in sh(b,c,a)} (-1)^{\sigma+nm} \\
& \eta_b(e_{\sigma(1)}\cdots;\cdots)\omega_{a+1}(e_{\sigma(m-2b+1)}\cdots;\lambda_c(e_{\sigma(n+m-2a-2b-1)}\cdots;\cdots),\cdots) \\
= & \sum_{a+c=k}\ \sum_{\sigma\in sh(n+m-2a-2,l-2c)}\ \sum_{\tau\in sh(a,c)} (-1)^{\sigma} \\
& (\omega\diamond\eta)_a(e_{\sigma(1)},\cdots;f_{\tau(1)},\cdots)\lambda_c(e_{\sigma(n+m-2a-1)},\cdots;f_{\tau(a+1)},\cdots) \\
& +(-1)^{nm}\sum_{a+b=k}\ \sum_{\sigma\in sh(m-2b,n+l-2a-2)}\ \sum_{\tau\in sh(b,a)} (-1)^{\sigma} \\
& \eta_b(e_{\sigma(1)},\cdots;f_{\tau(1)},\cdots)(\omega\diamond\lambda)_a(e_{\sigma(m-2b+1)},\cdots;f_{\tau(b+1)},\cdots) \\
= & \Big((\omega\diamond\eta)\cdot\lambda\Big)_k(\cdots)+(-1)^{nm}\Big(\eta\cdot(\omega\diamond\lambda)\Big)_k(\cdots)
\end{aligned}
$$

$$
\begin{aligned}
& (-1)^{n(m+l)+1}(\eta\lambda\diamond\omega)_k(e_1,\cdots,e_{n+m+l-2-2k};f_1,\cdots,f_k) \\
= & \sum_{a+c=k}\ \sum_{\sigma\in sh(m+l-2a-2,n-2c)}\ \sum_{\tau\in sh(c,a)} (-1)^{nm+nl+1+\sigma} \\
& (\eta\lambda)_{a+1}(e_{\sigma(1)},\cdots;\omega_c(e_{\sigma(m+l-2a-1)},\cdots;f_{\tau(1)},\cdots),f_{\tau(c+1)},\cdots) \\
= & \sum_{a+b+c=k}\ \sum_{\sigma\in sh(m-2a-2,n-2c,l-2b)}\ \sum_{\tau\in sh(c,a,b)} (-1)^{nm+nl+1+\sigma+nl} \\
& \eta_{a+1}(e_{\sigma(1)}\cdots;\omega_c(e_{\sigma(m-2a-1)}\cdots;\cdots),\cdots)\lambda_b(e_{\sigma(n+m-2a-2c-1)}\cdots;\cdots) \\
& + \sum_{a+b+c=k}\ \sum_{\sigma\in sh(m-2a,l-2b-2,n-2c)}\ \sum_{\tau\in sh(a,c,b)} (-1)^{nm+nl+1+\sigma} \\
& \eta_a(e_{\sigma(1)}\cdots;\cdots)\lambda_{b+1}(e_{\sigma(m-2a+1)}\cdots;\omega_c(e_{\sigma(m+l-2a-2b-1)}\cdots;\cdots),\cdots) \\
= & \sum_{a+b=k}\ \sum_{\sigma\in sh(m+n-2a-2,l-2b)}\ \sum_{\tau\in sh(a,b)} (-1)^{\sigma}(-1)^{nm+1} \\
& (\eta\diamond\omega)_a(e_{\sigma(1)},\cdots;f_{\tau(1)},\cdots)\lambda_b(e_{\sigma(n+m-2a-1)},\cdots;f_{\tau(a+1)},\cdots) \\
& + \sum_{a+b=k}\ \sum_{\sigma\in sh(m-2a,n+l-2b-2)}\ \sum_{\tau\in sh(a,b)} (-1)^{nm}(-1)^{\sigma} \\
& \eta_a(e_{\sigma(1)},\cdots;f_{\tau(1)},\cdots)\cdot(-1)^{nl+1}(\lambda\diamond\omega)_b(e_{\sigma(m-2a+1)},\cdots;f_{\tau(a+1)},\cdots) \\
= & (-1)^{nm+1}\Big((\eta\diamond\omega)\cdot\lambda\Big)_k(\cdots)+(-1)^{nm}(-1)^{nl+1}\Big(\eta\cdot(\lambda\diamond\omega)\Big)_k(\cdots)
\end{aligned}
$$

The sum of the three equations above is:

$$
\begin{aligned}
& \{\omega,\eta\lambda\}_k(e_1,\cdots,e_{n+m+l-2-2k};f_1,\cdots,f_k) \\
= & \Big((\omega\bullet\eta)\cdot\lambda\Big)_k(\cdots)+(-1)^{nm}\Big(\eta\cdot(\omega\bullet\lambda)\Big)_k(\cdots) \\
& +\Big((\omega\diamond\eta)\cdot\lambda\Big)_k(\cdots)+(-1)^{nm}\Big(\eta\cdot(\omega\diamond\lambda)\Big)_k(\cdots)
\end{aligned}
$$

$$\begin{aligned}
&+(-1)^{nm+1}\Big((\eta\diamond\omega)\cdot\lambda\Big)_k(\cdots)+(-1)^{nm}(-1)^{nl+1}\Big(\eta\cdot(\lambda\diamond\omega)\Big)_k(\cdots)\\
=\ &(\{\omega,\eta\}\cdot\lambda)_k(\cdots)+(-1)^{nm}(\eta\cdot\{\omega,\lambda\})_k(\cdots)
\end{aligned}$$

Thus $\{\omega,\bullet\}$ is a graded derivative, (2) is proved.
For (3), in order for $\{\omega,\eta\}$ to be a representable cochain, we need to prove

$$\{\omega,\eta\}_k(e_1,\cdots,e_{n+m-2k};f_1,\cdots,f_k)=(e_{n+m-2k},\bullet),\ \forall k.$$

$$\begin{aligned}
&\{\omega,\eta\}_k(e_1,\cdots,e_{n+m-2-2k};f_1,\cdots,f_k)\\
=\ &(-1)^{m+1}\sum_{a+b=k}\ \sum_{\sigma\in sh(n-2a-1,m-2b-1)}\ \sum_{\tau\in sh(a,b)}(-1)^{\sigma}\\
&\quad(\tilde{\omega_a}(e_{\sigma(1)},\cdots;f_{\tau(1)},\cdots),\tilde{\eta_b}(e_{\sigma(n-2a)},\cdots;f_{\tau(a+1)},\cdots))\\
&+\sum_{a+b=k}\ \sum_{\sigma\in sh(n-2a-2,m-2b)}\ \sum_{\tau\in sh(b,a)}(-1)^{\sigma}\\
&\quad\omega_{a+1}(e_{\sigma(1)},\cdots;\eta_b(e_{\sigma(n-2a-1)},\cdots;f_{\tau(1)},\cdots),f_{\tau(b+1)},\cdots)\\
&+(-1)^{nm+1}\sum_{a+b=k}\ \sum_{\sigma\in sh(m-2a-2,n-2b)}\ \sum_{\tau\in sh(b,a)}(-1)^{\sigma}\\
&\quad\eta_{a+1}(e_{\sigma(1)},\cdots;\omega_b(e_{\sigma(m-2a-1)},\cdots;f_{\tau(1)},\cdots),f_{\tau(b+1)},\cdots)\\
=\ &(-1)^{m+1}\sum_{a+b=k}\ \sum_{\sigma\in sh(n-2a-2,m-2b-1)}\ \sum_{\tau\in sh(b,a)}(-1)^{\sigma+m+1}\\
&\quad\omega_a(e_{\sigma(1)}\cdots e_{\sigma(n-2a-2)},e_{n+m-2-2k},\tilde{\eta_b}(e_{\sigma(n-2a-1)}\cdots;f_{\tau(1)}\cdots);f_{\tau(b+1)}\cdots)\\
&+(-1)^{m+1}\sum_{a+b=k}\ \sum_{\sigma\in sh(m-2b-2,n-2a-1)}\ \sum_{\tau\in sh(a,b)}(-1)^{\sigma+(n-1)m}\\
&\quad\eta_b(e_{\sigma(1)}\cdots e_{\sigma(m-2b-2)},e_{n+m-2-2k},\tilde{\omega_a}(e_{\sigma(m-2b-1)}\cdots;f_{\tau(1)}\cdots);f_{\tau(a+1)}\cdots)\\
&+\{(e_{n+m-2-2k},\bullet)\\
&\quad+\sum_{a+b=k}\ \sum_{\sigma\in sh(n-2a-2,m-2b-1)}\ \sum_{\tau\in sh(b,a)}(-1)^{\sigma}\\
&\quad\quad\omega_{a+1}(e_{\sigma(1)},\cdots;(\tilde{\eta_b}(e_{\sigma(n-2a-1)},\cdots;f_{\tau(1)},\cdots),e_{n+m-2-2k}),f_{\tau(b+1)},\cdots)\}\\
&+\{(e_{n+m-2-2k},\bullet)\\
&\quad+(-1)^{nm+1}\sum_{a+b=k}\ \sum_{\sigma\in sh(m-2a-2,n-2b-1)}\ \sum_{\tau\in sh(b,a)}(-1)^{\sigma}\\
&\quad\quad\eta_{a+1}(e_{\sigma(1)},\cdots;(\tilde{\omega_b}(e_{\sigma(m-2a-1)},\cdots;f_{\tau(1)},\cdots),e_{n+m-2-2k}),f_{\tau(b+1)},\cdots)\}\\
=\ &(e_{n+m-2-2k},\bullet)\\
&+\sum_{a+b=k}\ \sum_{\sigma\in sh(n-2a-2,m-2b-1)}\ \sum_{\tau\in sh(b,a)}(-1)^{\sigma}\omega_a(\cdots,e_{n+m-2-2k},\tilde{\eta_b}(\cdots);\cdots)\\
&+\sum_{a+b=k}\ \sum_{\sigma\in sh(m-2b-2,n-2a-1)}\ \sum_{\tau\in sh(a,b)}(-1)^{\sigma+nm+1}\eta_b(\cdots,e_{n+m-2-2k},\tilde{\omega_a}(\cdots);\cdots)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{a+b=k}\ \sum_{\sigma\in sh(n-2a-2,m-2b-1)}\ \sum_{\tau\in sh(b,a)}(-1)^{\sigma+1}\\
&\quad\{\omega_a(\cdots,e_{n+m-2-2k},\tilde{\eta}_b(\cdots);\cdots)+\omega_a(\cdots,\tilde{\eta}_b(\cdots),e_{n+m-2-2k};\cdots)\}\\
&+\sum_{a+b=k}\ \sum_{\sigma\in sh(m-2a-2,n-2b-1)}\ \sum_{\tau\in sh(b,a)}(-1)^{\sigma+nm}\\
&\quad\{\eta_a(\cdots,e_{n+m-2-2k},\tilde{\omega}_b(\cdots);\cdots)+\eta_a(\cdots,\tilde{\omega}_b(\cdots),e_{n+m-2-2k};\cdots)\}\\
=\ &(e_{n+m-2-2k},\bullet)
\end{aligned}
$$

Thus (3) is proved.

2)

$$
\begin{aligned}
&(\Theta\bullet\eta)_k(e_1,\cdots,e_{m+1-2k};f_1,\cdots,f_k)\\
=\ &(-1)^{m-1}\sum_{\sigma\in sh(2,m-2k-1)}(-1)^\sigma(\tilde{\Theta}_0(e_{\sigma(1)},e_{\sigma(2)}),\tilde{\eta}_k(e_{\sigma(3)}\cdots e_{\sigma(m+1-2k)};\cdots))\\
&+(-1)^{m-1}\sum_{\tau\in sh(1,k-1)}(\tilde{\Theta}_1(f_{\tau(1)}),\tilde{\eta_{k-1}}(e_1,\cdots,e_{m+1-2k};f_{\tau(2)},\cdots,f_{\tau(k)}))\\
=\ &(-1)^{m-1}\sum_{a<b}(-1)^{a+b+1}(e_a\circ e_b,\tilde{\eta}_k(e_1,\cdots,\hat{e_a},\cdots,\hat{e_b},\cdots e_{m+1-2k};f_1,\cdots f_k))\\
&+(-1)^{m-1}\sum_i(-1)(f_i,\tilde{\eta_{k-1}}(e_1,\cdots,e_{m+1-2k};f_1,\cdots,\hat{f_i},\cdots,f_k))\\
=\ &(-1)^m\sum_{a<b}(-1)^{a+b}\eta_k(e_1,\cdots,\hat{e_a},\cdots,\hat{e_b},\cdots,e_{m+1-2k},e_a\circ e_b;f_1,\cdots,f_k)\\
&+(-1)^m\sum_i\eta_{k-1}(e_1,\cdots,e_{m+1-2k},f_i;f_1,\cdots,\hat{f_i},\cdots,f_k)
\end{aligned}
$$

$$
\begin{aligned}
&(\Theta\diamond\eta)_k(e_1,\cdots,e_{m+1-2k};f_1,\cdots,f_k)\\
=\ &\sum_a(-1)^{a+1}\Theta_1(e_a)\circ\eta_k(e_1,\cdots,\hat{e_a},\cdots,e_{m+1-2k})\\
=\ &\sum_a(-1)^{a+1}(-1)(e_a,\eta_k(e_1,\cdots,\hat{e_a},\cdots e_{m+1-2k};f_1,\cdots,f_k))\\
=\ &\sum_a(-1)^a\rho(e_a)\eta_k(e_1,\cdots,\hat{e_a},\cdots,e_{m+1-2k};f_1,\cdots,f_k)
\end{aligned}
$$

$$
\begin{aligned}
&(-1)^{m+1}(\eta\diamond\Theta)_k(e_1,\cdots,e_{m+1-2k};f_1,\cdots,f_k)\\
=\ &(-1)^{m+1}\sum_{\sigma\in sh(m-2k-2,3)}(-1)^\sigma\\
&\eta_{k+1}(e_{\sigma(1)},\cdots,e_{\sigma(m-2k-2)})\circ\Theta_0(e_{\sigma(m-2k-1)},e_{\sigma(m-2k)},e_{\sigma(m-2k+1)})
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{m+1}\sum_a(-1)^{a+m+1}\eta_k(e_1,\cdots\hat{e_a},\cdots,e_{m+1-2k})\circ\Theta_1(e_a)\\
=\;&\sum_{a<b<c}(-1)^{a+b+c+1}\eta_{k+1}(e_1\cdots\hat{e_a}\cdots\hat{e_b}\cdots\hat{e_c}\cdots e_{m+1-2k};(e_a\circ e_b,e_c),f_1\cdots f_k)\\
&+\sum_a(-1)^a\sum_i\eta_k(e_1,\cdots,\hat{e_a},\cdots,e_{m+1-2k};-(e_a,f_i),f_1,\cdots,\hat{f_i},\cdots,f_k)\\
=\;&\sum_{a<b}(-1)^{a+b}\sum_{b<c<m+2-2k}(-1)^c\{\eta_k(\cdots,\hat{e_a},\cdots,\hat{e_b},\cdots,e_{c-1},e_a\circ e_b,e_c,\cdots;\cdots)\\
&\quad+\eta_k(\cdots,\hat{e_a},\cdots,\hat{e_b},\cdots,e_{c-1},e_c,e_a\circ e_b,\cdots;\cdots)\}\\
&+\sum_{i,a}(-1)^a\{\eta_{k-1}(\cdots e_{a-1},f_i,e_a,\cdots;\hat{f_i}\cdots)+\eta_{k-1}(\cdots e_{a-1},e_a,f_i,\cdots;\hat{f_i}\cdots)\}\\
=\;&\sum_{a<b}(-1)^{a+1}\{\eta_k(\cdots\hat{e_a}\cdots e_a\circ e_b,\cdots;\cdots)+(-1)^{b+m}\eta_k(\cdots\hat{e_a}\cdots\hat{e_b}\cdots e_a\circ e_b;\cdots)\}\\
&-\sum_i\eta_{k-1}(f_i,e_1,\cdots;\cdots\hat{f_i}\cdots)+(-1)^{m+1}\sum_i\eta_{k-1}(e_1\cdots e_{m+1-2k},f_i;\cdots\hat{f_i}\cdots)
\end{aligned}
$$

The sum of the equations above is

$$
\begin{aligned}
&\{\Theta,\eta\}_k(e_1,\cdots,e_{m+1-2k};f_1,\cdots,f_k)\\
=\;&\sum_a(-1)^a\rho(e_a)\eta_k(e_1,\cdots,\hat{e_a},\cdots,e_{m+1-2k};f_1,\cdots,f_k)\\
&+\sum_{a<b}(-1)^{a+1}\eta_k(\cdots,\hat{e_a},\cdots,e_a\circ e_b,\cdots;\cdots)\\
&-\sum_i\eta_{k-1}(f_i,e_1,\cdots,e_{m+1-2k};\cdots,\hat{f_i},\cdots)\\
=\;&-((d_0+\delta)\eta)_k(e_1,\cdots,e_{m+1-2k};f_1,\cdots,f_k)
\end{aligned}
$$

Thus 2) is proved. ■

If $(\bullet,\ ):\ S^\bullet(Z)\otimes L\to L^\vee$ is an isomorphism(i.e. the symmetric product of $S^\bullet(Z)\otimes L$ is strongly non-degenerate), any $\omega\in C^\bullet_{st}(L)$ is a representable cochain, so the bracket can be defined for any 2 cochains and $C^\bullet_{st}(L)$ is a graded-commutative Poisson algebra.

4.3 The derived bracket

Before stating the main theorem, we give the definition of fat Leibniz algebras:

Definition 4.6. Given a Leibniz algebra L, if the bilinear product $(\bullet,\bullet)$ is non-degenerate, or equivalently the induced map

$$(\bullet)^\flat : L\to Hom(L,Z)\ (2.3)$$

is injective, we call L a fat Leibniz algebra.

The omni Lie algebra $ol(V) = gl(V) \oplus V$ is obviously a fat Leibniz algebra.
Actually given any Leibniz algebra L with trivial center, there is associated a fat Leibniz algebra $\tilde{L}$:

Proposition 4.7. *Suppose L is a Leibniz algebra with trivial center, then $\tilde{L} \triangleq L/K$ is a fat Leibniz algebra, where K is the kernel of the bilinear product of L, i.e. $K = \{k \in L | (k, e) = 0, \ \forall e \in L\}$.*

Proof. Since the product of L is invariant:

$$\tau(e_1)(k, e_2) = (e_1 \circ k, e_2) + (k, e_1 \circ e_2), \quad \forall e_1, e_2 \in L, \ \forall k \in K,$$

it follows that

$$(e_1 \circ k, e_2) = 0, \quad \forall e_2 \in L,$$

so $e_1 \circ k \in K$. Furthermore since $e_1 \circ k + k \circ e_1 = (k, e_1) = 0$, so $k \circ e_1 = -e_1 \circ k$ is also in K. Thus K is an ideal of L.

The Leibniz bracket of L naturally induces a bracket on L/K:

$$\bar{e}_1 \circ \bar{e}_2 \triangleq \overline{e_1 \circ e_2},$$

where $\bar{e}$ is the equivalent class of $e \in L$ in L/K. Suppose there exists $\bar{k} \in L/K$ such that $(\bar{k}, \bar{e}) = 0$, $\forall \bar{e} \in L/K$, i.e. $(k, e) \in K$, $\forall e \in L$. Since (k, e) is in the left center of L, $(k, e) \in K$ implies that (k, e) is also in the right center of L. So $(k, e) = 0$ by the assumption that the center of L is trivial. It follows that k itself is in K, $\bar{k} = 0 \in L/K$. As a result, the bilinear product on L/K is non-degenerate. ■

Analogously to theorem 2.24, we have the following:

Theorem 4.8. *Suppose L is a Leibniz algebra with left center Z, $h \supseteq Z$ is an isotropic ideal in L acting trivially on Z. Θ and $\{\bullet, \bullet\}$ are defined as in 4.1.1 and 4.2.1. Then we have*

$$(e_1 \circ e_2)^\flat = -\{\{\Theta, e_1^\flat\}, e_2^\flat\}.$$

In particular, if L is a fat Leibniz algebra, then the Leibniz bracket can be represented as a derived bracket:

$$e_1 \circ e_2 = -\{\{\Theta, e_1^\flat\}, e_2^\flat\}^\sharp,$$

where $(\bullet)^\sharp : Im((\bullet)^\flat) \to L$ is the (partial) inverse map of $(\bullet)^\flat$, i.e.

$$((\phi)^\sharp)^\flat \triangleq \phi, \quad \forall \phi \in Im((\bullet)^\flat).$$

Proof. $\{\Theta, e_1^\flat\}$ is a 2 cochain:

$$\begin{aligned}
& \{\Theta, e_1^\flat\}_0(e_2, e_3) \\
= \; & \langle \tilde{\Theta}_0(e_2, e_3), \tilde{e_1^\flat}\rangle + \Theta_1(e_2) \circ e_1^\flat(e_3) - \Theta_1(e_3) \circ e_1^\flat(e_2) \\
= \; & (e_2 \circ e_3, e_1) - (e_2, (e_1, e_3)) + (e_3, (e_1, e_2)) \\
= \; & -(e_2 \circ e_1, e_3) + (e_3, e_1 \circ e_2 + e_2 \circ e_1) \\
= \; & (e_1 \circ e_2, e_3)
\end{aligned}$$

$$\{\Theta, e_1^\flat\}_1(f) = \langle \tilde{\Theta}_1(f), \tilde{e_1^\flat}\rangle = -(e_1, f)$$

We see that $\{\Theta, e_1^\flat\}$ is a representable cochain.
$\{\{\Theta, e_1^\flat\}, e_2^\flat\}$ is a 1 cochain:

$$\begin{aligned}
& \{\{\Theta, e_1^\flat\}, e_2^\flat\}_0(e_3) \\
= \; & \langle \widetilde{\{\Theta, e_1^\flat\}_0(e_3)}, \tilde{e_2^\flat}\rangle + \{\Theta, e_1^\flat\}_1 \circ e_2^\flat(e_3) \\
= \; & (e_1 \circ e_3, e_2) - (e_1, (e_2, e_3)) \\
= \; & -(e_1 \circ e_2, e_3)
\end{aligned}$$

So $(e_1 \circ e_2)^\flat(e_3) = -\{\{\Theta, e_1^\flat\}, e_2^\flat\}(e_3) = (e_1 \circ e_2, e_3)$.
The proof is finished. ■

Chapter 5

Equivariant Cohomology

5.1 The generalized action of a Lie algebra on a manifold

We know that an action of a Lie group G on M induces an infinitesimal action of its Lie algebra $\mathfrak{g}$ on M, which is a homomorphism of Lie algebras from $\mathfrak{g}$ to the space of sections of the tangent bundle TM. In generalized geometry, we consider the generalized tangent bundle $TM \oplus T^*M$ instead of TM. $TM \oplus T^*M$ can be endowed with the standard Courant algebroid structure, whose Dorfman bracket turns $\Gamma(TM \oplus T^*M)$ into a Leibniz algebra. So the following definition is naturally motivated:

Definition 5.1. The generalized action of a Lie algebra $\mathfrak{g}$ on a manifold M, is defined to be a homomorphism of Leibniz algebras from $\mathfrak{g}$ to the space of sections of the generalized tangent bundle $TM \oplus T^*M$.

Example 5.2. 1). Suppose ω is a symplectic 2-form on M, it induces a map $\omega^\flat: \ TM \to T^*M$. The graph of $\omega^\flat$, denoted by $\mathcal{G}_\omega$, is a Dirac structure in the standard Courant algebroid $TM \oplus T^*M$. Any map $\phi: \ \mathfrak{g} \to \Gamma(TM)$ induces a map $\bar{\phi}$ from $\mathfrak{g}$ to $\Gamma(TM \oplus T^*M)$:

$$\bar{\phi}(A) \triangleq \phi(A) + \omega^\flat(\phi(A)), \quad \forall A \in \mathfrak{g}.$$

It is easily checked that $\bar{\phi}$ is a generalized action of $\mathfrak{g}$ on M iff ϕ is a homomorphism of Lie algebras. Actually $\bar{\phi}$ is a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathcal{G}_\omega$ in this case.

2). Suppose Π is a Poisson bivector on M, it induces a map $\Pi^\sharp: \ T^*M \to TM$. The graph of $\Pi^\sharp$, denoted by $\mathcal{G}_\Pi$, is a Dirac structure in the standard Courant algebroid $TM \oplus T^*M$. Any map $\psi: \mathfrak{g} \to \Gamma(T^*M)$ induces a map $\bar{\psi}$ from $\mathfrak{g}$ to $\Gamma(TM \oplus T^*M)$:

$$\bar{\psi}(A) \triangleq \Pi^\sharp(\psi(A)) + \psi(A), \quad \forall A \in \mathfrak{g}.$$

It is easily checked that $\bar{\psi}$ is a generalized action of $\mathfrak{g}$ on M iff ψ is a homomorphism of Lie algebras from $\mathfrak{g}$ to $\Gamma(T^*M)$, where the Lie bracket $[\bullet, \bullet]_\Pi$ on $\Gamma(T^*M)$ is given by

$$[\alpha, \beta]_\Pi \triangleq L_{\Pi^\sharp(\alpha)}\beta - L_{\Pi^\sharp(\beta)}\alpha - d_M(\Pi(\alpha, \beta)), \quad \forall \alpha, \beta \in \Gamma(T^*M).$$

Actually $\bar{\psi}$ is a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathcal{G}_\Pi$ in this case.

In the following, we denote by E the standard Courant algebroid $TM \oplus T^*M$, and by $\mathcal{E}$ the space of its sections. So $\mathcal{E}$ is a Courant-Dorfman algebra with Dorfman bracket

$$(X + \xi) \circ (Y + \eta) = [X, Y] + L_X\eta - \iota_Y d_M \xi$$

and symmetric bilinear form

$$(X + \xi, Y + \eta) = \iota_X \eta + \iota_Y \xi.$$

Given a local chart $\{x^i\}_{1 \le i \le n}$ of M, take $\{\xi_i := \frac{\partial}{\partial x^i}, \xi^i := d_M x^i\}$ as a local basis of $\mathcal{E}$, then $g_{ij} \triangleq (\xi_i, \xi_j) = \delta_{i(n+j)}$(we assume that $\xi^i = \xi_{n+i},\ \xi_i = \xi^{n+i},\ \delta_{i(2n+i)} = 1$). If we denote by $\{p_i\}_{1 \le i \le n}$ the conjugates of $\{x^i\}$ with degree 2, the cubic Hamiltonian H of E can be written in a simple form:

$$H = p_i \xi^i.$$

Given an ordinary action of $\mathfrak{g}$ on M, we know that the de Rham differential $(\Omega^\bullet(M), d_M)$ can be realized as a $\mathfrak{g}$ differential algebra. Now given a generalized action $\varphi: \mathfrak{g} \to \mathcal{E},\ A \mapsto \hat{A}$, in the following we will define the interior product ι_A and Lie derivative L_A ($\forall A \in \mathfrak{g}$) on $C^\bullet_{st}(E)$, so that $(C^\bullet_{st}(E), d_{st})$ also becomes a $\mathfrak{g}$ differential algebra.

The interior product ι_A of $C^\bullet_{st}(E)$ by $\mathfrak{g}$ is defined to be:

$$\begin{array}{ccc} C^m_{st}(E) & \xrightarrow{\iota_A} & C^{m-1}_{st}(E) \\ \omega & \mapsto & \{\hat{A}^\flat, \omega\}. \end{array} \tag{5.1.1}$$

By theorem 2.25, if ω is viewed as a cochain in $C^\bullet(\mathcal{E}, R)$, then ι_A is simply the contraction map:

$$(\iota_A \omega)_k(e_1, \cdots, e_{n-1-2k}; f_1, \cdots f_k) \triangleq \omega_k(\hat{A}, e_1, \cdots, e_{n-1-2k}; f_1, \cdots, f_k).$$

From now on we will omit the denotations for isomorphism maps $(\bullet)^\flat$ and $(\bullet)^\sharp$ between $\mathcal{E}$ and $\check{\mathcal{E}}$ if it causes no confusion.

And we further define the Lie derivative of $\mathfrak{g}$ on $C^\bullet_{st}(E)$ as $L_A = d_{st}\circ\iota_A + \iota_A\circ d_{st}$, i.e.

$$\begin{aligned} & L_A\omega \\ = \;& d_{st}(\iota_A\omega) + \iota_A(d_{st}\omega) \\ = \;& -\{H,\{\hat{A},\omega\}\} - \{\hat{A},\{H,\omega\}\} \\ = \;& -(\{\{H,\hat{A}\},\omega\} + (-1)^3\{\hat{A},\{H,\omega\}\}) - \{\hat{A},\{H,\omega\}\} \\ = \;& -\{\{H,\hat{A}\},\omega\}. \end{aligned} \tag{5.1.2}$$

Then we have the following:

Proposition 5.3. *With the interior product ι_A and Lie derivative L_A defined above, $(C^\bullet_{st}(E), d_{st})$ becomes a $\mathfrak{g}$ differential algebra.*

Proof. Since$\{\bullet,\bullet\}$ is a graded Poisson bracket on $C^\bullet_{st}(E)$, ι_A and L_A are both graded derivatives on $C^\bullet_{st}(E)$. So we only need to prove $(C^\bullet_{st}(E), d_{st})$ is a $\mathfrak{g}$ differential complex, i.e. the following three formulas:

1). $[L_A, L_B] = L_{[A,B]}$,

2). $[L_A, \iota_B] = \iota_{[A,B]}$,

3). $[\iota_A, \iota_B] = 0$,

$\forall A, B \in \mathfrak{g}$, where $[\bullet,\bullet]$ on the left hand side is the commutator of operators on $C^\bullet_{st}(E)$.

1).

$$\begin{aligned} & [L_A, L_B]\omega \\ = \;& L_AL_B\omega - L_BL_A\omega \\ = \;& \{\{H,\hat{A}\},\{\{H,\hat{B}\},\omega\}\} - \{\{H,\hat{B}\},\{\{H,\hat{A}\},\omega\}\} \\ = \;& (\{\{\{H,\hat{A}\},\{H,\hat{B}\}\},\omega\} + (-1)^{2\cdot 2}\{\{H,\hat{B}\},\{\{H,\hat{A}\},\omega\}\}) - \{\{H,\hat{B}\},\{\{H,\hat{A}\},\omega\}\} \\ = \;& \{\{\{H,\hat{A}\},H\},\hat{B}\} + (-1)^{2\cdot 3}\{H,\{\{H,\hat{A}\},\hat{B}\}\},\omega\} \\ = \;& 0 + \{H,\hat{A}\circ\hat{B}\},\omega\} \\ = \;& -L_{[A,B]}\omega \end{aligned}$$

2).

$$\begin{aligned}
& [L_A, \iota_B]\omega \\
= & L_A\iota_B\omega - \iota_B L_A\omega \\
= & -\{\{H,\hat{A}\},\{\hat{B},\omega\}\} + \{\hat{B},\{\{H,\hat{A}\},\omega\}\} \\
= & -(\{\{\{H,\hat{A}\},\hat{B}\},\omega\} + (-1)^{2\cdot 1}\{\hat{B},\{\{H,\hat{A}\},\omega\}\}) + \{\hat{B},\{\{H,\hat{A}\},\omega\}\} \\
= & \{\hat{A}\circ\hat{B},\omega\} \\
= & \iota_{[A,B]}\omega
\end{aligned}$$

3).

$$\begin{aligned}
& [\iota_A, \iota_B]\omega \\
= & \iota_A\iota_B\omega + \iota_B\iota_A\omega \\
= & \{\hat{A},\{\hat{B},\omega\}\} + \{\hat{B},\{\hat{A},\omega\}\} \\
= & (\{\{\hat{A},\hat{B}\},\omega\} + (-1)^1\{\hat{B},\{\hat{A},\omega\}\}) + \{\hat{B},\{\hat{A},\omega\}\} \\
= & \{\{\hat{A},\hat{B}\},\omega\} \\
= & 0
\end{aligned}$$

The proof is finished. ■

Let $(C^\bullet_{\mathfrak{g}}(E), d_{\mathfrak{g}})$ be the Cartan model of equivariant cohomology for $(C^\bullet_{st}(E), d_{st})$. $C^m_{\mathfrak{g}}(E)$ is the space of invariant elements in

$$C^m(E,\mathfrak{g}) \triangleq \bigoplus_{i+2j=m} C^i_{st}(E)\otimes S^j(\mathfrak{g}^*).$$

Any $\eta \in C^m(E,\mathfrak{g})$ can be viewed as a sequence $(\eta_0, \eta_1, \cdots, \eta_{[\frac{m}{2}]})$, where η_j is a symmetric multilinear map:

$$\odot^j\mathfrak{g} \to C^{m-2j}_{st}(E).$$

According to Definition 2.39, by direct computation, we have the following:

$$\begin{aligned}
C^m_{\mathfrak{g}}(E) \quad = \quad & \{\eta \in C^m(E,\mathfrak{g}) | \\
& \sum_i L_{A_i}\eta_{k-1}(A_1,\cdots,\hat{A}_i,\cdots,A_k) + \sum_{i<j}\{(\hat{A}_i,\hat{A}_j),\eta_{k-2}(\cdots,\hat{A}_i,\cdots,\hat{A}_j,\cdots)\} = 0, \ \forall k\}
\end{aligned}$$

and

$$(d_{\mathfrak{g}}\eta)_k(A_1, \cdots, A_k) = d_{st}\eta_k(A_1, \cdots, A_k) + \sum_{1\leq i\leq k} \iota_{A_i}\eta_{k-1}(A_1, \cdots, \hat{A}_i, \cdots, A_k).$$

Next, we consider the specific example when the action $\varphi : \mathfrak{g} \to \mathcal{E}$ is the ordinary action sending $A \in \mathfrak{g}$ to the corresponding vector field $\hat{A} \in \Gamma(TM)$. Suppose G is a compact Lie group, and M is a manifold on which G acts freely. It is well-known that in this case the de Rham complex $\Omega^\bullet(M)$ is a $\mathfrak{g}$ differential algebra of type (C), so the corresponding equivariant cohomology is isomorphic to the de Rham cohomology of $N \triangleq M/G$. The following proposition tells that $C^\bullet_{st}(E)$ is also a $\mathfrak{g}$ differential algebra of type (C).

Proposition 5.4. *If G is a compact Lie group acting freely on M, then $(C^\bullet_{st}(E), d_{st})$ is a $\mathfrak{g}$ differential algebra of type (C). Thus by Theorem 2.41, the equivariant cohomology of $C^\bullet_{st}(E)$, i.e. the cohomology of the Cartan model $((C^\bullet_{st}(E) \otimes S^\bullet(\mathfrak{g}^*))^{\mathfrak{g}}, d_{\mathfrak{g}})$, is isomorphic to the cohomology of the basic complex $((C^\bullet_{st}(E))_{bas}, d_{st})$.*

Proof. Suppose $\{A_i\}_{1\leq i\leq l}$ is a basis in $\mathfrak{g}$. Since the G action is free, the corresponding vector fields $\{\hat{A}_i\}$ are independent in $\Gamma(TM)$. Let $\{\theta^i\}_{1\leq i\leq l}$ be their adjoint elements in $\Gamma(T^*M)$. By pseudo-metric of E, $\{\theta^i\}$ can be regarded as standard 1-cochains in $C^\bullet_{st}(E)$ (or $C^\bullet(\mathcal{E}, R)$). In particular, $\theta^i(\hat{A}_j) = \delta^i_j$. Actually such $\{\theta^i\}$s are in $1-1$ correspondence with maps $\phi : \mathfrak{g}^* \to C^1_{st}(E)$ satisfying

$$\iota_A(\phi(\mu)) = \langle A, \mu\rangle, \quad \forall A \in \mathfrak{g},\ \mu \in \mathfrak{g}^*. \tag{5.1.3}$$

The map ϕ corresponding to $\{\theta^i\}$ is defined by:

$$\phi(\mu^i) \triangleq \theta^i, \quad \forall i,$$

where $\{\mu^i\}$ is the adjoint basis in $\mathfrak{g}^*$ of $\{A_i\}$.

In order for $C^\bullet_{st}(E)$ to be of type (C), we only need to verify the existence of $\{\theta^i\}$ which span an invariant space in $C^1_{st}(E)$, or equivalently the existence of a map $\phi : \mathfrak{g}^* \to C^1_{st}(E)$ satisfying Equation 5.1.3 which is equivariant, i.e. $a \circ \phi \circ Ad^*_{a^{-1}} = \phi$, $\forall a \in G$, where $Ad^*_{a^{-1}}$ is the coadjoint representation of G on $\mathfrak{g}^*$.

Given any ϕ satisfying Equation 5.1.3, since G is compact, by averaging $a\circ\phi\circ Ad^*_{a^{-1}}$ over G, i.e. taking the integral

$$\int_G (a \circ \phi \circ Ad^*_{a^{-1}})da$$

with respect to Haar measure, we can obtain a new map $\tilde{\phi}$ which is equivariant and still satisfies Equation 5.1.3 (because any $a \circ \phi \circ Ad^*_{a^{-1}}$ satisfies Equation 5.1.3 too).

Thus the proof is finished. ■

Next we consider a more special case when M is a principal G bundle, and discuss what the basic complex of $C^\bullet_{st}(E)$ is.

Proposition 5.5. *If $M \to N$ is a principal G bundle, and $C^\bullet_{st}(E)$ is viewed as a $\mathfrak{g}$ differential algebra as in Proposition 5.3, then the standard complex of the standard Courant algebroid $TN \oplus T^*N$ is a subcomplex of $(C^\bullet_{st}(E))_{bas}$.*

Proof. Since $M \to N$ is a principal bundle, we can take local trivialization chart $\{x^i\}_{1\le i\le n}$ of M so that $x^1, \cdots, x^l$ are fiber coordinates and $x^{l+1}, \cdots, x^n$ are horizontal coordinates (i.e. $\{x^i\}_{i>l}$ is a local chart of N). $\{\xi_i := \frac{\partial}{\partial x^i}, \xi^i := dx^i\}_{i>l}$ is a local basis of sections of $TN \oplus T^*N$. Denote by $\{p_i\}_{i>l}$ the conjugates of $\{x^i\}_{i>l}$, then $\{x^i, \xi_i, \xi^i, p_i\}_{i>l}$ is a Darboux chart on $TN\oplus T^*N$. By local trivialization, $\{\xi_i, \xi^i, p_i\}_{i>l}$ can all be lifted to corresponding coordinates on M, for which we still use the same denotations, and they further extend to a Darboux chart $\{x^i, \xi_i, \xi^i, p_i\}_{1\le i\le n}$ on $TM \oplus T^*M$. Thus the standard complex of $TN \oplus T^*N$ is embedded in $C^\bullet_{st}(E)$ as a subcomplex (the cubic Hamiltonian of $TN \oplus T^*N$ is $H_N = \sum_{i>l} p_i\xi^i$, the difference $\sum_{1\le i\le l} p_i\xi^i$ of H_N and H acts trivially on $C^\bullet_{st}(TN \oplus T^*N)$). $\forall \omega \in C^\bullet_{st}(TN \oplus T^*N)$, ω doesn't contain any $\xi^j (1 \le j \le l)$, so $\iota_{A_i}\omega = 0$ since $\hat{A}_i$ is a $C^\infty(x^1, \cdots, x^l)$-linear combination of $\{\xi_j\}_{1\le j\le l}$, moreover $L_{A_i}\omega = \iota_{A_i}(d_{st}\omega) = 0$ since $d_{st}\omega$ is still in $C^\bullet_{st}(TN \oplus T^*N)$. Thus any cochain in $C^\bullet_{st}(TN \oplus T^*N)$ is basic in $C^\bullet_{st}(E)$, the proposition is proved. ■

However, a basic cochain in $C^\bullet_{st}(E)$ is not necessarily a cochain in $C^\bullet_{st}(TN\oplus T^*N)$. For example, when G is Abelian, we can take $\{\xi_i\}_{1\le i\le l}$ (with notations in the proof above)to be the corresponding vector fields of a basis of $\mathfrak{g}$, then any $p_i \in C^\bullet_{st}(E)$ $(1 \le i \le l)$ is a basic cochain, but not in $C^\bullet_{st}(TN \oplus T^*N)$. Note that $p_i = \{H, \xi_i\}$ is a coboundary. Actually we observe that for lower degrees $0, 1, 2$, any basic cocycle is cohomologous to some cocycle in $C^\bullet_{st}(TN \oplus T^*N)$. So this leads to the following conjecture:

Conjecture 5.6. *If $M \to N$ is a principal G bundle, the basic complex of $C^\bullet_{st}(E)$ is quasi-isomorphic to the standard complex $C^\bullet_{st}(TN \oplus T^*N)$.*

If this conjecture is true, the equivariant cohomology of $C^\bullet_{st}(E)$ would be isomorphic to the de Rham cohomology of N, hence recovering the classical isomorphism when we view $\Omega^\bullet(M)$ as a $\mathfrak{g}$ differential algebra.

5.2 Equivariant cohomology of Leibniz algebras

Throughout this section, let L be a Leibniz algebra with left center Z, let h be an isotropic Leibniz subalgebra in L (so h is actually a Lie algebra), and let (R, ρ)

be a left representation of L. R becomes an L-module by taking the symmetric extension, i.e. $(R, \rho, -\rho)$ is a representation of L. As in the former chapters, we denote by d_0 the coboundary differential of the corresponding Leibniz cohomology. Let $A^\bullet = \bigoplus_n A^n$ be a graded subspace of the corresponding Leibniz complex with A^n defined as follows:

$$\begin{aligned} A^n &= \{\alpha \in Hom(L^{\otimes n}, R)| \\ &\quad \alpha(e_1, \cdots e_k, e_{k+1,} \cdots e_n) + \alpha(e_1, \cdots e_{k+1}, e_{k,} \cdots e_n) = 0,\ \forall k,\ \forall e_1, \cdots, e_{k+1} \in h\} \end{aligned}$$

The following proposition implies that $(A^\bullet, d_0)$ is a subcomplex.

Proposition 5.7. *$(A^\bullet = \bigoplus_n A^n, d_0)$ is a cochain complex. Furthermore, if R is a commutative algebra on which L acts as derivations, then A becomes an h-differential algebra with interior product defined by*

$$\iota_f \alpha(e_1, \cdots, e_{n-1}) \triangleq \alpha(f, e_1, \cdots, e_{n-1})$$

(when $\alpha \in A^0$, $\iota_f \alpha$ is defined to be 0).

Proof. $\forall k,\ \forall e_1, \cdots, e_{k+1} \in h$, we have the following:

$$\begin{aligned}
& (d_0\alpha)(e_1, \cdots, e_k, e_{k+1,} \cdots, e_{n+1}) + (d_0\alpha)(e_1, \cdots, e_{k+1}, e_{k,} \cdots, e_{n+1}) \\
= & \sum_{a \neq k, k+1} (-1)^{a+1} \rho(e_a) \Big(\alpha(\cdots, \hat{e_a}, \cdots, e_k, e_{k+1}, \cdots) + \alpha(\cdots, \hat{e_a}, \cdots e_{k+1}, e_k, \cdots) \Big) \\
& + ((-1)^{k+1} + (-1)^k) \rho(e_k) \alpha(\cdots, \hat{e_k}, e_{k+1}, \cdots) \\
& + ((-1)^k + (-1)^{k+1}) \rho(e_{k+1}) \alpha(\cdots, e_k, \hat{e_{k+1}}, \cdots) \\
& + \sum_{a < b \neq k, k+1} (-1)^a \Big(\alpha(\cdots, \hat{e_a}, \cdots, e_a \circ e_b, \cdots, e_k, e_{k+1}, \cdots) + \alpha(\cdots, e_{k+1}, e_k, \cdots) \Big) \\
& + \sum_{a < k} (-1)^a \Big(\alpha(\cdots, \hat{e_a}, \cdots e_a \circ e_k, e_{k+1}, \cdots) + \alpha(\cdots, \hat{e_a}, \cdots, e_k, e_a \circ e_{k+1}, \cdots) \Big) \\
& + \sum_{a < k} (-1)^a \Big(\alpha(\cdots, \hat{e_a}, \cdots e_a \circ e_{k+1}, e_k, \cdots) + \alpha(\cdots, \hat{e_a}, \cdots, e_{k+1}, e_a \circ e_k, \cdots) \Big) \\
& + (-1)^k \alpha(\cdots, e_k \circ e_{k+1} + e_{k+1} \circ e_k, \cdots) \\
& + \sum_{b > k+1} ((-1)^k + (-1)^{k+1}) \alpha(\cdots, \hat{e_k}, e_{k+1}, \cdots, e_k \circ e_b, \cdots) \\
& + \sum_{b > k+1} ((-1)^{k+1} + (-1)^k) \alpha(\cdots, e_k, \hat{e_{k+1}}, \cdots, e_{k+1} \circ e_b, \cdots) \\
= & \ 0
\end{aligned}$$

So $A^\bullet$ is closed under d_0.

If R is a commutative algebra, $\forall\alpha \in A^n, \beta \in A^m$, define $\alpha\cdot\beta$ as:

$$\begin{aligned} & (\alpha\cdot\beta)(e_1,\cdots,e_{n+m}) \\ = & \sum_{\sigma\in sh(n,m)} (-1)^\sigma \alpha(e_{\sigma(1)},\cdots,e_{\sigma(n)})\beta(e_{\sigma(n+1)},\cdots,e_{\sigma(n+m)}) \end{aligned}$$

where $sh(n,m)$ means the shuffle permutation.

To see that this map defines a graded commutative algebra structure on $A^\bullet$, we need to check the following:

(1). $\forall\alpha \in A^n, \beta \in A^m$, $\alpha\cdot\beta \in A^{n+m}$.

$\forall k,\ \forall e_1,\cdots,e_{k+1} \in h$,

$$\begin{aligned} & (\alpha\cdot\beta)(\cdots,e_k,e_{k+1},\cdots) + (\alpha\cdot\beta)(\cdots,e_{k+1},e_k,\cdots) \\ = & \sum_{\sigma^{-1}(k),\sigma^{-1}(k+1)<n+1} (-1)^\sigma(\alpha(\cdots,e_k,e_{k+1},\cdots)+\alpha(\cdots,e_{k+1},e_k,\cdots))\beta(\cdots) \\ & + \sum_{\sigma^{-1}(k),\sigma^{-1}(k+1)>n} (-1)^\sigma\alpha(\cdots)(\beta(\cdots,e_k,e_{k+1},\cdots)+\beta(\cdots,e_{k+1},e_k,\cdots)) \\ & + \sum_{\sigma^{-1}(k)<n+1,\sigma^{-1}(k+1)>n} (-1)^\sigma \\ & \quad \Big(\alpha(\cdots,e_k,\cdots)\beta(\cdots,e_{k+1},\cdots)+\alpha(\cdots,e_{k+1},\cdots)\beta(\cdots,e_k,\cdots)\Big) \\ & + \sum_{\sigma^{-1}(k)>n,\sigma^{-1}(k+1)<n+1} (-1)^\sigma \\ & \quad \Big(\alpha(\cdots,e_{k+1},\cdots)\beta(\cdots,e_k,\cdots)+\alpha(\cdots,e_k,\cdots)\beta(\cdots,e_{k+1},\cdots)\Big) \\ = & \ 0. \end{aligned}$$

(2). Associativity.

$$\begin{aligned} & ((\alpha\cdot\beta)\cdot\gamma)(e_1,\cdots,e_{n+m+l}) \\ = & \sum_{\sigma\in sh(n+m,l)} (-1)^\sigma(\alpha\cdot\beta)(e_{\sigma(1)},\cdots,e_{\sigma(n+m)})\gamma(e_{\sigma(n+m+1)},\cdots,e_{\sigma(n+m+l)}) \\ = & \sum_{\sigma\in sh(n+m,l)} (-1)^\sigma\gamma(e_{\sigma(n+m+1)},\cdots,e_{\sigma(n+m+l)}) \cdot \\ & \quad \sum_{\tau\in sh(n,m)} (-1)^\tau\alpha(e_{\sigma(\tau(1))},\cdots,e_{\sigma(\tau(n))})\beta(e_{\sigma(\tau(n+1))},\cdots,e_{\sigma(\tau(n+m))}) \\ & \Big(\textit{let } \mu \textit{ be the permutation} \\ & ((\sigma\cdot\tau)(1),\cdots,(\sigma\cdot\tau)(n+m),\sigma(n+m+1),\cdots\sigma(n+m+l))\Big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mu \in sh(n,m,l)} (-1)^{\mu} \alpha(e_{\mu(1)}, \cdots) \beta(e_{\mu(n+1)}, \cdots) \gamma(e_{\mu(n+m+1)}, \cdots) \\
&= (\alpha \cdot (\beta \cdot \gamma))(e_1, \cdots, e_{n+m+l})
\end{aligned}
$$

(3). Graded commutativity.

$$
\begin{aligned}
& (\alpha \cdot \beta)(e_1, \cdots, e_{n+m}) \\
= & \sum_{\sigma \in sh(n,m)} (-1)^{\sigma} \alpha(e_{\sigma(1)}, \cdots, e_{\sigma(n)}) \beta(e_{\sigma(n+1)}, \cdots, e_{\sigma(n+m)}) \\
& (\textit{let } \sigma' \textit{ be the permutation } (\sigma(n+1), \cdots, \sigma(n+m), \sigma(1), \cdots, \sigma(n))) \\
= & \sum_{\sigma' \in sh(m,n)} (-1)^{nm} (-1)^{\sigma'} \beta(e_{\sigma'(1)}, \cdots, e_{\sigma'(m)}) \alpha(e_{\sigma'(m+1)}, \cdots, e_{\sigma'(n+m)}) \\
= & (-1)^{nm} (\beta \cdot \alpha)(e_1, \cdots, e_{n+m})
\end{aligned}
$$

The interior product ι_f as defined in the proposition is obviously well-defined ($\alpha \in A^n \implies \iota_f \alpha \in A^{n-1}$). Let Lie derivative be $L_f \triangleq \iota_f \circ d + d \circ \iota_f$. To prove that A becomes an h-differential algebra, we have to check the following:

1). ι_f and d_0 are graded derivative,

2). $\iota_f \circ \iota_g + \iota_g \circ \iota_f = 0,\ \forall f, g \in h$,

3). $\iota_{f \circ g} = L_f \iota_g - \iota_g L_f,\ \forall f, g \in h.$

Proof of 1):

$$
\begin{aligned}
& (\iota_f(\alpha \cdot \beta))(e_1, \cdots, e_{n+m-1}) \\
= & (\alpha \cdot \beta)(f, e_1, \cdots, e_{n+m-1}) \\
= & \sum_{\sigma \in sh(n,m), e_{\sigma(1)}=f} (-1)^{\sigma} \alpha(f, e_{\sigma(2)}, \cdots, e_{\sigma(n)}) \beta(e_{\sigma(n+1)}, \cdots, e_{\sigma(n+m)}) \\
& + \sum_{\sigma \in sh(n,m), e_{\sigma(n+1)}=f} (-1)^{\sigma} \alpha(e_{\sigma(1)}, \cdots, e_{\sigma(n)}) \beta(f, e_{\sigma(n+2)}, \cdots, e_{\sigma(n+m)}) \\
& (\textit{letting } \tau \textit{ be the permutation removing } f \textit{ from } \sigma) \\
= & \sum_{\tau \in sh(n-1,m)} (-1)^{\tau} (\iota_f \alpha)(e_{\tau(1)}, \cdots, e_{\tau(n-1)}) \beta(e_{\tau(n)}, \cdots, e_{\tau(n+m-1)}) \\
& + \sum_{\tau \in sh(n,m-1)} (-1)^{\tau+n} \alpha(e_{\tau(1)}, \cdots, e_{\tau(n)}) (\iota_f \beta)(e_{\tau(n+1)}, \cdots, e_{\tau(n+m-1)}) \\
= & \Big((\iota_f \alpha) \cdot \beta + (-1)^n (\alpha \cdot (\iota_f \beta))\Big)(e_1, \cdots, e_{n+m-1})
\end{aligned}
$$

$$
\begin{aligned}
&(d_0(\alpha\cdot\beta))(e_1,\cdots,e_{n+m+1})\\
=\ &\sum_a(-1)^{a+1}\rho(e_a)(\alpha\cdot\beta)(\cdots\hat{e_a}\cdots)+\sum_{a<b}(-1)^a(\alpha\cdot\beta)(\cdots\hat{e_a}\cdots\hat{e_b},e_a\circ e_b\cdots)\\
=\ &\sum_a(-1)^{a+1}\rho(e_a)\Big(\sum_{\sigma\in sh(n,m)\{\cdots,\hat{a},\cdots\}}(-1)^\sigma\alpha(e_{\sigma(1)}\cdots e_{\sigma(n)})\beta(e_{\sigma(n+1)}\cdots e_{\sigma(n+m)})\Big)\\
&+\sum_{a<b}(-1)^a\sum_{\sigma\in sh(n,m)\{\cdots\hat{a},\cdots\},\sigma^{-1}(b)<n+1}(-1)^\sigma\\
&\quad\alpha(e_{\sigma(1)},\cdots,\hat{e_b},e_a\circ e_b,\cdots e_{\sigma(n)})\beta(e_{\sigma(n+1)},\cdots e_{\sigma(n+m+1)})\\
&+\sum_{a<b}(-1)^a\sum_{\sigma\in sh(n,m)\{\cdots\hat{a},\cdots\},\sigma^{-1}(b)>n}(-1)^\sigma\\
&\quad\alpha(e_{\sigma(1)},\cdots e_{\sigma(n)})\beta(e_{\sigma(n+1)},\cdots,\hat{e_b},e_a\circ e_b,\cdots e_{\sigma(n+m+1)})\\
&(\textit{letting } \sigma_1 \textit{ be the permutation adding } a \textit{ to } \sigma \textit{ in front},\\
&\ \sigma_2 \textit{ be the permutation adding } a \textit{ to } \sigma \textit{ at back})\\
=\ &\sum_a\sum_{\sigma_1\in sh(n+1,m)}(-1)^{a+1}(-1)^{\sigma_1+\sigma_1^{-1}(a)-a}\\
&\quad\Big(\rho(e_a)\alpha(e_{\sigma_1(1)}\cdots\hat{e_a},e_{\sigma_1(\sigma_1^{-1}(a)+1)}\cdots e_{\sigma_1(n+1)})\Big)\beta(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)})\\
+\ &\sum_a\sum_{\sigma_2\in sh(n,m+1)}(-1)^{a+1}(-1)^{\sigma_2+\sigma_2^{-1}(a)-a}\\
&\quad\alpha(e_{\sigma_2(1)}\cdots e_{\sigma_2(n)})\Big(\rho(e_a)\beta(e_{\sigma_2(n+1)}\cdots\hat{e_a},e_{\sigma_2(\sigma_2^{-1}(a)+1)}\cdots e_{\sigma_2(n+m+1)})\Big)\\
&+\sum_{a<b}\sum_{\sigma_1\in sh(n+1,m),\ \sigma_1^{-1}(b)<n+2}(-1)^a(-1)^{\sigma_1+\sigma_1^{-1}(a)-a}\\
&\quad\alpha(e_{\sigma_1(1)}\cdots\hat{e_a},e_{\sigma_1(\sigma_1^{-1}(a)+1)}\cdots\hat{e_b},e_a\circ e_b\cdots)\beta(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)})\\
+\ &\sum_{a<b}\sum_{\sigma_2\in sh(n,m+1),\ \sigma_2^{-1}(b)>n+1}(-1)^a(-1)^{\sigma_2+\sigma_2^{-1}(a)-a}\\
&\quad\alpha(e_{\sigma_2(1)}\cdots)\beta(e_{\sigma_2(n+1)}\cdots\hat{e_a},e_{\sigma_2(\sigma_2^{-1}(a)+1)}\cdots\hat{e_b},e_a\circ e_b\cdots e_{\sigma_2(n+m+1)})\\
=\ &\sum_{\sigma_1}(-1)^{\sigma_1}\sum_{a_1:=\sigma_1^{-1}(a)<n+2}(-1)^{a_1+1}\\
&\quad\Big(\rho(e_{\sigma_1(a_1)})\alpha(e_{\sigma_1(1)}\cdots\widehat{e_{\sigma_1(a_1)}}\cdots e_{\sigma_1(n+1)})\Big)\beta(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)})\\
&+\sum_{\sigma_1}(-1)^{\sigma_1}\sum_{a_1:=\sigma_1^{-1}(a)<b_1:=\sigma_1^{-1}(b)<n+2}(-1)^{a_1}\\
&\quad\alpha(e_{\sigma_1(1)}\cdots\widehat{e_{\sigma_1(a_1)}}\cdots\widehat{e_{\sigma_1(b_1)}},e_{\sigma_1(a_1)}\circ e_{\sigma_1(b_1)}\cdots)\beta(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)})
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\sigma_2}(-1)^{\sigma_2+n}\alpha(e_{\sigma_2(1)},\cdots,e_{\sigma_2(n)})\cdot\sum_{a_2:=\sigma_2^{-1}(a)}\\
&\quad(-1)^{a_2-n+1}\rho(e_{\sigma_2(a_2)})\beta(e_{\sigma_2(n+1)},\cdots,\widehat{e_{\sigma_2(a_2)}},\cdots,e_{\sigma_2(n+m+1)})\\
&+\sum_{\sigma_2}(-1)^{\sigma_2+n}\alpha(e_{\sigma_2(1)},\cdots,e_{\sigma_2(n)})\cdot\sum_{n<a_2:=\sigma_2^{-1}(a)<b_2:=\sigma_2^{-1}(b)}\\
&\quad(-1)^{a_2-n}\beta(e_{\sigma_2(n+1)}\cdots\widehat{e_{\sigma_2(a_2)}}\cdots\widehat{e_{\sigma_2(b_2)}},e_{\sigma_2(a_2)}\circ e_{\sigma_2(b_2)}\cdots e_{\sigma_2(n+m+1)})\\
=\ &\Big((d_0\alpha)\cdot\beta+(-1)^n\alpha\cdot(d_0\beta)\Big)(e_1,\cdots,e_{n+m+1})
\end{aligned}
$$

Proof of 2):

$$
\begin{aligned}
&((\iota_f\circ\iota_g+\iota_g\circ\iota_f)\alpha)(e_1,\cdots,e_{n-2})\\
=\ &(\iota_g\alpha)(f,e_1,\cdots,e_{n-2})+(\iota_f\alpha)(g,e_1,\cdots,e_{n-2})\\
=\ &\alpha(g,f,e_1,\cdots,e_{n-2})+\alpha(f,g,e_1,\cdots,e_{n-2})\\
=\ &0
\end{aligned}
$$

Proof of 3):

$$
\begin{aligned}
&((L_f\iota_g-\iota_gL_f)\alpha)(e_1,\cdots,e_{n-1})\\
=\ &((d\circ\iota_g)\alpha)(f,e_1,\cdots,e_{n-1})\\
&+\sum_a(-1)^{a+1}\rho(e_a)((\iota_f\circ\iota_g)\alpha)(e_1,\cdots,\hat{e_a},\cdots e_{n-1})\\
&+\sum_{a<b}(-1)^a((\iota_f\circ\iota_g)\alpha)(e_1,\cdots,\hat{e_a},\cdots,\hat{e_b},e_a\circ e_b,\cdots e_{n-1})\\
&-((\iota_f\circ d+d\circ\iota_f)\alpha)(g,e_1,\cdots,e_{n-1})\\
=\ &\Big(\rho(f)\alpha(g,e_1,\cdots,e_{n-1})\\
&\quad+\sum_a(-1)^a\rho(e_a)\alpha(g,f,e_1,\cdots,\hat{e_a},\cdots,e_{n-1})\\
&\quad+\sum_a(-1)\alpha(g,e_1,\cdots,\hat{e_a},f\circ e_a,\cdots,e_{n-1})\\
&\quad+\sum_{a<b}(-1)^{a+1}\alpha(g,f,\cdots,\hat{e_a},\cdots,\hat{e_b},e_a\circ e_b,\cdots)\Big)\\
&+\sum_a(-1)^{a+1}\rho(e_a)\alpha(g,f,e_1,\cdots,\hat{e_a},\cdots,e_{n-1})\\
&+\sum_{a<b}(-1)^a\alpha(g,f,e_1,\cdots,\hat{e_a},\cdots,\hat{e_b},e_a\circ e_b,\cdots,e_{n-1})\\
&-(d\alpha)(f,g,e_1,\cdots,e_{n-1})\\
&-\Big(\rho(g)\alpha(f,e_1,\cdots,e_{n-1})\\
&\quad+\sum_a(-1)^a\rho(e_a)\alpha(f,g,e_1,\cdots,\hat{e_a},\cdots,e_{n-1})
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\sum_a(-1)\alpha(f,e_1,\cdots,\hat{e_a},g\circ e_a,\cdots,e_{n-1}) \\
&\quad +\sum_{a<b}(-1)^{a+1}\alpha(f,g,e_1,\cdots,\hat{e_a},\cdots,\hat{e_b},e_a\circ e_b,\cdots,e_{n-1})\Big) \\
&= \rho(f)\alpha(g,e_1,\cdots,e_{n-1})-\rho(g)\alpha(f,e_1,\cdots,e_{n-1}) \\
&\quad +\sum_a(-1)^{a+1}\rho(e_a)\alpha(f,g,e_1,\cdots,\hat{e_a},\cdots,e_{n-1}) \\
&\quad +\sum_a(-1)\alpha(g,e_1,\cdots,\hat{e_a},f\circ e_a,\cdots,e_{n-1}) \\
&\quad +\sum_a\alpha(f,e_1,\cdots,\hat{e_a},g\circ e_a,\cdots,e_{n-1}) \\
&\quad +\sum_{a<b}(-1)^{a}\alpha(f,g,e_1,\cdots,\hat{e_a},\cdots,\hat{e_b},e_a\circ e_b,\cdots,e_{n-1}) \\
&\quad -(d\alpha)(f,g,e_1,\cdots,e_{n-1}) \\
&= \alpha(f\circ g,e_1,\cdots,e_{n-1}) \\
&= (\iota_{f\circ g}\alpha)(e_1,\cdots,e_{n-1})
\end{aligned}
$$

Thus the proposition is proved. ■

The L-module structure of Z can be uniquely extended to $S^\bullet(Z)$ by Leibniz rule (the action on $S^0(Z)=\mathbb{R}$ are defined to be 0), so that $R=S^\bullet(Z)$ satisfies the condition in the proposition above.

Suppose $L=h\oplus X$ as vector space, and $\{\xi_i\}$ is a basis of h, denote by $\{\mu^i\}$ the dual basis of $h^\star$. If R is a unital algebra, any μ^i can be extended to a $\theta^i\in A^1$: $\theta^i(\xi_j)=(\mu^i,\xi_j)=\delta^i_j$, $\theta_i(x)=0$, $\forall x\in X$. In particular, when $h=Z$ and $R=S^\bullet(Z)$, it's easily seen that $L_{\xi_i}\theta^j=0$, so $A^\bullet$ is of type (C) in this case. Actually we have the following:

Proposition 5.8. *Suppose R is a unital commutative algebra on which L acts as derivations and L has a decomposition $h\oplus X$ of vector spaces such that $h\circ X\subseteq X$, then $A^\bullet$ is of type (C).*

Proof. It is easily seen from the assumption that the action of L on the unit of R is 0.

In order for $A^\bullet$ to be of type (C), we only need to prove $L_{\xi_i}\theta^j=-C^j_{ik}\theta^k$, where C^j_{ik} is the structure constant of h.

The proof is done in two cases:

$\forall \xi_l \in h$,

$$\begin{aligned}
& (L_{\xi_i}\theta^j)(\xi_l) \\
= \ & ((d \circ \iota_{\xi_i})\theta^j)(\xi_l) + ((\iota_{\xi_i} \circ d)\theta^j)(\xi_l) \\
= \ & 0 + (d\theta^j)(\xi_i, \xi_l) \\
= \ & \rho(\xi_i)\theta^j(\xi_l) - \rho(\xi_l)\theta^j(\xi_i) - \theta^j(\xi_i \circ \xi_l) \\
= \ & 0 - 0 - \theta^j(C_{il}^k \xi_k) \\
= \ & -C_{il}^j \\
= \ & (-C_{ik}^j \theta^k)(\xi_l).
\end{aligned}$$

$\forall x \in X$,

$$\begin{aligned}
& (L_{\xi_i}\theta^j)(x) \\
= \ & ((d \circ \iota_{\xi_i})\theta^j)(x) + ((\iota_{\xi_i} \circ d)\theta^j)(x) \\
= \ & 0 + (d\theta^j)(\xi_i, x) \\
= \ & \rho(\xi_i)\theta^j(x) - \rho(x)\theta^j(\xi_i) - \theta^j(\xi_i \circ x) \\
= \ & -\theta^j(\xi_i \circ x) \\
= \ & 0 \\
= \ & (-C_{ik}^j \theta^k)(x)
\end{aligned}$$

($\theta^j(\xi_i \circ x) = 0$ since $\xi_i \circ x \in X$.)

The proof is finished. ■

Since $A^\bullet$ is an h differential algebra, we can consider the complex of Cartan model $C_h^\bullet(A) = (A^\bullet \otimes S^\bullet(h^*))^h$ of the equivariant cohomology of $A^\bullet$. $C_h^\bullet(A)$ is a graded subspace of $C^\bullet(L)$. $\forall \omega \in C_h^n(A)$, $\exists \alpha_{n-2k}^{(i)} \in A^{n-2k}$, $\eta_k^{(i)} \in S^k(h^\star)$, such that $\omega_k = \sum_i \alpha_{n-2k}^{(i)} \otimes \eta_k^{(i)}$. $C_h^\bullet(A)$ is endowed with the equivariant coboundary differential

$$d = d_0 \otimes id - \iota_{\xi_i} \otimes \mu^i.$$

We will simply write $d_0 \otimes id$ as d_0 from now on.

$\forall \omega \in C_h^n(A)$, suppose $\omega_k = \alpha_{n-2k} \otimes \eta_k$(here we only consider monomials for simplicity, the result is obviously true for polynomials as well),

$$\begin{aligned}
& ((\iota_{\xi_i} \otimes \mu^i)\omega_k)(e_1, \cdots, e_{n-1-2k}; f_1, \cdots, f_{k+1}) \\
= & \sum_i (\iota_{\xi_i} \alpha_{n-2k})(e_1, \cdots, e_{n-1-2k})(\mu^i \cdot \eta_k)(f_1, \cdots, f_{k+1}) \\
= & \sum_i \alpha_{n-2k}(\xi_i, e_1, \cdots, e_{n-1-2k}) \cdot \sum_j \mu^i(f_j)\eta_k(\cdots, \hat{f}_j, \cdots) \\
= & \sum_{i,j} \alpha_{n-2k}(\mu^i(f_j)\xi_i, e_1, \cdots, e_{n-1-2k})\eta_k(\cdots, \hat{f}_j, \cdots) \\
= & \sum_j \alpha_{n-2k}(f_j, e_1, \cdots, e_{n-1-2k})\eta_k(\cdots, \hat{f}_j, \cdots) \\
= & \sum_j \omega_k(f_j, e_1, \cdots, e_{n-1-2k}; \cdots, \hat{f}_j, \cdots)
\end{aligned}$$

We see that $\iota_{\xi_i} \otimes \mu^i$ is exactly the operator δ as defined in the third chapter:

$$\begin{aligned}
& (\delta\omega)_k(e_1, \cdots, e_{n+1-2k}; f_1, \cdots, f_k) \\
= & \sum_{1 \le j \le k} \omega_{k-1}(f_j, e_1, \cdots, e_{n+1-2k}; f_1, \cdots, \hat{f}_j, \cdots f_k) \\
& \forall n, \ \forall \omega \in C^n(L), \ \forall k \le [\frac{n+1}{2}]
\end{aligned}$$

So the equivariant coboundary differential $d = d_0 - \delta$. $(d\omega)_k = (d_0\omega)_k - (\delta\omega)_k = d_0\omega_k - \delta\omega_{k-1}$.

Proposition 5.9.

$$\begin{aligned}
& (d^2\omega)_k = 0 \\
\Leftrightarrow & \sum_{a,i} \omega_{k-1}(e_1, \cdots, \hat{e_a}, f_i \circ e_a, \cdots; f_1, \cdots \hat{f}_i, \cdots) = 0
\end{aligned}$$

.

Proof. $d_0^2 = 0$ since d_0 is induced by the coboundary differential of Leibniz cohomology.

$\delta^2 = 0$ as proved in theorem 3.1.

$$\begin{aligned}
& ((d_0 \circ \delta + \delta \circ d_0)\omega)_k(e_1, \cdots e_{n+2-2k}; f_1, \cdots f_k) \\
= & \sum_a (-1)^{a+1} \rho(e_a)(\delta\omega)_k(e_1, \cdots, \hat{e_a}, \cdots e_{n+2-2k}; f_1, \cdots f_k) \\
& + \sum_{a<b} (-1)^a (\delta\omega)_k(e_1, \cdots, \hat{e_a}, \cdots e_a \circ e_b, \cdots e_{n+2-2k}; f_1, \cdots f_k) \\
& + \sum_i (d_0\omega)_{k-1}(f_i, e_1, \cdots e_{n+2-2k}; f_1, \cdots, \hat{f_i}, \cdots f_k) \\
= & \sum_{a,i} (-1)^{a+1} \rho(e_a)\omega_{k-1}(f_i, \cdots \hat{e_a}, \cdots; \cdots \hat{f_i}, \cdots) \\
& + \sum_{i,a<b} (-1)^a \omega_{k-1}(f_i, \cdots, \hat{e_a}, \cdots e_a \circ e_b, \cdots; \cdots \hat{f_i}, \cdots) \\
& + \sum_{i,a} (-1)^a \rho(e_a)\omega_{k-1}(f_i, \cdots, \hat{e_a}, \cdots; \cdots, \hat{f_i}, \cdots) \\
& + \sum_{i,a} (-1)\omega_{k-1}(\cdots, f_i \circ e_a, \cdots; \cdots, \hat{f_i}, \cdots) \\
& + \sum_{i,a<b} (-1)^{a+1} \omega_{k-1}(f_i, \cdots, \hat{e_a}, \cdots e_a \circ e_b, \cdots; \cdots, \hat{f_i}, \cdots) \\
= & -\sum_{i,a} \omega_{k-1}(e_1, \cdots, \hat{e_a}, f_i \circ e_a, \cdots; f_1, \cdots \hat{f_i}, \cdots)
\end{aligned}$$

So

$$\begin{aligned}
& (d^2\omega)_k(e_1, \cdots e_{n+2-2k}; f_1, \cdots f_k) \\
= & ((d_0^2 + \delta^2 + d_0 \circ \delta + \delta \circ d_0)\omega)_k(e_1, \cdots e_{n+2-2k}; f_1, \cdots f_k) \\
= & -\sum_{i,a} \omega_{k-1}(e_1, \cdots, \hat{e_a}, f_i \circ e_a, \cdots; f_1, \cdots \hat{f_i}, \cdots)
\end{aligned}$$

The proposition is proved. ■

So the complex of Cartan model is

$$\begin{aligned}
C_h^n(A) = & \{\omega \in \bigoplus_{p+2q=n} A^p \otimes S^q(h^*) \subset C^n(L) | \\
& \sum_{a,i} \omega_k(e_1, \cdots, \hat{e_a}, f_i \circ e_a, \cdots; f_1, \cdots \hat{f_i}, \cdots) = 0, \ \forall e \in L, \ f \in h\},
\end{aligned}$$

with coboundary differential $d_0 - \delta$.

It is obvious that $C_h^\bullet(A)$ also becomes a cochain complex under coboundary differential $d = d_0 + \delta$.

Actually we have the following:

Proposition 5.10. *$C_h^\bullet(A)$ becomes a cochain complex under coboundary differential $d_c = d_0 + c\delta$, where c is any nonzero constant. And the cohomology $H^\bullet(C_h^\bullet(A),\ d_c)$ doesn't depend on c.*

Proof. $d_c^2 = d_0^2 + c^2\delta^2 + c(d_0 \circ \delta + \delta \circ d_0) = 0$, so $(C_h^\bullet(A), d_c)$ is a cochain complex.

We only need to prove that $H^\bullet(C_h^\bullet(A),\ d_c) \cong H^\bullet(C_h^\bullet(A),\ d_1 = d_0 + \delta),\ \forall c$. In fact there exists an isomorphism ϕ of cochain complexes as follows:

$$\begin{array}{rcl} (C_h^\bullet(A),\ d_c) & \longrightarrow & (C_h^\bullet(A),\ d_1) \\ \lambda & \mapsto & \phi\lambda \\ (\lambda_0, \cdots, \lambda_k, \cdots) & \mapsto & (\frac{1}{c^0}\lambda_0, \cdots, \frac{1}{c^k}\lambda_k, \cdots) \end{array}$$

Since $(\phi(d_c\lambda))_k = \frac{1}{c^k}(d_c\lambda)_k = d_0(\frac{1}{c^k}\lambda_k) + \frac{1}{c^{k-1}}(\delta\lambda_{k-1}) = (d_1(\phi\lambda))_k$, ϕ is a cochain map.

ϕ is obviously a bijective map, thus the proposition is proved. ■

In particular, the cochain complex $(C_h^\bullet(A), d_1 = d_0 + \delta)$ is isomorphic to the complex of the Cartan model $(C_h^\bullet(A), d_{-1} = d_0 - \delta)$ for equivariant cohomology of A. This leads to the following definition:

Definition 5.11. The cochain complex $(C_h^\bullet(A),\ d = d_0 + \delta)$, denoted by $C_{eq}^\bullet(L, h, R)$, is called the equivariant (cochain) complex of L with respect to h and R. The resulting cohomology, denoted by $H_{eq}^\bullet(L, h, R)$, is called the equivariant cohomology of L with respect to h and R.

We will write $C_{eq}^\bullet(L, h, R)$ and $H_{eq}^\bullet(L, h, R)$ simply as $C_{eq}^\bullet(L)$ and $H_{eq}^\bullet(L)$ if it causes no confusion.

When (L, h, R) satisfy the condition in Proposition 5.8, applying Theorem 2.41, we have the following:

Proposition 5.12. *Suppose R is a unital algebra on which L acts as derivations, and L has a decomposition $h \oplus X$ of vector spaces such that $h \circ X \subseteq X$, then*

$$H_{eq}^\bullet(L, h, R) \cong H^\bullet(A_{bas}^\bullet).$$

In the following ,we consider a specific example: L is the omni Lie algebra $ol(V) = gl(V) \oplus V$, h is the isotropic Leibniz subalgebra of $ol(V)$ corresponding to a Lie bracket on V (i.e. h is the graph of $\sigma : V \to gl(V)$, where $\sigma(v_1)(v_2) \triangleq [v_1, v_2]_V$, see Theorem 2.6), and R is $S^\bullet(V)$ with L action ρ extended from the action on V by Leibniz rule.

Since h is the graph of $\sigma : V \to gl(V)$, $ol(V)$ can be decomposed as $h \oplus gl(V)$ as vector space. The Leibniz bracket of h and $gl(V)$ is obviously in $gl(V)$. So $(ol(V), h, S^\bullet(V))$ satisfy the condition in Proposition 5.8 and 5.12, we have the isomorphism

$$H^\bullet_{eq}(ol(V), h, S^\bullet(V)) \cong H^\bullet(A^\bullet_{bas}).$$

Let's compute the cohomology in degree 0 and 1:
Degree 0:
$A^0 = S^\bullet(V)$, and

$$\begin{aligned} A^0_{bas} &= \{\alpha \in S^\bullet(V) | \iota_f\alpha = 0, L_f\alpha = 0,\ \forall f \in h\} \\ &= \{\alpha \in S^\bullet(V) | \rho(f)\alpha = 0,\ \forall f \in h\} \\ &= \{\alpha \in S^\bullet(V) | [v,\alpha]_{S^\bullet(V)} = 0,\ \forall v \in V\}, \end{aligned}$$

where $[\bullet,\bullet]_{S^\bullet(V)}$ is the extended bracket of $[\bullet,\bullet]_V$ on $S^\bullet(V)$ by Leibniz rule.

The set of 0-coboundaries is null. And $\alpha \in A^0$ is a basic 0-cocycle iff $\rho(A)\alpha = 0,\ \forall A \in gl(V)$, it is easily seen that the only basic 0-cocycles are $S^0(V) = \mathbb{R}$. So

$$H^0_{eq}(ol(V), h, S^\bullet(V) \cong H^0(A^\bullet_{bas}) = \mathbb{R}. \tag{5.2.1}$$

Degree 1:
$A^1 = Hom(ol(V), S^\bullet(V))$, and

$$\begin{aligned} A^1_{bas} &= \{\alpha \in Hom(ol(V), S^\bullet(V)) | \iota_f\alpha = 0, L_f\alpha = 0,\ \forall f \in h \\ &= \{\alpha \in Hom(ol(V), S^\bullet(V)) | \alpha(f) = 0, \rho(f)\alpha(e) = \alpha(f \circ e),\ \forall f \in h, e \in ol(V)\} \\ &= \{\alpha \in Hom(ol(V), S^\bullet(V)) | \alpha(f) = 0, [v, \alpha(A)]_{S^\bullet(V)} = \alpha([\sigma(v), A]),\ \forall v \in V, A \in gl(V)\}. \end{aligned}$$

The set of basic 1-coboundaries is

$$\begin{aligned} &\{\alpha \in A^1_{bas} | \exists \beta \in A^0_{bas},\ \alpha = d\beta\} \\ = &\{\alpha \in Hom(ol(V), S^\bullet(V)) | \exists \beta \in A^0_{bas},\ \alpha(A + f) = \rho(A)\beta,\ \forall A \in gl(V),\ f \in h\}. \end{aligned}$$

The set of basic 1-cocycles is

$$\begin{aligned} &\{\alpha \in A^1_{bas} | d\alpha = 0\} \\ = &\{\alpha \in Hom(ol(V), S^\bullet(V)) | \iota_f\alpha = 0, d\alpha = 0,\ \forall f \in h\} \\ = &\{\alpha \in Hom(ol(V), S^\bullet(V)) | \alpha(f) = 0, [v, \alpha(A)]_{S^\bullet(V)} = \alpha([\sigma(v), A]), \\ &\quad \rho(A_1)\alpha(A_2) - \rho(A_2)\alpha(A_1) - \alpha([A_1, A_2] + A_1 v) = 0,\ \forall v \in V, A, A_1, A_2 \in gl(V)\}. \end{aligned}$$

And $H^1_{eq}(ol(V), h, S^\bullet(V)) \cong H^1(A^\bullet_{bas})$ is their quotient.

It is easily seen that , when the Lie bracket $[\bullet,\bullet]_V$ is 0 (i.e. h is V), basic 1-coboundaries and basic 1-cocycles are exactly 1-coboundaries and 1-cocycles in the Chevalley-Eilenberg complex of the Lie algebra $gl(V)$ with respect to its representation $S^\bullet(V)$. We know that the cohomology of the Lie algebra $gl(V)$ with respect to the representation $S^\bullet(V)$ is trivial, so

$$H^1_{eq}(ol(V), V, S^\bullet(V)) \cong H^1(A^1_{bas}) = 0.$$

Actually, we can prove that the equivariant cohomology in this case is 0 for higher degrees also:

Proposition 5.13. *When $h = V$, the equivariant complex $C^\bullet_{eq}(ol(V), V, S^\bullet(V))$ is acyclic.*

Proof. From the discussion above 5.2.1, we see that $H^0_{eq}(ol(V), V, S^\bullet(V)) = \mathbb{R}$.

So we need to prove that $H^n_{eq}(ol(V), V, S^\bullet(V)) = 0,\ \forall n \geq 1$, or equivalently, $H^n(A^\bullet_{bas}) = 0,\ \forall n \geq 1$.

Suppose $\alpha \in A^n_{bas}$ is a cocycle, it suffices to prove α is a coboundary.

Let $I \in gl(V)$ be the identity map, then $\forall e_1, \cdots, e_n \in ol(V)$,

$$\begin{aligned}
& (d\alpha)(e_1, \cdots, e_n, I) \\
= & \sum_a (-1)^{a+1}\rho(e_a)\alpha(e_1, \cdots, \hat{e_a}, \cdots, e_n, I) \\
& +(-1)^n \rho(I)\alpha(e_1, \cdots, e_n) \\
& +\sum_{a<b}(-1)^a \alpha(e_1, \cdots, \hat{e_a}, \cdots, \hat{e_b}, e_a \circ e_b, \cdots, e_n, I) \\
& +\sum_a (-1)^a \alpha(e_1, \cdots, \hat{e_a}, \cdots, e_n, e_a \circ I) \\
= & \sum_a (-1)^{a+1}\rho(e_a)\alpha(e_1, \cdots, \hat{e_a}, \cdots, e_n, I) \\
& +\sum_{a<b}(-1)^a \alpha(e_1, \cdots, \hat{e_a}, \cdots, \hat{e_b}, e_a \circ e_b, \cdots, e_n, I) \\
& +(-1)^n \alpha(e_1, \cdots, e_n).
\end{aligned}$$

Since α is a cocycle, it follows that:

$$\begin{aligned}
& \alpha(e_1, \cdots, e_n) \\
= & (-1)^{n+1}\Big(\sum_a (-1)^{a+1}\rho(e_a)\alpha(e_1, \cdots, \hat{e_a}, \cdots, e_n, I) \\
& +\sum_{a<b}(-1)^a \alpha(e_1, \cdots, \hat{e_a}, \cdots, \hat{e_b}, e_a \circ e_b, \cdots, e_n, I)\Big).
\end{aligned}$$

Let $\beta: \otimes^{n-1} ol(V) \to S^\bullet(V)$ be the map defined by

$$\beta(e_1, \cdots, e_{n-1}) \triangleq \alpha(e_1, \cdots, e_{n-1}, I).$$

Obviously $\beta \in A_{bas}^{n-1}$, and we have:

$$\begin{aligned}
& \alpha(e_1, \cdots, e_n) \\
= \ & (-1)^{n+1}\Big(\sum_a (-1)^{a+1}\rho(e_a)\alpha(e_1, \cdots, \hat{e_a}, \cdots, e_n, I) \\
& +\sum_{a<b}(-1)^a \alpha(e_1, \cdots, \hat{e_a}, \cdots, \hat{e_b}, e_a \circ e_b, \cdots, e_n, I)\Big) \\
= \ & (-1)^{n+1}\Big(\sum_a (-1)^{a+1}\rho(e_a)\beta(e_1, \cdots, \hat{e_a}, \cdots, e_n) \\
& +\sum_{a<b}(-1)^a \beta(e_1, \cdots, \hat{e_a}, \cdots, \hat{e_b}, e_a \circ e_b, \cdots, e_n)\Big) \\
= \ & (-1)^{n+1}(d\beta)(e_1, \cdots, e_n).
\end{aligned}$$

Thus $\alpha = d((-1)^{n+1}\beta)$ is a coboundary, the proposition is proved. ■

Appendix A

Appendix

A.1 NQ manifold

In this section we list some basic knowledge concerning graded manifolds, for more details we refer to [72, 73, 66, 62].

Definition A.1. A supermanifold of dimension $p|q$ is a locally ringed space $M = (M_0, \mathcal{O}_M)$, where M_0 is a smooth manifold of dimension p and $\mathcal{O}_M$ is a sheaf of $\mathbb{Z}_2$ graded $C^\infty(M_0)$-algebras on M_0 whose stalk $\mathcal{O}_x$ over each point $x \in M_0$ is local, together with a countable system of compatible charts covering M called atlas, where a chart on M means an isomorphism of locally ringed spaces

$$\phi: \ V = (V_0, \mathcal{O}_M|_{V_0}) \to U^{p|q} = (U_0, C^\infty(U_0) \otimes \wedge^\bullet(\mathbb{R}^q)$$

(V_0 is an open subset of M_0, U_0 is an open subset of $\mathbb{R}^p$).

$U^{p|q}$ is called a superdomain. $\mathcal{O}_M$ is called the sheaf of super functions on M.

Given any vector bundle $E \to M$, ΠE is a supermanifold with structure sheaf $\mathcal{O}_{\Pi E} = \Gamma_M(\wedge^\bullet E^*)$. Locally $\mathcal{O}_{\Pi E}$ is generated by $C^\infty(V_0)$ (V_0 is an open subset of M) and a local frame $\xi^1, \cdots, \xi^n$ of E^* (n is the rank of E). Π here means the parity reversal functor, which converts the fiber coordinates ξ^i into odd coordinates. ΠE is called an odd vector bundle. The fundamental classification theorem of smooth real supermanifolds asserts that any supermanifold M can be realized as an odd vector bundle.

Vector bundles on supermanifolds can be defined analogously to vector bundles on ordinary manifolds, using super vector spaces instead of ordinary vector spaces. In particular, the tangent bundle of a supermanifold is the space of graded derivations of the structure sheaf.

A graded manifold is a supermanifold with an additional grading:

Definition A.2. Let $M = (M_0, \mathcal{O}_M)$ be a fixed supermanifold with a fixed even vector field ϵ.

1). A chart (V_0, ϕ) of M is called $\mathbb{Z}$-graded iff its coordinates are eigenfunctions of ϵ with integer eigenvalues.

The structure sheaf of $\mathbb{Z}$-graded functions over this chart is the $\mathbb{Z}$-graded algebra generated by these coordinates, i.e. smooth functions in the coordinates of degree (or weight) 0, the free exterior algebra in the odd coordinates, and the free algebra of symmetric polynomials in the even coordinates not of degree 0, with the $\mathbb{Z}$-grading given by (eigenvalues of) ϵ. We say that a function $f \in \mathcal{O}_M$ is homogeneous of degree k if $\epsilon \cdot f = kf$, and denote by $|f| = k$.

2). A $\mathbb{Z}$-graded atlas of M is an open cover with $\mathbb{Z}$-graded charts such that the transition functions between them preserve the $\mathbb{Z}$-grading and are constituted of $\mathbb{Z}$-graded functions. Especially the number of coordinates in each degree is the same on all charts.

3). An (integer) graded manifold M is a supermanifold M together with a fixed vector field ϵ, called the Euler vector field, and a (maximal) $\mathbb{Z}$-graded atlas.

A graded manifold is said to be non-negatively graded if all coordinates are of non-negative integer degree.

Definition A.3. An N-manifold is a non-negatively graded supermanifold M, whose integer grading is compatible with parity (the underlying $\mathbb{Z}_2$-grading in the structure sheaf). In other words, even (parity) coordinates must have even weights and odd (parity) coordinates must have odd weights.

Vector bundles on graded manifolds (and N-manifolds) can be defined obviously. If M is a graded manifold and $E \to M$ is a graded vector bundle, we denote by $E[n]$ the graded vector bundle with fiber degrees shifted by n. For the case when M and E are N-manifolds, we need to shift the parity in the fibers as well so that $E[n]$ is still an N-manifold. Thus in this sense, $\Pi E = E[1]$.

Given N an N-manifold, TN and T^*N are both graded vector bundles, with Euler vector field acting as Lie derivative. The fiber coordinates of TN have the same degrees as the corresponding coordinates of N, while the fiber coordinates of T^*N have the opposite degrees.

Suppose $E \to M$ is an ordinary vector bundle, E could be viewed as a graded vector bundle over graded manifold M (with all coordinate weights equaling 0). So $E[1]$ is the shifted graded vector bundle with coordinates (x^i, ξ^a) of weights $|x^i| = 0,\ |\xi^a| = 1$, and $T^*E[1]$ is a graded vector bundle (over $E[1]$) with coordinates $(x^i, \xi^a, p_i, \theta_a)$ of weights $|x^i| = 0,\ |\xi^a| = 1,\ |p_i| = 0,\ |\theta_a| = -1$ (p_i and θ_a are conjugates of x^i and ξ^a respectively). Then shifted again, $T^*[2]E[1]$ is a graded vector bundle with coordinates$(x^i, \xi^a, p_i, \theta_a)$ of weights $|x^i| = 0,\ |\xi^a| = 1,\ |p_i| = 2,\ |\theta_a| = 1$.

Definition A.4. An NQ-manifold is an N-manifold endowed with an integrable (homological) vector field Q of weight $+1$, that is, $[\epsilon, Q] = Q$ and $[Q, Q] = 2Q^2 = 0$.

By definition, for an NQ-manifold N, all polynomial functions on N form a cochain complex under Q.

Analogously to the case of smooth manifolds, for a graded manifold M, there is a de Rham differential d_M (of degree 1) on $\Omega^\bullet(M) \triangleq \Gamma_M(S^\bullet(T^*[1]M))$, and the Schouten bracket $[\cdot,\cdot]_M$ (of degree -1) can be defined on $\Gamma_M(S^\bullet(T[-1]M))$.

Definition A.5. A graded Poisson manifold of degree m, $(M, \{\cdot,\cdot\})$ is a graded manifold M together with a bracket of degree m on its structure sheaf $\mathcal{O}_M$ satisfying, for all homogeneous functions $f, g, h \in \mathcal{O}_M$,

$$\begin{aligned}
\{f, g\} &= (-1)^{(|f|+m)(|g|+m)+1}\{f, g\} \\
\{f, gh\} &= \{f, g\}h + (-1)^{(|f|+m)|g|}\{f, h\} \\
\{f, \{g, h\}\} &= \{\{f, g\}, h\} + (-1)^{(|f|+m)(|g|+m)}\{g, \{f, h\}\}.
\end{aligned}$$

Or equivalently, a graded Poisson manifold of degree m is a graded manifold M with a homogeneous Poisson bivector $\Pi \in \Gamma_M(S^2(T[-1]M))$ of weight m, satisfying $[\Pi, \Pi]_M = 0$.

A graded pre-symplectic manifold (M, ω) of degree m is a graded manifold with a homogeneous d_M-closed 2-form ω of weight m.

A graded symplectic manifold (M, ω) of degree m is a graded pre-symplectic manifold with the additional condition that ω is non-degenerate. It is in particular a graded Poisson manifold of degree $-m$.

When we consider a symplectic NQ-manifold (M, ω), we have the additional requirement that $L_Q\omega = 0$.

A.2 Standard cohomology of Courant algebroid

In this section, we define standard cohomology of Courant algebroid in the language of NQ manifold.

In [62], Roytenberg proved the following:

Theorem A.6. *Symplectic NQ-manifolds of degree 2 are in 1-1 correspondence with Courant algebroids.*

Given a Courant algebroid $E \to M$, the corresponding symplectic NQ-manifold of degree 2 is constructed as follows (see [62] for more details):

As explained previously, $T^*[2]E[1]$ is a graded vector bundle with coordinates$(x^i, \xi^a, p_i, \theta_a)$ of weights $|x^i| = 0,\ |\xi^a| = 1,\ |p_i| = 2,\ |\theta_a| = 1$. Actually it becomes a graded symplectic manifold of degree 2 with symplectic form

$$\omega = dp_i dx^i + d\xi^a d\theta_a,$$

and it is a minimal symplectic realization of $(E \oplus E^*)[1]$ (whose graded Poisson structure is induced by the pseudo-metric on E). Denote by ι the isometric embedding

$$\begin{array}{rcl} E[1] & \overset{\iota}{\hookrightarrow} & (E \oplus E^*)[1] \\ \iota(e) & \triangleq & (e, \frac{1}{2}e^\flat). \end{array}$$

Then the required symplectic NQ-manifold is just the minimal symplectic realization $\mathbf{E}$ of $E[1]$ obtained by the pullback of $T^*[2]E[1] \to (E \oplus E^*)[1]$ along ι, i.e.

$$\begin{array}{ccc} \mathbf{E} & \longrightarrow & T^*[2]E[1] \\ \downarrow & & \downarrow \\ E[1] & \overset{\iota}{\longrightarrow} & (E \oplus E^*)[1]. \end{array}$$

A choice of local chart $\{x^i\}$ on M and a local basis $\{e_a\}$ of sections of E such that $g_{ab} \triangleq (e_a, e_b)$ ($\{\xi^a\}$ is the dual basis of $\{e_a\}$, thus $e_a^\flat = g_{ab}\xi^b$) are constants gives rise to an affine Darboux chart (x^i, ξ^a, p_i) on $\mathbf{E}$: the embedding of $\mathbf{E}$ into $T^*[2]E[1]$ is locally given by equations

$$\theta_a = \frac{1}{2} g_{ab}\xi^b.$$

So the symplectic form of $\mathbf{E}$ is

$$\Omega = dp_i dx^i + \frac{1}{2} d\xi^a g_{ab} d\xi^b.$$

The Q-structure on $\mathbf{E}$ is determined by a cubic Hamiltonian H, in an affine Darboux chart (x^i, ξ^a, p_i), H is of the form:

$$H = A_a^i(x) p_i \xi^a + \frac{1}{6} C_{abc}(x) \xi^a \xi^b \xi^c,$$

where $A_a^i(x) = \rho(e_a) \cdot x^i$ and $C_{abc}(x) = (e_a \circ e_b, e_c)$.

By direct computation, $\{H, H\} = 0$ due to the properties of Courant algebroid E. Furthermore, H satisfies the following properties:

Theorem A.7. *[62] With the notations above, we have*
1). $(e \circ e')^\flat = \{\{H, e^\flat\}, e'^\flat\}$,
2). $\rho(e) \cdot f = \{\{H, e^\flat\}, f\}$.

Since $(\bullet)^\flat$ is an isomorphism, the first equation above implies that the Dorfman bracket is a derived bracket.

Denote by A^n the space of homogeneous super functions of degree n on $\mathbf{E}$, the following definition is given by Roytenberg [62]:

Definition A.8. With the above notations, $(A^\bullet, Q = \{H, \cdot\})$ becomes a cochain complex, called the standard cochain complex of E. The corresponding cohomology is called the standard cohomology of the Courant algebroid E, and denoted by $H^\bullet_{st}(E)$.

As explained by Roytenberg [62], the standard cohomology $H^\bullet_{st}(E)$ in lower degrees have familiar structural interpretations:

"$H^0_{st}(E)$ is the space of smooth functions on M that are constant along the leaves of the anchor foliation; it is equal to $\mathbb{R}$ for transitive Courant algebroids. $H^1_{st}(E)$ is the space of sections of E acting trivially on E, modulo those of the form ∂f for some function f. Further, $H^2_{st}(E)$ is the space of linear vector fields on E preserving the Courant algebroid structure modulo those generated by sections of E as $e \circ \cdot$. $H^3_{st}(E)$ is the space of infinitesimal deformations of the Courant algebroid structure, modulo the trivial ones generated by $\Gamma(\mathbb{A})$ ($\mathbb{A}$ is the gauge Lie algebroid in the Atiyah sequence of the pseudo-Euclidean vector bundle E), while $H^4_{st}(E)$ houses the obstructions to extending an infinitesimal deformation to a formal one."

Example A.9. 1). When $E = TM \oplus T^*M$ is the standard Courant algebroid, we can take $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, dx^1, \cdots, dx^n\}$ as a local basis of sections of E. Then the cubic Hamiltonian H is quite simple:

$$H = \sum_{i=1}^{n} p_i x^i.$$

2). When E is an exact Courant algebroid with Severa class H, we still take $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, dx^1, \cdots, dx^n\}$ as a local basis of sections of E. The cubic Hamiltonian is of the form:

$$H = \sum_{i=1}^{n} p_i x^i + \sum_{a,b,c=1}^{n} \frac{1}{6} H_{abc}(x) \xi^a \xi^b \xi^c,$$

where $H_{abc}(x) = H(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^c})$, and $\{\xi^a\}$ is the dual of $\{\frac{\partial}{\partial x^a}\}$ in $\Gamma(E^*)$, if we identify the sections of E and E^*, then we can identify ξ^a with dx^a.

3). When E is a regular Courant algebroid, take a standard dissection $\Psi : F^* \oplus \mathcal{G} \oplus F \to E$. Suppose $\{dx^I\}_{1\le I\le m}$, $\{r_A\}_{1\le A\le l}$ and $\{\frac{\partial}{\partial x^I}\}_{1\le I\le m}$ are local bases of sections of F^*, $\mathcal{G}$ and F respectively ($\{r_A\}$ is a pseudo orthonormal basis of $\Gamma(\mathcal{G})$). Their images under the isomorphism map Ψ, $\{\xi^I := \Psi(dx^I),\ \xi_A := \Psi(r_A),\ \xi_I := \Psi(\frac{\partial}{\partial x^I})\}$ is a local basis of sections of E. Identifying the sections of E and E^*, the dual basis is $\{\xi_I,\ \xi^A,\ \xi^I\}$, with $\xi^A = \pm\xi_A$ (the sign equals $(\xi_A, \xi_A) = (r_A, r_A)_{\mathcal{G}}$). By lemma 2.15, the coefficient $C_{abc}(x)$ in the cubic Hamiltonian H equals 0 as long as any of ξ^a or ξ^b or ξ^c is in $\{\xi_I\}$. Thus H is of the form:

$$H = \sum_{I=1}^{m} p_I x^I + \sum_{a,b,c} \frac{1}{6} C_{abc}(x)\xi^a\xi^b\xi^c,$$

where the second sum is taken over all triples in $\{\xi^A\} \cup \{\xi^I\}$.

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